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Proper affine actions in non-swinging representations

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Abstract. For a semisimple real Lie group G with an irreducible representation ρ on a finite-dimensional real vector space V, we give a sufficient criterion on ρ for existence of a group of affine transformations of V whose linear part is Zariski-dense in $\rho(G)$ and that is free, nonabelian and acts properly discontinuously on V.

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Contents

1. Introduction

1.1. Background and motivation. The present paper is part of a larger effort to understand discrete groups Γ of affine transformations (subgroups of the affine group $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$) acting properly discontinuously on the affine space $\mathbb{R}^n$. The case where Γ consists of isometries (in other words, $\Gamma \subset \mathrm{O}_n(\mathbb{R}) \ltimes \mathbb{R}^n$) is well understood: a classical theorem by Bieberbach says that such a group always has an abelian subgroup of finite index.

We say that a group G acts *properly discontinuously* on a topological space X if for every compact $K \subset X$, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite. We define a *crystallographic* group to be a discrete group $\Gamma \subset \mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ acting properly discontinuously and such that the quotient space $\mathbb{R}^n / \Gamma$ is compact. In [4], Auslander conjectured that any crystallographic group is virtually solvable, that is, contains a solvable subgroup of finite index. Later, Milnor [20] asked whether this statement is actually true for any affine group acting properly discontinuously. The answer turned out to be negative: Margulis [18, 19] gave a nonabelian free group of affine transformations with linear part Zariski-dense in $\mathrm{SO}(2,1)$, acting properly discontinuously on $\mathbb{R}^3$. On the other hand, Fried and Goldman [13] proved the Auslander conjecture in dimension 3 (the cases $n = 1$ and 2 are easy). Recently, Abels, Margulis and Soifer [2] proved it in dimension $n \leq 6$. See [1] for a survey of already known results.

Margulis's breakthrough was soon followed by the construction of other counterexamples to Milnor's conjecture. The first advance was made by Abels, Margulis and Soifer [3]: they generalized Margulis's construction to subgroups of the affine group

$$\mathrm{SO}(2n+2, 2n+1) \ltimes \mathbb{R}^{4n+3},$$

for all values of n. The author further generalized this in his previous paper [22], by finding such subgroups in the affine group $G \ltimes \mathfrak{g}$, where G is any noncompact semisimple real Lie group, acting on its Lie algebra $\mathfrak{g}$ by the adjoint representation. Recently Danciger, Guéritaud and Kassel [11] found examples of affine groups acting properly discontinuously that were neither virtually solvable nor virtually free.

Proliferation of these counterexamples leads to the following question. Consider a semisimple real Lie group G; for every representation ρ of G on a finite-dimensional real vector space V, we may consider the affine group $G \ltimes V$. Which of those affine groups contain a nonabelian free subgroup with linear part Zariski-dense in G and acting properly discontinuously on V?

In this paper, we give a fairly general sufficient condition on the representation ρ for existence of such subgroups. Before stating this condition, we need to introduce a few classical notations.

1.2. Basic notations. For the remainder of the paper, we fix a semisimple real Lie group G; let $\mathfrak{g}$ be its Lie algebra. Let us introduce a few classical objects related to $\mathfrak{g}$ and G (defined for instance in Knapp's book [16], though our terminology and notation differ slightly from his).

We choose in $\mathfrak{g}$ the following items.

- A Cartan involution θ. Then we have the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, where we call $\mathfrak{k}$ the space of fixed points of θ and $\mathfrak{q}$ the space of fixed points of $-\theta$. We call K the maximal compact subgroup with Lie algebra $\mathfrak{k}$.
- A *Cartan subspace* $\mathfrak{a}$ compatible with θ (that is, a maximal abelian subalgebra of $\mathfrak{g}$ among those contained in $\mathfrak{q}$). We set $A := \exp \mathfrak{a}$.
- A system Σ^+ of positive restricted roots in $\mathfrak{a}^*$. Recall that a *restricted root* is a nonzero element $\alpha \in \mathfrak{a}^*$ such that the restricted root space

  $$\mathfrak{g}^\alpha := \{Y \in \mathfrak{g} \mid \text{for all } X \in \mathfrak{a},\ [X, Y] = \alpha(X)Y\}$$

  is nontrivial. They form a root system Σ; a system of positive roots Σ^+ is a subset of Σ contained in a half-space and such that $\Sigma = \Sigma^+ \sqcup -\Sigma^+$.

  We call Π be the set of simple restricted roots in Σ^+. We call

  $$\mathfrak{a}^{++} := \{X \in \mathfrak{a} \mid \text{for all } \alpha \in \Sigma^+,\ \alpha(X) > 0\}$$

  the (open) dominant Weyl chamber of $\mathfrak{a}$ corresponding to Σ^+, and

  $$\mathfrak{a}^+ := \{X \in \mathfrak{a} \mid \text{for all } \alpha \in \Sigma^+,\ \alpha(X) \geq 0\} = \overline{\mathfrak{a}^{++}}$$

  the closed dominant Weyl chamber.

Then we denote

- M the centralizer of $\mathfrak{a}$ in K, $\mathfrak{m}$ its Lie algebra;
- L the centralizer of $\mathfrak{a}$ in G, $\mathfrak{l}$ its Lie algebra (It is clear that $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$, and well known (see e.g. [16], Proposition 7.82a) that $L = MA$.);
- $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) the sum of the restricted root spaces $\mathfrak{g}^\alpha$ for α in Σ^+ (resp. in $-\Sigma^+$), and $N^+ := \exp(\mathfrak{n}^+)$ and $N^- := \exp(\mathfrak{n}^-)$ the corresponding Lie groups;
- $\mathfrak{p}^+ := \mathfrak{l} \oplus \mathfrak{n}^+$ and $\mathfrak{p}^- := \mathfrak{l} \oplus \mathfrak{n}^-$ the corresponding minimal parabolic subalgebras, $P^+ := LN^+$ and $P^- := LN^-$ the corresponding minimal parabolic subgroups;
- $W := N_G(A)/Z_G(A)$ the restricted Weyl group;
- w_0 the *longest element* of the Weyl group, that is, the unique element such that $w_0(\Sigma^+) = \Sigma^-$.

See Examples 2.3 and 2.4 in the author's previous paper [22] for working through these definitions in the cases $G = \mathrm{PSL}_n(\mathbb{R})$ and $G = \mathrm{PSO}^+(n, 1)$.

Finally, if ρ is a representation of G on a finite-dimensional real vector space V, we call:

- the *restricted weight space* in V corresponding to a form $\lambda \in \mathfrak{a}^*$ the space
$$V^\lambda := \{v \in V \mid \text{for all } X \in \mathfrak{a},\ \rho(X) \cdot v = \lambda(X)v\};$$
- a *restricted weight* of the representation ρ any form $\lambda \in \mathfrak{a}^*$ such that the corresponding weight space is nonzero.

Remark 1.1. The reader who is unfamiliar with the theory of noncompact semisimple real Lie groups may focus on the case where G is *split*, i.e. its Cartan subspace $\mathfrak{a}$ is actually a Cartan subalgebra (just a maximal abelian subalgebra, without any additional hypotheses). In that case the restricted roots are just roots, the restricted weights are just weights, and the restricted Weyl group is just the usual Weyl group. Also the algebra $\mathfrak{m}$ vanishes and M is a discrete group.

However, the case where G is split does not actually require the full strength of this paper, in particular because quasi-translations (see Section 4.5) then reduce to ordinary translations.

1.3. Statement of main result. Let ρ be an irreducible representation of G on a finite-dimensional real vector space V. Without loss of generality, we may assume that G is connected and acts faithfully. We may then identify the abstract group G with the linear group $\rho(G) \subset \mathrm{GL}(V)$. Let V_{Aff} be the affine space corresponding to V. The group of affine transformations of V_{Aff} whose linear part lies in G may then be written $G \ltimes_\rho V$ or simply $G \ltimes V$ (where V stands for the group of translations). Here is the main result of this paper.

Main Theorem. *Let G be a semisimple real Lie group, and let ρ be an irreducible representation of G on a finite-dimensional real vector space V that satisfies the following conditions*:

(i) *there exists a vector $v \in V$ such that*

  (*a*) *for all $l \in L$, $l(v) = v$, and*

  (*b*) *$\tilde{w}_0(v) \neq v$, where $\tilde{w}_0$ is any representative in G of $w_0 \in N_G(A)/Z_G(A)$;*

(ii) *there exists an element $X_0 \in \mathfrak{a}$ such that $-w_0(X_0) = X_0$ and for every nonzero restricted weight λ of ρ, we have $\lambda(X_0) \neq 0$.*

Then there exists a subgroup Γ in the affine group $G \ltimes_\rho V$ whose linear part is Zariski-dense in G and that is free, nonabelian and acts properly discontinuously on V_{Aff}.

Remark 1.2. Note that the choice of the representative $\tilde{w}_0$ in (i)(b) does not matter, precisely because by (i)(a) the vector v is fixed by $L = Z_G(A)$.

We call representations satisfying condition (ii) "*non-swinging*" representations (see Section 3.3 to understand why). This is only a technical assumption: if we remove it, the theorem remains true. This more general result is proved in the author's forthcoming paper [23].

Note that the previously-known examples do fall under the scope of this theorem.

Example 1.3. (1) For $G = \mathrm{SO}^+(2n+2, 2n+1)$, the standard representation (acting on $V = \mathbb{R}^{4n+3}$) satisfies these conditions (see Remark 3.11 and Examples 4.22.1.b and 10.2.1 for details). So Theorem A from [3] is a particular case of this theorem.

(2) If the real semisimple Lie group G is noncompact, the adjoint representation satisfies these conditions (see Remark 3.11 and Examples 4.22.3 and 10.2.2 for details). So the main theorem of [22] is a particular case of this theorem.

Remark 1.4. When G is compact, no representation can satisfy these conditions: indeed in that case L is the whole group G and condition (i)(a) fails. So for us, only noncompact groups are interesting. This is not surprising: indeed, any compact group acting on a vector space preserves a positive-definite quadratic form, and so falls under the scope of Bieberbach's theorem.

1.4. Strategy of the proof. The proof has a lot in common with the author's previous paper [22]. The main idea (which comes back to Margulis's seminal paper [18]) is to introduce, for some affine maps g, an invariant that measures the translation part of g along a particular affine subspace of V. The key part of the argument (just as in [18] and in [22]) is then to show that, under some conditions, the invariant of the product of two maps is roughly equal to the sum of their invariants (Proposition 8.1). Here are the two main difficulties that were not present in [22].

- The first one is that [22] crucially relies on the following fact: if two maps are $\mathbb{R}$-regular (i.e. the dimension of their centralizer is the lowest possible), in general position with respect to each other and strongly contracting (when acting on $\mathfrak{g}$), their product is still $\mathbb{R}$-regular. The natural generalization of the notion of an $\mathbb{R}$-regular map that is adapted to an arbitrary representation is that of a "generic" map, i.e. a map that has as few eigenvalues of modulus 1 (counted with multiplicity) as possible. Unfortunately, the corresponding statement is then no longer true in an arbitrary representation.

If the representation is “too large”, i.e. if it contains restricted weights that are not multiples of restricted roots (see Example 3.7.2), there are several different “types” of generic maps, depending on the region where their Jordan projection (see Definition 2.3) falls.

In order to ensure that the product of two generic maps g and h (that are in general position and strongly contracting) is still generic, we need to control the Jordan projection $\mathrm{Jd}(gh)$ of the product based on the Jordan projections $\mathrm{Jd}(g), \mathrm{Jd}(h)$ of the factors. To do this, we use ideas developed by Benoist in [5, 6]: when g and h are in general position and sufficiently contracting, he showed that $\mathrm{Jd}(gh)$ is approximately equal to $\mathrm{Jd}(g) + \mathrm{Jd}(h)$. So if we restrict all maps to have the same “type”, our argument works.

- Here is where the second difficulty comes: the argument of [5] only works for maps that are actually $\mathbb{R}$-regular in addition to being generic. In most representations this is automatically true: if every restricted root occurs as a restricted weight, then every generic map is in particular $\mathbb{R}$-regular. But when the representation is “too small”, this is not the case. (A surprising fact is that a handful of representations are actually “too large” and “too small” at the same time: see Example 3.7.4!)

  As an example, consider the subgroup G of $\mathrm{GL}_5(\mathbb{R})$ consisting of transformations preserving the quadratic form

$$x_1x_3 + x_2x_4 + x_5^2.$$

  This is a form of signature $(3, 2)$, so $G \simeq \mathrm{SO}(3, 2)$. Now take any real number $\lambda > 1$ and any $x \in \mathbb{R}$; then the element

$$g = \begin{pmatrix} \lambda & \lambda x & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & -\lambda^{-1}x & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G$$

  is generic in the standard representation (“pseudohyperbolic” in the terminology of [3] and [21]), but not $\mathbb{R}$-regular (when $x \neq 0$ it is not even semisimple!).

  Just as there are two different notions of being “generic” (the notion of $\mathbb{R}$-regularity, which is adapted to the adjoint representation, and the notion of being generic in ρ), there are also two different notions of being “in general position”, two different notions of being “strongly contracting” and so on. The results of [5] rely on the stronger version of every property.

If we had used them as such, our Proposition 6.17 about products of maps "of given type" (and the subsequent propositions that rely on it) would no longer include, as a particular case, the corresponding result for $G = \mathrm{SO}^+(n+1, n)$ (namely Lemma 5.6, point (1) in [3]). Instead, we would need to duplicate all definitions, and to always require that the maps we deal with satisfy both versions of the constraints. (In particular, we would probably lose the benefit of the unified treatment of the linear part and translation part, as outlined in Remark 5.3).

This weaker version is in theory sufficient for us, because it is known that "almost all" elements are $\mathbb{R}$-regular. So it is actually possible to construct the group Γ in such a way that its elements all have this additional property, and thus provide a working proof of the Main Theorem. But we felt that the simpler, stronger version of Proposition 6.17 was interesting in its own right.

To prove it, we needed a generalization of the results of [5]. Benoist's subsequent paper [6] does seem to provide such a generalization, by proving similar theorems with the hypothesis of $\mathbb{R}$-regularity replaced by what he calls "θ-proximality", for θ some subset of Π. This is quite close to what we are looking for; but unfortunately, the results of [6] rely on the assumption that the Jordan projections of the maps lie in a vector subspace of $\mathfrak{a}$ (see Remark 6.16 for details), which is unacceptably restrictive for us. So in Section 6 of this paper, we redeveloped this theory in a suitably general way. We did reuse some basic results from [6]; for example one of the key steps of our proof, Proposition 5.12 (about products of proximal maps), is very similar to Lemma 2.2.2 from [6].

1.5. Plan of the paper.

In Section 2, we give some background from representation theory.

In Section 3, we study the dynamics of elements of A. We choose one particular element $X_0 \in \mathfrak{a}$ with some nice properties, with the goal of eventually "modeling," in some sense, generators of the group Γ on $\exp(X_0)$.

In Section 4, we study the dynamics of elements g of the affine group $G \ltimes V$ that are of type X_0 (see Definition 4.14). This section culminates in the definition of the Margulis invariant of g, which measures the translation part of g along its "axis".

In Section 5, we study some quantitative properties of such elements g. In particular we define a quantitative measure of being "in general position", and a quantitative measure of being "strongly contracting"; both of these notions are tailored to the choice of ρ and of X_0. We also define analogous notions for proximal maps, and prove a theorem (Proposition 5.12) about products of proximal maps.

Section 6 is where most of the new ideas of this paper are exploited. Here we apply the theory of products of proximal maps to a selection of “fundamental representations” ρ_i (defined in Proposition 2.12). The goal is to show that the product of two strongly contracting maps of type X_0 in general position is still of type X_0.

In Section 7, we now apply the theory of products of proximal maps to suitable exterior powers of the maps g, in order to study the quantitative properties of products of elements of type X_0. This section follows Section 3.2 of [22] very closely.

Section 8 contains the key part of the proof. We prove that if we take two strongly contracting maps of type X_0 in general position, the Margulis invariant of their product is close to the sum of their Margulis invariants. This section follows Section 4 of [22] very closely.

The very short Section 9 uses induction to extend the results of the two previous sections to products of an arbitrary number of elements. We omit the proof, as it is a straightforward generalization of Section 5 in [22].

Section 10 contains the proof of the Main Theorem. It follows Section 6 of [22] quite closely, but there are a couple of additions.

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2. Algebraic preliminaries

In this section, we give some background about real finite-dimensional representations of semisimple real Lie groups.

In Subsection 2.1, for any element $g \in G$, we relate the eigenvalues and singular values of $\rho(g)$ (where ρ is some representation) to some “absolute” properties of g.

In Subsection 2.2, we enumerate some properties of restricted weights of a real finite-dimensional representation of a real semisimple Lie group.

2.1. Eigenvalues in different representations. The goal of this subsection is to prove Proposition 2.6, which expresses the eigenvalues and singular values of a given element $g \in G$ acting in a given representation ρ, exclusively in terms of the structure of g in the abstract group G (respectively its Jordan decomposition and its Cartan decomposition).

Proposition 2.1 (Jordan decomposition). *Let $g \in G$. There exists a unique decomposition of g as a product $g = g_h g_e g_u$, where*

- *g_h is conjugate in G to an element of A (**hyperbolic**);*
- *g_e is conjugate in G to an element of K (**elliptic**);*
- *g_u is conjugate in G to an element of N^+ (**unipotent**);*
- *these three maps commute with each other.*

Proof. This was proved by Kostant: see [17], Proposition 2.1. Alternatively, see [12], Theorem 2.19.24. Note however that technically, Kostant, Eberlein and our paper use three different sets of definitions of a hyperbolic, elliptic or unipotent element. That our definitions are equivalent to Kostant's (which are the ones used most commonly) is shown in [12], Theorem 2.19.16. That Eberlein's definitions are equivalent to Kostant's is shown in [12], Proposition 2.19.18. ☐

Proposition 2.2 (Cartan decomposition). *Let $g \in G$. Then there exists a decomposition of g as a product $g = k_1 a k_2$, with $k_1, k_2 \in K$ and $a = \exp(X)$ with $X \in \mathfrak{a}^+$. Moreover, the element X is uniquely determined by g.*

Proof. This is a classical result; see e.g. Theorem 7.39 in [16]. ☐

Definition 2.3. For every element $g \in G$, we define:

- the *Jordan projection* of g, written $\mathrm{Jd}(g)$, to be the unique element of the closed dominant Weyl chamber $\mathfrak{a}^+$ such that the hyperbolic part g_h (from the Jordan decomposition $g = g_h g_e g_u$ given above) is conjugate to $\exp(\mathrm{Jd}(g))$;
- the *Cartan projection* of g, written $\mathrm{Ct}(g)$, to be the element X from the Cartan decomposition given above.

To talk about singular values, we need to introduce a Euclidean structure. We are going to use a special one.

Lemma 2.4. *Let ρ_* be some real representation of G on some space V_*. There exists a K-invariant positive-definite quadratic form B_* on V_* such that all the restricted weight spaces are pairwise B_*-orthogonal.*

We want to reserve the plain notation ρ for the "default" representation, to be fixed once and for all at the beginning of Section 3. We use the notation ρ_* so as to encompass both this representation ρ and the representations ρ_i defined in Proposition 2.12.

Such quadratic forms have already been considered previously: see for example Lemma 5.33.a) in [8].

Example 2.5. If $\rho_* = \mathrm{Ad}$ is the adjoint representation, then B_* is the form B_θ given by

$$B_\theta(X, Y) = -B(X, \theta Y) \quad \text{for all } X, Y \in \mathfrak{g}$$

(see (6.13) in [16]), where B is the Killing form and θ is the Cartan involution.

Proof. This follows from the well-known fact that for any morphism $G \to H$ of reductive Lie groups (here we take $H = \mathrm{GL}(V_*)$), one can always find a Cartan involution of H that is compatible with a given Cartan involution of G. Alternatively, the form B_* can be constructed as a restriction of a positive-definite Hermitian form on $V_*^{\mathbb{C}}$ that is invariant by a suitable maximal compact subgroup of $G^{\mathbb{C}}$ (and such a Hermitian form can be found by the usual trick of averaging over the action of that compact group). □

Recall that the *singular values* of a map g in a Euclidean space are defined as the square roots of the eigenvalues of $g^* g$, where g^* is the adjoint map. The largest and smallest singular values of g then give respectively the operator norm of g and the reciprocal of the operator norm of g^{-1}.

Proposition 2.6. *Let $\rho_* : G \to \mathrm{GL}(V_*)$ be any representation of G on some vector space V_*; let $\lambda_*^1, \dots, \lambda_*^{d_*}$ be the list of all the restricted weights of ρ_*, repeated according to their multiplicities. Let $g \in G$; then*

(i) *the list of the moduli of the eigenvalues of $\rho_*(g)$ is given by*

$$\left(e^{\lambda_*^i(\mathrm{Jd}(g))}\right)_{1 \leq i \leq d_*};$$

(ii) *the list of the singular values of $\rho_*(g)$, with respect to a K-invariant Euclidean norm B_* on V_* that makes the restricted weight spaces of V_* pairwise orthogonal (such a norm exists by Lemma 2.4), is given by*

$$\left(e^{\lambda_*^i(\mathrm{Ct}(g))}\right)_{1 \leq i \leq d_*}.$$

Proof. (i) Let $g = g_h g_e g_u$ be the Jordan decomposition of g.

It is then well known that $\rho_*(g_e)$ and $\rho_*(g_u)$ are still respectively elliptic and unipotent in $\mathrm{GL}(V_*)$, and in particular have eigenvalues of modulus 1. Since g_h, g_e and g_u all commute with each other, we deduce that the eigenvalues of $\rho_*(g)$ are equal, in modulus, to those of $\rho_*(g_h)$.

On the other hand, g_h is by definition conjugate to $\exp(\mathrm{Jd}(g))$, so $\rho_*(g_h)$ has the same eigenvalues as $\rho_*(\exp(\mathrm{Jd}(g)))$.

Finally, since $\exp(\mathrm{Jd}(g))$ is in A (the group corresponding to the Cartan subspace), the list of the eigenvalues of $\rho_*(\exp(\mathrm{Jd}(g)))$ is by definition given by

$$\left(e^{\lambda_*^i(\mathrm{Jd}(g))}\right)_{1 \leq i \leq d_*}.$$

(ii) Let $g = k_1 \exp(\mathrm{Ct}(g)) k_2$ be the Cartan decomposition of g. Since $\rho_*(k_1)$ and $\rho_*(k_2)$ are B_*-orthogonal maps, the B_*-singular values of $\rho_*(g)$ coincide with those of the map $\exp(\mathrm{Ct}(g))$; since $\exp(\mathrm{Ct}(g))$, being an element of A, is self-adjoint, its singular values coincide with its eigenvalues. We conclude as in the previous point. ☐

2.2. Properties of restricted weights. In this subsection, we introduce a few properties of restricted weights of real finite-dimensional representations. (Proposition 2.7 is actually a general result about Coxeter groups.) The corresponding theory for ordinary weights is well known: see for example Chapter V in [16].

Let $\alpha_1, \dots, \alpha_r$ be an enumeration of the set Π of simple restricted roots generating Σ^+. For every i, we set

$$\alpha'_i := \begin{cases} 2\alpha_i & \text{if } 2\alpha_i \text{ is a restricted root,} \\ \alpha_i & \text{otherwise.} \end{cases} \tag{2.1}$$

For every index i such that $1 \leq i \leq r$, we define the i-th *fundamental restricted weight* ϖ_i by the relationship

$$2\frac{\langle \varpi_i, \alpha'_j \rangle}{\|\alpha'_j\|^2} = \delta_{ij} \tag{2.2}$$

for every j such that $1 \leq j \leq r$.

By abuse of notation, we will often allow ourselves to write things such as “for all i in some subset $\Pi' \subset \Pi$, ϖ_i satisfies...” (tacitly identifying the set Π' with the set of indices of the simple restricted roots that are inside).

In the following proposition, for any subset $\Pi' \subset \Pi$, we denote:

- by $W_{\Pi'}$ the Weyl subgroup of type Π':
$$W_{\Pi'} := \langle s_\alpha \rangle_{\alpha \in \Pi'}; \tag{2.3}$$
- by $\mathfrak{a}^+_{\Pi'}$ the fundamental domain for the action of $W_{\Pi'}$ on $\mathfrak{a}$:
$$\mathfrak{a}^+_{\Pi'} := \{X \in \mathfrak{a} \mid \text{for all } \alpha \in \Pi',\ \alpha(X) \geq 0\}, \tag{2.4}$$
which is a kind of prism whose base is the dominant Weyl chamber of $W_{\Pi'}$.

Proposition 2.7. *Take any $\Pi' \subset \Pi$, and let us fix $X \in \mathfrak{a}^+_{\Pi'}$. Let $Y \in \mathfrak{a}$. Then the following two conditions are equivalent*:

(i) *the vector Y is in $\mathfrak{a}^+_{\Pi'}$ and satisfies the system of linear inequalities*

$$\begin{cases} \varpi_i(Y) \leq \varpi_i(X) & \textit{for all } i \in \Pi', \\ \varpi_i(Y) = \varpi_i(X) & \textit{for all } i \in \Pi \setminus \Pi'; \end{cases}$$

(ii) *the vector Y is in $\mathfrak{a}^+_{\Pi'}$ and also in the convex hull of the orbit of X by $W_{\Pi'}$.*

Proof. For $\Pi' = \Pi$, this is well known: see e.g. [14], Proposition 8.44.

