

On normal subgroups of the braided Thompson groups

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Abstract. We inspect the normal subgroup structure of the braided Thompson groups V_{br} and F_{br} . We prove that every proper normal subgroup of V_{br} lies in the kernel of the natural quotient $V_{\text{br}} \twoheadrightarrow V$, and we exhibit some families of interesting such normal subgroups. For F_{br} , we prove that for any normal subgroup N of F_{br} , either N is contained in the kernel of $F_{\text{br}} \twoheadrightarrow F$, or else N contains $[F_{\text{br}}, F_{\text{br}}]$. We also compute the Bieri–Neumann–Strebel invariant $\Sigma^1(F_{\text{br}})$, which is a useful tool for understanding normal subgroups containing the commutator subgroup.

Mathematics Subject Classification (2010). Primary 20F65; Secondary 20F36, 20E07.

Keywords. Thompson group, braid group, BNS-invariant, finiteness properties.

Introduction

Thompson’s groups F , T , and V have spent the past fifty years appearing in a variety of contexts and serving as examples of groups with unique and unexpected properties. Some examples of such properties are that T and V are finitely presented, infinite and simple, and F is torsion-free and contains free abelian subgroups of arbitrarily high rank, but is finitely presented. While F is not simple it is true that $[F, F]$ is simple, and any proper quotient of F is abelian. Stronger than being finitely presented, all three groups are also of type F_∞ , meaning they admit classifying spaces with compact n -skeleta, for all $n \in \mathbb{N}$.

The braided Thompson groups V_{br} and F_{br} appeared more recently, but have proved to have many interesting properties. First, V_{br} was introduced independently by Brin [8] and Dehornoy [11], and serves as an “Artinification” of V . In particular it is a torsion-free group with V as a quotient, which contains copies of every braid group B_n , and is finitely presented. A subgroup F_{br} of V_{br} was introduced by Brady, Burillo, Cleary, and Stein [1]. This group is finitely presented, contains copies of every pure braid group PB_n and has F as a quotient. Both V_{br} and F_{br} are also of type F_∞ [2]. The fact that these groups are so vast as to contain every braid group, while still having such nice finiteness properties, makes them of considerable interest.

In this paper we analyze the normal subgroups of V_{br} and F_{br} . There is a natural normal subgroup P_{br} of V_{br} , which is the kernel of the map $V_{\text{br}} \twoheadrightarrow V$, and is also the kernel of $F_{\text{br}} \twoheadrightarrow F$. We prove the following Alternative for F_{br} :

Theorem 2.1. *Let N be a normal subgroup of F_{br} . Then either $N \leq P_{\text{br}}$ or else $[F_{\text{br}}, F_{\text{br}}] \leq N$.*

This has a corollary for V_{br} :

Corollary 2.8. *Any proper normal subgroup of V_{br} is contained in P_{br} .*

Note that, since V is simple, any $N \triangleleft V_{\text{br}}$ not contained in P_{br} satisfies $NP_{\text{br}} = V_{\text{br}}$, so the corollary could also be phrased: “Any normal subgroup of V_{br} either contains or is contained in P_{br} .” This was conjectured by Kai-Uwe Bux after the preprint [10].

A consequence of these results is that V_{br} and $[F_{\text{br}}, F_{\text{br}}]$ are perfect, but not F_{br} , which is somewhat analogous to the classical fact that V and $[F, F]$ are simple, but not F . Also, we obtain some pleasant statements for the braided versions that are also true (for sometimes trivial reasons) for the classical versions, like: any quotient of F_{br} is either abelian or else contains F , and any non-trivial quotient of V_{br} surjects onto V .

We further analyze normal subgroups of F_{br} containing the commutator subgroup by computing the Bieri–Neumann–Strebel invariant $\Sigma^1(F_{\text{br}})$. This is a geometric invariant of a finitely generated group G that determines which normal subgroups containing $[G, G]$ are themselves finitely generated. In general the BNS-invariant is considered to be quite difficult to compute. We state the result here, and see Section 3.1 for the notation and background.

Theorem 3.4. *The Bieri–Neumann–Strebel invariant $\Sigma^1(F_{\text{br}})$ for F_{br} consists of all points on the sphere $\Sigma(F_{\text{br}}) = S^3$ except for the points $[\phi_0]$ and $[\phi_1]$.*

For example our calculation of $\Sigma^1(F_{\text{br}})$ shows that for $[F_{\text{br}}, F_{\text{br}}] \leq N \leq F_{\text{br}}$, N fails to be finitely generated if and only if it is contained in either $\ker(\phi_0)$ or $\ker(\phi_1)$, with notation explained in Section 1.4.

Lastly we inspect normal subgroups of V_{br} and F_{br} contained in P_{br} . We classify how they arise, namely any such normal subgroup is the limit of a uniquely determined *complete coherent* sequence of normal subgroups of the PB_n . Details are given in Section 4, along with some examples, and some questions. Perhaps the most tantalizing question, which we have so far been unable to answer, is whether V_{br} and/or F_{br} is Hopfian; V and F are Hopfian, but we show that P_{br} is not, so it is not entirely clear what to expect.

The paper is organized as follows. In Section 1 we recall the relevant background on the braided Thompson groups. In Section 2 we prove the Alternative for F_{br} , Theorem 2.1. Normal subgroups of F_{br} containing the commutator subgroup are further investigated in Section 3, where the BNS-invariant $\Sigma^1(F_{\text{br}})$ is computed. Normal subgroups contained in P_{br} are discussed in Section 4.

Acknowledgments. I am grateful to Robert Bieri and Matt Brin for many helpful conversations, and Marco Marschler and Stefan Witzel for their comments and suggestions. Thanks are also due to the anonymous referee for many helpful suggestions, which in particular improved Section 4.

1. The braided Thompson groups

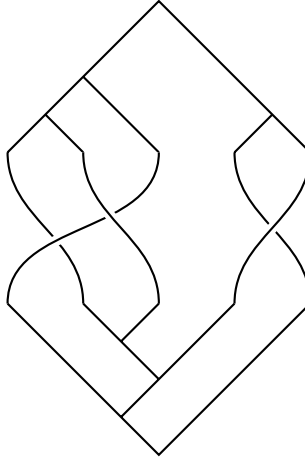
In this section we recall a model for elements of V_{br} and F_{br} , state some presentations, discuss the abelianizations of the groups (in fact V_{br} is perfect), and fix some notation for characters of F_{br} that will be used in Section 3.

1.1. Definitions and models. Elements of V_{br} are represented by *braided paired tree diagrams*, as in [1]. By a *tree* we will always mean a finite rooted binary tree. The *trivial tree* is just a single node. Vertices of a non-trivial tree have valency 3, except for the *leaves*, which have valency 1, and the *root*, which has valency 2. A non-leaf vertex u , together with the two edges and their vertices v, w connected to u and directed away from the root, form a *caret*. The vertices v and w are *children* of u . Our trees will always come equipped with a decision for each such u , as to which of v or w is the *left* or *right* child. This induces a numbering of the leaves of a tree, left to right, from 1 to n for some n .

A braided paired tree diagram is a triple (T_-, b, T_+) where T_{\pm} are trees, each with n leaves for some $n \in \mathbb{N}$, and b is an element of the braid group B_n . The model we will use for elements is split-braid-merge diagrams. We draw T_- (the splits) with the root on the top and the n leaves at the bottom, then the braid on n strands, and then T_+ (the merges) with the n leaves at the top and the root at the bottom. See Figure 1 for an example.

Two such triples are considered *equivalent* if they are connected via a finite sequence of *reductions and expansions*. An expansion of (T_-, b, T_+) amounts to adding a caret to some leaf of T_- , bifurcating the strand coming out of that leaf into two parallel strands, and then adding a caret to the leaf of T_+ at which that original strand ended. A reduction is the reverse of an expansion.

We will use expansions a lot in all that follows, so we make some relevant definitions here, following [16].

Figure 1. An element of V_{br} .

Definition 1.1 (cloning). Let $\kappa_k^n: B_n \rightarrow B_{n+1}$ be the injective function that takes a braid and bifurcates the k th strand into two parallel strands, where we number the strands at the bottom. We call κ_k^n the k th *cloning map*, and we say that the resulting strands are *clones*. Note that κ_k^n is not a group homomorphism, since the numbering of the strands may be different on the bottom and the top. When appropriate, we may also write κ_k for κ_k^n .

For a tree T with n leaves, let λ_k be a single-caret tree whose root is identified with the k th leaf of T . Denote by $T \cup \lambda_k$ the tree obtained by attaching this caret to that leaf. Let ρ_b be the image of b under the natural quotient $B_n \rightarrow S_n$. Now, for trees T_- and T_+ with n leaves and $b \in B_n$ we have the expansion

$$(T_-, b, T_+) = (T_- \cup \lambda_{\rho_b(k)}, \kappa_k^n(b), T_+ \cup \lambda_k).$$

We can iterate this. Let T be a tree with n leaves and let Φ be a forest with n roots. This is just a finite sequence of trees, and the *roots* of the forest are the roots of its trees. We can write Φ as an ordered union of carets, $\Phi = \lambda_{k_1} \cup \dots \cup \lambda_{k_r}$, and consider the tree $T \cup \Phi$ with $n + r$ leaves. For this to make sense we need $1 \leq k_1 \leq n$, then $1 \leq k_2 \leq n + 1$, and so forth up to $1 \leq k_r \leq n + (r - 1)$. Here by “ordered” union we just mean that the subscripts we need to use for the λ_{k_i} depend on the order in which the carets are attached, e.g., $\lambda_2 \cup \lambda_1 = \lambda_1 \cup \lambda_3$ are two strings of carets both representing the forest Φ consisting of two disjoint carets. Denote by κ_Φ^n the iterated cloning map

$$\kappa_\Phi := \kappa_{k_r} \circ \dots \circ \kappa_{k_1}.$$

This is well defined, that is, if Φ can be written as a different ordered union of carets, we still get the same cloning map.

In some ways it makes more sense to treat cloning maps as right maps, and write $(b)\kappa_k^n$ (as in [16]), since otherwise as seen above we have things like

$\kappa_{\lambda_{k_1}} \cup \lambda_{k_2} = \kappa_{\lambda_{k_2}} \circ \kappa_{\lambda_{k_1}}$, but this technical precision is outweighed by future notational awkwardness, so we will stick to writing $\kappa_k^n(b)$.

As an example of cloning, in Figure 2 we have an expansion of the form

$$(T_-, b, T_+) = (T_- \cup \lambda_1, \kappa_2^4(b), T_+ \cup \lambda_2).$$

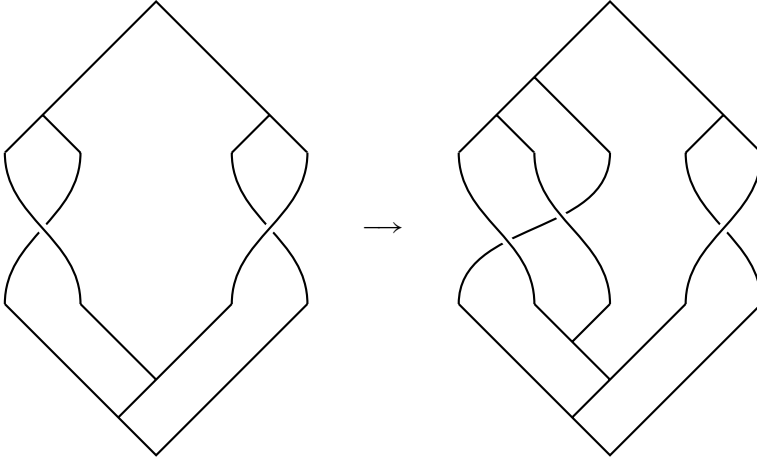


Figure 2. Expansion in V_{br} .

Recall that a braid b is *pure* if ρ_b is the trivial element of S_n . We will denote the subgroup of all pure braids by PB_n .

Observation 1.2. *If $b \in B_n$ then b is pure if and only if $\kappa_k^n(b)$ is pure for all k . That is, the property of a braid being pure is invariant under both expansion and reduction. Restricted to PB_n , the cloning maps $\kappa_k^n: PB_n \rightarrow PB_{n+1}$ are group homomorphisms, since the numbering of the strands is the same on the bottom and the top.*

The set of all equivalence classes of braided paired tree diagrams forms a group, V_{br} , with multiplication given by “stacking” the diagrams. By restricting to only considering pure braids, we obtain the subgroup F_{br} . Crucial to our model being useful is that one can always turn a product of split-braid-merge diagrams into a single split-braid merge diagram via finitely many reductions and expansions. There are some natural subgroups of V_{br} and F_{br} worth mentioning. First, V_{br} contains a copy of F (diagrams with no braiding), and “many” copies of every braid group B_n for $n \in \mathbb{N}$. In particular for any tree with n leaves, the set of triples (T, b, T) for $b \in B_n$ is isomorphic to B_n . Similarly, F_{br} contains F and many copies of every pure braid group PB_n for $n \in \mathbb{N}$, namely a copy of PB_n for every tree with n leaves.

1.2. The kernel P_{br} . The group V_{br} surjects onto V under the map that turns every braid b into the permutation ρ_b . The kernel of this map consists of elements represented by triples (T, p, T) where p is pure. Note that the two trees must *both* be T , if (T, p, T) is to become trivial under $V_{\text{br}} \twoheadrightarrow V$. We will denote this kernel by P_{br} , so we have a short exact sequence

$$1 \longrightarrow P_{\text{br}} \longrightarrow V_{\text{br}} \longrightarrow V \longrightarrow 1.$$

Of course $P_{\text{br}} \leq F_{\text{br}}$, and is the kernel of the natural quotient $F_{\text{br}} \twoheadrightarrow F$. The short exact sequence above restricts to

$$1 \longrightarrow P_{\text{br}} \longrightarrow F_{\text{br}} \longrightarrow F \longrightarrow 1,$$

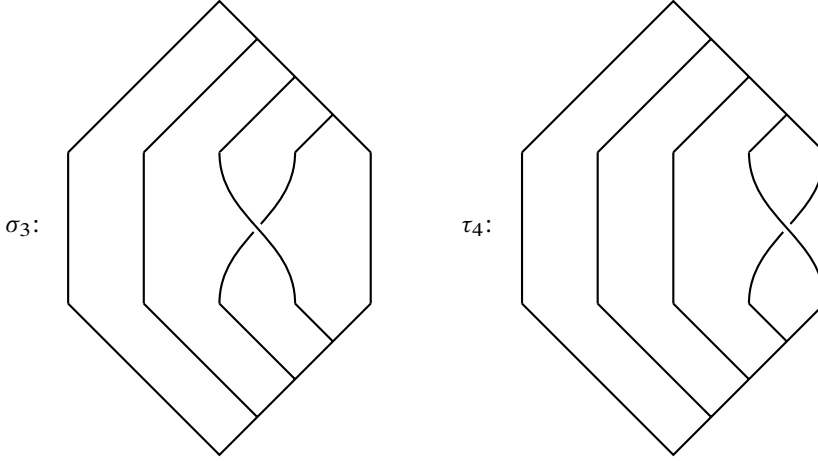
which splits, so $F_{\text{br}} = P_{\text{br}} \rtimes F$. The sequence for V_{br} does not split; for instance V has torsion but V_{br} is torsion-free.

The kernel P_{br} is a direct limit of copies of PB_n , arranged in a certain directed system. This is spelled out in detail in Section 1 of [4]. In short, for a tree T with n leaves, we have an isomorphic copy of PB_n , denoted PB_T , consisting of triples (T, p, T) for $p \in PB_n$. We write $T \leq T'$ if T' is obtained from T by an iterated process of adding carets to the leaves of T . This makes the set of PB_T into a directed system, with morphisms given by the inclusions induced by cloning maps. The limit of this system is exactly P_{br} . As a remark, the notation in [4] for P_{br} is PBV , and the inclusions induced by the cloning maps are denoted $\alpha_{n,T,i}$.

1.3. Presentations. Brady, Burillo, Cleary and Stein [1] give infinite and finite presentations for both V_{br} and F_{br} . For our purposes the infinite presentations are the more useful ones.

First we look at V_{br} . The generators are x_i ($0 \leq i$), σ_i and τ_i ($1 \leq i$). The relations are as follows:

$$\begin{aligned} \text{(A)} \quad x_j x_i &= x_i x_{j+1} && \text{for } 0 \leq i < j \\ \text{(b1)} \quad \sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for } 1 \leq i \leq j - 2 \\ \text{(b2)} \quad \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } 1 \leq i \\ \text{(b3)} \quad \sigma_i \tau_j &= \tau_j \sigma_i && \text{for } 1 \leq i \leq j - 2 \\ \text{(b4)} \quad \sigma_i \tau_{i+1} \sigma_i &= \tau_{i+1} \sigma_i \tau_{i+1} && \text{for } 1 \leq i \\ \\ \text{(c1)} \quad \sigma_i x_j &= x_j \sigma_i && \text{for } 1 \leq i < j \\ \text{(c2)} \quad \sigma_i x_i &= x_{i-1} \sigma_{i+1} \sigma_i && \text{for } 1 \leq i \\ \text{(c3)} \quad \sigma_i x_j &= x_j \sigma_{i+1} && \text{for } 1 \leq j \leq i - 2 \\ \text{(c4)} \quad \sigma_{i+1} x_i &= x_{i+1} \sigma_{i+1} \sigma_{i+2} && \text{for } 1 \leq i \\ \\ \text{(d1)} \quad \tau_i x_j &= x_j \tau_{i+1} && \text{for } 1 \leq j \leq i - 2 \\ \text{(d2)} \quad \tau_i x_{i-1} &= \sigma_i \tau_{i+1} && \text{for } 1 \leq i \\ \text{(d3)} \quad \tau_i &= x_{i-1} \tau_{i+1} \sigma_i && \text{for } 1 \leq i \end{aligned}$$


 Figure 3. Examples of generators of V_{br} .

The elements x_i are the standard generators of F . The σ_i are given by (R_{i+2}, a_i, R_{i+2}) , where R_{i+2} is the *all-right tree* with $i + 2$ leaves, and $a_i \in B_{i+2}$ is the braid that crosses strand i across strand $i + 1$. An all-right tree is one in which, for every caret but the first, that caret's root is the previous caret's right leaf. The τ_i are given by (R_{i+1}, b_i, R_{i+1}) , where $b_i \in B_{i+1}$ crosses strand i across strand $i + 1$. The important difference is that in σ_i the last strand is not used, and in τ_i it is. See Figure 3 for some examples.

Now we look at F_{br} . The generators are x_i ($0 \leq i$), $\alpha_{i,j}$ and $\beta_{i,j}$ ($1 \leq i < j$). The relations are as follows:

- (A) $x_j x_i = x_i x_{j+1}$ for $0 \leq i < j$
- (B1) $\alpha_{r,s}^{-1} \alpha_{i,j} \alpha_{r,s} = \alpha_{i,j}$ for $1 \leq r < s < i < j$
 or $1 \leq i < r < s < j$
- (B2) $\alpha_{r,s}^{-1} \alpha_{i,j} \alpha_{r,s} = \alpha_{r,j} \alpha_{i,j} \alpha_{r,j}^{-1}$ for $1 \leq r < s = i < j$
- (B3) $\alpha_{r,s}^{-1} \alpha_{i,j} \alpha_{r,s} = (\alpha_{i,j} \alpha_{s,j}) \alpha_{i,j} (\alpha_{i,j} \alpha_{s,j})^{-1}$ for $1 \leq r = i < s < j$
- (B4) $\alpha_{r,s}^{-1} \alpha_{i,j} \alpha_{r,s} = (\alpha_{r,j} \alpha_{s,j} \alpha_{r,j}^{-1} \alpha_{s,j}^{-1}) \alpha_{i,j} (\alpha_{r,j} \alpha_{s,j} \alpha_{r,j}^{-1} \alpha_{s,j}^{-1})^{-1}$ for $1 \leq r < i < s < j$
- (B5) $\alpha_{r,s}^{-1} \beta_{i,j} \alpha_{r,s} = \beta_{i,j}$ for $1 \leq r < s < i < j$
 or $1 \leq i < r < s < j$
- (B6) $\alpha_{r,s}^{-1} \beta_{i,j} \alpha_{r,s} = \beta_{r,j} \beta_{i,j} \beta_{r,j}^{-1}$ for $1 \leq r < s = i < j$
- (B7) $\alpha_{r,s}^{-1} \beta_{i,j} \alpha_{r,s} = (\beta_{i,j} \beta_{s,j}) \beta_{i,j} (\beta_{i,j} \beta_{s,j})^{-1}$ for $1 \leq r = i < s < j$
- (B8) $\alpha_{r,s}^{-1} \beta_{i,j} \alpha_{r,s} = (\beta_{r,j} \beta_{s,j} \beta_{r,j}^{-1} \beta_{s,j}^{-1}) \beta_{i,j} (\beta_{r,j} \beta_{s,j} \beta_{r,j}^{-1} \beta_{s,j}^{-1})^{-1}$ for $1 \leq r < i < s < j$
- (C) $\beta_{i,j} = \beta_{i,j+1} \alpha_{i,j}$ for $1 \leq i < j$

- | | |
|---|--------------------------|
| (D1) $\alpha_{i,j} x_{k-1} = x_{k-1} \alpha_{i+1,j+1}$ | for $1 \leq k < i < j$ |
| (D2) $\alpha_{i,j} x_{k-1} = x_{k-1} \alpha_{i+1,j+1} \alpha_{i,j+1}$ | for $1 \leq k = i < j$ |
| (D3) $\alpha_{i,j} x_{k-1} = x_{k-1} \alpha_{i,j+1}$ | for $1 \leq i < k < j$ |
| (D4) $\alpha_{i,j} x_{k-1} = x_{k-1} \alpha_{i,j+1} \alpha_{i,j}$ | for $1 \leq i < k = j$ |
| (D5) $\alpha_{i,j} x_{k-1} = x_{k-1} \alpha_{i,j}$ | for $1 \leq i < j < k$ |
| (D6) $\beta_{i,j} x_{k-1} = x_{k-1} \beta_{i+1,j+1}$ | for $1 \leq k < i < j$ |
| (D7) $\beta_{i,j} x_{k-1} = x_{k-1} \beta_{i+1,j+1} \beta_{i,j+1}$ | for $1 \leq k = i < j$ |
| (D8) $\beta_{i,j} x_{k-1} = x_{k-1} \beta_{i,j+1}$ | for $1 \leq i < k < j$ |
| (D9) $\beta_{i,j} x_{k-1} = x_{k-1} \beta_{i,j}$ | for $1 \leq i < j < k$. |

These generators can be written in terms of the generators for V_{br} as follows:

$$\alpha_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1}^2 \sigma_{j-2}^{-1} \cdots \sigma_i^{-1} \beta_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \tau_{j-1}^2 \sigma_{j-2}^{-1} \cdots \sigma_i^{-1}.$$

Pictorially, $\alpha_{i,j}$ is an all-right tree of splits, out to $j+1$ strands, then strand i braids around strand j and goes back to position i , and then there is an all-right tree of merges. For $\beta_{i,j}$ the only difference is that we go out to j strands, and so we use the last strand. See Figure 4 for some examples.