Now let Π' be an arbitrary subset of Π. We may translate everything by the vector

$$\sum_{i \in \Pi \setminus \Pi'} \varpi_i(X) H_i$$

(where $(H_i)_{i \in \Pi}$ is the basis of $\mathfrak{a}$ dual to the basis $(\varpi_i)_{i \in \Pi}$ of $\mathfrak{a}^*$), which is obviously fixed by $W_{\Pi'}$. Thus we reduce the problem to the case where

$$\varpi_i(X) = 0 \quad \text{for all } i \in \Pi \setminus \Pi'. \tag{2.5}$$

Now let Σ' be the intersection of Σ with the vector space $\mathfrak{a}_{\Pi'}$ determined by this system of equations, which is also the linear span of $(\alpha_i)_{i \in \Pi'}$. Then Σ' is a root system that has

- Π' as a simple root system;
- $W_{\Pi'}$ as the Weyl group;
- $\mathfrak{a}^+_{\Pi'} \cap \mathfrak{a}_{\Pi'}$ as the dominant Weyl chamber.

This reduces the problem to the case $\Pi' = \Pi$. ☐

Proposition 2.8. *Every restricted weight of every representation of* $\mathfrak{g}$ *is a linear combination of fundamental restricted weights with integer coefficients.*

Proof. This is a particular case of Proposition 5.8 in [9]. For a correction concerning the proof, see also Remark 5.2 in [10]. ☐

Proposition 2.9. *If* ρ_* *is an irreducible representation of* $\mathfrak{g}$*, there is a unique restricted weight* λ_* *of* ρ_**, called its* ***highest restricted weight****, such that no element of the form* $\lambda_* + \alpha_i$ *with* $1 \leq i \leq r$ *is a restricted weight of* ρ_**.*

Remark 2.10. In contrast to the situation with non-restricted weights, the highest restricted weight is not always of multiplicity 1; nor is a representation uniquely determined by its highest restricted weight.

Proof. This easily follows from the existence and uniqueness of the ordinary (non-restricted) highest weight, given for example in [16], Theorem 5.5 (d). ☐

Proposition 2.11. *Let* ρ_* *be an irreducible representation of* $\mathfrak{g}$*; let* λ_* *be its highest restricted weight. Let* Λ_{λ_*} *be the restricted root lattice shifted by* λ_**:*

$$\Lambda_{\lambda_*} := \{\lambda_* + c_1\alpha_1 + \cdots + c_r\alpha_r \mid c_1, \ldots, c_r \in \mathbb{Z}\}.$$

Then the set of restricted weights of ρ_* *is exactly the intersection of the lattice* Λ_{λ_*} *with the convex hull of the orbit* $\{w(\lambda_*) \mid w \in W\}$ *of* λ_* *by the restricted Weyl group.*

Proof. Once again, this follows from the corresponding result for non restricted weights (see e.g. [14], Theorem 10.1) by passing to the restriction. In the case of restricted weights, one of the inclusions is stated in [15], Proposition 4.22. ☐

Theorem 7.2 in [24] yields as a special case the following result:

Proposition 2.12. *For every index i such that $1 \leq i \leq r$, there exists an irreducible representation ρ_i of G on a space V_i whose highest restricted weight is equal to $n_i \varpi_i$ (for some positive integer n_i) and has multiplicity* 1.

Here is a result describing the restricted weights of these representations.

Lemma 2.13. *Fix an index i such that $1 \leq i \leq r$. Then*

(i) *ρ_i has $n_i \varpi_i - \alpha_i$ as a restricted weight*;

(ii) *all restricted weights of ρ_i other than $n_i \varpi_i$ have the form*

$$n_i \varpi_i - \alpha_i - \sum_{j=1}^{r} c_j \alpha_j,$$

with $c_j \geq 0$ for every j.

Proof. (i) We have

$$\begin{aligned} s_{\alpha_i}(n_i \varpi_i) &= s_{\alpha'_i}(n_i \varpi_i) \\ &= n_i \varpi_i - 2n_i \frac{\langle \varpi_i, \alpha'_i \rangle}{\langle \alpha'_i, \alpha'_i \rangle} \alpha'_i \\ &= n_i \varpi_i - n_i \alpha'_i \end{aligned} \tag{2.6}$$

(recall that α'_i is equal to $2\alpha_i$ if $2\alpha_i$ is a restricted root and to α_i otherwise). By Proposition 2.11, $s_{\alpha_i}(n_i \varpi_i)$ is a restricted weight of ρ_i (because it is the image of a restricted weight of ρ_i by an element of the Weyl group) and then $n_i \varpi_i - \alpha_i$ is also a restricted weight of ρ_i (as a convex combination of two restricted weights of ρ_i, that belongs to the restricted root lattice shifted by $n_i \varpi_i$).

(ii) Let λ be some restricted weight of ρ_i. By Proposition 2.11 taken together with Proposition 2.7, we already know that it can be written as

$$\lambda = n_i \varpi_i - \sum_{j=1}^{r} c'_j \alpha_j,$$

where all coefficients c'_j are nonnegative integers. It remains to show that if $\lambda \neq n_i \varpi_i$, then necessarily $c'_i > 0$.

Assume that $c'_i = 0$. By Proposition 8.42 in [14], we lose no generality in assuming that λ is dominant. Let $\Pi_i := \Pi \setminus \{i\}$; by Proposition 2.7, it follows that λ is then in the convex hull of the orbit of $n_i \varpi_i$ by W_{Π_i}. But clearly W_{Π_i} fixes ϖ_i, hence also $n_i \varpi_i$. The conclusion follows. ☐

3. Choice of a reference Jordan projection

For the remainder of the paper, we fix ρ an irreducible representation of G on a finite-dimensional real vector space V. For the moment, ρ may be any representation; but in the course of the paper, we shall gradually introduce several assumptions on ρ (namely Assumptions 3.2, 3.10, 4.23 and 10.1) that will ensure that ρ satisfies the hypotheses of the Main Theorem.

We denote by Ω the set of restricted weights of ρ. For any $X \in \mathfrak{a}$, we define $\Omega_X^{>}$ (resp. $\Omega_X^{<}$, $\Omega_X^{=}$, $\Omega_X^{\geq}$, $\Omega_X^{\leq}$) to be the set of all restricted weights of ρ that take a positive (resp. negative, zero, nonnegative, nonpositive) value on X:

$$\Omega_X^{\geq} := \{\lambda \in \Omega \mid \lambda(X) \geq 0\}, \quad \Omega_X^{>} := \{\lambda \in \Omega \mid \lambda(X) > 0\},$$

$$\Omega_X^{\leq} := \{\lambda \in \Omega \mid \lambda(X) \leq 0\}, \quad \Omega_X^{<} := \{\lambda \in \Omega \mid \lambda(X) < 0\},$$

$$\Omega_X^{=} := \{\lambda \in \Omega \mid \lambda(X) = 0\}.$$

The goal of this section is to study these sets, and to choose a vector $X_0 \in \mathfrak{a}^+$ for which the corresponding sets have some nice properties. The motivation for their study is that they parametrize the dynamical spaces (defined in Subsection 4.3) of $\exp(X_0)$ (obviously), and actually of any element $g \in G$ whose Jordan projection "has the same type" as X_0 (see Proposition 4.16).

In Subsection 3.1, we introduce the notion of a generic vector $X \in \mathfrak{a}$, and impose a first constraint on ρ: that 0 be a restricted weight.

In Subsection 3.2, we introduce an equivalence relation on the set of generic vectors that identifies elements with the same dynamics, and give several examples.

In Subsection 3.3, we introduce the notion of a symmetric vector $X \in \mathfrak{a}$, and ensure that ρ does not exclude generic vectors from being symmetric.

In Subsection 3.4, we define parabolic subgroups and subalgebras of type X; we also associate to every $X \in \mathfrak{a}^+$ a set Π_X of simple restricted roots and a subgroup W_X of the restricted Weyl group.

In Subsection 3.5, we prove Proposition 3.19, which shows that every equivalence class of generic vectors has a representative that has "as much symmetry" as the whole equivalence class, called an "extreme" representative.

At the end of this section, we shall fix once and for all an extreme, symmetric, generic vector $X_0 \in \mathfrak{a}^+$, which will serve as a reference Jordan projection (see the definition at the beginning of Subsection 4.4).

3.1. Generic elements.

We say that an element $X \in \mathfrak{a}$ is *generic* if

$$\Omega_X^{=} \subset \{0\}.$$

Remark 3.1. This is indeed the generic case: it happens as soon as X avoids a finite collection of hyperplanes, namely the kernels of all nonzero restricted weights of ρ.

Assumption 3.2. From now on, we assume that 0 is a restricted weight of ρ:

$$0 \in \Omega \quad \text{or equivalently} \quad \dim V^0 > 0.$$

Remark 3.3. By Proposition 2.11, this is the case if and only if the highest restricted weight of ρ is a $\mathbb{Z}$-linear combination of restricted roots.

Remark 3.4. We lose no generality in assuming this property, because it comes as a consequence of condition (i)(a) of the Main Theorem (which is also Assumption 4.23, see below). Indeed, any nonzero vector fixed by L is in particular fixed by $A \subset L$, which means that it belongs to the zero restricted weight space.

Remark 3.5. Conversely, this assumption provides the bare minimum without which the conclusion of the Main Theorem is certain to fail. In fact without this assumption, the group $G \ltimes V$ cannot even have *any* infinite Zariski-dense subgroup acting properly. Indeed, let Γ be such a subgroup; using a lemma due to Selberg, we lose no generality in assuming Γ to be torsion-free. On the other hand, the linear part of a generic element g of such a group does not have 1 as an eigenvalue. This means that g has a fixed point, which is a contradiction.

In that case, for generic X we actually have

$$\Omega_X^{=} = \{0\}.$$

3.2. Types of elements of $\mathfrak{a}$. For two vectors $X, Y \in \mathfrak{a}$, we say that Y *has the same type* as X if

$$\begin{cases} \Omega_Y^{>} = \Omega_X^{>}; \\ \Omega_Y^{<} = \Omega_X^{<}, \end{cases} \tag{3.1}$$

i.e. if every restricted weight takes the same sign on both of them. This implies that all five sets $\Omega^{\geq}$, $\Omega^{\leq}$, $\Omega^{=}$, $\Omega^{>}$ and $\Omega^{<}$ coincide for X and Y, hence that $\exp(X)$ and $\exp(Y)$ have the same dynamical spaces (see Subsection 4.3).

This is an equivalence relation on $\mathfrak{a}$, which partitions $\mathfrak{a}$ into finitely many equivalence classes. We are only interested in generic equivalence classes. Some generic $X \in \mathfrak{a}$ being fixed, we call

$$\mathfrak{a}_{\rho,X} := \left\{ Y \in \mathfrak{a} \;\middle|\; \begin{array}{l} \text{for all } \lambda \in \Omega_X^{>},\ \lambda(Y) > 0; \\ \text{for all } \lambda \in \Omega_X^{<},\ \lambda(Y) < 0 \end{array} \right\} \tag{3.2}$$

its equivalence class in $\mathfrak{a}$. If X is dominant, we additionally call

$$\mathfrak{a}_{\rho,X}^{+} := \mathfrak{a}_{\rho,X} \cap \mathfrak{a}^{+} \tag{3.3}$$

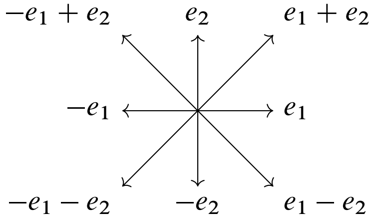
its equivalence class in the closed dominant Weyl chamber $\mathfrak{a}^{+}$.

Remark 3.6. Every equivalence class is a convex cone. Also, these equivalence classes actually coincide with connected components of the set of generic vectors.

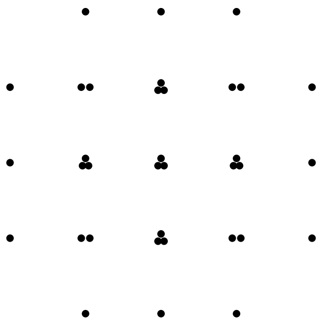
Example 3.7. (1) Let G be any noncompact semisimple real Lie group and let $\rho = \mathrm{Ad}$ be its adjoint representation (so that $V = \mathfrak{g}$).

- A vector $X \in \mathfrak{a}$ is generic if and only if it lies in one of the open Weyl chambers. In particular a vector X in $\mathfrak{a}^+$ is generic if and only if it lies in $\mathfrak{a}^{++}$.
- All elements of $\mathfrak{a}^{++}$ have the same type; so there is only one generic equivalence class in $\mathfrak{a}^+$. Specifically, for any vector $X \in \mathfrak{a}^{++}$, we have $\mathfrak{a}_{\mathrm{Ad},X} = \mathfrak{a}^+_{\mathrm{Ad},X} = \mathfrak{a}^{++}$.

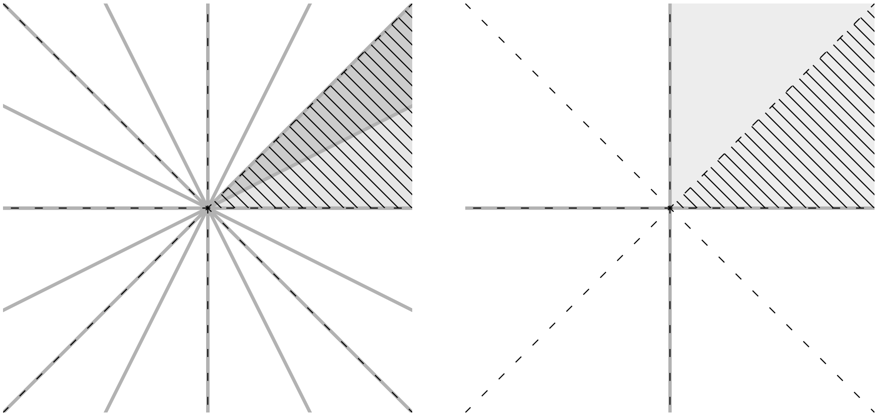
(2) Take $G = \mathrm{SO}^+(3,2)$. The root system is then B_2:



As this group is split, the roots are also the restricted roots. Let ρ be the representation with highest weight $2e_1 + e_2$ (in the notations of [16], Appendix C). This is a representation of dimension 35, whose weights (also restricted weights) are as follows:



(the number of dots at each node represents multiplicity.) Then every equivalence class in $\mathfrak{a}$ is contained in some Weyl chamber: see Figure 1a. The dominant Weyl chamber $\mathfrak{a}^+$ is split into two equivalence classes by the line of slope $\frac{1}{2}$, kernel of the weight $-e_1 + 2e_2$.



(a) ρ of highest weight $2e_1 + e_2$ (b) ρ of highest weight e_1

Figure 1. Equivalence classes and Weyl chambers for two different representations of $G = \mathrm{SO}^+(3,2)$. Dashed lines represent walls of Weyl chambers. Thick gray lines represent kernels of nonzero weights, which separate the different equivalence classes. The dominant Weyl chamber $\mathfrak{a}^+$ is hatched. All equivalence classes in $\mathfrak{a}$ that intersect $\mathfrak{a}^+$ are shaded, with different shades if there are more than one.

(3) Take $G = \mathrm{SO}^+(3,2)$ and ρ the standard representation on $V = \mathbb{R}^5$. Using once again the notations of [16], Appendix C, its highest weight is e_1 and its weights are $\pm e_1$, $\pm e_2$ and 0 (of course all with multiplicity 1). Then (see Figure 1b):

- A vector $X \in \mathfrak{a}^+$ is generic if and only if it avoids the “horizontal” wall of the dominant Weyl chamber (the one normal to e_2).
- All such vectors have the same type. So for a generic $X \in \mathfrak{a}^+$, the equivalence class $\mathfrak{a}^+_{\rho,X}$ is the half-open dominant Weyl chamber, with the diagonal wall included and the horizontal wall excluded.
- The whole equivalence class $\mathfrak{a}_{\rho,X}$ is then an open quadrant of the plane $\mathfrak{a}$, consisting of two half-open Weyl chambers glued back-to-back along their shared diagonal wall.

(4) Suppose that G and ρ are such that the set of the restricted weights of ρ neither contains all restricted roots of G, nor is contained in the set of restricted roots of G and their multiples. Then both phenomena occur at the same time: equivalence classes in $\mathfrak{a}$ neither contain nor are contained in the Weyl chambers.

Examples are not immediate to come up with: the author even mistakenly believed for some time that no such representations existed. However, here is one such example.

- Take $G = \mathrm{PSp}_4(\mathbb{R})$ (which is a split form), following the notation convention of [16]: this is a group of rank 4 with a standard representation of dimension 8 (most people would call it $\mathrm{PSp}_8(\mathbb{R})$ instead). In the notations of [16], Appendix C, its roots are all the possible expressions of the form $\pm e_j \pm e_i$ or $\pm 2e_i$.

- Take ρ to be the representation with highest weight $e_1 + e_2 + e_3 + e_4$. It has
  - the 16 weights of the form $\pm e_1 \pm e_2 \pm e_3 \pm e_4$, with multiplicity 1;
  - the 24 weights of the form $\pm e_i \pm e_j$, with multiplicity 1;
  - the zero weight with multiplicity 2,

  for a total dimension of 42.

The reader may check that there are then three different "types" of generic vectors in the dominant Weyl chamber $\mathfrak{a}^+$, with e.g. the following representatives:

(a) $X = (4, 2, 1, 0)$;

(b) $X = (5, 3, 2, 1)$;

(c) $X = (4, 3, 2, 0)$.

In cases (a) and (c), we notice that X lies on the wall normal to $2e_4$; its equivalence class then contains a whole slice of that wall.

3.3. Swinging. We start this subsection with the following observation: if the Jordan projection of g is X, then the Jordan projection of g^{-1} is $-w_0(X)$, where w_0 is the "longest element" of the Weyl group that interchanges positive and negative restricted roots (see Section 1.2).

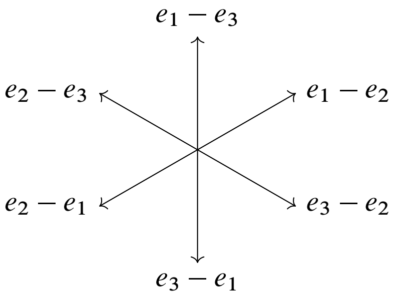
We would like to ensure that for every element g of the group Γ we are trying to construct, the element g itself and its inverse g^{-1} have similar dynamics. To do that, we would like X and $-w_0(X)$ to be of the same type. Replacing if necessary X and $-w_0(X)$ by their midpoint, we lose no generality in assuming they are actually equal.

Definition 3.8. We say that an element $X \in \mathfrak{a}$ is *symmetric* if it is invariant by $-w_0$:

$$-w_0(X) = X.$$

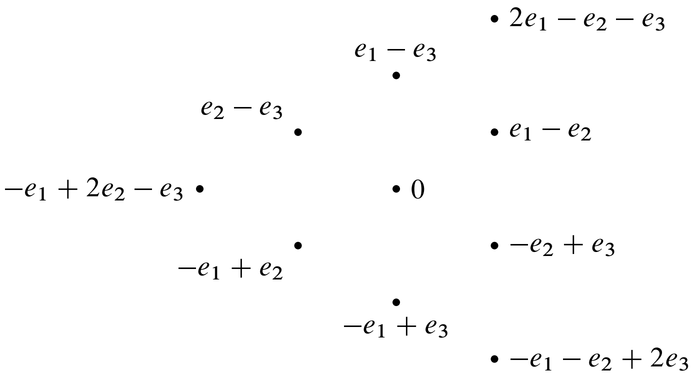
Unfortunately, it is not always possible to find a vector X that is both symmetric and generic, as shown by the following example.

Example 3.9. Take $G = \mathrm{SL}_3(\mathbb{R})$. It is a split form, so its restricted root system is the same as its root system, namely A_2:



For this group, $-w_0$ is the map that exchanges the two simple positive roots $e_1 - e_2$ and $e_2 - e_3$ (we use the notations of [16], Appendix C); in the picture above, it corresponds to the reflection about the vertical axis. So a vector $X \in \mathfrak{a}^+$ is symmetric if and only if it lies on that vertical axis (which bisects the dominant Weyl chamber $\mathfrak{a}^+$).

Now consider the representation ρ of G with highest weight $2e_1 - e_2 - e_3$. Note that this is three times the first fundamental weight, so ρ is actually the third symmetric product $S^3\mathbb{R}^3$ of the standard representation. Here are its weights:



We see that any symmetric vector necessarily annihilates the weight $-e_1 + 2e_2 - e_3$, hence it cannot be generic.

We call this phenomenon “swinging”. Here is the picture to have in mind: when we apply the involution $-w_0$ to some generic X, the annihilator of X (i.e. the hyperplane of $\mathfrak{a}^*$ consisting of linear forms that vanish on X) “swings” past the weight $-e_1 + 2e_2 - e_3$, thus switching it from the set $\Omega^{>}$ to the set $\Omega^{<}$.

From now on, we assume that this issue does not arise:

Assumption 3.10 (“no swinging”). From now on, we assume that ρ is such that there exists a symmetric generic element of $\mathfrak{a}$.

This is precisely condition (ii) from the Main Theorem.

Remark 3.11. (1) It is well known that when the restricted root system of G has any type other than A_n (with $n \geq 2$), D_{2n+1} or E_6, we actually have $w_0 = -\mathrm{Id}$. For those groups, *every* vector $X \in \mathfrak{a}$ is symmetric, and so every representation satisfies this condition.

(2) For the remaining groups, a straightforward linear algebra manipulation shows that this condition is equivalent to the following: no nonzero restricted weight of ρ must fall into the linear subspace

$$\{\lambda \in \mathfrak{a}^* \mid w_0\lambda = \lambda\} \tag{3.4}$$

(the “axis of symmetry” of w_0 in $\mathfrak{a}^*$). For example, this is always true for the adjoint representation (any restricted root fixed by w_0 would need to be positive and negative at the same time). Heuristically, this seems to hold when the highest restricted weight is “small”, but to quickly fail when it gets “large enough.”

3.4. Parabolic subgroups and subalgebras. A parabolic subgroup (or subalgebra) is usually defined in terms of a subset Π' of the set Π of simple restricted roots. We find it more convenient however to use a slightly different language. To every such subset corresponds a facet of the Weyl chamber, given by intersecting the walls corresponding to elements of Π'. We may exemplify this facet by picking some element X in it that does not belong to any subfacet. Conversely, for every $X \in \mathfrak{a}^+$, we define the corresponding subset

$$\Pi_X := \{\alpha \in \Pi \mid \alpha(X) = 0\}. \tag{3.5}$$

The parabolic subalgebras and subgroups of type Π_X can then be very conveniently rewritten in terms of X, as follows.

Remark 3.12. The set Π_X actually encodes the “type” of X with respect to the adjoint representation.

Definition 3.13. For every $X \in \mathfrak{a}^+$, we define

- $\mathfrak{p}_X^+$ and $\mathfrak{p}_X^-$ the parabolic subalgebras of type X, and $\mathfrak{l}_X$ their intersection:

$$\mathfrak{p}_X^+ := \mathfrak{l} \oplus \bigoplus_{\alpha(X) \geq 0} \mathfrak{g}^\alpha;$$

$$\mathfrak{p}_X^- := \mathfrak{l} \oplus \bigoplus_{\alpha(X) \leq 0} \mathfrak{g}^\alpha;$$

$$\mathfrak{l}_X := \mathfrak{l} \oplus \bigoplus_{\alpha(X) = 0} \mathfrak{g}^\alpha;$$

- P_X^+ and P_X^- the corresponding parabolic subgroups, and L_X their intersection:
$$\begin{aligned} P_X^+ &:= N_G(\mathfrak{p}_X^+); \\ P_X^- &:= N_G(\mathfrak{p}_X^-); \\ L_X &:= P_X^+ \cap P_X^-. \end{aligned}$$

An object closely related to these parabolic subgroups (see formula (4.4), the Bruhat decomposition for parabolic subgroups) is the stabilizer of X in the Weyl group:

Definition 3.14. For any $X \in \mathfrak{a}^+$, we set
$$W_X := \{w \in W \mid wX = X\}.$$

Remark 3.15. The group W_X is also closely related to the set Π_X. Indeed, it follows immediately that a simple restricted root α belongs to Π_X if and only if the corresponding reflection s_α belongs to W_X. Conversely, it is well known (Chevalley's lemma, see e.g. [16], Proposition 2.72) that these reflections actually generate the group W_X.

Thus W_X is actually the same thing as W_{Π_X} (i.e. the group $W_{\Pi'}$ as defined in (2.3), with $\Pi' = \Pi_X$).

Example 3.16. To help understand the conventions we are taking, here are the extreme cases.

(1) If X lies in the open Weyl chamber $\mathfrak{a}^{++}$, then
    - $P_X^+ = P^+$ is the minimal parabolic subgroup; $P_X^- = P^-$; $L_X = L$;
    - $\Pi_X = \emptyset$;
    - $W_X = \{\mathrm{Id}\}$.

(2) If $X = 0$, then
    - $P_X^+ = P_X^- = L_X = G$;
    - $\Pi_X = \Pi$;
    - $W_X = W$.

3.5. Extreme vectors. Besides W_X, we are also interested in the group
$$W_{\rho,X} := \{w \in W \mid wX \text{ has the same type as } X\}, \tag{3.6}$$
which is the stabilizer of X "up to type". It obviously contains W_X. The goal of this subsection is to show that in every equivalence class, we can actually choose X in such a way that both groups coincide.

Example 3.17. In Example 3.7.3 ($G = \mathrm{SO}^+(3,2)$ acting on $V = \mathbb{R}^5$), the group $W_{\rho,X}$ corresponding to any generic X is a two-element group. If we take X to be generic not only with respect to ρ but also with respect to the adjoint representation (in other terms if X is in an open Weyl chamber), then the group W_X is trivial. If however we take as X any element of the diagonal wall of the Weyl chamber, we have indeed $W_X = W_{\rho,X}$.

Definition 3.18. We call an element $X \in \mathfrak{a}^+$ *extreme* if $W_X = W_{\rho,X}$, i.e. if it satisfies the following property:

$$wX \text{ has the same type as } X \iff wX = X \quad \text{for all } w \in W.$$

Proposition 3.19. *For every generic $X \in \mathfrak{a}^+$, there exists a generic $X' \in \mathfrak{a}^+$ that has the same type as X and that is extreme.*

If moreover X is symmetric, then X' is still symmetric.

Remark 3.20. The following statement will never be used in the paper (so we leave it without proof), but might help to understand what is going on: for every generic X, we have

$$\mathfrak{a}_{\rho,X} = W_{\rho,X}\mathfrak{a}^+_{\rho,X} = W_{X'}\mathfrak{a}^+_{\rho,X}.$$

Also, it can be shown that a representative X' of a given equivalence class $\mathfrak{a}^+_{\rho,X}$ is extreme if and only if it lies in every wall of the Weyl chamber that "touches" $\mathfrak{a}^+_{\rho,X}$ (or, equivalently, passes through $\mathfrak{a}_{\rho,X}$), hence the term "extreme".

Proof. To construct an element that has the same type as X but has the whole group $W_{\rho,X}$ as stabilizer, we simply average over the action of this group: we set

$$X' = \sum_{w \in W_{\rho,X}} wX. \tag{3.7}$$

(As multiplication by positive scalars does not change anything, we have written it as a sum rather than an average for ease of manipulation.) Then obviously:

- by definition every wX for $w \in W_{\rho,X}$ has the same type as X; since the equivalence class $\mathfrak{a}_{\rho,X}$ is a convex cone, their sum X' also has the same type as X;
- in particular X' is generic;
- by construction whenever wX has the same type as X, we have $wX' = X'$; conversely if w fixes X', then wX has the same type as $wX' = X'$ which has the same type as X. So X' is extreme.

Let us now show that $X' \in \mathfrak{a}^+$, i.e. that for every $\alpha \in \Pi$, we have $\alpha(X') \geq 0$.

- If $s_\alpha X' = X'$, then obviously $\alpha(X') = 0$.
- Otherwise, since X' is extreme, it follows that $s_\alpha X'$ does not even have the same type as X'. Since X' is generic, this means that there exists a restricted weight λ of ρ such that
$$\begin{cases} \lambda(X') > 0, \\ s_\alpha(\lambda)(X') < 0. \end{cases} \tag{3.8}$$
By definition, the same inequalities then hold for any Y with the same type as X' (or as X):
$$\begin{cases} \lambda(Y) > 0, \\ s_\alpha(\lambda)(Y) < 0. \end{cases}$$
In particular the form $\lambda - s_\alpha(\lambda)$, which is a multiple of α, takes a positive value on every such Y; hence α never vanishes on the equivalence class $\mathfrak{a}_{\rho,X}$. By hypothesis $X \in \mathfrak{a}^+$, so $\alpha(X) \geq 0$. Since $\mathfrak{a}_{\rho,X}$ is connected, we conclude that $\alpha(X') > 0$.

Finally, assume that X is symmetric, i.e. $-w_0(X) = X$. Then since w_0 belongs to the Weyl group, it induces a permutation on Ω, hence we have:

$$w_0\Omega_X^{>} = \Omega_{w_0X}^{>} = \Omega_{-X}^{>} = \Omega_X^{<}, \tag{3.9}$$

so that w_0 swaps the sets $\Omega_X^{>}$ and $\Omega_X^{<}$. Now by definition we have

$$W_{\rho,X} = \mathrm{Stab}_W(\Omega_X^{>}) \cap \mathrm{Stab}_W(\Omega_X^{<}), \tag{3.10}$$

hence w_0 normalizes $W_{\rho,X}$. Obviously the map $X \mapsto -X$ commutes with everything, so $-w_0$ also normalizes $W_{\rho,X}$. We conclude that

$$\begin{aligned} -w_0(X') &= \sum_{w \in W_{\rho,X}} -w_0(w(X)) \\ &= \sum_{w' \in W_{\rho,X}} w'(-w_0(X)) \\ &= X', \end{aligned} \tag{3.11}$$

so that X' is still symmetric. ☐

Remark 3.21. In practice, it can be shown that if G is simple, the set Π_X for extreme, symmetric, generic X can actually only be one of the following:

(a) empty;

(b) the set of long simple restricted roots;

(c) the whole set Π.

Case (a) accounts for the vast majority of representations. Case (b) obviously only occurs when the restricted root system has a non-simply-laced Dynkin diagram (G_2, F_4, B_n, C_n or BC_n), and then only occurs in finitely many representations of each group. Case (c) only occurs in trivial situations, namely when either $\dim \mathfrak{a} = 0$ (i.e. the group G is compact) or the representation is trivial.

The proof of this fact mostly relies on the following two observations.

- As soon as Ω is large enough to include some simple restricted root α, no set $\Pi_{X'}$ may contain α. Indeed in that case, $\alpha(X')$ never vanishes for generic X'.
- The Weyl group acts transitively on the set of restricted roots of the same length; so as soon as Ω contains one restricted root of a given length, it contains all of them.

For the remainder of the paper, we fix some symmetric generic vector X_0 in the closed dominant Weyl chamber $\mathfrak{a}^+$ that is extreme.

4. Dynamics of maps of type X_0

Now we take an element g in the affine group $\rho(G) \ltimes V$ such that the Jordan projection of its linear part has the same type as X_0. The goal of this section is to understand the dynamics of g acting on the affine space corresponding to V, in particular its "dynamical spaces" defined in Subsection 4.3. There is a lot of parallelism between this section and Section 2 in [22].

In Subsection 4.1, we introduce the dynamical subspaces of X_0. We also show that the stabilizers in G of those subspaces (except for the neutral one) are precisely the parabolic subgroups introduced in Subsection 3.4.

In Subsection 4.2, we introduce some formalism that reduces the study of the affine space V_{Aff} corresponding to V to the study of a vector space called A. We also introduce affine equivalents of linear notions defined previously.

In Subsection 4.3, we define the linear and affine dynamical subspaces associated to an element of the affine group $\rho(G) \ltimes V$. This is very similar to Section 2.1 in [22].

In Subsection 4.4, we give a description of the dynamical subspaces of an element $g \in \rho(G) \ltimes V$ whose Jordan projection has the same type as X_0.

In Subsection 4.5, we show that the action of any such element on its affine neutral space is a "quasi-translation", and explain what that means. This is a generalization of Section 2.4 in [22].

In Subsection 4.6, we introduce a family of canonical identifications between different affine neutral spaces, and use them to define the "Margulis invariant" for any such element g, which is a vector measuring its translation part along a subspace of its affine neutral space. This is a generalization of Section 2.5 in [22].

4.1. Reference dynamical spaces. Recall that X_0 is some generic, symmetric, extreme vector in the closed dominant Weyl chamber $\mathfrak{a}^+$, chosen once and for all.

Definition 4.1. We define the following subspaces of V:

- $V_0^> := \bigoplus_{\lambda(X_0)>0} V^\lambda$, the *reference expanding space*;
- $V_0^< := \bigoplus_{\lambda(X_0)<0} V^\lambda$, the *reference contracting space*;
- $V_0^= := \bigoplus_{\lambda(X_0)=0} V^\lambda$, the *reference neutral space*;
- $V_0^\geq := \bigoplus_{\lambda(X_0)\geq 0} V^\lambda$, the *reference noncontracting space*;
- $V_0^\leq := \bigoplus_{\lambda(X_0)\leq 0} V^\lambda$, the *reference nonexpanding space*.

In other terms, $V_0^\geq$ is the direct sum of all restricted weight spaces corresponding to weights in $\Omega_{X_0}^\geq$, and similarly for the other spaces.

Clearly these are precisely the dynamical spaces (see Subsection 4.3) associated to the map $\exp(X_0)$ (acting on V by ρ).

Remark 4.2. Note that since X_0 is generic, $V_0^=$ is actually just the zero restricted weight space:

$$V_0^= = V^0;$$

moreover by Assumption 3.2, zero *is* a restricted weight, so this space is nontrivial.

Example 4.3. (1) For $G = \mathrm{SO}^+(p,q)$ acting on $V = \mathbb{R}^{p+q}$ (where $p \geq q$), there is only one generic type. The spaces $V_0^>$ and $V_0^<$ are some maximal totally isotropic subspaces (transverse to each other), $V_0^\geq$ and $V_0^\leq$ are their respective orthogonal complements, and $V_0^=$ is the $(p-q)$-dimensional space orthogonal to both $V_0^>$ and $V_0^<$.

(2) If G is any semisimple real Lie group acting on $V = \mathfrak{g}$ (its Lie algebra) by the adjoint representation, then the reference noncontracting space $\mathfrak{g}_0^\geq$ is obviously equal to $\mathfrak{p}_{X_0}^+$. There is once again only one generic type, given by any $X_0 \in \mathfrak{a}^{++}$; we then have $\Pi_{X_0} = \emptyset$, so that $\mathfrak{p}_{X_0}^+ = \mathfrak{p}^+$ is actually the (reference) *minimal* parabolic subalgebra. We have similar identities for the other dynamical spaces, namely:

$$\mathfrak{g}_0^\geq = \mathfrak{p}^+; \quad \mathfrak{g}_0^> = \mathfrak{n}^+;$$

$$\mathfrak{g}_0^\leq = \mathfrak{p}^-; \quad \mathfrak{g}_0^< = \mathfrak{n}^-;$$

$$\mathfrak{g}_0^= = \mathfrak{l}.$$

Let us now understand what happens when we apply an element of G to one of those subspaces. The motivation for this, as well as the explanation of the term “reference subspace”, comes from Corollary 4.17.

Proposition 4.4. *We have*

(i) $\mathrm{Stab}_W(V_0^{\geq}) = \mathrm{Stab}_W(V_0^{>}) = \mathrm{Stab}_W(V_0^{\leq}) = \mathrm{Stab}_W(V_0^{<}) = W_{X_0}$,

(ii) $\mathrm{Stab}_G(V_0^{\geq}) = \mathrm{Stab}_G(V_0^{>}) = P_{X_0}^{+}$,

(iii) $\mathrm{Stab}_G(V_0^{\leq}) = \mathrm{Stab}_G(V_0^{<}) = P_{X_0}^{-}$.

Remark 4.5. Note that every restricted weight space is invariant by $Z_G(A) = L$: indeed, take some $\lambda \in \mathfrak{a}^*$, $v \in V^\lambda$, $l \in L$, $X \in \mathfrak{a}$; then we have:

$$X \cdot l(v) = l(\mathrm{Ad}(l^{-1})(X) \cdot v) = l(X \cdot v) = \lambda(X) l(v). \tag{4.1}$$

Moreover, the group $N_G(A)$ permutes these spaces. So if we have a direct sum of several restricted weight spaces, it makes sense to talk about its image by an element of W; and we have the obvious identity

$$wV^\lambda = V^{w\lambda} \quad \text{for all } w \in W, \lambda \in \mathfrak{a}^*. \tag{4.2}$$

Proof of Proposition 4.4. (i) First note that since X_0 is generic, the only restricted weight that vanishes on X_0 is the zero weight, so we have indeed

$$\mathrm{Stab}_W(\Omega_0^{\geq}) = \mathrm{Stab}_W(\Omega_0^{>}) = \mathrm{Stab}_W(\Omega_0^{\leq}) = \mathrm{Stab}_W(\Omega_0^{<}),$$

hence

$$\mathrm{Stab}_W(V_0^{\geq}) = \mathrm{Stab}_W(V_0^{>}) = \mathrm{Stab}_W(V_0^{\leq}) = \mathrm{Stab}_W(V_0^{<}).$$

Moreover, this group is obviously included in $W_{\rho,X_0} = \mathrm{Stab}_W(\mathfrak{a}_{\rho,X_0})$, which is also equal to W_{X_0} since X_0 is extreme. Conversely, let $w \in W_{X_0}$; then X_0 is fixed by w, and so is (say) the set $\Omega_{X_0}^{\geq}$ of restricted weights nonnegative on X_0. It follows that $\mathrm{Stab}_W(V_0^{\geq})$ contains W_{X_0}.

(ii) We first show that both $\mathrm{Stab}_G\, V_0^{>}$ and $\mathrm{Stab}_G\, V_0^{\geq}$ contain the group P^+.

- The group L stabilizes every restricted weight space V^λ, as noted in Remark 4.5 above.
- Let α be a positive restricted root and λ a restricted weight such that the value $\lambda(X_0)$ is positive (resp. nonnegative). Then clearly we have

  $$\mathfrak{g}^\alpha \cdot V^\lambda \subset V^{\lambda+\alpha},$$

  and $(\lambda+\alpha)(X_0)$ is still positive (resp. nonnegative). Hence $\mathfrak{n}^+$ stabilizes $V_0^{>}$ and $V_0^{\geq}$.
- The statement follows as $P^+ = L\exp(\mathfrak{n}^+)$.

Now take any element $g \in G$. Let us apply the Bruhat decomposition: we may write

$$g = p_1 w p_2,$$

with p_1, p_2 some elements of the minimal parabolic subgroup P^+ and w some element of the restricted Weyl group W (see e.g. [16], Theorem 7.40). (Technically we need to replace $w \in W = N_G(A)/Z_G(A)$ by some representative $\tilde{w} \in N_G(A)$; but by the remark preceding this proof, we may ignore this distinction.) From the statement that we just proved it immediately follows that

$$\mathrm{Stab}_G(V_0^{\gtrapprox}) = \mathrm{Stab}_G(V_0^{>}) = P^+ \, \mathrm{Stab}_W(V_0^{=}) P^+ = P^+ W_{X_0} P^+. \tag{4.3}$$

On the other hand, we have the Bruhat decomposition for parabolic subgroups:

$$P_{X_0}^+ := \mathrm{Stab}_G(\mathfrak{p}_{X_0}^+) = P^+ W_{X_0} P^+. \tag{4.4}$$

This can be shown by applying a similar reasoning to the adjoint representation: indeed in that case the space $V_0^{\gtrapprox}$ corresponding to the same X_0 is just $\mathfrak{p}_{X_0}^+$. (There is just a small difficulty due to the fact that X_0 is not, in general, generic with respect to the adjoint representation.)

The conclusion follows.

(iii) Replacing P^+ and $P_{X_0}^+$ respectively by P^- and $P_{X_0}^-$, the same reasoning applies. ☐

4.2. Extended affine space.

Let V_{Aff} be an affine space whose underlying vector space is V.

Definition 4.6 (extended affine space). We choose once and for all a point p_0 in V_{Aff} which we take as an origin; we call $\mathbb{R}p_0$ the one-dimensional vector space formally generated by this point, and we set $A := V \oplus \mathbb{R}p_0$ the *extended affine space* corresponding to V. (We hope that A, the extended affine space, and A, the group corresponding to the Cartan space, occur in sufficiently different contexts that the reader will not confuse them.) Then V_{Aff} is the affine hyperplane “at height 1” of this space, and V is the corresponding vector hyperplane:

$$V = V \times \{0\} \subset V \times \mathbb{R}p_0; \quad V_{\mathrm{Aff}} = V \times \{1\} \subset V \times \mathbb{R}p_0.$$

Definition 4.7 (linear and affine group). Any affine map g with linear part $\ell(g)$ and translation vector v, defined on V_{Aff} by

$$g \colon x \longmapsto \ell(g)(x) + v,$$

can be extended in a unique way to a linear map defined on A, given by the matrix

$$\begin{pmatrix} \ell(g) & v \\ 0 & 1 \end{pmatrix}.$$

From now on, we identify the abstract group G with the group $\rho(G) \subset \mathrm{GL}(V)$, and the corresponding affine group $G \ltimes V$ with a subgroup of $\mathrm{GL}(A)$.

Definition 4.8 (affine subspaces). We define an *extended affine subspace* of A to be a vector subspace of A not contained in V. For every k, there is a one-to-one correspondence between $k+1$-dimensional extended affine subspaces of A and k-dimensional affine subspaces of V_{Aff}. For any extended affine subspace of A denoted by A_1 (or A_2, A' and so on), we denote by V_1 (or V_2, V' and so on) the space $A \cap V$ (which is the linear part of the corresponding affine space $A \cap V_{\mathrm{Aff}}$).

Definition 4.9 (translations). By abuse of terminology, elements of the normal subgroup $V \lhd G \ltimes V$ will still be called *translations*, even though we shall see them mostly as endomorphisms of A (so that they are formally transvections). For any vector $v \in V$, we denote by τ_v the corresponding translation.

Definition 4.10 (reference affine dynamical spaces). We now give a name for (the vector extensions of) the affine subspaces of V_{Aff} parallel respectively to $V_0^{\geqq}$, $V_0^{\leqq}$ and $V_0^{=}$ and passing through the origin: we set

$$\begin{aligned} A_0^{\geqq} &:= V_0^{\geqq} \oplus \mathbb{R}p_0, && \text{the } \textit{reference affine noncontracting space};\\ A_0^{\leqq} &:= V_0^{\leqq} \oplus \mathbb{R}p_0, && \text{the } \textit{reference affine nonexpanding space};\\ A_0^{=} &:= V_0^{=} \oplus \mathbb{R}p_0, && \text{the } \textit{reference affine neutral space}. \end{aligned}$$

These are obviously the affine dynamical spaces (see next subsection) corresponding to the map $\exp(X_0)$, seen as an element of $G \ltimes V$ by identifying G with the stabilizer of p_0 in $G \ltimes V$.

Definition 4.11 (affine Jordan projection). Finally, we extend the notion of Jordan projection to the whole group $G \ltimes V$, by setting

$$\mathrm{Jd}(g) := \mathrm{Jd}(\ell(g)) \quad \text{for all } g \in G \ltimes V.$$

Remark 4.12. (1) It is tempting to try to define an "affine Jordan decomposition", by observing that any affine map $g \in G \ltimes V$ may be written as $g = \tau_v g_h g_e g_u$, with g_h (resp. g_e, g_u) conjugate in $G \ltimes V$ to an element of A (resp. of K, of N^+) and v some element of V. Unfortunately, we can neither require that τ_v commute with the other three factors, nor (as erroneously claimed in the author's previous paper [22]) determine v in a unique fashion. The trouble comes from unipotent elements; to understand the problem, examine the affine transformation

$$g \colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So we must be a little more careful; see the proof of Proposition 4.16 for a more detailed study of conjugacy classes in $G \ltimes V$.

(2) We do not extend similarly the Cartan projection to $G \ltimes V$, for the following reason. While eigenvalues of an element of $G \ltimes V$ depend only on the eigenvalues of its linear part, the same statement does not hold for its singular values.

4.3. Definition of dynamical spaces. For every $g \in G \ltimes V$, we define its *linear dynamical spaces* as follows:

- $V_g^{>}$, the *expanding space* associated to g:

  the largest vector subspace of V stable by g such that all eigenvalues λ of the restriction of g to that subspace satisfy $|\lambda| > 1$;
- $V_g^{<}$, the *contracting space* associated to g:

  the same thing with $|\lambda| < 1$;
- $V_g^{=}$, the *neutral space* associated to g:

  the same thing with $|\lambda| = 1$;
- $V_g^{\geq}$, the *noncontracting space* associated to g:

  the same thing with $|\lambda| \geq 1$;
- $V_g^{\leq}$, the *nonexpanding space* associated to g:

  the same thing with $|\lambda| \leq 1$.

Equivalently, $V_g^{>}$ is the direct sum of all the generalized eigenspaces E^{λ} of g associated to eigenvalues λ of modulus larger than 1 (defined as $E^{\lambda} = \ker(g - \lambda\,\mathrm{Id})^n$ where $n = \dim V$), and similarly for the four other spaces. We then obviously have

$$V = \overbrace{V_g^{>} \oplus \underbrace{V_g^{=}}^{}}^{V_g^{\geq}}\!\!\underbrace{\phantom{V_g^{=}} \oplus V_g^{<}}_{V_g^{\leq}}. \tag{4.5}$$

Also note that the restriction of g from A to V is just its linear part, so that the linear dynamic subspaces of g only depend on $\ell(g)$.

For every $g \in G \ltimes V$, we define its *affine dynamical subspaces*:

- $A_g^{\geq}$, the *affine noncontracting space* associated to g,
- $A_g^{\leq}$, the *affine nonexpanding space* associated to g,
- and $A_g^{=}$, the *affine neutral space* associated to g,

in the same way as the linear dynamical subspaces, but with V replaced everywhere by A.

Remark 4.13. (1) Note that if we defined in the same way $A_g^{>}$ (resp. $A_g^{<}$), it would actually be contained in V and so just be equal to $V_g^{>}$ (resp. $V_g^{<}$). Indeed an element of $G \ltimes V$ can never act on a vector in $A \setminus V$ (i.e. an element of A with a nonzero component along $\mathbb{R}p_0$) with an eigenvalue other than 1.

(2) Thus the affine analog of the decomposition (4.5) is now:

$$A = \overbrace{V_g^{>} \oplus \rlap{$\underbrace{\phantom{A_g^{=} \oplus V_g^{<}}}_{A_g^{\leq}}$} A_g^{=}}^{A_g^{\geq}} \oplus V_g^{<} \tag{4.6}$$

(pay attention to the distribution of A's and V's).

(3) From this identity, it immediately follows that neither $A_g^{=}$, $A_g^{\geq}$ nor $A_g^{\leq}$ are contained in V.

(4) Finally, it is obvious that the intersections of these three spaces with V are respectively $V_g^{=}$, $V_g^{\geq}$ and $V_g^{\leq}$. Thus this notation is consistent with the convention outlined above.

In purely affine terms, these spaces may be understood as follows:

- $A_g^{=} \cap V_{\mathrm{Aff}}$ is the unique g-invariant affine space parallel to $V_g^{=}$ (the "axis" of g);
- $A_g^{\geq} \cap V_{\mathrm{Aff}}$ is the unique affine space parallel to $V_g^{\geq}$ and containing $A_g^{=} \cap V_{\mathrm{Aff}}$, and similarly for $A_g^{\leq} \cap V_{\mathrm{Aff}}$.

4.4. Description of dynamical spaces. We shall now characterize the dynamical subspaces of those elements of $G \ltimes V$ that satisfy the following property.

Definition 4.14. We say that an element $g \in G \ltimes V$ is *of type* X_0 if $\mathrm{Jd}(g)$ has the same type as X_0, i.e. if

$$\mathrm{Jd}(g) \in \mathfrak{a}_{\rho, X_0}.$$

Example 4.15. (1) For $G = \mathrm{SO}^{+}(p, q)$ acting on $V = \mathbb{R}^{p+q}$ (where $p \geq q$), there is only one generic type. For every $g \in G$, we have

$$\dim V_g^{>} = \dim V_g^{<} \leq q. \tag{4.7}$$

An element $g \in G$ is of generic type if and only if equality is attained. Such elements have been called *pseudohyperbolic* in the previous literature ([3, 21]).

(2) If G is any semisimple real Lie group acting on $V = \mathfrak{g}$ (its Lie algebra) by the adjoint representation, there is only one generic type and an element $g \in G$ is of that type if and only if $\mathrm{Jd}(g) \in \mathfrak{a}^{++}$. Such elements are called $\mathbb{R}$-*regular* or (particularly in [8]) *loxodromic*.

Here is a partial description of the dynamical spaces of an element of type X_0.

Proposition 4.16. *Let $g \in G \ltimes V$ be a map of type X_0. In that case*:

(i) *there exists a map $\phi \in G \ltimes V$, called a **canonizing map** for g, such that*

$$\begin{cases} \phi(A^{\geq}_{g}) = A^{\geq}_{0}, \\ \phi(A^{\leq}_{g}) = A^{\leq}_{0}; \end{cases}$$

(ii) *the space $V^{>}_{g}$ is uniquely determined by $A^{\geq}_{g}$. The space $V^{<}_{g}$ is uniquely determined by $A^{\leq}_{g}$.*

(Compare this with Claim 2.5 in [22].)