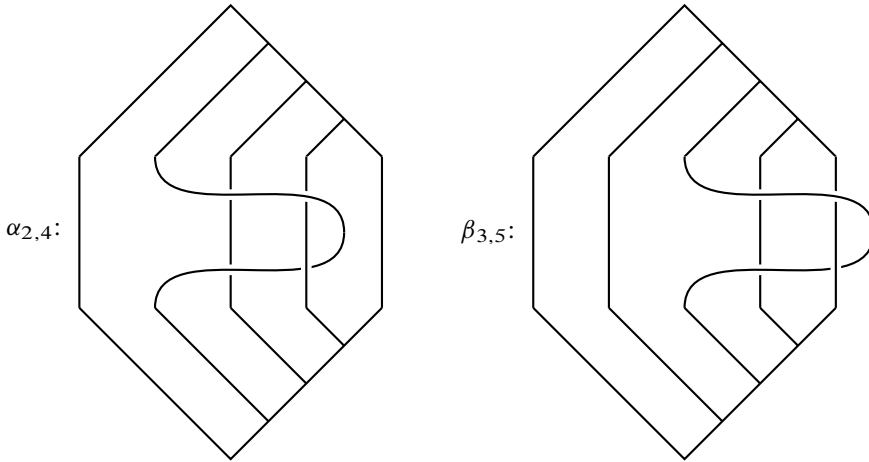


Figure 4. Examples of generators of F_{br} .

Observation 1.3. *The group P_{br} is generated by the conjugates of $\alpha_{i,j}$ and $\beta_{i,j}$ ($1 \leq i < j$) by elements of F .*

Proof. We need to generate every PB_T . If T has n leaves and R_n is the all-right tree with n leaves, then PB_T is conjugate to PB_{R_n} by the element (R_n, T) of F . But PB_{R_n} is generated by the $\alpha_{i,j}$ for $1 \leq i < j < n$ and $\beta_{i,j}$ for $1 \leq i < j \leq n$. \square

It will be convenient in Section 3 to use a fact about F_{br} that does not seem to have been recorded before, namely that it is an ascending HNN-extension of a certain subgroup. For $n \geq 0$ let $F(n)$ be the subgroup of F generated by all the x_i with $i \geq n$. In particular $F(0) = F$. It is well known that F is an ascending HNN-extension of $F(1)$ with stable element x_0 . Now define $F_{\text{br}}(n)$ to be the subgroup of F_{br} generated by all the $\alpha_{i,j}$ and $\beta_{i,j}$ for $1 \leq i < j$ and all the x_i for $i \geq n$.

Lemma 1.4. *We have that F_{br} is an ascending HNN-extension of $F_{\text{br}}(1)$ with stable element x_0 .*

Proof. An initial proof involved establishing a presentation for $F_{\text{br}}(1)$, and was more involved; the following faster proof is inspired by helpful discussions with Robert Bieri and Matt Brin.

Let $\theta: F_{\text{br}}(1) \hookrightarrow F_{\text{br}}(1)$ be the monomorphism given by right conjugation by x_0 . There is an epimorphism Ψ from the abstract HNN-extension $F_{\text{br}}(1)*_{\theta}$, with stable element t , to F_{br} given by specializing t to x_0 , so Ψ is the identity map on $F_{\text{br}}(1)$. We need to check that Ψ is injective. Let $g \in \ker(\Psi)$. Since g is an element of $F_{\text{br}}(1)*_{\theta}$, and since the HNN-extension is ascending, we can write g in the form $g = t^n h t^m$ for $h \in F_{\text{br}}(1)$, $n \geq 0$ and $m \leq 0$. Also, there is an epimorphism $F_{\text{br}}(1)*_{\theta} \twoheadrightarrow \mathbb{Z}$ that reads 0 on $F_{\text{br}}(1)$ and 1 on t , and factors through Ψ ; this tells us that $m = -n$. Hence h is itself in $\ker(\Psi)$. But $\Psi|_{F_{\text{br}}(1)}$ is the identity, so $h = 1$ and we are done. \square

1.4. Abelianization and characters. The first observation of this subsection is about V_{br} , and ensures that any future discussion about abelian quotients and characters will be uninteresting for V_{br} .

Observation 1.5. *The group V_{br} is perfect.*

This is an easy exercise in abelianizing the presentation for V_{br} from the previous subsection and checking that every generator becomes trivial.

One can also abelianize the presentation for F_{br} without too much difficulty. For reference we will describe the steps in the following lemma. Here, bars indicate the images of elements in the abelianization.

Lemma 1.6. *The abelianization of F_{br} is generated by \bar{x}_0 , \bar{x}_1 , $\bar{\beta}_{1,3}$ and $\bar{\alpha}_{1,2}$.*

Proof. We start with generators \bar{x}_i ($0 \leq i$), $\bar{\alpha}_{i,j}$ ($1 \leq i < j$) and $\bar{\beta}_{i,j}$ ($1 \leq i < j$). Relation (A) tells us that $\bar{x}_i = \bar{x}_1$ for all $i \geq 2$. Relations (D1), (D3), (D6), and (D8) tell us that each $\bar{\alpha}_{k,\ell}$ equals $\bar{\alpha}_{i,j}$ for some

$$(i, j) \in \{(1, 2), (1, 3), (2, 3), (2, 4)\},$$

with a similar statement for the $\bar{\beta}_{k,\ell}$. So far we have reduced down to ten generators. We will show that six of them are redundant, leaving the four in the statement of the lemma.

From (D4) we see that $\bar{\alpha}_{1,3} = 0$ and $\bar{\alpha}_{2,4} = 0$. Then (D2) says that $\bar{\alpha}_{2,3} = \bar{\alpha}_{1,2}$. From (C) and (D7) we see that $\bar{\beta}_{i+1,j+1} = \bar{\alpha}_{i,j}$ for all $1 \leq i < j$, which implies that $\bar{\beta}_{2,4} = 0$ and $\bar{\beta}_{2,3} = \bar{\alpha}_{1,2}$. Finally, (C) says that $\bar{\beta}_{1,2} = \bar{\beta}_{1,2} + \bar{\alpha}_{1,2}$. \square

In order to prove that these four generators are in fact linearly independent, and so F_{br} abelianizes to \mathbb{Z}^4 , we first describe four discrete characters of F_{br} , denoted ϕ_0, ϕ_1, ω_0 and ω_1 , which will be dual to certain combinations of these generators. Recall that a *character* of a group is a homomorphism from the group to the additive real numbers, and a character is *discrete* if its image is isomorphic to \mathbb{Z} .

First, note that F_{br} acts on $[0, 1]$ via the map $\pi: F_{\text{br}} \rightarrow F$. A standard basis for $\text{Hom}(F, \mathbb{R}) \cong \mathbb{R}^2$ is $\{\chi_0, \chi_1\}$, where $\chi_i(f) := \log_2(f'(i))$, see [3]. Hence we get two linearly independent characters for F_{br} by composing, namely:

$$\phi_0 := \chi_0 \circ \pi \quad \text{and} \quad \phi_1 := \chi_1 \circ \pi.$$

The values of ϕ_0 and ϕ_1 can be read off a representative triple (T_-, p, T_+) for an element of F_{br} . For a tree T , thought of as a metric graph with edge lengths all 1, let r be the root, ℓ_ℓ the leftmost leaf and ℓ_r the rightmost leaf. Let $L(T)$ be the length of the reduced edge path from r to ℓ_ℓ , and $R(T)$ the length of the reduced edge path from r to ℓ_r . Then

$$\phi_0(T_-, p, T_+) = L(T_+) - L(T_-),$$

$$\phi_1(T_-, p, T_+) = R(T_+) - R(T_-).$$

The characters ϕ_0 and ϕ_1 both have P_{br} in their kernels. To find the missing two dimensions in what we will eventually see is $\text{Hom}(F_{\text{br}}, \mathbb{R}) \cong \mathbb{R}^4$, we now look at characters that can detect braiding. First, let ω_0 be the character that takes an element (T_-, p, T_+) and reads off the total winding number of the first and last strands of p around each other. This is invariant under reduction and expansion, and so is well defined. Finally, let ω_1 be the character that reads off the sum of the total winding numbers of adjacent strands of p , i.e., 1 and around 2, plus 2 around 3, etc. This is again invariant under reduction and expansion, so is well defined.

As an example of these measurements, one can compute that $\phi_0(g) = 1$, $\phi_1(g) = 0$, $\omega_0(g) = 1$ and $\omega_1(g) = -1$ for the element g pictured in Figure 5.

Lemma 1.7. *The characters $(\phi_0, \phi_1, \omega_0, \omega_1)$ form a basis for $\text{Hom}(F_{\text{br}}, \mathbb{R}) \cong \mathbb{R}^4$, the elements $(\bar{x}_1 - \bar{x}_0, -\bar{x}_1, \bar{\beta}_{1,3}, \bar{\alpha}_{1,2})$ form a basis for the abelianization \mathbb{Z}^4 of F_{br} , and these bases are dual.*

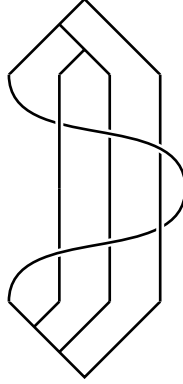


Figure 5. For the pictured element g , we have $\phi_0(g) = 1$, $\phi_1(g) = 0$, $\omega_0(g) = 1$ and $\omega_1(g) = -1$.

Proof. Set $f_1 := \phi_0$, $f_2 := \phi_1$, $f_3 := \omega_0$ and $f_4 := \omega_1$. Also set $e_1 := \bar{x}_1 - \bar{x}_0$, $e_2 := -\bar{x}_1$, $e_3 := \bar{\beta}_{1,3}$ and $e_4 := \bar{\alpha}_{1,2}$. Then for $1 \leq i, j \leq 4$ one can check that $f_i(e_j) = \delta_{ij}$, the Kronecker delta. Here we have extended the definitions of the characters to accepting inputs from the abelianization, which is fine since they vanish on the commutator subgroup. This proves that the e_i are linearly independent, so form a basis. From this it follows that $\text{Hom}(F_{\text{br}}, \mathbb{R}) \cong \mathbb{R}^4$, and since the f_i are linearly independent they form a basis. \square

As a remark, most winding number measurements are *not* invariant under reduction and expansion, for instance the total winding number of the first and second strands; on $\beta_{1,2}$ this measurement reads 1, but when we bifurcate the first strand, we get a braid in which the first and second strands do not wind (indeed are parallel), so the measurement reads 0. Of course now the second and third strands wind, and ω_1 still reads 1, as it is a sum over all pairs of adjacent strands. The important point is that the designations “second,” “third,” etc. are not well behaved under cloning, but “next,” “first” and “last” are.