Proof. (i) We start with the obvious decomposition

$$g = \tau_v \ell(g), \tag{4.8}$$

where $\ell(g) \in G$ is the linear part of g (seen as an element of $G \ltimes V$ by identifying G with the stabilizer of the origin p_0) and $v \in V$ is its translation part. We then observe that we may rewrite this as

$$g = \tau_{v'} \tau_w^{-1} \ell(g) \tau_w \tag{4.9}$$

for some $w \in V$, where v' is now actually an element of $V^{=}_{g}$. Indeed, for any translation vector $v \in V$ and linear map $f \in G$, we have

$$f \tau_v = \tau_{f(v)} f. \tag{4.10}$$

The statement then follows from the fact that the map induced by $\ell(g) - \mathrm{Id}$ on $V^{>}_{g} \oplus V^{<}_{g}$ does not have 0 as an eigenvalue, hence is surjective. (In fact, this argument shows that we could even require v' to lie in the actual characteristic space corresponding to the eigenvalue 1.)

Now let $\ell(g) =: g_h g_e g_u$ be the Jordan decomposition of $\ell(g)$, so that

$$\tau_w g \tau_w^{-1} = \tau_{v'} g_h g_e g_u; \tag{4.11}$$

let $\phi_\ell \in G$ be any map that conjugates g_h to $\exp(\mathrm{Jd}(g))$, i.e. such that $\phi_\ell g_h \phi_\ell^{-1} = \exp(\mathrm{Jd}(g))$; and let $\phi := \phi_\ell \tau_w$.

Calling $g' := \phi g \phi^{-1}$ and $\tau_{v''}$, g'_e, g'_u the respective conjugates of the maps $\tau_{v'}$, g_e, g_u by ϕ_ℓ (so that $v'' = \phi_\ell(v')$), we then have

$$g' = \tau_{v''} \exp(\mathrm{Jd}(g)) g'_e g'_u, \tag{4.12}$$

where $g'_e \in G$ is elliptic, $g'_u \in G$ is unipotent, both of them commute with $\exp(\mathrm{Jd}(g))$, and $v'' \in V^{=}_{g'}$.

As already seen in the proof of Proposition 2.6, g'_e and g'_u have all eigenvalues of modulus 1 and commute with $\exp(\mathrm{Jd}(g))$. Hence the linear dynamical spaces of g' coincide with those of $\exp(\mathrm{Jd}(g))$.

Now since $\exp(\mathrm{Jd}(g)) \in G$ fixes p_0, the space $A^{=}_{\exp(\mathrm{Jd}(g))}$ is equal to $V^{=}_{\exp(\mathrm{Jd}(g))} \oplus \mathbb{R}p_0$; and since $v'' \in V^{=}_{g'}$, that space is still invariant by g'. It follows that we have

$$A^{=}_{g'} = A^{=}_{\exp(\mathrm{Jd}(g))} = V^{=}_{\exp(\mathrm{Jd}(g))} \oplus \mathbb{R}p_0. \tag{4.13}$$

By taking the direct sum with $V^{>}$ and with $V^{<}$, we deduce that all the affine dynamical spaces of g' coincide with those of $\exp(\mathrm{Jd}(g))$.

Now since g is of type X_0, by definition, $\mathrm{Jd}(g)$ is a vector in $\mathfrak{a}$ that has the same type as X_0. It follows that the affine dynamical subspaces of $\exp(\mathrm{Jd}(g))$ coincide with those of $\exp(X_0)$, which are the reference subspaces. We conclude that

$$\begin{cases} A^{\geq}_{g'} = A^{\geq}_0; \\ A^{\leq}_{g'} = A^{\leq}_0. \end{cases}$$

Since obviously $A^{\geq}_{g'} = \phi(A^{\geq}_g)$ and similarly for $A^{\leq}$, the conclusion follows.

(ii) Suppose that g_1 and g_2 are two maps of type X_0 such that $A^{\geq}_{g_1} = A^{\geq}_{g_2}$. Define $g'_1 = \phi_1 g_1 \phi_1^{-1}$ and $g'_2 = \phi_2 g_2 \phi_2^{-1}$ as in the previous point; then we have

$$A^{\geq}_{g_{1,2}} = \phi_1^{-1}(A^{\geq}_0) = \phi_2^{-1}(A^{\geq}_0).$$

In other terms, the transition map $\phi_2 \circ \phi_1^{-1}$ stabilizes $A^{\geq}_0$.

Clearly the linear part of $\phi_2 \circ \phi_1^{-1}$ then stabilizes $V^{\geq}_0$. It follows from Proposition 4.4 that it also stabilizes $V^{>}_0$. Since the latter space is contained in V, the translation part of $\phi_2 \circ \phi_1^{-1}$ acts trivially on it; so the affine map $\phi_2 \circ \phi_1^{-1}$ itself also stabilizes $V^{>}_0$.

Now obviously we also have $V^{>}_{g_1} = \phi_1^{-1}(V^{>}_{g'_1})$, and similarly for g_2 and ϕ_2. But it also follows from the previous point that

$$V^{>}_{g'_1} = V^{>}_{g'_2} = V^{>}_0.$$

We conclude that $V^{>}_{g_1} = V^{>}_{g_2}$ as required.

The same proof works for $A^{\leq}$ and $V^{<}$. □

This immediately allows us to describe the remaining dynamical spaces of g:

Corollary 4.17. *Let $g \in G \ltimes V$ be a map of type X_0. Then if $\phi \in G \ltimes V$ is* ***any*** *canonizing map of g, we have*

$$\phi(A^{\geq}_g) = A^{\geq}_0, \quad \phi(V^{\geq}_g) = V^{\geq}_0, \quad \phi(V^{>}_g) = V^{>}_0,$$
$$\phi(A^{\leq}_g) = A^{\leq}_0, \quad \phi(V^{\leq}_g) = V^{\leq}_0, \quad \phi(V^{<}_g) = V^{<}_0,$$
$$\phi(A^{=}_g) = A^{=}_0, \quad \phi(V^{=}_g) = V^{=}_0.$$

In other terms, if ϕ is a canonizing map of g then all eight dynamical spaces of the conjugate $\phi g \phi^{-1}$ coincide with the reference dynamical spaces. This explains why we called them "reference" spaces.

Proof. The equalities for $A^{\geq}$ and $A^{\leq}$ hold by definition of a canonizing map. The equality for $A^{=}$ follows by taking the intersection. The equalities for $V^{\geq}$, $V^{\leq}$ and $V^{=}$ follow by taking the linear part. The equalities for $V^{>}$ and $V^{<}$ follow from Proposition 4.16 (ii). ☐

4.5. Quasi-translations. Let us now investigate the action of a map $g \in G \ltimes V$ of type X_0 on its affine neutral space $A_g^{=}$. The goal of this subsection is to prove that it is "almost" a translation (Proposition 4.20).

We fix on V a Euclidean form B satisfying the conditions of Lemma 2.4 for the representation ρ.

Definition 4.18. We call *quasi-translation* any affine automorphism of $A_0^{=}$ induced by an element of $L \ltimes V_0^{=}$.

Let us explain and justify this terminology. First note that the action of L on $V_0^{=}$ preserves B: indeed, the action of M does so because $M \subset K$, and the action of A on this space is just trivial. The following statement is then immediate:

Proposition 4.19. *Let V_0^t be the set of fixed points of L in $V_0^{=}$:*

$$V_0^t := \{v \in V_0^{=} \mid \text{for all } l \in L,\ lv = v\}.$$

(*Note that this is also the set of fixed points of M*). *Let V_0^r be the B-orthogonal complement of V_0^t in $V_0^{=}$, and let $\mathrm{O}(V_0^r)$ denote the set of B-preserving automorphisms of V_0^r. Then any quasi-translation is an element of*

$$(\mathrm{O}(V_0^r) \ltimes V_0^r) \times V_0^t.$$

In other words, quasi-translations are affine isometries of $V_0^{=}$ that preserve the directions of V_0^r and V_0^t and act by a pure translation on the V_0^t component. You may think of a quasi-translation as a kind of "screw displacement"; the superscripts t and r respectively stand for "translation" and "rotation".

We now claim that any map of type X_0 acts on its affine neutral space by quasi-translations:

Proposition 4.20. *Let $g \in G \ltimes V$ be a map of type X_0, and let $\phi \in G \ltimes V$ be any canonizing map for g. Then the restriction of the conjugate $\phi g \phi^{-1}$ to $A_0^{=}$ is a quasi-translation.*

Let us actually formulate an even more general result, which will have another application in the next subsection.

Lemma 4.21. *Any map $f \in G \ltimes V$ stabilizing both $A_0^{\geqq}$ and $A_0^{\leqq}$ acts on $A_0^{=}$ by quasi-translation.*

Proof. We begin by showing that any element of $\mathfrak{l}_{X_0} = \mathfrak{p}_{X_0}^+ \cap \mathfrak{p}_{X_0}^-$ acts on $V_0^=$ in the same way as some element of $\mathfrak{l}$. Recall that by definition

$$\mathfrak{l}_{X_0} = \mathfrak{l} \oplus \bigoplus_{\alpha(X_0)=0} \mathfrak{g}^\alpha;$$

hence it is sufficient to show that for every restricted root α such that $\alpha(X_0) = 0$, we have $\mathfrak{g}^\alpha \cdot V_0^= = 0$. Indeed, since $V_0^= = V^0$ (because X_0 is generic), we have

$$\mathfrak{g}^\alpha \cdot V_0^= \subset V^\alpha.$$

On the other hand, we know by Proposition 4.4 that for such α, the action of $\mathfrak{g}^\alpha$ stabilizes both $V_0^{\geqq}$ and $V_0^{\leqq}$; it follows that the image $\mathfrak{g}^\alpha \cdot V_0^=$ lies in both of these spaces, hence in their intersection $V_0^=$, which is also V^0. Since α is nonzero, we have $V^0 \cap V^\alpha = 0$, which yields the desired equality.

Let $P_{X_0,e}^+$ and $P_{X_0,e}^-$ denote the identity components of $P_{X_0}^+$ and $P_{X_0}^-$ respectively; by integrating the previous statement, it follows that any element of $P_{X_0,e}^+ \cap P_{X_0,e}^-$ acts on $V_0^=$ in the same way as some element of L.

Now it follows from [16] Proposition 7.82 (d) (using 7.83 (e)) that

$$L_{X_0} = P_{X_0}^+ \cap P_{X_0}^- \subset M(P_{X_0,e}^+ \cap P_{X_0,e}^-). \tag{4.14}$$

(Here we are using the assumption that G is connected.) We deduce that any element of L_{X_0} acts on $V_0^=$ in the same way as some element of L.

Finally, any $f \in G \ltimes V$ stabilizing both $A_0^{\geqq}$ and $A_0^{\leqq}$ has linear part stabilizing both $V_0^{\geqq}$ and $V_0^{\leqq}$ (hence lying in L_{X_0}, by Proposition 4.4), and translation part contained both in $V_0^{\geqq}$ and in $V_0^{\leqq}$ (in other words, in $V_0^=$). The conclusion follows. □

Proof of Proposition 4.20. The proposition follows immediately from this lemma by taking $f = \phi g \phi^{-1}$. Indeed, by definition the "canonized" map $\phi g \phi^{-1}$ has $A_0^{\geqq}$ and $A_0^{\leqq}$ as dynamical spaces; in particular it stabilizes them. □

Example 4.22. (1) For $G = \mathrm{SO}^+(p, q)$ acting on $V = \mathbb{R}^{p+q}$ (with $p \geq q$), we have

- $M \simeq \mathrm{SO}_{p-q}(\mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})^{q-1}$,
- $V_0^= = V^0 \simeq \mathbb{R}^{p-q}$,

and the action of M on $V_0^=$ is as follows: the connected factor $\mathrm{SO}_{p-q}(\mathbb{R})$ acts in the obvious way; the discrete factor $(\mathbb{Z}/2\mathbb{Z})^{q-1}$ acts trivially. We may then distinguish two cases.

a. If $p - q \geq 2$, then the action of M is transitive. The space V_0^t is trivial and $V_0^r = V_0^=$. Any affine isometry of $V_0^=$ may be a quasi-translation.

b. If $p - q = 1$, then the group M is trivial. We have on the contrary $V_0^t = V_0^=$ and V_0^r is trivial. A quasi-translation is just a translation.

(We exclude the case $p = q$ because in that case $V^0 = 0$, which violates Assumption 3.2.)

(2) More generally if G is split, then we have $\mathfrak{m} = 0$. The group M is in general a nontrivial finite group; however, it can be shown (by considering the complexification of G) that we still always have $V_0^t = V_0^=$, and a quasi-translation is still just a translation.

(3) If G is any semisimple real Lie group acting on $V = \mathfrak{g}$ (its Lie algebra) by the adjoint representation, then:

- $\mathfrak{g}_0^= = \mathfrak{g}^0 = \mathfrak{l}$;
- $\mathfrak{g}_0^t$ is the direct sum of $\mathfrak{a}$ and of the center of $\mathfrak{m}$;
- $\mathfrak{g}_0^r$ is the semisimple part of $\mathfrak{m}$ (in other terms, its derived subalgebra).

The example of $G = \mathrm{SO}^+(4,1)$ (acting on $\mathfrak{so}(4,1)$, not on $\mathbb{R}^5$) shows that V_0^t and V_0^r can both be nontrivial at the same time.

We would like to treat quasi-translations a bit like translations; for this, we need to have at least a nontrivial space V_0^t. So from now on, we exclude cases like 1.a. in the list of examples we just considered (Example 4.22):

Assumption 4.23. The representation ρ is such that

$$\dim V_0^t > 0.$$

This is precisely condition (i)(a) from the Main Theorem.

4.6. Canonical identifications and the Margulis invariant. The main goal of this subsection is to associate to every map $g \in G \ltimes V$ of type X_0 a vector in V_0^t, called its "Margulis invariant" (see Definition 4.31). The two propositions (4.27 and 4.29) and and the lemma (4.30) that lead up to this definition are important as well, and will be often used subsequently.

Corollary 4.17 has shown us that the “geometry” of any map g of type X_0 (namely the position of its dynamical spaces) is entirely determined by the pair of spaces

$$(A_g^{\geq}, A_g^{\leq}) = \phi(A_0^{\geq}, A_0^{\leq}).$$

In fact, such pairs of spaces play a crucial role. Let us begin with a definition; its connection with the observation we just made will become clear after Proposition 4.27.

Definition 4.24. We define a *parabolic space* to be any subspace of V that is the image of $V_0^{\geq}$ by some element of G.

We define an *affine parabolic space* to be any subspace of A that is the image of $A_0^{\geq}$ by some element of $G \ltimes V$.

We say that two parabolic spaces (or two affine parabolic spaces) are *transverse* if their intersection has the lowest possible dimension.

Remark 4.25. Since X_0 is symmetric, $V_0^{\leq}$ (resp. $A_0^{\leq}$) is in particular a parabolic space (resp. an affine parabolic space).

A subspace $A^{\geq} \subset A$ is an affine parabolic space if and only if it is not contained in V and its linear part $V^{\geq} = A^{\geq} \cap V$ is a parabolic space.

Clearly $V_0^{\geq}$ and $V_0^{\leq}$ are transverse, and so are $A_0^{\geq}$ and $A_0^{\leq}$. So two parabolic spaces (resp. affine parabolic spaces) are transverse if and only if their intersection has the same dimension as $V_0^{=}$ (resp. $A_0^{=}$).

Example 4.26. (1) For $G = \mathrm{SO}^+(p, q)$ acting on $V = \mathbb{R}^{p+q}$ (let us assume $p \geq q$), a subspace $F \subset \mathbb{R}^{p+q}$ is a parabolic space if and only if $F^{\perp}$ is a maximal isotropic subspace. Equivalently, F is a parabolic space if and only if F contains $F^{\perp}$ and is minimal for that property (namely p-dimensional). Two parabolic spaces are transverse if and only if their intersection has dimension $p - q$. Pairs of transverse parabolic spaces were called *frames* in [21].

(2) If G is any semisimple real Lie group acting on $V = \mathfrak{g}$ (its Lie algebra) by the adjoint representation, a parabolic space is just an arbitrary minimal parabolic subalgebra of $\mathfrak{g}$ (hence the name “parabolic space”).

Proposition 4.27. *A pair of parabolic spaces* (*resp. of affine parabolic spaces*) *is transverse if and only if it may be sent to* $(V_0^{\geq}, V_0^{\leq})$ (*resp. to* $(A_0^{\geq}, A_0^{\leq})$) *by some element of* G (*resp. of* $G \ltimes V$).

In particular, it follows from Proposition 4.16 that for any map $g \in G \ltimes V$ of type X_0, the pair $(A_g^{\geq}, A_g^{\leq})$ is a transverse pair of affine parabolic spaces.

This Proposition, as well as its proof, is very similar to Claim 2.8 in [22].

Proof. Let us prove the linear version; the affine version follows immediately. Let (V_1, V_2) be any pair of parabolic spaces. By definition, for $i = 1, 2$, we may write $V_i = \phi_i(V_0^{\gtrsim})$ for some $\phi_i \in G$. Let us apply the Bruhat decomposition to the map $\phi_1^{-1}\phi_2$: we may write

$$\phi_1^{-1}\phi_2 = p_1 w p_2, \tag{4.15}$$

where p_1, p_2 belong to the minimal parabolic subgroup P^+, and w is an element of the restricted Weyl group W (or, technically, some representative thereof). Let $\phi := \phi_1 p_1 = \phi_2 p_2^{-1} w^{-1}$; since P^+ stabilizes $V_0^{\gtrsim}$, we have

$$V_1 = \phi(V_0^{\gtrsim}) \quad \text{and} \quad V_2 = \phi(wV_0^{\gtrsim}). \tag{4.16}$$

Thus V_1 and V_2 are transverse if and only if $wV_0^{\gtrsim}$ is transverse to $V_0^{\gtrsim}$, which means that the dimension of their intersection, which is also equal to the sum of the multiplicities of restricted weights contained in the intersection

$$\Omega_{X_0}^{\gtrsim} \cap w\Omega_{X_0}^{\gtrsim},$$

is the smallest possible.

Clearly, this last intersection always contains $\{0\}$. Since X_0 is generic, it can actually be equal to $\{0\}$ if only we can choose w so as to have

$$w\Omega_{X_0}^{\gtrsim} = \Omega_{X_0}^{\lesssim}. \tag{4.17}$$

Since X_0 is symmetric, this identity (4.17) *is* realized in particular for $w = w_0$. This means that V_1 and V_2 are transverse if and only if w satisfies (4.17), in which case we have indeed $V_1 = \phi(V_0^{\gtrsim})$ and $V_2 = \phi(V_0^{\lesssim})$ as required. □

Remark 4.28. It follows from Proposition 4.4 that the set of all parabolic spaces can be identified with the flag variety $G/P_{X_0}^+$, by identifying every parabolic space $\phi(V_0^{\gtrsim})$ with the coset $\phi P_{X_0}^+$.

In this interpretation, two parabolic spaces $V_1 = \phi_1(V_0^{\gtrsim})$ and $V_2 = \phi_2(V_0^{\gtrsim}) = \phi_2 \circ w_0(V_0^{\lesssim})$ are then transverse if and only if the corresponding pair of cosets

$$(\phi_1 P_{X_0}^+,\ \phi_2 w_0 P_{X_0}^-)$$

is in the G-orbit of the point $(P_{X_0}^+, P_{X_0}^-)$ in $G/P_{X_0}^+ \times G/P_{X_0}^-$, also known as the *open G-orbit* in $G/P_{X_0}^+ \times G/P_{X_0}^-$, since it can be shown that it is indeed the unique open G-orbit in that space.

Consider a transverse pair of affine parabolic spaces. Their intersection may be seen as a sort of "abstract affine neutral space". We now introduce a family of "canonical identifications" between those spaces. Unfortunately, these identifications have an inherent ambiguity: they are only defined up to quasi-translation.

Proposition 4.29. *Let* (A_1, A_2) *be a pair of transverse affine parabolic spaces. Then any map* $\phi \in G \ltimes V$ *such that* $\phi(A_1, A_2) = (A_0^{\geqslant}, A_0^{\leqslant})$ *gives, by restriction, an identification of the intersection* $A_1 \cap A_2$ *with* $A_0^{=}$*, which is unique up to quasi-translation.*

Here by $\phi(A_1, A_2)$ we mean the pair $(\phi(A_1), \phi(A_2))$. Note that if $A_1 \cap A_2$ is obtained in another way as an intersection of two affine parabolic spaces, the identification with $A_0^{=}$ will, in general, no longer be the same, not even up to quasi-translation: there could also be an element of the Weyl group involved.

Compare this with Corollary 2.14 in [22].

Proof. The existence of such a map ϕ follows from Proposition 4.27. Now let ϕ and ϕ' be two such maps, and let f be the map such that

$$\phi' = f \circ \phi \tag{4.18}$$

(i.e. $f := \phi' \circ \phi^{-1}$). Then by construction f stabilizes both $A_0^{\geqslant}$ and $A_0^{\leqslant}$. It follows from Lemma 4.21 that the restriction of f to $A_0^{=}$ is a quasi-translation. ☐

Let us now explain why we call these identifications "canonical". The following lemma, while seemingly technical, is actually crucial: it tells us that the identifications defined in Proposition 4.29 commute (up to quasi-translation) with the projections that naturally arise if we change one of the parabolic subspaces in the pair while fixing the other.

Lemma 4.30. *Take any affine parabolic space* A_1*.*

Let A_2 *and* A'_2 *be any two affine parabolic spaces both transverse to* A_1*.*

Let ϕ *(respectively* ϕ'*) be an element of* $G \ltimes V$ *that sends the pair of spaces* (A_1, A_2) *(respectively* (A_1, A'_2)*) to* $(A_0^{\geqslant}, A_0^{\leqslant})$*; these two maps exist by Proposition 4.27.*

Let W_1 *be the inverse image of* $V_0^{>}$ *by any map* ϕ *such that* $A_1 = \phi^{-1}(A_0^{\geqslant})$ *(this image is unique by Proposition 4.4).*

Let

$$\psi \colon A_1 \longrightarrow A_1 \cap A'_2$$

be the projection parallel to W_1*.*

Then the map $\overline{\psi}$ *defined by the commutative diagram*

$$\begin{tikzcd}
A_0^{=} \arrow[r, "\overline{\psi}"] & A_0^{=} \\
A_1 \cap A_2 \arrow[u, "\phi"] \arrow[r, "\psi"] & A_1 \cap A'_2 \arrow[u, "\phi'"']
\end{tikzcd}$$

is a quasi-translation.

The space W_1 is, in some sense, the "abstract linear expanding space" corresponding to the "abstract affine noncontracting space" A_1: more precisely, for any map $g \in G \ltimes V$ of type X_0 such that $A_g^{\geq} = A_1$, we have $V_g^{>} = W_1$ (by Proposition 4.16 (ii)).

The projection ψ is well defined because $A_0^{\geq} = V_0^{>} \oplus A_0^{=} = V_0^{>} \oplus (A_0^{\geq} \cap A_0^{\leq})$, and so $A_1 = \phi'^{-1}(A_0^{\geq}) = W_1 \oplus (A_1 \cap A_2')$.

This statement generalizes Lemma 2.18 in [22]. The proof is similar, but care must be taken to replace minimal parabolics by parabolics of type X_0.

Proof. Without loss of generality, we may assume that $\phi = \mathrm{Id}$ (otherwise we simply replace the three affine parabolic spaces by their images under ϕ^{-1}.) Then we have $A_1 = A_0^{\geq}$, $A_2 = A_0^{\leq}$ and $A_2' = \phi'^{-1}(A_0^{\leq})$, where ϕ' can be any map stabilizing the space $A_0^{\geq}$. We want to show that the map $\overline{\psi} = \phi' \circ \psi$ (considered as a map from $A_0^{=}$ to itself) is a quasi-translation.

We know that ϕ' lies in the stabilizer $\mathrm{Stab}_{G \ltimes V}(A_0^{\geq})$; by Proposition 4.4, the latter is equal to $P_{X_0}^{+} \ltimes V_0^{\geq}$. We now introduce the algebra

$$\mathfrak{n}_{X_0}^{+} := \bigoplus_{\alpha(X_0)>0} \mathfrak{g}^{\alpha} \tag{4.19}$$

and the group $N_{X_0}^{+} := \exp \mathfrak{n}_{X_0}^{+}$. We then have the Langlands decomposition

$$P_{X_0}^{+} = L_{X_0} N_{X_0}^{+} \tag{4.20}$$

(see e.g. [16], Proposition 7.83). Since L_{X_0} stabilizes $V_0^{>}$, this generalizes to the "affine Langlands decomposition"

$$P_{X_0}^{+} \ltimes V_0^{\geq} = (L_{X_0} \ltimes V_0^{=})(N_{X_0}^{+} \ltimes V_0^{>}). \tag{4.21}$$

Thus we may write $\phi' = l \circ n$ with $l \in L_{X_0} \ltimes V_0^{=}$ and $n \in N_{X_0}^{+} \ltimes V_0^{>}$.

We shall use the following fact: every element n of the group $N_{X_0}^{+} \ltimes V_0^{>}$ stabilizes the space $V_0^{>}$ and induces the identity map on the quotient space $A_0^{\geq}/V_0^{>}$. Indeed, when the element n lies in $N_{X_0}^{+}$, since $N_{X_0}^{+}$ is connected, this follows from the fact that $\mathfrak{n}_{X_0}^{+} \cdot V_0^{\geq} \subset V_0^{>}$ (which, in turn, follows from the obvious fact that if $\lambda(X_0) \geq 0$ and $\alpha(X_0) > 0$, then $(\lambda + \alpha)(X_0) > 0$). When n is a pure translation by a vector of $V_0^{>}$, this is obvious.

By definition, ψ also stabilizes $V_0^{>}$ and induces the identity on $A_0^{\geq}/V_0^{>}$; hence so does the map $n \circ \psi$. But we also know that $n \circ \psi$ is defined on $A_1 \cap A_2 = A_0^{=}$, and sends it onto

$$n \circ \psi(A_1 \cap A_2) = n(A_1 \cap A_2') = l^{-1}(A_0^{=}) = A_0^{=}.$$

Hence the map $n \circ \psi$ is the identity on $A_0^{=}$. It follows that $\overline{\psi} = \phi' \circ \psi = l \circ n \circ \psi = l$ (in restriction to $A_0^{=}$); by Lemma 4.21, $\overline{\psi}$ is a quasi-translation as required. □

Now let g be a map of type X_0. We already know that it acts on its neutral affine space by quasi-translation; now the canonical identifications we have just introduced allow us to compare the actions of different elements on their respective neutral affine spaces, as if they were both acting on the same space $A_0^=$. However there is a catch: since the identifications are only canonical up to quasi-translation, we lose information about the rotation part; only the translation part along V_0^t remains.

Formally, we make the following definition. Let π_t denote the projection from $V_0^=$ onto V_0^t parallel to V_0^r.

Definition 4.31. Let $g \in G \ltimes V$ be a map of type X_0. Take any point x in the affine space $A_g^= \cap V_{\mathrm{Aff}}$ and any map $\phi \in G$ such that $\phi(V_g^{\geq}, V_g^{\leq}) = (V_0^{\geq}, V_0^{\leq})$. Then the vector

$$M(g) := \pi_t(\phi(g(x) - x)) \in V_0^t.$$

is called the *Margulis invariant* of g.

This vector does not depend on the choice of x or ϕ: indeed, composing ϕ with a quasi-translation does not change the V_0^t-component of the image. See Proposition 2.16 in [22] for a detailed proof of this claim (for $V = \mathfrak{g}$).

5. Quantitative properties

In this section, we define and study two important quantitative properties of maps of type X_0:

- C-non-degeneracy, which means that the geometry of the map is not too close to a degenerate case;
- and contraction strength, which measures the extent to which the map g is "much more contracting" on its contracting space than on its affine nonexpanding space.

In Subsection 5.1, we define these and several other quantitative properties. Several definitions coincide with those from Section 2.6 in [22] or generalize them.

In the very short Subsection 5.2 (which is a straightforward generalization of Section 2.7 from [22]), we compare these properties for an affine map and its linear part.

In Subsection 5.3, we define analogous quantitative properties for proximal maps, and relate properties of a product of a (sufficiently contracting and nondegenerate) pair of proximal maps to the properties of the factors. This is almost the same thing as Section 3.1 in [22], but with one additional result.

5.1. Definitions. We endow the extended affine space A with a Euclidean norm (written simply $\|\cdot\|$) whose restriction to V coincides with the norm B defined in Lemma 2.4 and that makes p_0 orthogonal to V. Then the subspaces $V_0^>$, $V_0^<$, V_0^r, V_0^t and $\mathbb{R}p_0$ are pairwise orthogonal, and the restriction of this norm to V_0^r is invariant by quasi-translations. For any linear map g acting on A, we write $\|g\| := \sup_{x\neq 0} \frac{\|g(x)\|}{\|x\|}$ its operator norm.

Consider a Euclidean space E (for the moment, the reader may suppose that $E = A$; later we will also need the case $E = \Lambda^p A$ for some integer p). We introduce on the projective space $\mathbb{P}(E)$ a metric by setting, for every $\overline{x}, \overline{y} \in \mathbb{P}(E)$,

$$\alpha(\overline{x}, \overline{y}) := \arccos \frac{|\langle x, y\rangle|}{\|x\|\|y\|} \in [0, \tfrac{\pi}{2}], \tag{5.1}$$

where x and y are any vectors representing respectively $\overline{x}$ and $\overline{y}$ (obviously, the value does not depend on the choice of x and y). This measures the angle between the lines $\overline{x}$ and $\overline{y}$. For shortness' sake, we will usually simply write $\alpha(x, y)$ with x and y some actual vectors in $E \setminus \{0\}$.

For any vector subspace $F \subset E$ and any radius $\varepsilon > 0$, we shall denote the ε-neighborhood of F in $\mathbb{P}(E)$ by:

$$B_{\mathbb{P}}(F, \varepsilon) := \{x \in \mathbb{P}(E) \mid \alpha(x, \mathbb{P}(F)) < \varepsilon\}. \tag{5.2}$$

(You may think of it as a kind of "conical neighborhood".)

Consider a metric space $(\mathcal{M}, \delta)$; let X and Y be two subsets of $\mathcal{M}$. We shall denote the ordinary, minimum distance between X and Y by

$$\delta(X, Y) := \inf_{x\in X} \inf_{y\in Y} \delta(x, y), \tag{5.3}$$

as opposed to the Hausdorff distance, which we shall denote by

$$\delta^{\mathrm{Haus}}(X, Y) := \max\big(\sup_{x\in X} \delta(\{x\}, Y),\ \sup_{y\in Y} \delta(\{y\}, X)\big). \tag{5.4}$$

Finally, we introduce the following notation. Let X and Y be two positive quantities, and $p_1, \dots, p_k$ some parameters. Whenever we write

$$X \lesssim_{p_1,\dots,p_k} Y,$$

we mean that there is a constant K, depending on nothing but $p_1, \dots, p_k$, such that $X \leq KY$. (If we do not write any subscripts, this means of course that K is an "absolute" constant — or at least, that it does not depend on any "local" parameters; we consider the "global" parameters such as the choice of G and of the Euclidean norms to be fixed once and for all.) Whenever we write

$$X \asymp_{p_1,\dots,p_k} Y,$$

we mean that $X \lesssim_{p_1,\dots,p_k} Y$ and $Y \lesssim_{p_1,\dots,p_k} X$ at the same time.

Definition 5.1. Take a pair of affine parabolic spaces (A_1, A_2). An *optimal canonizing map* for this pair is a map $\phi \in G \ltimes V$ satisfying

$$\phi(A_1, A_2) = (A_0^{\geq}, A_0^{\leq})$$

and minimizing the quantity $\max\left(\|\phi\|, \|\phi^{-1}\|\right)$. By Proposition 4.27 and a compactness argument, such a map exists if and only if A_1 and A_2 are transverse.

We define an *optimal canonizing map* for a map $g \in G \ltimes V$ of type X_0 to be an optimal canonizing map for the pair $(A_g^{\geq}, A_g^{\leq})$.

Let $C \geq 1$. We say that a pair of affine parabolic spaces (A_1, A_2) (resp. a map g of type X_0) is C*-non-degenerate* if it has an optimal canonizing map ϕ such that

$$\|\phi\| \leq C \quad \text{and} \quad \|\phi^{-1}\| \leq C.$$

Now take g_1, g_2 two maps of type X_0 in $G \ltimes V$. We say that the pair (g_1, g_2) is C*-non-degenerate* if every one of the four possible pairs $(A_{g_i}^{\geq}, A_{g_j}^{\leq})$ is C-non-degenerate.

The point of this definition is that there are a lot of calculations in which, when we treat a C-non-degenerate pair of spaces as if they were perpendicular, we err by no more than a (multiplicative) constant depending on C. The following result will often be useful:

Lemma 5.2. *Let $C \geq 1$. Then any map $\phi \in \mathrm{GL}(E)$ such that $\|\phi^{\pm 1}\| \leq C$ induces a C^2-Lipschitz continuous map on $\mathbb{P}(E)$.*

This is exactly Lemma 2.20 from [22].

Remark 5.3. The set of transverse pairs of extended affine spaces is characterized by two open conditions: there is of course transversality of the spaces, but also the requirement that each space not be contained in V. What we mean here by "degeneracy" is failure of one of these two conditions. Thus the property of a pair (A_1, A_2) being C-non-degenerate actually encompasses two properties.

First, it implies that the spaces A_1 and A_2 are transversal in a quantitative way. More precisely, this means that some continuous function that would vanish if the spaces were not transversal is bounded below. An example of such a function is the smallest non identically vanishing of the "principal angles" defined in the proof of Lemma 7.2 (iv).

Second, it implies that both A_1 and A_2 are "not too close" to the space V (in the same sense). In purely affine terms, this means that the affine spaces $A_1 \cap V_{\mathrm{Aff}}$ and $A_2 \cap V_{\mathrm{Aff}}$ contain points that are not too far from the origin.

Both conditions are necessary, and appeared in the previous literature (such as [19] and [3]). However, they were initially treated separately. The idea of encompassing both in the same concept of "C-non-degeneracy" seems to have been first introduced in the author's previous paper [22].

Definition 5.4. Let $g \in \mathrm{GL}(E)$, let $n = \dim E$, and let p be an integer such that $1 \leq p < n$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of g ordered by nondecreasing modulus. Then we define the *p-th spectral gap* of g to be the quotient

$$\kappa_p(g) := \frac{|\lambda_{p+1}|}{|\lambda_p|}. \tag{5.5}$$

Note that we chose the convention where the gap is a number *smaller* than or equal to 1.

When $E = A$, we will most often use the p-th spectral gap for $p = \dim A_0^{\gtrapprox}$. In this case we will omit the index:

$$\kappa(g) := \kappa_{\dim A_0^{\gtrapprox}}(g). \tag{5.6}$$

Also, we denote the *spectral radius* of g, i.e. the largest modulus of any eigenvalue, by:

$$r(g) := |\lambda_1|. \tag{5.7}$$

(The usual notation, $\rho(g)$, is already taken to mean "g in the representation ρ".)

Definition 5.5. Let $s > 0$. For a map $g \in G \ltimes V$ of type X_0, we say that g is *s-contracting* if we have

$$\frac{\|g(x)\|}{\|x\|} \leq s \frac{\|g(y)\|}{\|y\|} \quad \text{for all } (x, y) \in V_g^{<} \times A_g^{\gtrapprox}. \tag{5.8}$$

(Note that by Corollary 4.17 the spaces $V_g^{<}$ and $A_g^{\gtrapprox}$ always have the same dimensions as $V_0^{<}$ and $A_0^{\gtrapprox}$ respectively, hence they are nonzero.)

We define the *strength of contraction* of g to be the smallest number $s(g)$ such that g is $s(g)$-contracting. In other words, we have

$$s(g) = \left\| g|_{V_g^{<}} \right\| \left\| g^{-1}|_{A_g^{\gtrapprox}} \right\|. \tag{5.9}$$

Remark 5.6. This strength of contraction $s(g)$ is defined as a kind of "mixed gap": it measures the gap between *singular* values of the restrictions of g to some sums of its *eigen*spaces. It turns out that this definition is the most convenient for our purposes.

However, if the map g from the above definition is C-non-degenerate, then we may pretend that $s(g)$ is a "purely singular" gap, as long as we do not care about multiplicative constants. Indeed, let $g' = \phi g \phi^{-1}$, where ϕ is an optimal canonizing map for g; then it is easy to see that we have

$$s(g) \asymp_C s(g'). \tag{5.10}$$

On the other hand, since $V_{g'}^{<} = V_0^{<}$ and $A_{g'}^{\gtrapprox} = A_0^{\gtrapprox}$ are orthogonal (by convention), every singular value of g' is either a singular value of $g'|_{V_0^{<}}$ or of $g'|_{A_0^{\gtrapprox}}$. It follows that $s(g')$ is the quotient between two actual singular values of g', and two *consecutive* singular values if $s(g)$ is small enough. See the proof of Lemma 5.1 (iii) for a more detailed discussion.

Remark 5.7. The spectral gap and contraction strength are somewhat related. Take some affine map $g \in G \ltimes V$ of type X_0; then since the norm of any linear map is at least equal to its spectral radius, we obviously have

$$s(g) \geq \kappa(g). \tag{5.11}$$

On the other hand, for any map $g \in G \ltimes V$, we have

$$\log s(g^N) = N \log \kappa(g) + \underset{N \to \infty}{\mathcal{O}} (\log N). \tag{5.12}$$

If g is of type X_0, then $\kappa(g) < 1$, so that

$$s(g^N) \xrightarrow[N \to \infty]{} 0. \tag{5.13}$$

5.2. Affine and linear case. For any map $f \in G \ltimes V$, we denote by $\ell(f)$ the linear part of f, seen as an element of $G \ltimes V$ by identifying G with the stabilizer of the "origin" p_0. In other words, for every vector $(x, t) \in V \oplus \mathbb{R}p_0 = A$, we set

$$\ell(f)(x, t) = f(x, 0) + (0, t). \tag{5.14}$$

(Seeing G as a subgroup of $G \ltimes V$ allows us to avoid introducing new definitions of C-non-degeneracy and contraction strength for elements of G.)

Lemma 5.8. *Let $C \geq 1$, and take any C-non-degenerate map g (or C-non-degenerate pair of maps (g, h)) of type X_0 in $G \ltimes V$. Then*

(i) *the map $\ell(g)$ (resp. the pair $(\ell(g), \ell(h))$) is still C-non-degenerate*;
(ii) *we have $s(\ell(g)) \leq s(g)$;*
(iii) *suppose that $s(g^{-1}) \leq 1$. Then we actually have $s(g) \asymp_C s(\ell(g)) \| g|_{A_g^{=}} \|$.*

Proof. The proof is exactly the same as the proof of Lemma 2.25 in [22], *mutatis mutandis*. ☐

5.3. Proximal maps. Let E be a Euclidean space. The goal of this section is to show Proposition 5.12. We begin with a few definitions.

Definition 5.9. Let $\gamma \in \mathrm{GL}(E)$; let $\lambda_1, \dots, \lambda_n$ be its eigenvalues repeated according to multiplicity and ordered by nonincreasing modulus. We define the *proximal spectral gap* of γ as its first spectral gap:

$$\tilde{\kappa}(\gamma) := \kappa_1(\gamma) = \frac{|\lambda_2|}{|\lambda_1|}.$$

We say that γ is *proximal* if $\tilde{\kappa}(\gamma) < 1$. We may then decompose E into a direct sum of a line E^s_γ, called its *attracting space*, and a hyperplane E^u_γ, called its *repelling space*, both stable by γ and such that:

$$\begin{cases} \gamma|_{E^s_\gamma} = \lambda_1 \,\mathrm{Id}; \\ \text{for every eigenvalue } \lambda \text{ of } \gamma|_{E^u_\gamma}, \ |\lambda| < |\lambda_1|. \end{cases}$$

Definition 5.10. Consider a line E^s and a hyperplane E^u of E, transverse to each other. An *optimal canonizing map* for the pair (E^s, E^u) is a map $\phi \in GL(E)$ satisfying

$$\phi(E^s) \perp \phi(E^u)$$

and minimizing the quantity $\max(\|\phi\|, \|\phi^{-1}\|)$.

We define an *optimal canonizing map* for a proximal map $\gamma \in \mathrm{GL}(E)$ to be an optimal canonizing map for the pair (E^s_γ, E^u_γ).

Let $C \geq 1$. We say that the pair formed by a line and a hyperplane (E^s, E^u) (resp. that a proximal map γ) is *C-non-degenerate* if it has an optimal canonizing map ϕ such that $\left\|\phi^{\pm 1}\right\| \leq C$.

Now take γ_1, γ_2 two proximal maps in $\mathrm{GL}(E)$. We say that the pair (γ_1, γ_2) is *C-non-degenerate* if every one of the four possible pairs $(E^s_{\gamma_i}, E^u_{\gamma_j})$ is C-non-degenerate.

Definition 5.11. Let $\gamma \in \mathrm{GL}(E)$ be a proximal map. We define the *proximal strength of contraction* of γ by

$$\tilde{s}(\gamma) := \frac{\| \gamma|_{E^u_\gamma} \|}{\| \gamma|_{E^s_\gamma} \|} = \frac{\| \gamma|_{E^u_\gamma} \|}{r(\gamma)}$$

(where $r(\gamma)$ is the spectral radius of γ, equal to $|\lambda_1|$ in the notations of the previous definition). We say that γ is *$\tilde{s}$-contracting* if $\tilde{s}(\gamma) \leq \tilde{s}$.

Be careful that the meaning of some of these terms changes depending on the context: they mean different things for maps of type X_0 (see Definitions 5.4 and 5.5 above) and for proximal maps. We tried to at least keep the notations unambiguous: compare the definitions of $\tilde{s}$ and $\tilde{\kappa}$ with those of s and κ.

In the following proposition, the notation $\tilde{s}_{5.12}(C)$ might puzzle the reader. This constant is in fact indexed by the number of the proposition where it appears, a convention that we will follow throughout the paper.

Proposition 5.12. *For every $C \geq 1$, there is a positive constant $\tilde{s}_{5.12}(C)$ with the following property. Take a C-non-degenerate pair of proximal maps γ_1, γ_2 in $\mathrm{GL}(E)$, and suppose that both γ_1 and γ_2 are $\tilde{s}_{5.12}(C)$-contracting. Then $\gamma_1\gamma_2$ is proximal, and we have*

(i) $\alpha(E^s_{\gamma_1\gamma_2}, E^s_{\gamma_1}) \lesssim_C \tilde{s}(\gamma_1)$;

(ii) $\tilde{s}(\gamma_1\gamma_2) \lesssim_C \tilde{s}(\gamma_1)\tilde{s}(\gamma_2)$.

(iii) $r(\gamma_1\gamma_2) \asymp_C \|\gamma_1\|\|\gamma_2\|$.

Note that since we have $r(\gamma_1\gamma_2) \leq \|\gamma_1\gamma_2\| \leq \|\gamma_1\|\|\gamma_2\|$, what the last point really says is that *all three* values have the same order of magnitude.

Similar results have appeared in the literature for a long time, e.g. Lemma 5.7 in [3], Proposition 6.4 in [5] or Lemma 2.2.2 in [6].

Proof. The first two points have already been proved in the author's previous paper: see Proposition 3.4 in [22]. To prove (iii), we start with the following observation. Let $\eta = \frac{\pi}{2C^2}$; then by Lemma 5.2 we have:

$$\alpha(E^s_{\gamma_1}, E^u_{\gamma_2}) \geq \eta.$$

On the other hand, we have already seen in the proof of Proposition 3.4 in [22] that we have

$$E^s_{\gamma_1\gamma_2} \in B\big(E^s_{\gamma_1}, \tfrac{\eta}{3}\big).$$

The triangular inequality immediately gives us

$$\alpha(E^s_{\gamma_1\gamma_2}, E^u_{\gamma_2}) \geq \frac{2\eta}{3}. \tag{5.15}$$

Take any nonzero $x \in E^s_{\gamma_1\gamma_2}$. We are going to show the estimates

$$\frac{\|\gamma_2(x)\|}{\|x\|} \asymp_C \|\gamma_2\|; \tag{5.16a}$$

$$\frac{\|\gamma_1(\gamma_2(x))\|}{\|\gamma_2(x)\|} \asymp_C \|\gamma_1\|. \tag{5.16b}$$

Since by definition, we have $\gamma_1(\gamma_2(x)) = \lambda x$ for some $\lambda \in \mathbb{R}$ having absolute value $r(\gamma_1\gamma_2)$, the estimate (iii) follows by multiplying (5.16a) and (5.16b) together.

Let us first show (5.16a). Let ϕ be an optimal canonizing map for γ_2; since γ_2 is C-non-degenerate, we lose no generality by replacing γ_2 and x respectively by $\gamma_2' := \phi\gamma_2\phi^{-1}$ and $x' := \phi(x)$. Obviously we have:

$$\|\gamma_2'(x')\| \leq \|\gamma_2'\|\|x'\|. \tag{5.17}$$

To show the other inequality, let us decompose

$$x' =: \underbrace{x_s'}_{\in E^s_{\gamma_2'}} + \underbrace{x_u'}_{\in E^u_{\gamma_2'}}. \tag{5.18}$$

Then we have

$$\|\gamma_2'(x')\| \geq \|\gamma_2'(x_s')\| - \|\gamma_2'(x_u')\|. \tag{5.19}$$

For the first term, we have

$$\begin{aligned}\|\gamma_2'(x_s')\| &= r(\gamma_2') \cdot \|x_s'\| \\ &= \|\gamma_2'\| \cdot \sin\alpha(\phi(E^s_{\gamma_1\gamma_2}), E^u_{\gamma_2'}) \cdot \|x'\| \\ &\geq \|\gamma_2'\| \cdot \sin\frac{\alpha(E^s_{\gamma_1\gamma_2}, E^u_{\gamma_2})}{C^2} \cdot \|x'\| \quad \text{by Lemma 5.2} \\ &\geq \|\gamma_2'\| \cdot \sin\frac{1}{C^2}\frac{2\eta}{3} \cdot \|x'\| \quad \text{by (5.15).}\end{aligned} \tag{5.20}$$

For the second term, we have

$$\begin{aligned}\|\gamma_2'(x_u')\| &\leq \left\| \gamma_2'|_{E^u_{\gamma_2'}} \right\| \|x_u'\| \\ &\leq \left\| \gamma_2'|_{E^u_{\gamma_2'}} \right\| \|x'\| \\ &= \|\gamma_2'\| \; \tilde{s}(\gamma_2') \; \|x'\| \\ &\leq \|\gamma_2'\| \; C^2 \tilde{s}(\gamma_2) \; \|x'\|.\end{aligned} \tag{5.21}$$

Plugging those two estimates into (5.19), we obtain

$$\|\gamma_2'(x')\| \geq \|\gamma_2'\| \Big(\sin \frac{2\eta}{3C^2} - C^2 \tilde{s}(\gamma_2) \Big) \|x'\|. \tag{5.22}$$

We may assume that $\tilde{s}(\gamma_2) \leq \frac{1}{2}\frac{1}{C^2} \sin \frac{2\eta}{3C^2}$. Since by construction η depends only on C, we conclude that

$$\|\gamma_2'(x')\| \gtrsim_C \|\gamma_2'\| \; \|x'\|. \tag{5.23}$$

Putting together (5.17) and (5.23), we get (5.16a) as required.

Now to show (5.16b), simply notice that

$$\gamma_2(E^s_{\gamma_1\gamma_2}) = E^s_{\gamma_2\gamma_1} \tag{5.24}$$

(since $\gamma_2\gamma_1$ is the conjugate of $\gamma_1\gamma_2$ by γ_2), so that $\gamma_2(x) \in E^s_{\gamma_2\gamma_1}$. Hence we may follow the same reasoning as for (5.16a), simply exchanging the roles of γ_1 and γ_2. ☐

6. Additivity of Jordan projections

The goal of this section is to prove Proposition 6.17, which says that the product of two sufficiently contracting maps of type X_0 and in general position is still of type X_0. As it is a purely linear property, we forget about translation parts and work exclusively in the linear group G for the duration of this section. We proceed in four stages.

We start with Proposition 6.1, which shows that if an element of G is of type X_0 and strongly contracting in the default representation ρ, it is proximal and strongly contracting in some of the fundamental representations ρ_i defined in Proposition 2.12.

We continue with Proposition 6.7, which relates C-non-degeneracy in V and C'-non-degeneracy in the spaces V_i.

We then prove Proposition 6.11 (and a reformulated version, Corollary 6.13), which constrains the Jordan projection of gh in terms of the Cartan projections of g and h.

Finally, we use Corollary 6.13 to prove Proposition 6.17.

Proposition 6.1. *For every* $C \geq 1$*, there is a positive constant* $s_{6.1}(C)$ *with the following property. Let* $g \in G$ *be a* C*-non-degenerate map of type* X_0 *such that* $s(g) \leq s_{6.1}(C)$*. Then for every* $i \in \Pi \setminus \Pi_{X_0}$*, the map* $\rho_i(g)$ *is proximal and we have*

$$\tilde{s}(\rho_i(g)) \lesssim_C s(g).$$

Remark 6.2. Note that since all Euclidean norms on a finite-dimensional vector space are equivalent, this estimate makes sense even though we did not specify any norm on V_i. In the course of the proof, we shall choose one that is convenient for us.

Recall that "$i \in \Pi \setminus \Pi_{X_0}$" is a notation shortcut for "i such that $\alpha_i \in \Pi \setminus \Pi_{X_0}$."

Remark 6.3. Note that we have excluded the indices i that lie in Π_{X_0}. The latter should be thought of as a kind of "exceptional set"; indeed, recall (Remark 3.21) that it is often empty.

To pave the way for proving the proposition, let us prove a few lemmas that lead to a relation between the contraction strength of an element of G and its Cartan projection.

Lemma 6.4. *For every* $i \in \Pi \setminus \Pi_{X_0}$*, we may find two restricted weights* $\lambda_i^{\geq} \in \Omega_{X_0}^{\geq}$ *and* $\lambda_i^{<} \in \Omega_{X_0}^{<}$ *such that*

$$\lambda_i^{\geq} - \lambda_i^{<} = \alpha_i.$$

(Recall that $\Omega_{X_0}^{\geq}$ is the set of restricted weights that take nonnegative values on X_0, and $\Omega_{X_0}^{<}$ is its complement in Ω.)