We should point out that we now have an easy algorithm to check whether an element of F_{br} is in $[F_{\text{br}}, F_{\text{br}}]$, namely if and only if it lies in the kernels of ϕ_0 , ϕ_1 , ω_0 and ω_1 .

2. An alternative result

This section is almost entirely about F_{br} , with implications for V_{br} relegated to the end. The upshot for V_{br} is Corollary 2.8, which says that every proper normal subgroup is contained in P_{br} .

The main result of this section is the following:

Theorem 2.1 (F_{br} alternative). *Let N be a normal subgroup of F_{br} . Then either $N \leq P_{\text{br}}$ or else $[F_{\text{br}}, F_{\text{br}}] \leq N$.*

By an ‘‘Alternative result’’ we mean a statement that any subgroup (or here any normal subgroup) must have one of two ‘‘quite different’’ forms. For example, the classical Tits Alternative for a group says that any subgroup either contains a non-abelian free group, or else is virtually solvable (note that Thompson’s groups do not satisfy the Tits Alternative).

A quick corollary to Theorem 2.1 is the following:

Corollary 2.2. *The commutator subgroup $[F_{\text{br}}, F_{\text{br}}]$ is perfect.*

Proof. We just need to show that $[[F_{\text{br}}, F_{\text{br}}], [F_{\text{br}}, F_{\text{br}}]]$ is not contained in P_{br} . But $[[F_{\text{br}}, F_{\text{br}}], [F_{\text{br}}, F_{\text{br}}]]$ contains $[[F, F], [F, F]]$, so this is clear. \square

Note that $[F_{\text{br}}, F_{\text{br}}]$ is not simple, as it has $[P_{\text{br}}, P_{\text{br}}]$ as a proper non-trivial normal subgroup.

There are various Alternatives known for F of the form, ‘‘every subgroup of F either has property \mathcal{P} , or else contains a copy of G ,’’ for some property \mathcal{P} and some subgroup G . Examples of this phenomenon include

- (1) for \mathcal{P} the property of being abelian, G is \mathbb{Z}^∞ [9];
- (2) for \mathcal{P} the property of being solvable, G is Bleak’s group W [5];
- (3) the Brin–Sapir Conjecture is that this phenomenon occurs for \mathcal{P} being the property of being elementary amenable and G being F [7, Conjecture 3].

In general, understanding the subgroups of F is an active and ongoing endeavor.

Returning to the task at hand, to prove Theorem 2.1, we begin with a technical proposition about the normal closure of elements of F in F_{br} .

Proposition 2.3. *Let $1 \neq f \in F \leq F_{\text{br}}$. The normal closure $N := \langle\langle f \rangle\rangle$ of f in F_{br} contains $[F_{\text{br}}, F_{\text{br}}]$.*

The first part of our proof is inspired by the proof of Lemma 20 in [10], which says that the normal closure of $[F, F]$ in V_{br} is all of V_{br} .

Proof. First note that since F has trivial center, without loss of generality $f \in [F, F]$, and since $[F, F]$ is simple, $[F, F] \leq N$. Elements of the form $x_i x_j^{-1}$ and $x_i^{-1} x_j$ for $i, j \geq 1$ are in $[F, F]$, and hence in N . This tells us that for any $i \geq 1$, the element $\alpha_{i,i+1} x_i x_{i+2}^{-1} \alpha_{i,i+1}^{-1}$ is in N . Applying (D4) and (D5), we get

$$N \ni x_i \alpha_{i,i+2} \alpha_{i,i+1} x_{i+2}^{-1} \alpha_{i,i+1}^{-1} = x_i \alpha_{i,i+2} x_{i+2}^{-1}$$

and so $\alpha_{i,i+2} x_{i+2}^{-1} x_i \in N$, whence $\alpha_{i,i+2} \in N$. This holds for all $i \geq 1$, and thanks to (D3), we also get that $\alpha_{i,j} \in N$ for all $1 \leq i < j - 1$.

The next goal is to force enough $\beta_{i,j}$ to be in N . Running a similar trick as above, we start with $\beta_{i,i+2}x_{i-1}x_{i+2}^{-1}\beta_{i,i+2}^{-1}$ being in N , use (D7) and (D9), and get that N contains $\beta_{i+1,i+3}\beta_{i,i+3}\beta_{i,i+2}^{-1}$ for all $i \geq 2$. Using (C) we get that N contains $\beta_{i+1,i+3}\beta_{i,i+2}\alpha_{i,i+2}^{-1}\beta_{i,i+2}^{-1}$, and then since $\alpha_{i,i+2}$ is in N , so is $\beta_{i+1,i+3}$. Using (D6) and (D8) then, we see that N contains every $\beta_{i,j}$ for $2 \leq i < j - 1$.

It now suffices to prove that upon modding out $[F, F]$ and all the $\alpha_{i,j}$ for $1 \leq i < j - 1$ and $\beta_{i,j}$ for $2 \leq i < j - 1$, the presentation becomes abelian. Denote elements of this quotient by putting hats on the elements (we have reserved bars for the abelianization). That all the \hat{x}_i commute follows since we have modded out $[F, F]$. Note that $\hat{\alpha}_{r,r+2} = \hat{1}$ for all $r \geq 1$, so by (B1) and (B2) we see that all the $\hat{\alpha}_{i,i+1}$ commute. Also, since $\hat{\beta}_{r,r+2} = \hat{1}$ for all $r \geq 2$, by (C) we have $\hat{\beta}_{i,i+1} = \hat{\alpha}_{i,i+1}$ for $i \geq 2$. Next we claim that every \hat{x}_{k-1} commutes with every $\hat{\alpha}_{i,i+1}$ (for $k, i \geq 1$). If $k \geq i + 1$ this follows from (D4) or (D5). If $k \leq i$ then (D1) and (D2) say

$$\hat{\alpha}_{i,i+1}\hat{x}_{k-1} = \hat{x}_{k-1}\hat{\alpha}_{i+1,i+2}.$$

Multiplying on the right by \hat{x}_{i+2}^{-1} , and using (D5) and the fact that $\hat{x}_{k-1}\hat{x}_{i+2}^{-1} = \hat{1}$, this becomes

$$\hat{\alpha}_{i,i+1} = \hat{\alpha}_{i+1,i+2},$$

so the claim is proved.

The last thing to show is that $\hat{\beta}_{1,j}$ commutes with all the other generators. If $j > 2$, then (D8), (D9), and conjugation by $\hat{x}_1\hat{x}_{j+1}^{-1} = \hat{1}$ tell us that $\hat{\beta}_{1,j} = \hat{\beta}_{1,j+1}$. So we only need to look at $\hat{\beta}_{1,2}$ and $\hat{\beta}_{1,3}$. First note that by (C), $\hat{\beta}_{1,2} = \hat{\beta}_{1,3}\hat{\alpha}_{1,2}$, so if $\hat{\beta}_{1,3}$ commutes with everything (including $\hat{\alpha}_{1,2}$), then so will $\hat{\beta}_{1,2}$. Now, $\hat{\beta}_{1,3}$ commutes with every \hat{x}_{k-1} , by (D7)–(D9), and using that identification of $\hat{\beta}_{1,3}$ with every $\hat{\beta}_{1,j}$ for $j \geq 3$. We also need an *ad hoc* argument that $\hat{\beta}_{1,3}$ commutes with \hat{x}_2 , which is easily checked (and holds even without the hats). Using (B5) we get that $\hat{\beta}_{1,3}$ commutes with every $\hat{\alpha}_{i,i+1} = \hat{\beta}_{i,i+1}$ for $i \geq 2$. Lastly, this fact plus (B7) tells us that $\hat{\beta}_{1,3}$ commutes with $\hat{\alpha}_{1,2}$. \square

Thanks to this proposition, we see that “catching” a non-trivial element of F is a way to blow up a normal subgroup to contain $[F_{\text{br}}, F_{\text{br}}]$. To prove the Alternative then, the goal is to start with an element of $F_{\text{br}} \setminus P_{\text{br}}$ and “catch” a non-trivial element of F in its normal closure. First we need some technical lemmas.

Let $g = (T, p, T) \in P_{\text{br}}$. The tree T defines a partition of $[0, 1]$ into dyadic subintervals; let $X(T)$ be the set of endpoints of said subintervals. Each subinterval corresponds to a leaf of T , and hence to a strand of p . If the endpoint $x \in X(T)$ lies in $(0, 1)$ and is such that the two subintervals on either side of x correspond to strands of p that are clones, then call x *inessential*. Otherwise call x *essential*. Let $X_{\text{ess}}(T, p)$ be the set of essential endpoints of the subintervals determined by T . The next observation justifies denoting this set by $X_{\text{ess}}(g)$.

Observation 2.4. *The set $X_{\text{ess}}(g)$ defined above is an invariant of g .*

Proof. We need to show that $X_{\text{ess}}(T, p)$ is invariant under reduction and expansion. Let $T' = T \cup \lambda_k$. We have $g = (T', \kappa_k(p), T')$. On the level of subintervals, all we have done is cut the k th subinterval in half, and so $|X(T', \kappa_k(p))| = |X(T, p)| + 1$. Let x' be the element of $X(T', \kappa_k(p)) \setminus X(T, p)$, i.e., the midpoint of the k th subinterval. Since the strands in $\kappa_k(p)$ on either side of x' are clones, we know that x' is inessential. This shows that $X_{\text{ess}}(T, p) = X_{\text{ess}}(T', \kappa_k(p))$, which tells us that the set is invariant under reduction and expansion. \square

Note that $X_{\text{ess}}(g) = \{0, 1\}$ if and only if $g = 1$ in P_{br} .

Proposition 2.5 (commuting condition). *Let $g \in P_{\text{br}}$ and $f \in F$. If f fixes $X_{\text{ess}}(g)$ then $[g, f] = 1$.*

Proof. Choose a tree T , say with n leaves, such that $g = (T, p, T)$ for some $p \in PB_n$. The tree gives us a subdivision of $[0, 1]$, say with endpoints $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$. For $1 \leq i \leq n$ let $I_i := [x_{i-1}, x_i]$. Let $0 = x_{i_0} < x_{i_1} < \dots < x_{i_{r-1}} < x_{i_r} = 1$ be precisely the essential endpoints, so $|X_{\text{ess}}(g)| = r - 1$. For $1 \leq s \leq r$ define

$$J_s := \bigcup_{i_{s-1} \leq i \leq i_s} I_i,$$

so the J_s are the closures of the connected components of $[0, 1] \setminus X_{\text{ess}}(g)$. The J_s partition the set of intervals I_i , and hence partition the leaves of T . For a given J_s , the strands of p indexed by the subintervals contained in J_s are all clones of each other.

Now, the fact that f fixes $X_{\text{ess}}(g)$ means that it can be represented by a tree pair of the form $(T \cup \Phi, T \cup \Phi')$, where Φ is a forest whose roots are identified with the leaves of T , as is Φ' , such that a certain important property holds. To state the property we need some setup. Write Φ as $\Phi_1 \cup \dots \cup \Phi_r$, where Φ_s is the subforest whose roots are precisely those roots of Φ identified with the leaves of T lying in J_s . One might call Φ_s the subforest of Φ with “support” in J_s . Similarly define Φ'_s for $1 \leq s \leq r$. Now, the important property of Φ and Φ' , which we get since f fixes $X_{\text{ess}}(g)$, is that for each $1 \leq s \leq r$, the leaves of Φ_s are in bijection with the leaves of Φ'_s . This bijection preserves the order on the leaves, and is induced by f . The “paired forest diagram” (Φ_s, Φ'_s) describes how f acts on the interval J_s .

A consequence of all the above is that $\kappa_\Phi(p) = \kappa_{\Phi'}(p)$ (see Definition 1.1). Indeed, for any J_s the strands of p indexed in J_s are clones of each other, and applying κ_Φ further clones this block of strands into a number of strands equal to the number of leaves of Φ_s . This is true of $\kappa_{\Phi'}$ as well, since Φ_s and Φ'_s have

the same number of leaves. Then since this holds for all s , we conclude that $\kappa_\Phi(p) = \kappa_{\Phi'}(p)$.

The following calculation finishes the proof:

$$\begin{aligned}
 fgf^{-1} &= (T \cup \Phi, T \cup \Phi')(T, p, T)(T \cup \Phi', T \cup \Phi) \\
 &= (T \cup \Phi, T \cup \Phi')(T \cup \Phi', \kappa_{\Phi'}(p), T \cup \Phi')(T \cup \Phi', T \cup \Phi) \\
 &= (T \cup \Phi, \kappa_{\Phi'}(p), T \cup \Phi) \\
 &= (T \cup \Phi, \kappa_\Phi(p), T \cup \Phi) \\
 &= (T, p, T) = g.
 \end{aligned}$$

□

Figure 6 gives an indication of what is really happening in Proposition 2.5.

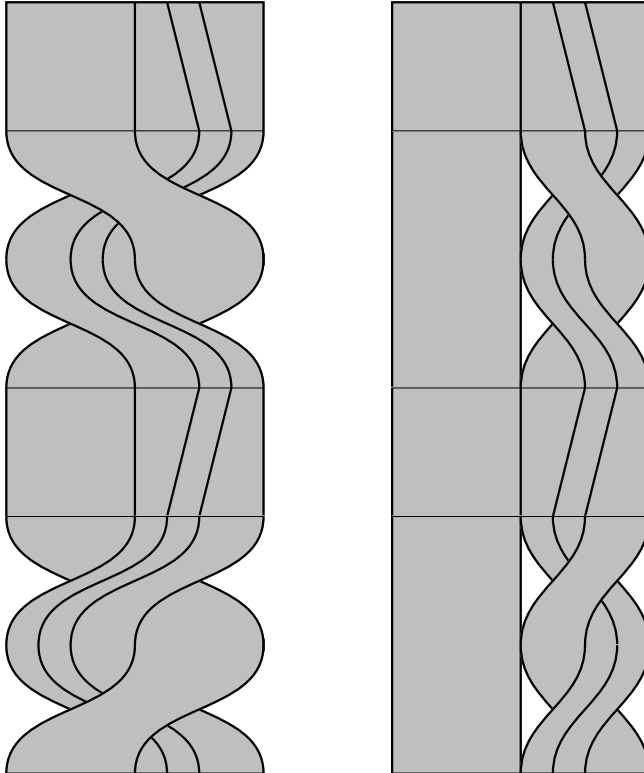


Figure 6. The picture on the left is the commutator $[x_1, \beta_{1,2}]$. Since $X_{\text{ess}}(\beta_{1,2}) = \{0, \frac{1}{2}, 1\}$, and x_1 fixes this set, this commutator is trivial, and one can check that the resulting “ribbon diagram” indeed represents the trivial element. The picture on the right demonstrates that $[x_1, \beta_{2,3}]$ is not trivial, as the ribbon diagram does not represent the trivial element, and indeed $X_{\text{ess}}(\beta_{2,3}) = \{0, \frac{1}{2}, \frac{3}{4}, 1\}$, which is not fixed by x_1 .

Lemma 2.6. *Let $g \in P_{\text{br}}$ and $1 \neq f \in F$. Then there exists $h \in F$ such that $[h, g] = 1$ but $[h, f] \neq 1$.*

Proof. Order the elements of $X_{\text{ess}}(g)$ by $0 = x_0 < x_1 < \cdots < x_{r-1} < x_r = 1$. By Proposition 2.5, it suffices to find $h \in F$ that fixes $X_{\text{ess}}(g)$ but does not commute with f .

First suppose f fixes $X_{\text{ess}}(g)$. Let $1 \leq i \leq r$. Suppose that every h with support in $[x_{i-1}, x_i]$ commutes with f . Then $f|_{[x_{i-1}, x_i]}$ must be trivial. Since $f \neq 1$, this cannot happen for every i , so we conclude that there exists h with support in $[x_{i-1}, x_i]$ for some i such that $[h, f] \neq 1$, and since the support of h is disjoint from $X_{\text{ess}}(g)$, also $[h, g] = 1$.

Now suppose f does not fix $X_{\text{ess}}(g)$. There exists an element $h \in F$ whose fixed point set is precisely $X_{\text{ess}}(g)$, so in particular $[h, g] = 1$. If f were to commute with h then it would necessarily stabilize its fixed point set, so instead we conclude that $[h, f] \neq 1$. \square

Proof of Theorem 2.1. Let $t \in F_{\text{br}} \setminus P_{\text{br}}$. Write $t = gf$ for $g \in P_{\text{br}}$ and $1 \neq f \in F$. By Lemma 2.6 we can choose $h \in F$ such that h commutes with g but not f . In particular, the normal closure of t contains $(f^{-1}g^{-1})(ghfh^{-1}) = f^{-1}hfh^{-1}$. This is a non-trivial element of F (even of $[F, F]$), so by Proposition 2.3, the normal closure of t contains $[F_{\text{br}}, F_{\text{br}}]$. \square

Remark 2.7. Since P_{br} contains every pure braid group, there is no hope of classifying all subgroups of F_{br} in any real sense. At least we do know that every finitely generated subgroup of P_{br} must lie in some PB_n , since P_{br} is a direct limit of copies of the PB_n . Another interesting fact is that, since P_{br} contains every pure braid group, it also contains every right-angled Artin group, by a result of Kim and Koberda [12]. In particular, F_{br} and V_{br} are examples of finitely presented (even type F_∞) groups that contain every right-angled Artin group.

Now that we have an Alternative for F_{br} , we can derive one for V_{br} . The two options for a normal subgroup turn out to be that it is either contained in P_{br} , or else equals all of V_{br} . To prove this we will quote a result from [10] that says that the normal closure of $[F, F]$ in V_{br} is all of V_{br} . The proof is similar to the first part of the proof of our Proposition 2.3.

Corollary 2.8 (alternative for V_{br}). *Let N be a proper normal subgroup of V_{br} . Then N is contained in P_{br} .*

Proof. Suppose N is not contained in P_{br} , so by Theorem 2.1 N contains $[F_{\text{br}}, F_{\text{br}}]$. In particular N contains $[F, F]$, and so by [10, Lemma 20], $N = V_{\text{br}}$. \square

The fact that V is simple tells us that either $N \leq P_{\text{br}}$ or $NP_{\text{br}} = V_{\text{br}}$, so the content of the corollary is that in the latter case in fact $N = V_{\text{br}}$.

Corollary 2.9. P_{br} is characteristic in V_{br} and F_{br} .

Proof. The statement for V_{br} is immediate from Corollary 2.8. The statement for F_{br} follows from Theorem 2.1 once we observe that $[F_{\text{br}}, F_{\text{br}}]$ is characteristic. \square

3. Normal subgroups over $[F_{\text{br}}, F_{\text{br}}]$

Theorem 2.1 tells us that normal subgroups of F_{br} either contain $[F_{\text{br}}, F_{\text{br}}]$, or else live in P_{br} . The latter situation is rather complicated, and we will discuss some examples in Section 4, along with some general results. Normal subgroups over $[F_{\text{br}}, F_{\text{br}}]$ are more tractable though. In this section we compute the Bieri–Neumann–Strebel invariant $\Sigma^1(F_{\text{br}})$, which sheds some light on such subgroups, for instance by characterizing which of them are finitely generated.

3.1. The BNS invariant. The Bieri–Neumann–Strebel (BNS) invariant of a finitely generated group G , introduced in [6], is a geometric invariant $\Sigma^1(G)$ that, among other things, provides a means of understanding normal subgroups of G containing the commutator subgroup $[G, G]$. For instance $\Sigma^1(G)$ tells us when such a normal subgroup is finitely generated or not.

Historically, $\Sigma^1(G)$ has proved to be difficult to compute in general. Some groups for which Σ^1 is interesting and has been successfully computed include right-angled Artin groups [14], pure braid groups [13], pure loop braid groups [15] and Thompson’s group F , see [6, 3].

The BNS-invariant $\Sigma^1(G)$ of a finitely generated group G is defined as follows. Consider characters $\chi: G \rightarrow \mathbb{R}$ of G . Two characters χ and χ' are *equivalent* if there exists $c \in \mathbb{R}^{>0}$ such that $\chi(g) = c\chi'(g)$ for all $g \in G$. The equivalence classes of non-trivial characters form the *character sphere* $\Sigma(G)$ of G . It is a d -sphere if the torsion-free rank of $G/[G, G]$ is $d + 1$. Now pick a finite generating set S for G and let $\Gamma(G, S)$ be the Cayley graph. For a character $\chi: G \rightarrow \mathbb{R}$ let $\Gamma(G, S)^{\chi \geq 0}$ be the full subgraph of $\Gamma(G, S)$ spanned by those vertices $g \in G$ with $\chi(g) \geq 0$. The BNS-invariant $\Sigma^1(G)$ is a subset of $\Sigma(G)$, which does not depend on S , defined by:

$$\Sigma^1(G) := \{[\chi] \in \Sigma(G) \mid \Gamma(G, S)^{\chi \geq 0} \text{ is connected}\}.$$

The following is one of the main applications of $\Sigma^1(G)$.

Citation 3.1 ([6, Theorem B1]). *Let G be a finitely generated group and let $N \triangleleft G$ with G/N abelian. Then N is finitely generated if and only if for every $[\chi] \in \Sigma(G)$ such that $\chi(N) = 0$, we have $[\chi] \in \Sigma^1(G)$.*

Another important fact is that $\Sigma^1(G)$ is invariant under automorphisms of G . As is standard, we denote the complement $\Sigma(G) \setminus \Sigma^1(G)$ by $\Sigma^1(G)^c$.

3.2. Tools. In this subsection we establish some terminology and notation, and cite some useful results that we will use to calculate $\Sigma^1(F_{\text{br}})$ in the following section. First we collect some definitions. Let G be a group, and let $I, J \subseteq G$. We say that J *dominates* I if every element of I commutes with some element of J . The *commuting graph* $C(J)$ of J is the graph with vertex set J and an (unoriented) edge between a and b if a and b commute. We say that $g \in G$ *survives* under a character χ if $\chi(g) \neq 0$. Otherwise we say it *dies*, or that χ *kills* it. If g survives under χ we will also sometimes call it *χ -hyperbolic*.