Proof. Fix some $i \in \Pi \setminus \Pi_{X_0}$. Since X_0 is extreme, $s_{\alpha_i}(X_0)$ then does not have the same type as X_0. Since X_0 is generic, we may then find a restricted weight λ of ρ such that

$$\lambda(X_0) > 0 \quad \text{and} \quad s_{\alpha_i}(\lambda)(X_0) < 0 \tag{6.1}$$

(we already made this observation in (3.8)). Since λ is a restricted weight, by Proposition 2.8, the number

$$n_\lambda := \frac{\langle \lambda, \alpha_i \rangle}{2\langle \alpha_i, \alpha_i \rangle} \tag{6.2}$$

is an integer. We have, on the one hand:

$$n_\lambda \alpha_i(X_0) = (\lambda - s_{\alpha_i}(\lambda))(X_0) > 0;$$

on the other hand, $\alpha_i(X_0) \geq 0$ (because $X_0 \in \mathfrak{a}^+$); hence n_λ is positive.

By Proposition 2.11, every element of the sequence

$$\lambda,\ \lambda-\alpha_i,\quad \dots,\quad \lambda-n_\lambda\alpha_i$$

is a restricted weight of ρ. We may then simply take $\lambda_i^{\gtreqqless}$ to be the last term of this sequence that still belongs to $\Omega^{\gtreqqless}_{X_0}$, and take $\lambda_i^{<} := \lambda_i^{\gtreqqless}-\alpha_i$ to be the immediately following term of the sequence. ☐

Lemma 6.5 (Cartan decomposition in L_{X_0}). *Let $g \in L_{X_0}$. Then there exist two elements k_1 and k_2 in $K \cap L_{X_0}$ and a unique element* $\mathrm{Ct}_{X_0}(g) \in \mathfrak{a}^+_{\Pi_{X_0}}$ *such that*

$$g = k_1 \exp(\mathrm{Ct}_{X_0}(g))k_2.$$

(Recall (2.4) that $\mathfrak{a}^+_{\Pi_{X_0}} = \{X \in \mathfrak{a} \mid \text{for all } \alpha \in \Pi_{X_0},\ \alpha(X) \geq 0\}$.)

Proof. By Proposition 7.82 (a) in [16], L_{X_0} is the centralizer of the intersection of the kernels of simple roots in Π_{X_0}:

$$L_{X_0} = Z_G(\{X \in \mathfrak{a} \mid \text{for all } \alpha \in \Pi_{X_0},\ \alpha(X) = 0\}). \tag{6.3}$$

By Proposition 7.25 in [16], it follows:

- that L_{X_0} is reductive;
- that $K \cap L_{X_0}$ is a maximal compact subgroup in L_{X_0}.

Obviously $\mathfrak{a} \subset \mathfrak{l}_{X_0}$ is a Cartan subspace of $\mathfrak{l}_{X_0}$, and $\mathfrak{a}^+_{\Pi_{X_0}}$ is a Weyl chamber for L_{X_0}. So this result is just the Cartan decomposition in the reductive group L_{X_0} (see Theorem 7.39 in [16]). ☐

Lemma 6.6. *For every $C \geq 1$, there is a constant $k_{6.6}(C)$ with the following property. Let $g \in G$ be a C-non-degenerate map of type X_0 such that* $\log s(g) \leq -k_{6.6}(C)$. *Then we have*

$$\min_{\lambda\in\Omega^{\gtreqqless}_{X_0}} \lambda(\mathrm{Ct}(g)) - \max_{\lambda\in\Omega^{\lesseqgtr}_{X_0}} \lambda(\mathrm{Ct}(g)) \geq -\log s(g) - k_{6.6}(C).$$

Note that the first term on the left-hand side is certainly nonpositive, since $0 \in \Omega^{\gtreqqless}_{X_0}$.

Proof. Let us first focus on the particular case where g satisfies

$$\begin{cases} V_g^{\geq} = V_0^{\geq}; \\ V_g^{\leq} = V_0^{\leq}. \end{cases}$$

In this case, we shall prove that the statement holds with $k_{6.6}(C) = 0$. In fact, we shall even prove that in this case, if $\log s(g) \leq 0$, we actually have the equality

$$\min_{\lambda\in\Omega^{\gtreqqless}_{X_0}} \lambda(\mathrm{Ct}(g)) - \max_{\lambda\in\Omega^{\lesseqgtr}_{X_0}} \lambda(\mathrm{Ct}(g)) \;=\; -\log s(g). \tag{6.4}$$

By construction, obviously g stabilizes $V_0^{\geq}$ and $V_0^{\leq}$; hence (using Proposition 4.4) it also stabilizes $V_0^{<}$, and we have

$$g \in P_{X_0}^{+} \cap P_{X_0}^{-} = L_{X_0}. \tag{6.5}$$

By Lemma 6.5, we then have

$$g = k_1 \exp(\mathrm{Ct}_{X_0}(g)) k_2 \tag{6.6}$$

with $k_1, k_2 \in K \cap L_{X_0}$. In particular both k_1 and k_2 stabilize both $V_0^{\geq}$ and $V_0^{<}$. Hence so does the L_{X_0}-Cartan projection $\mathrm{Ct}_{X_0}(g)$, and we have

$$\begin{cases} \| g|_{V_0^{\geq}} \| = \left\| \exp(\mathrm{Ct}_{X_0}(g))\big|_{V_0^{\geq}} \right\|; \\ \| (g)^{-1}\big|_{V_0^{<}} \| = \left\| \exp(\mathrm{Ct}_{X_0}(g))^{-1}\big|_{V_0^{<}} \right\|. \end{cases} \tag{6.7}$$

Now we know that $\exp(\mathrm{Ct}_{X_0}(g))$ (seen in the default representation ρ) is self-adjoint (by choice of the Euclidean structure B), hence its singular values coincide with its eigenvalues. (Moreover $V_0^{\geq}$ and $V_0^{<}$ are orthogonal.) As $\exp(\mathrm{Ct}_{X_0}(g)) \in A$, obviously it acts on every restricted weight space V^{λ} with the eigenvalue

$$\exp(\lambda(\mathrm{Ct}_{X_0}(g))).$$

This almost gives us the identity we want, but with $\mathrm{Ct}_{X_0}(g)$ instead of $\mathrm{Ct}(g)$:

$$\min_{\lambda \in \Omega_{X_0}^{\geq}} \lambda(\mathrm{Ct}_{X_0}(g)) - \max_{\lambda \in \Omega_{X_0}^{<}} \lambda(\mathrm{Ct}_{X_0}(g)) \;=\; -\log s(g). \tag{6.8}$$

Note that, in contrast to the identity (6.4) that we are trying to prove, this identity holds for all values of $s(g)$. To conclude, it remains to show that if we assume $\log s(g) \leq 0$, then we actually have $\mathrm{Ct}_{X_0}(g) = \mathrm{Ct}(g)$.

Indeed if $\log s(g) \leq 0$, then the left-hand side of (6.8) must be nonnegative. By Lemma 6.4, it then follows that in particular, we have

$$\lambda_i^{\geq}(\mathrm{Ct}_{X_0}(g)) \geq \lambda_i^{<}(\mathrm{Ct}_{X_0}(g)) \quad \text{for all } i \in \Pi \setminus \Pi_{X_0}, \tag{6.9}$$

hence

$$\alpha_i(\mathrm{Ct}_{X_0}(g)) \geq 0 \quad \text{for all } i \in \Pi \setminus \Pi_{X_0}. \tag{6.10}$$

On the other hand we also have

$$\alpha_i(\mathrm{Ct}_{X_0}(g)) \geq 0 \quad \text{for all } i \in \Pi_{X_0}, \tag{6.11}$$

since $\mathrm{Ct}_{X_0}(g) \in \mathfrak{a}^{+}_{\Pi_{X_0}}$ by construction. Joining both systems of inequalities, we obtain that

$$\mathrm{Ct}_{X_0}(g) \in \mathfrak{a}^{+}.$$

This shows that (6.6) actually gives a Cartan decomposition of g in the whole group G. By uniqueness of Cartan projection, we conclude that $\mathrm{Ct}_{X_0}(g) = \mathrm{Ct}(g)$ as desired.

Now let us deal with arbitrary g. Let ϕ be an optimal canonizing map for g, and let $g' = \phi g \phi^{-1}$. Then it is easy to see that we have

$$s(g') \asymp_C s(g)$$

(we already mentioned this in Remark 5.6), and the difference $\mathrm{Ct}(g') - \mathrm{Ct}(g)$ is bounded by a constant that depends only on C. Taking a suitable value of $k_{6.6}(C)$, the general result for g then follows from the particular result applied to g'. ☐

Proof of Proposition 6.1. Let $s_{6.1}(C)$ be a positive constant small enough to satisfy all the constraints that will appear in the course of the proof. Let us fix $i \in \Pi \setminus \Pi_{X_0}$, and let $g \in G$ be a map satisfying the hypotheses. Let us prove the two estimates

$$\tilde{\kappa}(\rho_i(g)) = \exp(\alpha_i(\mathrm{Jd}(g)))^{-1} \leq \kappa(g), \tag{6.12}$$

which will show that $\rho_i(g)$ is proximal; and then the two estimates

$$\tilde{s}(\rho_i(g)) \asymp_C \exp(\alpha_i(\mathrm{Ct}(g)))^{-1} \lesssim_C s(g), \tag{6.13}$$

whose combination completes the proof.

- Let us start with the right part of (6.12). Lemma 6.4 gives us two restricted weights $\lambda_i^{\geq}$ and $\lambda_i^{<}$ of ρ such that:

  $$\begin{cases} \lambda_i^{\geq}(X_0) \geq 0; \\ \lambda_i^{<}(X_0) < 0, \end{cases}$$

  and whose difference is α_i. Now since g is of type X_0, by definition, any restricted weight of ρ has the same sign when evaluated at $\mathrm{Jd}(g)$ or at X_0. Thus we also have

  $$\begin{cases} \lambda_i^{\geq}(\mathrm{Jd}(g)) \geq 0; \\ \lambda_i^{<}(\mathrm{Jd}(g)) < 0. \end{cases}$$

  From Proposition 2.6, it then follows that

  $$\kappa(g) \geq \exp(\alpha_i(\mathrm{Jd}(g)))^{-1}$$

  as desired.

- Similarly we may establish the right part of (6.13). By using once again the restricted weights $\lambda_i^{\geq}$ and $\lambda_i^{<}$ given by Lemma 6.4, it follows from Lemma 6.6 that

  $$\alpha_i(\mathrm{Ct}(g)) \geq -\log s(g) - k_{6.6}(C),$$

  provided we take $s_{6.1}(C) \leq \exp(-k_{6.6}(C))$. By negating both sides and exponentiating, the desired estimate follows immediately.

- Let us now prove the left part of (6.12). By Proposition 2.6 (i), the list of the moduli of the eigenvalues of $\rho_i(g)$ is precisely

  $$(e^{\lambda_i^j(\mathrm{Jd}(g))})_{1 \leq j \leq d_i},$$

  where d_i is the dimension of V_i and $(\lambda_i^j)_{1 \leq j \leq d_i}$ is the list of restricted weights of ρ_i repeated according to their multiplicity.

  Up to reordering that list, we may suppose that

  $$\lambda_i^1 = n_i \varpi_i$$

  is the highest restricted weight of ρ_i. We may also suppose that

  $$\lambda_i^2 = n_i \varpi_i - \alpha_i;$$

  indeed it is also a restricted weight of ρ_i by Lemma 2.13 (i). Now take any $j > 2$. Since by hypothesis, the restricted weight $n_i \varpi_i$ has multiplicity 1, we have $\lambda_i^j \neq \lambda_i^1$. By Lemma 2.13 (ii), it follows that this restricted weight has the form

  $$\lambda_i^j = n_i \varpi_i - \alpha_i - \sum_{i'=1}^{r} c_{i'} \alpha_{i'},$$

  with $c_{i'} \geq 0$ for every index i'.

  Finally, since by definition $\mathrm{Jd}(g)$ lies in $\mathfrak{a}^+$, for every index i' we have $\alpha_{i'}(\mathrm{Jd}(g)) \geq 0$. It follows that for every $j > 2$, we have

  $$\lambda_i^1(\mathrm{Jd}(g)) \geq \lambda_i^2(\mathrm{Jd}(g)) \geq \lambda_i^j(\mathrm{Jd}(g)). \tag{6.14}$$

  In other words, among the moduli of the eigenvalues of $\rho_i(g)$, the largest is

  $$\exp(\lambda_i^1(\mathrm{Jd}(g))) = \exp(n_i \varpi_i(\mathrm{Jd}(g))),$$

  and the second largest is

  $$\exp(\lambda_i^2(\mathrm{Jd}(g))) = \exp(n_i \varpi_i(\mathrm{Jd}(g)) - \alpha_i(\mathrm{Jd}(g))).$$

  It follows that

  $$\tilde{\kappa}(\rho_i(g)) = \exp(\alpha_i(\mathrm{Jd}(g)))^{-1}$$

  as desired.

- Let us finish with the left part of (6.13). We start with the following observation: for every $C \geq 1$, the set
$$\{\phi \in G \mid \|\phi\| \leq C,\ \|\phi^{-1}\| \leq C\} \tag{6.15}$$
is compact. It follows that the continuous map
$$\phi \longmapsto \max\left(\|\rho_i(\phi)\|,\ \|\rho_i(\phi^{-1})\|\right) \tag{6.16}$$
is bounded on that set, by some constant C'_i that depends only on C (and on the choice of a norm on V_i, to be made soon). Let ϕ be the optimal canonizing map of g, and let $g' = \phi g \phi^{-1}$; then we get
$$\tilde{s}(\rho_i(g)) \asymp_C \tilde{s}(\rho_i(g')). \tag{6.17}$$
Now let us choose, on the space V_i where the representation ρ_i acts, a K-invariant Euclidean form B_i such that all the restricted weight spaces for ρ_i are pairwise B_i-orthogonal (this is possible by Lemma 2.4 applied to ρ_i). Then $\tilde{s}(\rho_i(g'))$ is simply the quotient of the two largest singular values of $\rho_i(g')$. By Proposition 2.6 (ii) (giving the singular values of an element of G in a given representation) and by a calculation analogous to the previous point, we have
$$\tilde{s}(\rho_i(g')) = \exp(\alpha_i(\mathrm{Ct}(g)))^{-1}. \tag{6.18}$$
The desired estimate follows by combining (6.17) with (6.18). ☐

Proposition 6.7. *Let (g_1, g_2) be a C-non-degenerate pair of elements of G of type X_0. Then for every $i \in \Pi \setminus \Pi_{X_0}$, the pair $(\rho_i(g_1), \rho_i(g_2))$ is a C'_i-non-degenerate pair of proximal maps in $\mathrm{GL}(V_i)$, where C'_i is some constant that depends only on C and i.*

Before proving this proposition, we need a couple of lemmas.

Lemma 6.8. *Let $i \in \Pi \setminus \Pi_{X_0}$.*

(i) *The restricted weight space $V_i^{n_i \varpi_i}$ is stable by $\rho_i(P^+_{X_0})$.*

(ii) *The direct sum of all restricted weight spaces V_i^{λ} with $\lambda \neq n_i \varpi_i$ is stable by $\rho_i(P^-_{X_0})$.*

Proof. (i) Let us first prove that this space is stable by $\mathfrak{p}^+_{X_0}$. By definition, we have:
$$\mathfrak{p}^+_{X_0} = \mathfrak{l} \oplus \bigoplus_{\beta(X_0) \geq 0} \mathfrak{g}^{\beta};$$
Since $\mathfrak{l}$ centralizes $\mathfrak{a}$, it preserves the restricted weight space decomposition; so clearly $\mathfrak{l}$ stabilizes $V_i^{n_i \varpi_i}$.

Now let β be a root such that $\beta(X_0) \geq 0$; let us write

$$\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha.$$

By definition of the set Π_{X_0}, we then have

$$c_\alpha \geq 0 \quad \text{for } \alpha \in \Pi \setminus \Pi_{X_0}. \tag{6.19}$$

Now we know that

$$\mathfrak{g}^\beta \cdot V_i^{n_i \varpi_i} \subset V_i^{n_i \varpi_i + \beta}.$$

The latter space is actually zero. Indeed, otherwise, $n_i \varpi_i + \beta$ would have to be a restricted root. But from Lemma 2.13, we know that this would imply

$$c_{\alpha_i} \leq -1,$$

which contradicts the inequality above, since i (or, technically, the root α_i) is in $\Pi \setminus \Pi_{X_0}$. It follows that for every β such that $\beta(X_0) \geq 0$, the space $V_i^{n_i \varpi_i}$ is stable by $\mathfrak{g}^\beta$; we conclude that it is stable by $\mathfrak{p}^+_{X_0}$.

By integration, we deduce that this space is also stable by $P^+_{X_0,e}$. Now we know (it follows from [16], Proposition 7.82 (d)) that $P^+_{X_0} = MP^+_{X_0,e}$. Since M centralizes $\mathfrak{a}$, it preserves the restricted weight space decomposition, so it stabilizes $V_i^{n_i \varpi_i}$. We conclude that $P^+_{X_0}$ stabilizes $V_i^{n_i \varpi_i}$.

(ii) The proof is completely analogous. □

In the following lemma, we denote by $\mathcal{PS}$ the set of all parabolic spaces of V; we also identify the projective space $\mathbb{P}(V_i)$ with the set of vector lines in V_i and the projective space $\mathbb{P}(V_i^*)$ with the set of vector hyperplanes of V_i.

Remark 6.9. Recall (Remark 4.28) that by Proposition 4.4, the manifold $\mathcal{PS}$ is diffeomorphic to $G/P^+_{X_0}$ (in a G-equivariant way).

Lemma 6.10.

(i) *For every $i \in \Pi \setminus \Pi_{X_0}$, there exists a unique pair of continuous maps*

$$\Phi_i^s \colon \mathcal{PS} \longrightarrow \mathbb{P}(V_i) \quad \textit{and} \quad \Phi_i^u \colon \mathcal{PS} \longrightarrow \mathbb{P}(V_i^*)$$

such that for every map $g \in G$ of type X_0, we have

$$\begin{cases} E^s_{\rho_i(g)} = \Phi_i^s(V_g^{\geq}); \\ E^u_{\rho_i(g)} = \Phi_i^u(V_g^{\leq}). \end{cases}$$

(ii) *Moreover, these maps have the following property: whenever $V_1, V_2 \in \mathcal{PS}$ are transverse, we have $\Phi_i^s(V_1) \not\in \Phi_i^u(V_2)$.*

Proof. Take any g of type X_0; then from the inequality (6.14) ranking the values of different restricted weights of ρ_i evaluated at $\mathrm{Jd}(g)$, we deduce that we have

$$\begin{cases} E^s_{\rho_i(\exp(\mathrm{Jd}(g)))} = V_i^{n_i\varpi_i}; \\ E^u_{\rho_i(\exp(\mathrm{Jd}(g)))} = \bigoplus_{\lambda \neq n_i\varpi_i} V_i^\lambda. \end{cases} \tag{6.20}$$

Now take any $\phi \in G$; applying the defining identities of $\Phi_i^{s,u}$ to the conjugate $\phi\exp(\mathrm{Jd}(g))\phi^{-1}$, we deduce that these two maps, if they exist, must necessarily satisfy

$$\begin{cases} \Phi_i^s(\phi(V_0^{\geq})) = \rho_i(\phi)(V_i^{n_i\varpi_i}); \\ \Phi_i^u(\phi(V_0^{\leq})) = \rho_i(\phi)\big(\bigoplus_{\lambda \neq n_i\varpi_i} V_i^\lambda\big). \end{cases} \tag{6.21}$$

We may take this as a definition of Φ_i^s and Φ_i^u; it remains to check that it is not ambiguous. Clearly it is enough to check that whenever some $\phi \in G$ stabilizes the space $V_0^{\geq}$ (resp. $V_0^{\leq}$), it also stabilizes the line $V_i^{n_i\varpi_i}$ (resp. hyperplane $\bigoplus_{\lambda \neq n_i\varpi_i} V_i^\lambda$). Since $i \in \Pi \setminus \Pi_{X_0}$, this follows from Lemma 6.8 and Proposition 4.4. That the maps Φ_i^s and Φ_i^u thus defined are continuous is then obvious.

As for property (ii), it now follows from Proposition 4.27, which says that G acts transitively on the set of transverse pairs of parabolic spaces. ☐

Proof of Proposition 6.7. Let us fix some $i \in \Pi \setminus \Pi_{X_0}$ and some $C \geq 1$. Then the set of C-non-degenerate pairs of parabolic spaces is compact. On the other hand, the function

$$(V_1, V_2) \longmapsto \alpha(\Phi_i^s(V_1), \Phi_i^u(V_2))$$

is continuous, and (by Lemma 6.10 (ii)) takes positive values on that set. Hence it is bounded below. So there is a constant $C_i' \geq 1$, depending only on C, such that whenever a pair (V_1, V_2) of parabolic spaces is C-non-degenerate, the pair $(\Phi_i^s(V_1), \Phi_i^u(V_2))$ is C_i'-non-degenerate.

The conclusion then follows by Lemma 6.10 (i). ☐

Proposition 6.11. *For every $C \geq 1$, there are positive constants $s_{6.11}(C)$ and $k_{6.11}(C)$ with the following property. Take any C-non-degenerate pair (g, h) of elements of G of type X_0 such that $s(g) \leq s_{6.11}(C)$ and $s(h) \leq s_{6.11}(C)$. Then we have*

(i) $\varpi_i(\mathrm{Jd}(gh) - \mathrm{Ct}(g) - \mathrm{Ct}(h)) \leq 0$ *for all* $i \in \Pi$;

(ii) $\varpi_i(\mathrm{Jd}(gh) - \mathrm{Ct}(g) - \mathrm{Ct}(h)) \geq -k_{6.11}(C)$ *for all* $i \in \Pi \setminus \Pi_{X_0}$.

See Figure 2 for a picture explaining both this proposition and the corollary below.

Remark 6.12. Though we shall not use it, a very important particular case is $g = h$. We then obviously have $\mathrm{Jd}(gh) = 2\,\mathrm{Jd}(g)$ and $\mathrm{Ct}(g) + \mathrm{Ct}(h) = 2\,\mathrm{Ct}(g)$, so that the inequalities (i) and (ii) give a relationship between the Cartan and Jordan projections of a C-non-degenerate, sufficiently contracting map of type X_0.

Before proving the proposition, let us give a more palatable (though slightly weaker) reformulation.

Corollary 6.13. *For every $C \geq 1$, there exists a positive constant $k_{6.13}(C)$ with the following property. For any pair (g, h) satisfying the hypotheses of Proposition 6.11, we have*

$$\mathrm{Jd}(gh) \in \mathrm{Conv}(W_{X_0} \cdot \mathrm{Ct}'(g, h)), \tag{6.22}$$

where Conv *denotes the convex hull and* $\mathrm{Ct}'(g, h)$ *is some vector in* $\mathfrak{a}$ *satisfying*

$$\| \mathrm{Ct}'(g, h) - \mathrm{Ct}(g) - \mathrm{Ct}(h) \| \leq k_{6.13}(C). \tag{6.23}$$

In fact, we can already give an explicit expression for this vector $\mathrm{Ct}'(g, h)$:

Definition 6.14. We define $\mathrm{Ct}'(g, h)$ to be the unique solution of the linear system

$$\begin{cases} \varpi_i(\mathrm{Ct}'(g, h)) = \varpi_i(\mathrm{Jd}(gh)) & \text{for all } i \in \Pi \setminus \Pi_{X_0}; \\ \varpi_i(\mathrm{Ct}'(g, h)) = \varpi_i(\mathrm{Ct}(g) + \mathrm{Ct}(h)) & \text{for all } i \in \Pi_{X_0}. \end{cases} \tag{6.24}$$

(This works since $(\varpi_i)_{i \in \Pi}$ is a basis of $\mathfrak{a}^*$.)

Remark 6.15. Note that the vector $\mathrm{Ct}'(g, h)$ might *not* lie in the closed dominant Weyl chamber $\mathfrak{a}^+$ (even though it is very close to the vector $\mathrm{Ct}(g) + \mathrm{Ct}(h)$ which does).

It remains to check that the vector $\mathrm{Ct}'(g, h)$ thus defined satisfies indeed the required conditions.