The point of all this terminology is a useful criterion to determine if a character is in $\Sigma^1(G)$:

Citation 3.2 (survivors dominating generators, [13, Lemma 1.9]). *Let G be a group and χ a character of G . Suppose there are sets $I, J \subseteq G$ such that I generates G , every element of J survives under χ , J dominates I , and $C(J)$ is connected. Then $[\chi] \in \Sigma^1(G)$.*

We will also make use of the following standard result, cf. [13, Lemma 1.3]:

Citation 3.3 (quotients). *Let $\pi: G \twoheadrightarrow H$ be an epimorphism of groups. Let χ be a character of H and let $\phi := \chi \circ \pi$ be the corresponding character of G . If $[\phi] \in \Sigma^1(G)$ then $[\chi] \in \Sigma^1(H)$.*

3.3. The BNS invariant $\Sigma^1(F_{\text{br}})$. The answer is:

Theorem 3.4. *The Bieri–Neumann–Strebel invariant $\Sigma^1(F_{\text{br}})$ for F_{br} consists of all points on the sphere $\Sigma(F_{\text{br}}) = S^3$ except for the points $[\phi_0]$ and $[\phi_1]$, where ϕ_0 and ϕ_1 are as defined in Section 1.4.*

We will prove the theorem by looking at various cases. First we take care of the points $[\pm\phi_i]$. Here we will appeal to symmetry under an automorphism $\rho: F_{\text{br}} \rightarrow F_{\text{br}}$ that switches the roles of ϕ_0 and ϕ_1 . The automorphism ρ takes (T_-, p, T_+) , viewed as a split-braid-merge diagram from top to bottom living in 3-space, and rotates it 180 degrees about an axis passing through the roots of both trees.

Observation 3.5. *We have $[\phi_i] \in \Sigma^1(F_{\text{br}})^c$ and $[-\phi_i] \in \Sigma^1(F_{\text{br}})$, for $i = 0, 1$.*

Proof. First note that we need only check the statements for $\pm\phi_0$, since the automorphism ρ switches the roles of ϕ_0 and ϕ_1 .

We know that $[\phi_0] \in \Sigma^1(F_{\text{br}})^c$ by Citation 3.3, since we have an epimorphism $F_{\text{br}} \twoheadrightarrow F$ and the induced character $[\chi_0]$ on F is in $\Sigma^1(F)^c$, see [3].

For the other statement, recall from Lemma 1.4 that F_{br} is an ascending HNN-extension of $F_{\text{br}}(1)$ by x_0 . We have $-\phi_0(F_{\text{br}}(1)) = 0$ and $-\phi_0(x_0) = 1$. Moreover, $F_{\text{br}}(1)$ is finitely generated, by arguments similar to those in [1] that show F_{br} is finitely generated. Hence by [3, Theorem 2.1(1)], we have $[-\phi_0] \in \Sigma^1(F_{\text{br}})$. \square

Now that we have handled $[\pm\phi_i]$, the strategy for the remaining characters is as follows. First we look at characters of F_{br} that do not kill P_{br} . We first suppose that χ has non-zero ω_1 component. Next we suppose that χ does have zero ω_1 component and non-zero ω_0 component (see Section 1.4 for definitions). Then we consider the case when χ does kill P_{br} , but has non-zero ϕ_0 and ϕ_1 components. In each of these three cases, we are able to apply Citation 3.2 to conclude that $[\chi] \in \Sigma^1(F_{\text{br}})$.

Case 1. First we assume that the ω_1 component of $[\chi]$ is non-zero. We will use Citation 3.2 to show that $[\chi]$ is guaranteed to be in $\Sigma^1(F_{\text{br}})$. The main trick is that every $\alpha_{i,i+1}$ is χ -hyperbolic.

Lemma 3.6 (non-zero ω_1). *Let χ be any character with non-zero ω_1 component. Then $[\chi] \in \Sigma^1(F_{\text{br}})$.*

Proof. Let

$$J_1 := \{\alpha_{i,i+1} \mid i \geq 1\}$$

and

$$I_1 := \{\alpha_{i,j} \mid 1 \leq i < j\} \cup \{\beta_{i,j} \mid 1 \leq i < j - 2\} \cup \{x_2, x_2x_0^{-1}\}.$$

We claim that J_1 dominates I_1 , $C(J_1)$ is connected, I_1 generates F_{br} , and every element of J_1 survives under χ . First note that any $\alpha_{i,j}$ commutes with $\alpha_{j+1,j+2}$ (B1), that any $\beta_{i,j}$ with $i < j - 2$ commutes with $\alpha_{i+1,i+2}$ (B5), that x_2 commutes with $\alpha_{1,2}$ (D5), and that $x_2x_0^{-1}$ commutes with $\alpha_{4,5}$ (D1). This tells us that J_1 dominates I_1 . Now observe that every element of J_1 commutes with $\alpha_{1,2}$ except for $\alpha_{2,3}$ (B1), but $\alpha_{2,3}$ commutes with $\alpha_{4,5}$, so $C(J_1)$ is connected. That I_1 generates F_{br} is routine to check in light of (C), since this ensures that $\beta_{i,j}$ with $i < j \leq i + 2$ can be obtained using I_1 , and since x_2 and $x_2x_0^{-1}$ generate F . Finally, every element of J_1 survives under ω_1 and dies under ϕ_0 , ϕ_1 and ω_0 , so necessarily survives under χ . \square

Case 2. Next suppose that the ω_0 component is non-zero. We will again use Citation 3.2, but with a different dominating set J_0 and generating set I_0 .

First we need to discuss central elements of PB_n . For any n , the center $Z(PB_n)$ is cyclic, generated by an element Δ_n that can be visualized as spinning the n strands around in lockstep by 360 degrees. For any tree T with n leaves, the element

$$\delta(T) := (T, \Delta_n, T)$$

commutes with every element of the form (T, p, T) for $p \in PB_n$. However, it is important that we use the same tree T ; there is no guarantee that $\delta(T)$ will commute with an element of the form (T', p, T') if T' is not T .

Another important observation about the elements $\delta(T)$ is that they survive under ω_0 ; indeed, ω_0 of such an element is 1. As a remark, ω_1 of such an element equals $n - 1$, but this will not matter in what follows.

Lemma 3.7. *Any element of $[F, F] \leq F_{\text{br}}$ commutes with some conjugate of an element of the form $\beta_{1,j}$.*

Proof. Conjugates of $\beta_{1,j}$ by elements of F amount to subdividing $[0, 1]$ into j subintervals and then braiding the first and last ones around each other. In particular, the essential endpoints of the subdivision are 0, the first endpoint after 0, the last endpoint before 1, and 1; see the definition before Observation 2.4. Given an element $f \in [F, F]$, so the support of f in $[0, 1]$ is bounded away from 0 and 1, we can choose a subdivision in which these four essential points are disjoint from the support of f . Then f will commute with the conjugate of the $\beta_{1,j}$ corresponding to this subdivision by Proposition 2.5. \square

Proposition 3.8 (non-zero ω_0). *Let χ be any character with non-zero ω_0 component. Then $[\chi] \in \Sigma^1(F_{\text{br}})$.*

Proof. Thanks to Lemma 3.6 we may assume that χ has ω_1 component zero. In particular, every $\delta(T)$ is χ -hyperbolic.

Let J_0 be the set of all conjugates of $\beta_{1,j}$ for $j \geq 2$ and all $\delta(T)$ for all trees T and all $n \geq 1$. Then every element of J_0 survives under ω_0 and dies under ϕ_0 and ϕ_1 , and since χ has ω_1 component zero this tells us that every element of J_0 survives under χ . We next claim that $C(J_0)$ is connected, and in fact that it is connected with diameter 2. Indeed, given any two elements x, y of J_0 there exists a tree T with n leaves such that $x = (T, p, T)$ and $y = (T, q, T)$ for $p, q \in PB_n$. Then x and y both commute with $\delta(T)$.

Now we need to find a generating set I_0 for F_{br} that is dominated by J_0 . Since every element of P_{br} commutes with some $\delta(T)$, we may as well include all of P_{br} in I_0 . We just need to add elements to I_0 that are dominated by J_0 until we have generated $F \leq F_{\text{br}}$. By Lemma 3.7, we can add all of $[F, F]$ to I_0 . Also, note that $\beta_{1,2}$ commutes with any element of F fixing $1/2$, by Proposition 2.5, and F is generated by such elements together with $[F, F]$, so we are done. \square

Case 3. Now suppose that χ kills P_{br} , so its ω_0 and ω_1 components are both zero. Also assume, for this case, that the ϕ_0 and ϕ_1 components of χ are not both zero. We will find yet another pair of sets J_F and I_F such that Citation 3.2 applies. As a remark, the proof recovers the fact that the restrictions of such characters to F are in $\Sigma^1(F)$, originally proved in [6].

Lemma 3.9. *Let χ be a character that kills F_{br} and whose ϕ_0 and ϕ_1 components are both non-zero. Then $[\chi] \in \Sigma^1(F_{\text{br}})$.*

Proof. Let J_F be the set of all elements of $F \leq F_{\text{br}}$ whose support has precisely one of the endpoints of $[0, 1]$ as a limit point. Since elements with disjoint supports commute, it is straightforward to verify that $C(J_F)$ is connected. Also, any element of J_F survives under χ by our hypothesis on χ .

Now define I'_F to be $[F, F] \cup J_F$. It is straightforward to check that J_F dominates I'_F and that I'_F generates F , so we recover the fact that $[\chi|_F] \in \Sigma^1(F)$. Now let I_F be the union of I'_F with the set of elements of the form $\alpha_{i,j}$ and $\beta_{i,j}$ ($1 \leq i < j$), so I_F generates F_{br} . Any $\alpha_{i,j}$ or $\beta_{i,j}$ commutes with x_j by (D5) and (D9), which is in J_F , so J_F dominates I_F . \square

We can now put the cases together and compute $\Sigma^1(F_{\text{br}})$.

Proof of Theorem 3.4. Let $[\chi] \in \Sigma(F_{\text{br}})$, say

$$\chi = a\phi_0 + b\phi_1 + c\omega_0 + d\omega_1.$$

If $c \neq 0$ or $d \neq 0$ then $[\chi] \in \Sigma^1(F_{\text{br}})$ by Lemma 3.6 and Proposition 3.8, so assume $c = 0$ and $d = 0$. If a and b are both non-zero then $[\chi] \in \Sigma^1(F_{\text{br}})$ by Lemma 3.9. The four remaining points of $\Sigma(F_{\text{br}})$ are $[\pm\phi_i]$ for $i = 0, 1$, which are handled by Observation 3.5. \square

An immediate application is that we know exactly when normal subgroups of F_{br} containing $[F_{\text{br}}, F_{\text{br}}]$ are finitely generated.

Corollary 3.10. *Let N be a normal subgroup of F_{br} , and suppose that N contains $[F_{\text{br}}, F_{\text{br}}]$. Then N is finitely generated if and only if $N \not\subseteq \ker(\phi_0)$ and $N \not\subseteq \ker(\phi_1)$.*

Proof. First note that by assumption F_{br}/N is abelian. If N is contained in the kernel of either ϕ_0 or ϕ_1 , then N is not finitely generated, by Citation 3.1. Now suppose that N is not contained in either kernel. Let χ be any non-trivial character of F_{br} such that $\chi(N) = 0$, so $[\chi] \notin \{[\phi_0], [\phi_1]\}$. In particular, $[\chi] \in \Sigma^1(F_{\text{br}})$, so N is finitely generated by Citation 3.1. \square

We can combine this result with a proposition from the previous section:

Corollary 3.11. *Let $f, g \in F$ be elements such that $\chi_0(f) \neq 0$, $\chi_1(f) = 0$, $\chi_0(g) = 0$ and $\chi_1(g) \neq 0$. Then the normal closure $\langle\langle f, g \rangle\rangle$ in F_{br} is finitely generated.*

Proof. The normal closure contains the commutator subgroup by Proposition 2.3. Hence it is finitely generated by Corollary 3.10. \square

4. Normal subgroups under P_{br}

Before discussing normal subgroups of V_{br} and F_{br} contained in P_{br} , we should generally inspect the subgroups of P_{br} . Since P_{br} is a direct limit of copies of pure braid groups, we know that every subgroup of P_{br} is a direct limit of subgroups of the PB_n , so our first goal is to pin down what can happen. For each $n \in \mathbb{N}$ and each tree T with n leaves, recall from Section 1.2 that PB_T denotes the copy of PB_n consisting of triples of the form (T, p, T) , and P_{br} is the direct limit of the PB_T .

Given a family of subgroups G_T of the PB_T , one for each tree T , we can consider the subgroup of P_{br} generated by all the G_T . If we want any hope of recovering the family from the subgroup it generates, and hence of classifying the subgroups of P_{br} , we need some conditions on the family. The criteria to check this are as follows. Let $(G_T)_T$ be a family of subgroups with $G_T \leq PB_T$ for each T . We will call the family *coherent* if whenever $T \leq T'$, the inclusion $PB_T \rightarrow PB_{T'}$ restricts to an inclusion $G_T \rightarrow G_{T'}$, i.e., $G_T \leq G_{T'}$ as subgroups of P_{br} . If moreover $G_{T'} \cap PB_T$ equals G_T we will call the family *complete*; the condition here that is not immediate is $G_{T'} \cap PB_T \subseteq G_T$. The point is that we can recover a complete coherent family of subgroups of the PB_T from the subgroup they generate in P_{br} , as the next proposition makes precise.

Proposition 4.1 (subgroups of P_{br}). *The subgroups of P_{br} are in one-to-one correspondence with the complete coherent families of subgroups of the PB_T .*

Proof. Every coherent family yields a subgroup of P_{br} , namely the subgroup generated by the subgroups in the family. Also, given a subgroup G of P_{br} , the family $(PB_T \cap G)_T$ is coherent and complete for trivial reasons. The only thing to check then is that two distinct complete coherent families yield distinct subgroups of P_{br} . Let $(G_T)_T$ be coherent and complete, and let G be the subgroup of P_{br} generated by the G_T . We claim that $PB_T \cap G \subseteq G_T$ for all T , after which we will be done, since the reverse inclusion is immediate.

Let $g \in PB_T \cap G$. Since G is generated by the G_T , we can write g as a product $g = g_1 \cdots g_r$ where each g_i lies in some G_{T_i} . Let T' be a common upper bound for $\{T\} \cup \{T_i\}_{i=1}^r$, so g and all the g_i lie in $PB_{T'}$. Since the family is coherent all the g_i even lie in $G_{T'}$. This implies $g \in G_{T'}$, and now since the family is complete and $g \in PB_T$, we conclude that $g \in G_T$. \square

A consequence of the proof is that given any subgroup $G \leq P_{\text{br}}$, the unique complete coherent family that generates G is $(PB_T \cap G)_T$.

For $n(T)$ the number of leaves of T , denote by

$$\psi_T: PB_T \longrightarrow PB_{n(T)}$$

the isomorphism $(T, p, T) \mapsto p$.

Lemma 4.2 (families of normal subgroups). *Let $(G_T)_T$ be a complete coherent family of subgroups $G_T \leq PB_T$, and let $G \leq P_{\text{br}}$ be the subgroup generated by the G_T . Then G is normal in F_{br} if and only if each G_T is normal in PB_T , and for every T and S with $n(T) = n(S)$ we have $\psi_T(G_T) = \psi_S(G_S)$ in PB_n .*

Proof. First suppose each G_T is normal in PB_T and $\psi_T(G_T) = \psi_S(G_S)$ for all T, S with $n(T) = n(S)$. An element of G is a triple $g = (T, p, T)$ for some T and $p \in \psi_T(G_T)$. Let $h = (S, q, U)$ be an arbitrary element of F_{br} . Since the family $(G_T)_T$ is coherent, we can expand T, S and U until without loss of generality $h = (S, q, T)$, so $hgh^{-1} = (S, qpq^{-1}, S)$. Now, G_T is normal in PB_T , so $qpq^{-1} \in \psi_T(G_T)$, and so by hypothesis is also in $\psi_S(G_S)$, which means that $(S, qpq^{-1}, S) \in G_S$. We conclude that $hgh^{-1} \in G$.

Now suppose G is normal in F_{br} . It is immediate that G_T is normal in PB_T . Let T and S both have n leaves, and let $(T, p, T) \in G_T$, so $(T, p, T) \in G$. Since G is normal we also have $(S, p, S) \in G$. But $G_S = PB_S \cap G$, so $p \in \psi_S(G_S)$. This shows $\psi_T(G_T) \subseteq \psi_S(G_S)$, and the reverse inclusion follows by the same argument. \square

In conclusion, the normal subgroups of F_{br} contained in P_{br} are obtained precisely by choosing a normal subgroup $G_n \triangleleft PB_n$ for each n such that for every $1 \leq k \leq n$, we have:

$$\kappa_k^n(PB_n) \cap G_{n+1} = \kappa_k^n(G_n). \quad (4.1)$$

Indeed, this equation ensures that the family $(G_T)_T$ given by $G_T := G_{n(T)}$ is complete and coherent. We will call $(G_n)_{n \in \mathbb{N}}$ a *complete coherent sequence of normal subgroups*.

Given such a sequence $(G_n)_{n \in \mathbb{N}}$, we will denote the corresponding subgroup of P_{br} by $\mathcal{K}(G_*)$ (following the notation in [16]), so $\mathcal{K}(G_*) \triangleleft F_{\text{br}}$. In terms of triples, we have:

$$\mathcal{K}(G_*) = \{(T, p, T) \mid T \text{ has } n \text{ leaves and } p \in G_n\}.$$

It is straightforward to decide when $\mathcal{K}(G_*)$ is even normal in V_{br} .

Lemma 4.3. *Let $(G_n)_{n \in \mathbb{N}}$ be a sequence satisfying Equation 4.1, so $\mathcal{K}(G_*)$ is normal in F_{br} . Then $\mathcal{K}(G_*)$ is normal in V_{br} if and only if each $G_n \leq PB_n$ is normal in B_n .*

Proof. First suppose $\mathcal{K}(G_*)$ is normal in V_{br} . Let T be arbitrary, and let $p \in G_n$. For any $b \in B_n$, we have

$$(T, bpb^{-1}, T) = (T, b, T)(T, p, T)(T, b^{-1}, T) \in \mathcal{K}(G_*)$$

and $(T, bpb^{-1}, T) \in PB_T$, so in fact $(T, bpb^{-1}, T) \in G_T$, which implies that $bpb^{-1} \in G_n$.

Now suppose that G_n is normal in B_n for all n . Let $g \in \mathcal{K}(G_*)$ and $h \in V_{\text{br}}$. Choose $T, S, p \in PB_{n(T)}$ and $b \in B_{n(T)}$ such that $g = (T, p, T)$ and $h = (S, b, T)$; in other words expand the triples until they have a common bottom tree. Now

$$hgh^{-1} = (S, b, T)(T, p, T)(T, b^{-1}, S) = (S, bpb^{-1}, S)$$

is in G_S since $bpb^{-1} \in G_{n(S)}$. We conclude that $\mathcal{K}(G_*)$ is normal in V_{br} . \square

Here is one family of examples of complete coherent sequences of normal subgroups $(G_n)_{n \in \mathbb{N}}$. Let $n \in \mathbb{N}$ and $1 \leq m \leq n$. Call a pure braid $p \in PB_n$ *m-loose* if it becomes trivial upon deleting all but any m strands. For example, every pure braid is 1-loose, and a pure braid p is 2-loose if and only if every total winding number between two strands is zero, if and only if $p \in [PB_n, PB_n]$. Let $\Lambda_n(m)$ be the subgroup of PB_n consisting of all *m-loose* braids, so $\Lambda_n(m)$ is normal in B_n . We have

$$\begin{aligned} PB_n &= \Lambda_n(1) \\ &> \Lambda_n(2) \\ &= [PB_n, PB_n] \\ &> \Lambda_n(3) \\ &\vdots \\ &> \Lambda_n(n-1) \\ &> \Lambda_n(n) \\ &= \{1\}. \end{aligned}$$

As a remark, $\Lambda_n(n-1)$ is the group of *Brunnian braids*, i.e., braids that become trivial upon removing any single strand.

Lemma 4.4. *For $1 \leq k \leq n$ and $1 \leq m \leq n$, we have $\kappa_k^n(PB_n) \cap \Lambda_{n+1}(m) = \kappa_k^n(\Lambda_n(m))$, i.e., Equation 4.1 is satisfied.*

Proof. One direction is trivial: if $p \in \Lambda_n(m)$, then $\kappa_k^n(p) \in \Lambda_{n+1}(m)$. Now suppose that $\kappa_k^n(p) \in \Lambda_{n+1}(m)$. Let $1 \leq i_1 < i_2 < \dots < i_m \leq n$ be the numbering of m arbitrary strands of p , and let $S := \{i_1, \dots, i_m\}$. Let $\pi_S: PB_n \rightarrow PB_m$ be the map that deletes all those strands not numbered by elements of S . Define

$$S(k) := \{i_1 + \varepsilon_1, \dots, i_m + \varepsilon_m\},$$

where ε_j is 0 if $i_j \leq k$ and is 1 if $k < i_j$. Then $\pi_{S(k)} \circ \kappa_k^n = \pi_S$. By assumption, $\pi_{S(k)}(\kappa_k^n(p)) = 1$, and so $\pi_S(p) = 1$. We conclude that $p \in \Lambda_n(m)$. \square

We emphasize that m -looseness is about deleting *all but* any m strands to get the trivial braid. If we instead considered deleting any m strands to get the trivial braid, then this would *not* give a coherent sequence. For instance, if $1 \neq p \in \Lambda_n(n-1)$ (so p is Brunnian), then the lemma says $\kappa_k^n(p)$ is in $\Lambda_{n+1}(n-1)$, but it is not in $\Lambda_{n+1}(n)$ (so not Brunnian), since deleting one of the cloned strands will bring us back to p , not to 1.