Proof of Corollary 6.13. The estimate (6.23) immediately follows from the inequalities of Proposition 6.11. On the other hand, we may now rewrite Proposition 6.11 without the epsilons: combining Proposition 6.11 (i) with the definition of $\mathrm{Ct}'(g, h)$, we get

$$\begin{cases} \varpi_i(\mathrm{Jd}(gh) - \mathrm{Ct}'(g, h)) \leq 0 & \text{for all } i \in \Pi; \\ \varpi_i(\mathrm{Jd}(gh) - \mathrm{Ct}'(g, h)) = 0 & \text{for all } i \in \Pi \setminus \Pi_{X_0}. \end{cases} \tag{6.25}$$

Let us now show the inequalities

$$\alpha_i(\mathrm{Ct}'(g, h)) \geq 0 \quad \text{for all } i \in \Pi_{X_0}. \tag{6.26}$$

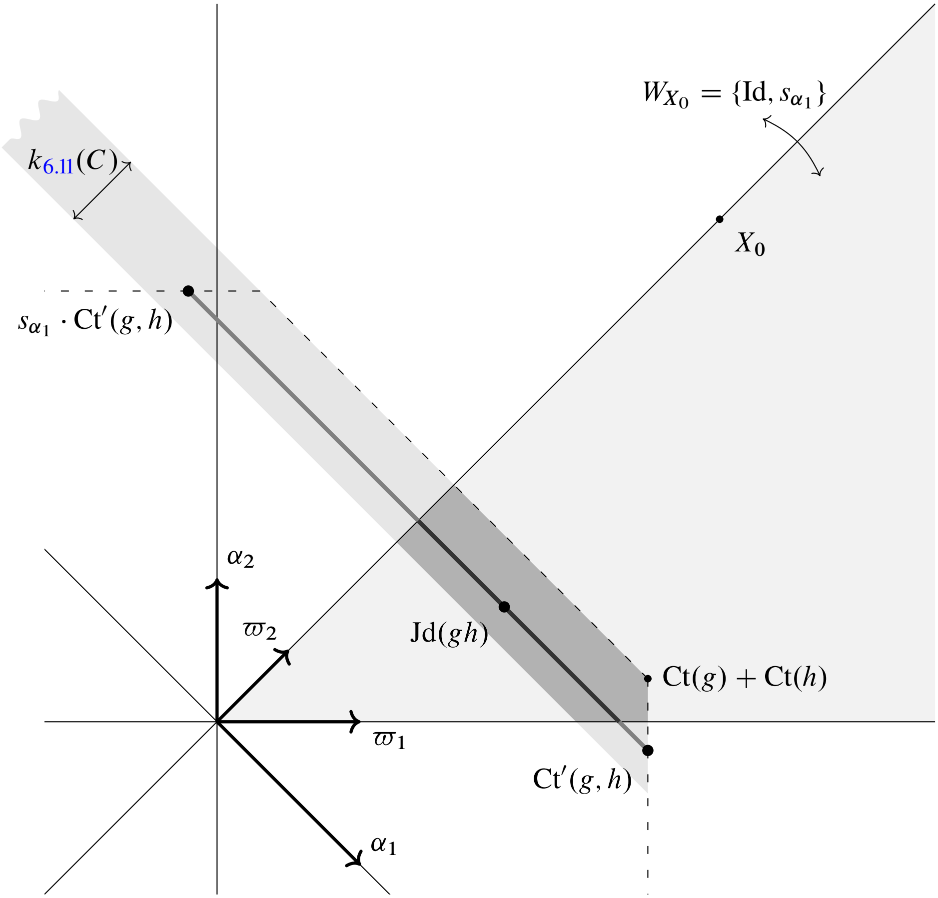


Figure 2. This picture represents the situation of Example 3.7.3, namely $G = \mathrm{SO}^+(3,2)$ acting on $\mathbb{R}^5$. We have chosen a generic, symmetric, extreme vector X_0. The set Π_{X_0} is then $\{\alpha_1\}$ (or $\{1\}$ with the usual abuse of notations), and the group W_{X_0} is generated by the single reflection s_{α_1}. Proposition 6.11 states that $\mathrm{Jd}(gh)$ lies in the shaded “infinite trapezoid”. Corollary 6.13 states that it lies on the thick line segment. In any case it lies by definition in the dominant open Weyl chamber (the shaded sector).

Let $(H_i)_{i \in \Pi}$ be the basis of $\mathfrak{a}$ dual to $(\varpi_i)_{i \in \Pi}$, i.e. the unique basis such that the identity

$$\varpi_i \Big(\sum_{j \in \Pi} c_j H_j \Big) = c_i \tag{6.27}$$

holds for any $i \in \Pi$ and any tuple $(c_j) \in \mathbb{R}^{\Pi}$. By definition of the fundamental restricted weights ϖ_i, it then follows that we also have the identity

$$\lambda(H_i) = \frac{2\langle \lambda, \alpha'_i \rangle}{\|\alpha'_i\|^2} \quad \text{for all } i \in \Pi,\ \lambda \in \mathfrak{a}^*. \tag{6.28}$$

By decomposing the vector $\mathrm{Ct}(g) + \mathrm{Ct}(h) - \mathrm{Ct}'(g,h)$ in the basis $(H_i)_{i \in \Pi}$ and by plugging the formula (6.27) into the second line of the defining system (6.24), we find that we may write

$$\mathrm{Ct}'(g,h) = \mathrm{Ct}(g) + \mathrm{Ct}(h) - \sum_{j \in \Pi \setminus \Pi_{X_0}} c_j H_j; \tag{6.29}$$

by combining the first line of the defining system (6.24) with Proposition 6.11 (i), we also obtain that $c_j \geq 0$ for every $j \in \Pi \setminus \Pi_{X_0}$.

Finally, take any index $i \in \Pi_{X_0}$. Then we have

$$\begin{aligned} \alpha_i \Big(\sum_{j \in \Pi \setminus \Pi_{X_0}} c_j H_j \Big) &= \sum_{j \in \Pi \setminus \Pi_{X_0}} c_j \alpha_i(H_j) \\ &= \sum_{j \in \Pi \setminus \Pi_{X_0}} c_j \frac{2\langle \alpha_i, \alpha'_j \rangle}{\|\alpha'_j\|^2} \\ &\leq 0; \end{aligned} \tag{6.30}$$

indeed since j varies in $\Pi \setminus \Pi_{X_0}$ and $i \in \Pi_{X_0}$, we have $i \neq j$ hence $\langle \alpha_i, \alpha_j \rangle \leq 0$; and α'_j is by construction a positive multiple of α_j. We conclude that

$$\alpha_i(\mathrm{Ct}'(g,h)) \geq \alpha_i(\mathrm{Ct}(g)) + \alpha_i(\mathrm{Ct}(h)) \geq 0$$

(since $\mathrm{Ct}(g), \mathrm{Ct}(h) \in \mathfrak{a}^+$), which gives us (6.26).

Now the system of inequalities (6.26) is equivalent to saying that

$$\mathrm{Ct}'(g,h) \in \mathfrak{a}^+_{X_0}, \tag{6.31}$$

where $\mathfrak{a}^+_{X_0}$ is a fundamental domain for the action of the Weyl subgroup W_{X_0} on $\mathfrak{a}$, more specifically the one that contains the dominant Weyl chamber $\mathfrak{a}^+$. The statement (6.22) then follows from this and from (6.25), by applying Proposition 2.7 which characterizes convex hulls of orbits of W_{X_0}. ☐

Proof of Proposition 6.11. Let $i \in \Pi$. We know (see (6.14) above) that for any vector $X \in \mathfrak{a}^+$, the number $n_i \varpi_i(X)$ is the largest eigenvalue of $\rho_i(X)$. From Proposition 2.6, it then follows that:

$$\begin{cases} n_i \varpi_i(\mathrm{Ct}(g)) = \log \|\rho_i(g)\|; \\ n_i \varpi_i(\mathrm{Jd}(g)) = \log r(\rho_i(g)) \end{cases} \tag{6.32}$$

(recall that r denotes the spectral radius).

(i) is straightforward from here: indeed,

$$\begin{aligned} n_i \varpi_i(\mathrm{Jd}(gh)) &= \log r(\rho_i(gh)) \\ &\leq \log \|\rho_i(gh)\| \\ &\leq \log \|\rho_i(g)\| \|\rho_i(h)\| \\ &= n_i \varpi_i(\mathrm{Ct}(g) + \mathrm{Ct}(h)). \end{aligned} \tag{6.33}$$

(ii) Assume that $i \in \Pi \setminus \Pi_{X_0}$. By Proposition 6.1, we know that the maps $\rho_i(g)$ and $\rho_i(h)$ are proximal. By Proposition 6.7, they form a C'_i-non-degenerate pair, for some C'_i that depends only on C. By Proposition 6.1, if we take $s_{6.11}(C)$ small enough, we may then assume that both $\rho_i(g)$ and $\rho_i(h)$ are $\tilde{s}_{5.12}(C'_i)$-contracting. We may then apply Proposition 5.12 (iii) to these two maps: we get

$$r(\rho_i(g)\rho_i(h)) \asymp_C \|\rho_i(g)\| \|\rho_i(h)\|.$$

Now from Proposition 2.6, it follows that we have:

$$\begin{cases} r(\rho_i(gh)) = \exp(n_i \varpi_i(\mathrm{Jd}(gh))); \\ \|\rho_i(g)\| = \exp(n_i \varpi_i(\mathrm{Ct}(g))); \\ \|\rho_i(h)\| = \exp(n_i \varpi_i(\mathrm{Ct}(h))). \end{cases}$$

Taking the logarithm, we deduce that there exists $\varepsilon_i(C)$ such that for sufficiently contracting g and h, we have

$$n_i \varpi_i(\mathrm{Jd}(gh) - \mathrm{Ct}(g) - \mathrm{Ct}(h)) \in [-\varepsilon_i(C),\ \varepsilon_i(C)]. \tag{6.34}$$

Taking

$$k_{6.11}(C) := \max_{i \in \Pi \setminus \Pi_{X_0}} \frac{1}{n_i} \varepsilon_i(C), \tag{6.35}$$

the conclusion follows. ☐

Remark 6.16. Corollary 6.13 generalizes a result given by Benoist in [6]. More specifically, by taking together Lemma 4.1 and Lemma 4.5.2 from that paper, we obtain that under suitable conditions, the vector

$$\mathrm{Jd}(gh) - \mathrm{Ct}(g) - \mathrm{Ct}(h)$$

(which is $\lambda(gh) - \mu(g) - \mu(h)$ in Benoist's notations) is bounded. This seems to be stronger than our result; but in fact, it also relies on stronger assumptions. More precisely, there are two possible ways to interpret Benoist's result in the context of our paper.

- Either we may take his set θ to be our $\Pi \setminus \Pi_{X_0}$. In that case, [6] uses the additional assumption that g and h are "of type θ", which is very restrictive: it means that their Jordan projections must lie in the intersection of the kernels of all roots in Π_{X_0} (which is also the space of fixed points of W_{X_0}). To control the Cartan projections of g and h, [6] uses the assumption that g and h actually belong to a whole Zariski-dense subgroup of G, all of whose elements are of type θ. As Benoist remarks in the second paragraph of Remark 3.2.1 in [6], the latter assumption only makes sense for p-adic groups; in the case of real groups which is of interest to us, as shown in the appendix of [7], this is actually impossible unless $\theta = \Pi$.
- Or we may take θ to be the whole set Π. But in that case, [6] needs the assumption that g and h are proximal (and in general position) in *all* representations ρ_i, which is stronger than the hypotheses we have made.

Proposition 6.17. *For every $C \geq 1$, there is a positive constant $s_{6.17}(C) \leq 1$ with the following property. Take any C-non-degenerate pair (g, h) of maps of type X_0 in G such that $s(g^{\pm 1}) \leq s_{6.17}(C)$ and $s(h^{\pm 1}) \leq s_{6.17}(C)$. Then gh is still of type X_0.*

Proof. Let $C \geq 1$, and let (g, h) be a C-non-degenerate pair of maps in $G \ltimes V$ of type X_0, such that

$$s(g^{\pm 1}) \leq s_{6.17}(C) \quad \text{and} \quad s(h^{\pm 1}) \leq s_{6.17}(C)$$

for some positive constant $s_{6.17}(C)$ to be specified later.

Lemma 6.6 then gives us

$$\begin{aligned} \max_{\lambda \in \Omega^{<}_{X_0}} \lambda(\mathrm{Ct}(g)) &\leq \log s(g) + k_{6.6}(C) + \min_{\lambda \in \Omega^{\geq}_{X_0}} \lambda(\mathrm{Ct}(g)) \\ &\leq \log s(g) + k_{6.6}(C). \end{aligned} \tag{6.36}$$

(Indeed by Assumption 3.2, $\lambda = 0$ is a restricted weight that is certainly contained in $\Omega^{\geq}_{X_0}$, so the minimum above is nonpositive.)

Taking $s_{6.17}(C)$ small enough, we may assume that

$$\lambda(\mathrm{Ct}(g)) < -\frac{1}{2}(\max_{\lambda\in\Omega} \|\lambda\|)\, k_{6.13}(C) \quad \text{for all } \lambda \in \Omega^{<}_{X_0}. \tag{6.37}$$

Of course a similar estimate holds for h:

$$\lambda(\mathrm{Ct}(h)) < -\frac{1}{2}(\max_{\lambda\in\Omega} \|\lambda\|)\, k_{6.13}(C) \quad \text{for all } \lambda \in \Omega^{<}_{X_0}. \tag{6.38}$$

Now let λ be any restricted weight that does not vanish on X_0. We distinguish two cases:

- Suppose that $\lambda(X_0) < 0$. Recall Corollary 6.13; for the key vector $\mathrm{Ct}'(g,h)$ that it involves, we will use the value given by Definition 6.14. Then on the one hand, we deduce from (6.23) that:

$$\begin{aligned}|\lambda(\mathrm{Ct}'(g,h)) - \lambda(\mathrm{Ct}(g)) - \lambda(\mathrm{Ct}(h))| &\leq \|\lambda\|\,\|\mathrm{Ct}'(g,h) - \mathrm{Ct}(g) - \mathrm{Ct}(h)\| \\ &\leq (\max_{\lambda\in\Omega}\|\lambda\|)\, k_{6.13}(C).\end{aligned} \tag{6.39}$$

  Adding together the three estimates (6.37), (6.38) and (6.39), we get

$$\lambda(\mathrm{Ct}'(g,h)) < 0; \tag{6.40}$$

  and this is true for *any* $\lambda \in \Omega^{<}_{X_0}$.

  On the other hand, we have (6.22) which says that

$$\mathrm{Jd}(gh) \in \mathrm{Conv}(W_{X_0} \cdot \mathrm{Ct}'(g,h)).$$

  Now since $\Omega^{<}_{X_0}$ is stable by W_{X_0}, it follows from (6.40) that we still have

$$\lambda(w(\mathrm{Ct}'(g,h))) = w^{-1}(\lambda)(\mathrm{Ct}'(g,h)) < 0$$

  for any $w \in W_{X_0}$. Thus λ takes negative values on every point of the orbit $W_{X_0} \cdot \mathrm{Ct}'(g,h)$; hence it also takes negative values on every point of its convex hull. In particular, we have

$$\lambda(\mathrm{Jd}(gh)) < 0. \tag{6.41}$$

- Suppose that $\lambda(X_0) > 0$. Since the set of restricted weights Ω is invariant by W, the form $w_0(\lambda)$ is still a restricted weight; since by hypothesis X_0 is symmetric (i.e. $-w_0(X_0) = X_0$), we then have

$$w_0(\lambda)(X_0) < 0.$$

We may thus apply the previous point to the weight $w_0(\lambda)$ and to the map $(gh)^{-1} = h^{-1}g^{-1}$ (since g^{-1} and h^{-1} verify the same hypotheses as g and h); this gives us

$$w_0(\lambda)(\mathrm{Jd}((gh)^{-1})) < 0.$$

Since $\mathrm{Jd}((gh)^{-1}) = -w_0(\mathrm{Jd}(gh))$, we conclude that

$$\lambda(\mathrm{Jd}(gh)) > 0. \tag{6.42}$$

We conclude that gh is indeed of type X_0. ☐

Remark 6.18. If we assume that both g and g^{-1} are sufficiently contracting, then clearly Lemma 6.6 implies that $\mathrm{Ct}'(g, g)$ and then $\mathrm{Ct}(g)$ also has the same type as X_0. Conversely, we may show (by a version of Lemma 6.6 with the inequality going both ways) that if $\mathrm{Ct}(g)$ has the same type as X_0 and is "far enough" from the borders of $\mathfrak{a}_{\rho,X_0}$, then g and g^{-1} are strongly contracting.

7. Products of maps of type X_0

The goal of this section is to prove Proposition 7.4, which not only says that a product of a C-non-degenerate, sufficiently contracting pair of maps of type X_0 is itself of type X_0, but allows us to control the geometry and contraction strength of the product. To do this, we proceed almost exactly as in Section 3.2 in [22]: we reduce the problem to Proposition 5.12, by considering the action of $G \ltimes V$ on a suitable exterior power $\Lambda^p A$ (rather than on the spaces V_i as in the previous section).

There is however one crucial difference from [22]: while it is still true that when g is of type X_0, its exterior power $\Lambda^p g$ is proximal, the converse no longer holds. Filling that gap is what the whole previous section was about.

Remark 7.1. The reader might wonder why we did not (developing upon the final remark from the previous section) prove an additivity theorem for Cartan projections similar to Proposition 6.11, and use it to estimate $s(gh)$ in terms of $s(g^{\pm 1})$ and $s(h^{\pm 1})$. Since we need to study the action on the spaces V_i anyway, this would seemingly allow us to forgo the additional introduction of $\Lambda^p A$.

The reason is that this approach only works for *linear* maps g and h: for $g \in G \ltimes V$, the Cartan projection is only defined for $\ell(g)$ and only gives information about the singular values of $\ell(g)$, not those of g. So while possible, this approach would force us, on the other hand, to abandon the unified treatment of quantitative properties of affine maps (as outlined in Remark 5.3).

We introduce the integers:

$$\begin{aligned} p &:= \dim A_0^{\geq} = \dim V_0^{\geq} + 1; \\ q &:= \dim V_0^{<}; \\ d &:= \dim A = \dim V + 1 = q + p. \end{aligned} \tag{7.1}$$

For every $g \in G \ltimes V$, we may define its exterior power $\Lambda^p g \colon \Lambda^p A \to \Lambda^p A$. The Euclidean structure of A induces in a canonical way a Euclidean structure on $\Lambda^p A$.

Lemma 7.2. (i) *Let $g \in G \ltimes V$ be a map of type X_0. Then $\Lambda^p g$ is proximal, and the attracting* (*resp. repelling*) *space of $\Lambda^p g$ depends on nothing but $A_g^{\geq}$* (*resp. $V_g^{<}$*):

$$\begin{cases} E^s_{\Lambda^p g} = \Lambda^p A_g^{\geq}, \\ E^u_{\Lambda^p g} = \{x \in \Lambda^p A \mid x \wedge \Lambda^q V_g^{<} = 0\}. \end{cases}$$

(ii) *For every $C \geq 1$, whenever (g_1, g_2) is a C-non-degenerate pair of maps of type X_0, $(\Lambda^p g_1, \Lambda^p g_2)$ is a C^p-non-degenerate pair of proximal maps.*
For every $C \geq 1$, for every C-non-degenerate map $g \in G \ltimes V$ of type X_0, we have

$$s(g) \lesssim_C \tilde{s}(\Lambda^p g). \tag{7.2}$$

If in addition $s(g) \leq 1$, we have

$$s(g) \asymp_C \tilde{s}(\Lambda^p g). \tag{7.3}$$

(*Recall the Definitions 5.5 and 5.11 of the two different notions of "contraction strength" $s(g)$ and $\tilde{s}(\gamma)$, respectively.*)

(iii) *For any two p-dimensional subspaces A_1 and A_2 of A, we have*

$$\alpha^{\mathrm{Haus}}(A_1, A_2) \asymp \alpha(\Lambda^p A_1,\ \Lambda^p A_2).$$

This is similar to Lemma 3.8 in [22], except for point (i) which here is weaker than there.

Proof. For (i), let $g \in G \ltimes V$ be a map of type X_0. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of g (acting on A) counted with multiplicity and ordered by nondecreasing modulus; then $|\lambda_{q+1}| = 1$ and $|\lambda_q| < 1$. On the other hand, we know that the eigenvalues of $\Lambda^p g$ counted with multiplicity are exactly the products of the form $\lambda_{i_1} \cdots \lambda_{i_p}$, where $1 \leq i_1 < \cdots < i_p \leq d$. As the two largest of them (by modulus) are $\lambda_{q+1} \cdots \lambda_d$ and $\lambda_q \lambda_{q+2} \cdots \lambda_d$, it follows that $\Lambda^p g$ is proximal.

As for the expression of E^s and E^u, it follows immediately by considering a basis that trigonalizes g.

For (ii), (iii), and (iv), the proof is exactly the same as for the corresponding points in Lemma 3.8 in [22], *mutatis mutandis.* □

We also need the following technical lemma, which generalizes Lemma 3.9 in [22]:

Lemma 7.3. *There is a constant* $\varepsilon > 0$ *with the following property. Let* A_1, A_2 *be any two affine parabolic spaces such that*

$$\begin{cases} \alpha^{\mathrm{Haus}}(A_1, A_0^{\geq}) \leq \varepsilon, \\ \alpha^{\mathrm{Haus}}(A_2, A_0^{\leq}) \leq \varepsilon. \end{cases}$$

Then they form a 2*-non-degenerate pair.*

(Of course the constant 2 is arbitrary; we could replace it by any number larger than 1.)

Proof. The proof is exactly the same as the proof of Lemma 3.9 in [22], *mutatis mutandis*. ☐

Proposition 7.4. *For every* $C \geq 1$*, there is a positive constant* $s_{7.4}(C) \leq 1$ *with the following property. Take any* C*-non-degenerate pair* (g, h) *of maps of type* X_0 *in* $G \ltimes V$*; suppose that we have* $s(g^{\pm 1}) \leq s_{7.4}(C)$ *and* $s(h^{\pm 1}) \leq s_{7.4}(C)$*. Then* gh *is of type* X_0*,* $2C$*-non-degenerate, and we have*

(i) $\begin{cases} \alpha^{\mathrm{Haus}}(A_{gh}^{\geq}, A_g^{\geq}) \lesssim_C s(g), \\ \alpha^{\mathrm{Haus}}(A_{gh}^{\leq}, A_h^{\leq}) \lesssim_C s(h^{-1}); \end{cases}$

(ii) $s(gh) \lesssim_C s(g)s(h)$.

(This generalizes Proposition 3.6 in [22].)

Before giving the proof, let us first formulate a particular case:

Corollary 7.5. *Under the same hypotheses, we have*

$$\begin{cases} \alpha^{\mathrm{Haus}}(V_{gh}^{\geq}, V_g^{\geq}) \lesssim_C s(\ell(g)), \\ \alpha^{\mathrm{Haus}}(V_{gh}^{\leq}, V_h^{\leq}) \lesssim_C s(\ell(h)^{-1}). \end{cases}$$

Proof. This follows from Lemma 5.8. The proof is the same as for Corollary 3.7 in [22]. ☐

Proof of Proposition 7.4. Let us fix some positive constant $s_{7.4}(C)$, small enough to satisfy all the constraints that will appear in the course of the proof. Let (g, h) be a pair of maps satisfying the hypotheses.

First note that by Lemma 5.8, we have

$$s(\ell(g)^{\pm 1}) \leq s(g^{\pm 1}) \leq s_{7.4}(C) \tag{7.4}$$

and similarly for h. If we take $s_{7.4}(C) \leq s_{6.17}(C)$, then Proposition 6.17 tells us that $\ell(gh)$, hence gh (indeed the Jordan projection depends only on the linear part), is of type X_0.

The remaining part of the proof works exactly like the proof of Proposition 3.6 in [22], namely by applying Proposition 5.12 to the maps $\gamma_1 = \Lambda^p g$ and $\gamma_2 = \Lambda^p h$. Taking into account the central position occupied in the paper by the proposition we are currently proving, let us reproduce these details nevertheless. Let us check that γ_1 and γ_2 satisfy the required hypotheses:

- By Lemma 7.2 (i), γ_1 and γ_2 are proximal.
- By Lemma 7.2 (ii), the pair (γ_1, γ_2) is C^p-non-degenerate.
- Since we have supposed $s_{7.4}(C) \leq 1$, it follows by Lemma 7.2 (iii) that $\tilde{s}(\gamma_1) \lesssim_C s(g)$ and $\tilde{s}(\gamma_2) \lesssim_C s(h)$. If we choose $s_{7.4}(C)$ sufficiently small, then γ_1 and γ_2 are $\tilde{s}_{5.12}(C^p)$-contracting, i.e. sufficiently contracting to apply Proposition 5.12.

Thus we may apply Proposition 5.12. It remains to deduce from its conclusions the conclusions of Proposition 7.4.

- We already know that gh is of type X_0.
- From Proposition 5.12 (i), using Lemma 7.2 (i), (iii), and (iv), we get

  $$\alpha^{\mathrm{Haus}}(A^{\geq}_{gh},\ A^{\geq}_{g}) \lesssim_C s(g),$$

  which shows the first line of Proposition 7.4 (i).
- By applying Proposition 5.12 to $\gamma_2^{-1}\gamma_1^{-1}$ instead of $\gamma_1\gamma_2$, we get in the same way the second line of Proposition 7.4 (i).
- Let ϕ be an optimal canonizing map for the pair $(A^{\geq}_{g}, A^{\leq}_{h})$. By hypothesis, we have $\left\|\phi^{\pm 1}\right\| \leq C$. But if we take $s_{7.4}(C)$ sufficiently small, the two inequalities that we have just shown, together with Lemma 7.3, allow us to find a map ϕ' with $\|\phi'\| \leq 2$, $\|\phi'^{-1}\| \leq 2$ and

  $$\phi' \circ \phi(A^{\geq}_{gh}, A^{\leq}_{gh}) = (A^{\geq}_{0}, A^{\leq}_{0}).$$

  It follows that the composition map gh is $2C$-non-degenerate.
- The last inequality, namely Proposition 7.4 (ii), now is deduced from Proposition 5.12 (ii) by using Lemma 7.2 (iii). □

8. Additivity of Margulis invariants

Proposition 8.1 below is the key ingredient of the proof of the Main Theorem. It explains how the Margulis invariant behaves under group operations (inverse and composition).