We now have a concrete family of normal subgroups of V_{br} contained in P_{br} , namely:

$$\mathcal{K}(\Lambda_*(m)) := \{(T, p, T) \mid T \text{ has } n \text{ leaves and } p \in \Lambda_n(m)\}.$$

We have $\mathcal{K}(\Lambda_*(1)) = P_{\text{br}}$ and $\mathcal{K}(\Lambda_*(2)) = [P_{\text{br}}, P_{\text{br}}]$. As m grows, we find a descending chain of normal subgroups

$$\dots \triangleleft \mathcal{K}(\Lambda_*(3)) \triangleleft \mathcal{K}(\Lambda_*(2)) = [P_{\text{br}}, P_{\text{br}}] \triangleleft P_{\text{br}} = \mathcal{K}(\Lambda_*(1)) \triangleleft F_{\text{br}}$$

with

$$\bigcap_{m \in \mathbb{N}} \mathcal{K}(\Lambda_*(m)) = \{1\}.$$

As a non-example of a complete coherent sequence, consider the sequence of normal subgroups of B_n given by the centers $Z(B_n) \leq PB_n$, generated by Δ_n . Upon cloning, $\kappa_k^n(\Delta_n) \leq PB_{n+1}$ is not contained in $Z(B_{n+1})$, so this sequence is not coherent. Indeed, these subgroups, when considered as subgroups of V_{br} , normally generate all of P_{br} ; in fact the single element $\beta_{1,2} = (R_2, \Delta_2, R_2)$ already normally generates P_{br} in V_{br} .

When thinking of normal subgroups of (pure) braid groups, an obvious question is whether the coherent sequence $(PB_n^{(m)})_{n \in \mathbb{N}}$ of m th derived subgroups is complete for fixed $m > 2$. When $m = 2$ we have $PB_n^{(2)} = \Lambda_n(2)$, so the answer is yes, but the $m = 3$ case is already unclear. Concretely, if g is a product of commutators of products of commutators, and g happens to feature a cloned strand, so $g = \kappa_k(h)$ for some h , then is h a product of commutators of products of commutators? All of these questions hold as well for the sequence of m th terms of upper or lower central series, for fixed m , and for all the corresponding versions for the braid groups B_n .

4.1. Quotients. Given a complete coherent sequence of normal subgroups $(G_n)_{n \in \mathbb{N}}$, with limit $\mathcal{K}(G_*)$, we can consider the quotients $F_{\text{br}}/\mathcal{K}(G_*)$ and $V_{\text{br}}/\mathcal{K}(G_*)$, which are somewhat straightforward to describe. The quotient map $\pi: V_{\text{br}} \twoheadrightarrow V_{\text{br}}/\mathcal{K}(G_*)$ takes a triple (T, b, S) to a triple (T, bG_n, S) , where $n = n(T) = n(S)$. In particular the quotient can be described as the set of such triples, up to reduction and expansion, which are well defined since $(G_n)_{n \in \mathbb{N}}$ is complete and coherent. In the future we believe these quotients could be further inspected using the ‘‘cloning systems’’ framework from [16].

Example 4.5 ($F_{\text{br}}/\mathcal{K}(\Lambda_*(2))$). Note that since $\Lambda_n(2) = [PB_n, PB_n]$, we have $PB_n/\Lambda_n(2) = H_1(PB_n) = \mathbb{Z}^{\binom{n}{2}}$. Heuristically an element $\vec{v} \in H_1(PB_n)$ is a record of the total winding numbers of each pair of strands (hence the $\binom{n}{2}$) of a representative of \vec{v} in PB_n . Fix a basis $(e_{i,j} \mid 1 \leq i < j \leq n)$ for $\mathbb{Z}^{\binom{n}{2}}$. If \vec{v} is the image of $p \in PB_n$ in the abelianization $H_1(PB_n)$, then the coefficient of $e_{i,j}$ in \vec{v} is the total winding number of strands i and j in p .

The quotient $F_{\text{br}}/\mathcal{K}(\Lambda_*(2))$ is described as follows. An element of the quotient is represented by a triple (T, \vec{v}, S) where T and S are trees with n leaves and $\vec{v} \in H^1(PB_n)$. We consider such triples up to reduction and expansion, as in F_{br} ; now expansion is described as follows. If we expand the k th leaf of T by attaching a caret, call the new tree T' , then we correspondingly replace \vec{v} with an element $\kappa_k^n(\vec{v})$ in $\mathbb{Z}^{\binom{n+1}{2}}$; the map κ_k^n is defined on the basis vectors as follows:

$$\kappa_k^n(e_{i,j}) := \begin{cases} e_{i+1,j+1} & \text{if } 1 \leq k < i < j, \\ e_{i+1,j+1} + e_{i,j+1} & \text{if } 1 \leq k = i < j, \\ e_{i,j+1} & \text{if } 1 \leq i < k < j, \\ e_{i,j+1} + e_{i,j} & \text{if } 1 \leq i < k = j, \\ e_{i,j} & \text{if } 1 \leq i < j < k. \end{cases}$$

This should be compared to the relations (D1)–(D9) in the presentation of F_{br} from Section 1.3, which also specify how to write $\kappa_k^n(\alpha_{i,j})$ and $\kappa_k^n(\beta_{i,j})$ as products of generators, e.g., $\kappa_k^n(\alpha_{i,j}) = \alpha_{i+1,j+1}\alpha_{i,j+1}$ if $k = i$, and so forth.

One can show that, since no $H^1(PB_n)$ contains a non-abelian free group (being abelian), neither does $F_{\text{br}}/\mathcal{K}(\Lambda_*(2))$. In particular $F_{\text{br}}/\mathcal{K}(\Lambda_*(2))$ is not isomorphic to F_{br} . We take this as evidence that none of the $F_{\text{br}}/\mathcal{K}(\Lambda_*(m))$ should be isomorphic to F_{br} . On the other hand, the $m = 2$ case is somewhat unique; for $m > 2$, $PB_n/\Lambda_n(m)$ does contain non-abelian free groups. Since $PB_n/\Lambda_n(m)$ embeds into $F_{\text{br}}/\mathcal{K}(\Lambda_*(m))$ for any n , this is a proper quotient of F_{br} that contains F and contains non-abelian free groups.

Question 4.6. *Are any of the quotients $F_{\text{br}}/\mathcal{K}(\Lambda_*(m))$ (for $m > 2$) isomorphic to F_{br} itself? Are any of the quotients $V_{\text{br}}/\mathcal{K}(\Lambda_*(m))$ (for $m \geq 2$) isomorphic to V_{br} ?*

Note that for V_{br} the above question includes the $m = 2$ case, since the group $V_{\text{br}}/\mathcal{K}(\Lambda_*(2))$ does contain free subgroups (by virtue of V , unlike F , containing free subgroups). More generally, one can ask:

Question 4.7. *Are F_{br} and/or V_{br} Hopfian?*

This question becomes especially intriguing in light of the following:

Proposition 4.8. *The group P_{br} is not Hopfian.*

Proof. For a non-trivial tree T with n leaves, define T_L to be the subtree of T whose root is the left child of the root of T . Let T_R be the tree whose root is the right child of the root of T . For $0 \leq m \leq n$, we have an epimorphism

$$\phi_{n,m}: PB_n \longrightarrow PB_m$$

given by forgetting all the strands of a pure braid except for the first m of them. Now define a map

$$\phi_L: P_{\text{br}} \longrightarrow P_{\text{br}}$$

sending (T, p, T) to $(T_L, \phi_{n,n(T_L)}(p), T_L)$. It is straightforward to check that this is well defined under reduction and expansion, and is a surjective homomorphism. It is also not injective; indeed the kernel contains every generator $\alpha_{i,j}$ and $\beta_{i,j}$ for $1 \leq i < j$, since these were defined using all-right trees. \square

The normal subgroups $\ker(\phi_{n,m})$ do not form a coherent sequence, and in fact once they are considered inside F_{br} or V_{br} , they normally generate all of P_{br} . This can be seen by noting that we catch every $\alpha_{i,j}$ and $\beta_{i,j}$ for $1 \leq i < j$, and the conjugates of these elements in F_{br} generate all of P_{br} . Hence this does not give any direct hints about Question 4.7.

References

- [1] T. Brady, J. Burillo, S. Cleary, and M. Stein, Pure braid subgroups of braided Thompson's groups. *Publ. Mat.* **52** (2008), no. 1, 57–89. [Zbl 1185.20043](#) [MR 2384840](#)
- [2] K.-U. Bux, M. G. Fluch, M. Marschler, S. Witzel, and M. C. B. Zaremsky, The braided Thompson's groups are of type F_∞ . *J. Reine Angew. Math.* **718** (2016), 59–101. With an appendix by M. C. B. Zaremsky. [Zbl 06626587](#) [MR 3545879](#)
- [3] R. Bieri, R. Geoghegan, and D. H. Kochloukova, The sigma invariants of Thompson's group F . *Groups Geom. Dyn.* **4** (2010), no. 2, 263–273. [Zbl 1214.20048](#) [MR 2595092](#)
- [4] J. Burillo and J. González-Meneses, Bi-orderings on pure braided Thompson's groups. *Q. J. Math.* **59** (2008), no. 1, 1–14. [Zbl 1188.20031](#) [MR 2392498](#)
- [5] C. Bleak, A minimal non-solvable group of homeomorphisms. *Groups Geom. Dyn.* **3** (2009), no. 1, 1–37. [Zbl 1188.20037](#) [MR 2466019](#)
- [6] R. Bieri, W. D. Neumann, and R. Strebel, A geometric invariant of discrete groups. *Invent. Math.* **90** (1987), no. 3, 451–477. [Zbl 0642.57002](#) [MR 0914846](#)
- [7] M. G. Brin, Elementary amenable subgroups of R. Thompson's group F . *Internat. J. Algebra Comput.* **15** (2005), no. 4, 619–642. [Zbl 1110.20018](#) [MR 2160570](#)

- [8] M. G. Brin, The algebra of strand splitting. I. A braided version of Thompson's group V . *J. Group Theory*, 10(6):757–788, 2007. [Zbl 1169.20021](#) [Zbl 2364825](#)
- [9] M. G. Brin and C. C. Squier, Groups of piecewise linear homeomorphisms of the real line. *Invent. Math.* **79** (1985), no. 3, 485–498. [Zbl 0563.57022](#) [MR 0782231](#)
- [10] K.-U. Bux and D. Sonkin, Some remarks on the braided Thompson group BV . Preprint 2008. [arXiv:0807.0061](#) [math.GR]
- [11] P. Dehornoy, The group of parenthesized braids. *Adv. Math.* **205** (2006), no. 2, 354–409. [Zbl 1160.20027](#) [MR 2258261](#)
- [12] S.-h. Kim and T. Koberda, Anti-trees and right-angled Artin subgroups of braid groups. *Geom. Topol.* **19** (2015), no. 6, 3289–3306. [Zbl 1351.20021](#) [MR 3447104](#)
- [13] N. Koban, J. McCammond, and J. Meier, The BNS-invariant for the pure braid groups. *Groups Geom. Dyn.* **9** (2015), no. 3, 665–682. [Zbl 1326.20045](#) [MR 3420539](#)
- [14] J. Meier and L. VanWyk, The Bieri–Neumann–Strebel invariants for graph groups. *Proc. London Math. Soc.* (3) **71** (1995), no. 2, 263–280. [Zbl 0835.20037](#) [MR 1337468](#)
- [15] L. A. Orlandi-Korner, The Bieri–Neumann–Strebel invariant for basis-conjugating automorphisms of free groups. *Proc. Amer. Math. Soc.* **128** (2000), no. 5, 1257–1262. [Zbl 0963.20016](#) [MR 1712889](#)
- [16] S. Witzel and M. C. B. Zaremsky, Thompson groups for systems of groups, and their finiteness properties. To appear in *Groups Geom. Dyn.* Preprint 2014. [arXiv:1405.5491](#) [math.GR]

Received April 23, 2014

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