The first point is easy to prove, but still important. It is a generalization of Proposition 4.1 (i) in [22]; as the general case is slightly harder, we have now given more details.

The proof of the second point occupies the remainder of this section. We prove it by reducing it successively to Lemma 8.6 (which is proved using the technical lemma 8.7), then to Lemma 8.9. The proof follows very closely that of Proposition 4.2 (ii) in [22], and we have actually omitted the proofs of Lemmas 8.7 and 8.9. We did repeat the proof of the proposition itself (to help the reader figure out precisely what is to be changed), as well as the proof of Lemma 8.6 (to clear up a small confusion in the original proof: see Remark 8.8).

Proposition 8.1. (i) *For every map $g \in G \ltimes V$ of type X_0, we have*

$$M(g^{-1}) = -w_0(M(g)).$$

(ii) *For every $C \geq 1$, there are positive constants $s_{8.1}(C) \leq 1$ and $k_{8.1}(C)$ with the following property. Let $g, h \in G \ltimes V$ be a C-non-degenerate pair of maps of type X_0, with $g^{\pm 1}$ and $h^{\pm 1}$ all $s_{8.1}(C)$-contracting. Then gh is of type X_0, and we have*:

$$\|M(gh) - M(g) - M(h)\| \leq k_{8.1}(C).$$

Remark 8.2. To justify the slight abuse of notations $w_0(M(g))$, recall Remark 1.2, that we may now reformulate as follows: w_0 induces a linear involution on V_0^t (which is the space of fixed points by L), and this involution does not depend on the choice of a representative of w_0 in G.

Let $C \geq 1$. We choose some positive constant $s_{8.1}(C) \leq 1$, small enough to satisfy all the constraints that will appear in the course of the proof. For the remainder of this section, we fix $g, h \in G \ltimes V$ a C-non-degenerate pair of maps of type X_0 such that $g^{\pm 1}$ and $h^{\pm 1}$ are $s_{8.1}(C)$-contracting.

The following remark will be used throughout this section.

Remark 8.3. We may suppose that the pairs $(A^{\geq}_{gh}, A^{\leq}_{gh})$, $(A^{\geq}_{hg}, A^{\leq}_{hg})$, $(A^{\geq}_{g}, A^{\leq}_{gh})$, and $(A^{\geq}_{hg}, A^{\leq}_{g})$ are all $2C$-non-degenerate. Indeed, recall that (by Proposition 7.4), we have

$$\begin{cases} \alpha^{\mathrm{Haus}}(A^{\geq}_{gh},\ A^{\geq}_{g}) \lesssim_C s(g), \\ \alpha^{\mathrm{Haus}}(A^{\leq}_{gh},\ A^{\leq}_{h}) \lesssim_C s(h^{-1}), \end{cases}$$

and similar inequalities with g and h interchanged. On the other hand, by hypothesis, $(A^{\geq}_{g}, A^{\leq}_{h})$ is C-non-degenerate. If we choose $s_{8.1}(C)$ sufficiently small, these four statements then follow from Lemma 7.3.

Proof of Proposition 8.1. (i) Let ϕ be a canonizing map for g. Since $V^{\geq}_{g^{-1}} = V^{\leq}_{g}$ and vice-versa (obviously) and since $V^{\leq}_{0} = w_0 V^{\geq}_{0}$ and vice-versa (because X_0 is symmetric), it follows that $w_0\phi$ is a canonizing map for g^{-1}.

It remains to show that w_0 commutes with π_t. Indeed, it is well known that the group W, that we defined as the quotient $N_G(A)/Z_G(A)$, is also equal to the quotient $N_K(A)/Z_K(A)$ (see [16], formulas (7.84a) and (7.84b)); hence

$$N_G(A) = WZ_G(A) = WZ_K(A)A = N_K(A)A \subset KA. \tag{8.1}$$

Let $\tilde{w}_0$ be any representative of w_0 in $N_G(A)$. We already know that both $V^{=}_{0} = V^{0}$ (Remark 4.5) and V^{t}_{0} (Remark 8.2) are invariant by $\tilde{w}_0$. Now by definition the group A acts trivially on V^0, and by construction K acts on V^0 by orthogonal transformations (indeed the Euclidean structure was chosen in accordance with Lemma 2.4); hence V^{r}_{0}, which is the orthogonal complement of V^{t}_{0} in V^0, is also invariant by $\tilde{w}_0$.

The desired formula now immediately follows from the definition of the Margulis invariant.

(ii) The proof of this point is a straightforward generalization of the proof of Proposition 4.1 (ii) in [22].

If we take $s_{8.1}(C) \leq s_{7.4}(C)$, then Proposition 7.4 ensures that gh is of type X_0.

To estimate $M(gh)$, we decompose $gh\colon A^{=}_{gh} \to A^{=}_{gh}$ into a product of several maps.

- We begin by decomposing the product gh into its factors. We have the commutative diagram

$$\begin{tikzcd} A^{=}_{gh} & A^{=}_{hg} \arrow[l, "g"] & A^{=}_{gh} \arrow[l, "h"] \arrow[ll, bend right, "gh"'] \end{tikzcd} \tag{8.2}$$

  Indeed, since hg is the conjugate of gh by h and vice-versa, we have $h(A^{=}_{gh}) = A^{=}_{hg}$ and $g(A^{=}_{hg}) = A^{=}_{gh}$.

- Next we factor the map $g\colon A^{=}_{hg} \to A^{=}_{gh}$ through the map $g\colon A^{=}_{g} \to A^{=}_{g}$, which is better known to us. We have the commutative diagram

$$\begin{tikzcd}
A^{=}_{gh} \arrow[dr, "\pi_g"'] & & & A^{=}_{hg} \arrow[lll, "g"'] \arrow[dl, "\pi_g"] \\
& A^{=}_{g} & A^{=}_{g} \arrow[l, "g"] &
\end{tikzcd} \tag{8.3}$$

  where π_g is the projection onto $A^{=}_{g}$ parallel to $V^{>}_{g} \oplus V^{<}_{g}$. (It commutes with g because $A^{=}_{g}$, $V^{>}_{g}$ and $V^{<}_{g}$ are all invariant by g.)

- Finally, we decompose again both arrows labelled π_g on the last diagram into two factors. For any two maps u and v of type X_0, we introduce the notation

$$A^{=}_{u,v} := A^{\geq}_{u} \cap A^{\leq}_{v}.$$

  We call P_1 (resp. P_2) the projection onto $A^{=}_{g,gh}$ (resp. $A^{=}_{hg,g}$), still parallel to $V^{>}_{g} \oplus V^{<}_{g}$. To justify this definition, we must check that $A^{=}_{g,gh}$ (and similarly $A^{=}_{hg,g}$) is supplementary to $V^{>}_{g} \oplus V^{<}_{g}$. Indeed, by Remark 8.3, $A^{\leq}_{gh}$ is transverse to $A^{\geq}_{g}$, hence (by Proposition 4.16 (ii)) supplementary to $V^{>}_{g}$; thus $A^{\geq}_{g} = V^{>}_{g} \oplus A^{=}_{g,gh}$ and $A = V^{<}_{g} \oplus A^{\geq}_{g} = V^{<}_{g} \oplus V^{>}_{g} \oplus A^{=}_{g,gh}$. Then we have the commutative diagrams

$$\begin{tikzcd}
A^{=}_{gh} \arrow[r, "P_1"] \arrow[rr, bend right, "\pi_g"'] & A^{=}_{g,gh} \arrow[r, "\pi_g"] & A^{=}_{g}
\end{tikzcd} \tag{8.4a}$$

  and

$$\begin{tikzcd}
A^{=}_{hg} \arrow[r, "P_2"] \arrow[rr, bend right, "\pi_g"'] & A^{=}_{hg,g} \arrow[r, "\pi_g"] & A^{=}_{g}
\end{tikzcd} \tag{8.4b}$$

The second and third step can be repeated with h instead of g. The way to adapt the second step is straightforward; for the third step, we factor $\pi_h : A^{=}_{hg} \to A^{=}_{h}$ through $A^{=}_{h,hg}$ and $\pi_h : A^{=}_{gh} \to A^{=}_{h}$ through $A^{=}_{gh,h}$.

Combining these three decompositions, we get the lower half of Diagram 3. (We left out the expansion of h; we leave drawing the full diagram for especially patient readers.) Let us now interpret all these maps as endomorphisms of $A^{=}_{0}$. To do this, we choose some optimal canonizing maps

$$\phi_g, \quad \phi_{gh}, \quad \phi_{hg}, \quad \phi_{g,gh}, \quad \phi_{hg,g}$$

respectively of g, of gh, of hg, of the pair $(A_g^{\geq}, A_{gh}^{\leq})$ and of the pair $(A_{hg}^{\geq}, A_g^{\leq})$. This allows us to define $\overline{g_{gh}}$, $\overline{h_{gh}}$, $\overline{g_{g,gh}}$, $\overline{g_=}$, $\overline{P_1}$, $\overline{P_2}$, $\overline{\psi_1}$, $\overline{\psi_2}$ to be the maps that make the whole Diagram 3 commutative.

$$\begin{tikzcd}
A_0^= \arrow[dr, "\overline{P_1}"'] & & & & & & A_0^= \arrow[llllll, "\overline{g_{gh}}"'] \arrow[dl, "\overline{P_2}"] & A_0^= \arrow[l, "\overline{h_{gh}}"'] \\
& A_0^= \arrow[dr, "\overline{\psi_1}"'] & & & & A_0^= \arrow[llll, "\overline{g_{g,gh}}"'] \arrow[dl, "\overline{\psi_2}"] & & \\
& & A_0^= & & A_0^= \arrow[ll, "\overline{g_=}"'] & & & \\
A_{gh}^= \arrow[uuu, dashed, "\phi_{gh}"] \arrow[dr, "P_1"] & & & & & & A_{hg}^= \arrow[llllll, "g"'] \arrow[uuu, dashed, "\phi_{hg}"] \arrow[dl, "P_2"'] & A_{gh}^= \arrow[l, "h"'] \arrow[uuu, dashed, "\phi_{gh}"] \\
& A_{g,gh}^= \arrow[uuu, dashed, "\phi_{g,gh}"] \arrow[dr, "\pi_g"] & & & & A_{hg,g}^= \arrow[uuu, dashed, "\phi_{hg,g}"] \arrow[dl, "\pi_g"'] & & \\
& & A_g^= \arrow[uuu, dashed, "\phi_g"] & & A_g^= \arrow[uuu, dashed, "\phi_g"] \arrow[ll, "g"'] & & &
\end{tikzcd}$$

Diagram 3

Now let us define

$$\begin{cases} M_{gh}(g) := \pi_t(\overline{g_{gh}}(x) - x), \\ M_{gh}(h) := \pi_t(\overline{h_{gh}}(x) - x), \end{cases} \tag{8.5}$$

for any $x \in V_{\mathrm{Aff},0}^=$, where $V_{\mathrm{Aff},0}^= := A_0^= \cap V_{\mathrm{Aff}}$ is the affine space parallel to $V_0^=$ and passing through the origin. Since gh is the conjugate of hg by g and vice-versa, the elements of $G \ltimes V$ (defined in an obvious way) whose restrictions to $A_0^=$ are $\overline{g_{gh}}$ and $\overline{h_{gh}}$ stabilize the spaces $A_0^{\geq}$ and $A_0^{\leq}$. By Lemma 4.21, $\overline{g_{gh}}$ and $\overline{h_{gh}}$ are thus quasi-translations. It follows that these values $M_{gh}(g)$ and $M_{gh}(h)$ do not depend on the choice of x. Compare this to the definition of a Margulis invariant (Definition 4.31): we have $M(gh) = \pi_t(\overline{g_{gh}} \circ \overline{h_{gh}}(x) - x)$ for any $x \in V_{\mathrm{Aff},0}^=$.

It immediately follows that

$$M(gh) = M_{gh}(g) + M_{gh}(h). \tag{8.6}$$

We may now estimate each of the two terms separately: if we show that $\|M_{gh}(g) - M(g)\| \lesssim_C 1$ and $\|M_{gh}(h) - M(h)\| \lesssim_C 1$, we are done. These two estimates follow immediately from Lemma 8.6 below. (Note that while the vectors $M_{gh}(g)$ and $M_{gh}(h)$ are elements of V_0^t, the maps $\overline{g_{gh}}$ and $\overline{h_{gh}}$ are extended affine isometries acting on the whole subspace $A_0^{=}$.) ☐

Remark 8.4. In contrast to actual Margulis invariants, the values $M_{gh}(g)$ and $M_{gh}(h)$ *do* depend on our choice of canonizing maps. Choosing other canonizing maps would force us to subtract some constant from the former and add it to the latter.

Definition 8.5. We shall say that a linear bijection f between two subspaces of the extended affine space A is $K(C)$*-bounded* if it is bounded by a constant depending only on C, that is, $\|f\| \lesssim_C 1$ and $\|f^{-1}\| \lesssim_C 1$. We say that two automorphisms f_1, f_2 of $A_0^{=}$ (depending somehow on g and h) are $K(C)$*-almost equivalent*, and we write $f_1 \approx_C f_2$, if they satisfy the condition

$$\|f_1 - \xi \circ f_2 \circ \xi'\| \lesssim_C 1$$

for some $K(C)$-bounded quasi-translations ξ, ξ'. This is indeed an equivalence relation.

Lemma 8.6. *The maps* $\overline{g_{gh}}$ *and* $\overline{h_{gh}}$ *are* $K(C)$*-almost equivalent to* $\overline{g_=}$ *and* $\overline{h_=}$*, respectively.*

To show this, we use the following property.

Lemma 8.7. *All the non-horizontal* (*i.e. vertical or diagonal*) *arrows in Diagram* 3 *represent* $K(C)$*-bounded, bijective maps.*

Note that Lemma 8.7 alone does not imply Lemma 8.6: indeed, while the maps $\overline{\psi}_1$ and $\overline{\psi}_2$ are quasi-translations by Lemma 4.30, the maps $\overline{P}_1$ and $\overline{P}_2$ need not be. This issue will be addressed in Lemma 8.9.

Proof of Lemma 8.7. The proof is exactly the same as the proof of Lemma 4.6 in [22], *mutatis mutandis*. ☐

Proof of Lemma 8.6. We shall concentrate on the estimate $\overline{g_{gh}} \approx_C \overline{g_=}$; the proof of the estimate $\overline{h_{gh}} \approx_C \overline{h_=}$ is analogous.

We now use Lemma 4.30 which shows that canonical identifications commute up to quasi-translation with suitable projections; it implies that the maps $\overline{\psi_1}$ and $\overline{\psi_2}$ are quasi-translations. Hence $\overline{g_{g,gh}}$ is also a quasi-translation.

We would like to pretend that $\overline{g_{gh}}$ and $\overline{g_{g,gh}}$ are actually translations. To do that, we modify slightly the upper right-hand corner of Diagram 3. We set

$$\begin{cases} \phi'_{hg} := \ell(\overline{g_{gh}}) \circ \phi_{hg}, \\ \phi'_{hg,g} := \ell(\overline{g_{g,gh}}) \circ \phi_{hg,g}, \end{cases} \tag{8.7}$$

where ℓ stands for the linear part as defined in Section 5.2, and we define $\overline{P'_2}$, $\overline{\psi'_2}$, $\overline{g'_{gh}}$, $\overline{g'_{g,gh}}$ so as to make the new diagram commutative (see Diagram 4). The factors $\ell(\overline{g_{gh}})$ and $\ell(\overline{g_{g,gh}})$ we introduced (the short horizontal arrows in Diagram 4) have norm 1: indeed, being quasi-translations of $A_0^{=}$ fixing p_0, they are orthogonal linear transformations (by Lemma 4.21). Thus Lemma 8.7 still holds for Diagram 4; but now, the modified maps $\overline{g'_{gh}}$ and $\overline{g'_{g,gh}}$ are translations by construction.

$$\begin{tikzcd}
A_0^{=} \arrow[dr, "\overline{P_1}"'] & & & & A_0^{=} \arrow[llll, "\overline{g'_{gh}}"'] \arrow[dl, "\overline{P'_2}"'] & \textcolor{gray}{A_0^{=}} \arrow[l, dashed, "\ell(\overline{g_{gh}})"] \arrow[dl, gray, "\overline{P_2}"] \\
& A_0^{=} \arrow[dr, "\overline{\psi_1}"'] & & A_0^{=} \arrow[ll, "\overline{g'_{g,gh}}"'] \arrow[d, "\overline{\psi'_2}"'] & \textcolor{gray}{A_0^{=}} \arrow[l, dashed, "\ell(\overline{g_{g,gh}})"] \arrow[dl, gray, "\overline{\psi_2}"] & \\
& & A_0^{=} & A_0^{=} \arrow[l, "\overline{g_{=}}"'] & & \\
A_{gh}^{=} \arrow[uuu, dashed, "\phi_{gh}"] \arrow[dr, "P_1"] & & & & & A_{hg}^{=} \arrow[lllll, "g"'] \arrow[dl, "P_2"'] \arrow[uuul, dashed, "\phi'_{hg}"] \arrow[uuu, dashed, gray, "\phi_{hg}"'] \\
& A_{g,gh}^{=} \arrow[uuu, dashed, "\phi_{g,gh}"] \arrow[dr, "\pi_g"] & & & A_{hg,g}^{=} \arrow[uuul, dashed, "\phi'_{hg,g}"] \arrow[uuu, dashed, gray, "\phi_{hg,g}"'] \arrow[dl, "\pi_g"'] & \\
& & A_g^{=} \arrow[uuu, dashed, "\phi_g"] & A_g^{=} \arrow[l, "g"'] \arrow[uuu, dashed, "\phi_g"] & &
\end{tikzcd}$$

Diagram 4

We may write:

$$\overline{g'_{gh}} = (\overline{P_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P_1}) \circ (\overline{P_1}^{-1} \circ \overline{P'_2}). \tag{8.8}$$

Then, since $\overline{g'_{gh}}$ and $\overline{g'_{g,gh}}$ are translations, $\overline{P_1}^{-1} \circ \overline{P'_2}$ is also a translation. By Lemma 8.7 (applied to Diagram 4), it is the composition of two $K(C)$-bounded maps, hence $K(C)$-bounded. Thus we have

$$\overline{g'_{gh}} \approx_C \overline{P_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P_1}. \tag{8.9}$$

Since $\ell(\overline{g_{gh}})$, $\ell(\overline{g_{g,gh}})$, $\overline{\psi_1}$ and $\overline{\psi_2}$ are $K(C)$-bounded quasi-translations, $\overline{g_{gh}}$ is $K(C)$-almost equivalent to $\overline{g'_{gh}}$ and $\overline{g_=}$ is $K(C)$-almost equivalent to $\overline{g'_{g,gh}}$. It remains to check that the map $\overline{g'_{g,gh}}$ is $K(C)$-almost equivalent to its conjugate $\overline{P_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P_1}$.

This follows from Lemma 8.9 below. Indeed, let $\overline{P''_1}$ be the quasi-translation constructed in Lemma 8.9. Let $v \in V_0^=$ be the translation vector of $\overline{g'_{g,gh}}$, so that

$$\overline{g'_{g,gh}} =: \tau_v. \tag{8.10}$$

Keep in mind that while we call the map τ_v a “translation,” it is formally a transvection: its matrix in a suitable basis is $\left(\begin{smallmatrix} \mathrm{Id} & v \\ 0 & 1 \end{smallmatrix}\right)$. Then we have

$$\begin{aligned} \|\overline{P_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P_1} - \overline{P''_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P''_1}\| &= \|\tau_{\overline{P_1}^{-1}(v)} - \tau_{\overline{P''_1}^{-1}(v)}\| \\ &= \|\overline{P_1}^{-1}(v) - \overline{P''_1}^{-1}(v)\| \\ &\leq \|(\overline{P_1}^{-1} - \overline{P''_1}^{-1})|_{V_0^=}\| \, \|v\| \end{aligned} \tag{8.11}$$

(as $v \in V_0^=$).

Remark 8.8. While the corresponding calculation in [22] does not technically contain any explicit falsehoods (the inequality just happens to be slightly weaker than what it should be), it implicitly relies on the false “identity” $\tau_u - \tau_v = \tau_{u-v}$. Here we have corrected this confusion.

Now by Lemma 4.21, we know that the quasi-translation $\overline{P''_1}$ restricted to $V_0^=$ is a linear map preserving the Euclidean norm. We also know that the map $\rho \mapsto \rho^{-1}$ (defined on $\mathrm{GL}(V_0^=)$) is Lipschitz-continuous on a neighborhood of the orthogonal group (which is compact). Finally, by Lemma 5.8, $s(\ell(g))$ does not exceed $s(g)$ which is by hypothesis smaller than or equal to $s_{8.1}(C)$. Taking $s_{8.1}(C)$ small enough, we may deduce from Lemma 8.9 that

$$\|(\overline{P_1}^{-1} - \overline{P''_1}^{-1})|_{V_0^=}\| \lesssim_C s(\ell(g)). \tag{8.12}$$

On the other hand, we have $\|v\| \leq \|\tau_v\| = \|\overline{g'_{g,gh}}\| \lesssim_C \|g|_{A_g^=}\|$, since $\overline{g'_{g,gh}}$ is the composition of $g|_{A_g^=}$ with several $K(C)$-bounded maps. It follows that

$$\|\overline{P_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P_1} - \overline{P''_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P''_1}\| \lesssim_C s(\ell(g))\|g|_{A_g^=}\|. \tag{8.13}$$

By Lemma 5.8 (iii), we have $s(\ell(g))\|g|_{A_g^=}\| \lesssim_C s(g)$; and we know that $s(g) \leq 1$. Finally we get

$$\|\overline{P_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P_1} - \overline{P''_1}^{-1} \circ \overline{g'_{g,gh}} \circ \overline{P''_1}\| \lesssim_C 1. \tag{8.14}$$

To complete the proof of Lemma 8.6, and hence also the proof of Proposition 8.1, it remains only to prove Lemma 8.9. ☐

Lemma 8.9. *The linear part of the map $\overline{P_1}$ is "almost" a quasi-translation. More precisely, there is a quasi-translation $\overline{P''_1}$ such that*

$$\|(\overline{P_1} - \overline{P''_1})|_{V_0^=}\| \lesssim_C s(\ell(g)).$$

Recall that $\ell(g)$ is the map with the same linear part as g, but with no translation part: see subsection 5.2. We use the double prime because the relationship between $\overline{P''_1}$ and $\overline{P_1}$ is not the same as the relationship between $\overline{P'_2}$ and $\overline{P_2}$.

Proof. The proof is exactly the same as the proof of Lemma 4.7 in [22], *mutatis mutandis*. ☐

9. Margulis invariants of words

We have already studied how contraction strengths (Proposition 7.4) and Margulis invariants (Proposition 8.1) behave when we take the product of two C-non-degenerate, sufficiently contracting maps of type X_0. The goal of this section is to generalize these results to words of arbitrary length on a given set of generators. It is a straightforward generalization of Section 5 in [22] (we slightly changed the notations).

Definition 9.1. Take k generators $g_1, \dots, g_k$. Consider a word $g = g_{i_1}^{\sigma_1} \cdots g_{i_l}^{\sigma_l}$ with length $l \geq 1$ on these generators and their inverses (for every m we have $1 \leq i_m \leq k$ and $\sigma_m = \pm 1$). We say that g is *reduced* if for every m such that $1 \leq m \leq l-1$, we have $(i_{m+1}, \sigma_{m+1}) \neq (i_m, -\sigma_m)$. We say that g is *cyclically reduced* if it is reduced and also satisfies $(i_1, \sigma_1) \neq (i_l, -\sigma_l)$.

Proposition 9.2. *For every $C \geq 1$, there is a positive constant $s_{9.2}(C) \leq 1$ with the following property. Take any family of maps $g_1, \dots, g_k \in G \ltimes V$ satisfying the following hypotheses.*

(H1) *Every g_i is of type X_0.*

(H2) *Any pair taken among the maps $\{g_1, \dots, g_k, g_1^{-1}, \dots, g_k^{-1}\}$ is C-non-degenerate, except of course if it has the form (g_i, g_i^{-1}) for some i.*

(H3) *For every i, we have $s(g_i) \leq s_{9.2}(C)$ and $s(g_i^{-1}) \leq s_{9.2}(C)$.*

Take any nonempty cyclically reduced word $g = g_{i_1}^{\sigma_1} \cdots g_{i_l}^{\sigma_l}$ (*with* $1 \leq i_m \leq k$, $\sigma_m = \pm 1$ *for every* m). *Then* g *is of type* X_0, $2C$*-non-degenerate, and we have*

$$\left\| M(g) - \sum_{m=1}^{l} M(g_{i_m}^{\sigma_m}) \right\| \leq l k_{8.1}(2C)$$

(*where* $k_{8.1}(2C)$ *is the constant introduced in Proposition* 8.1).

The proof proceeds by induction, with Proposition 7.4 and Proposition 8.1 providing the induction step. However, there is a subtlety (already dealt with in [22]). When we suppose that the pair (g, h) is C-non-degenerate, we can only conclude that gh is $2C$-non-degenerate; this would break the induction if we used a direct approach. To guarantee $2C$-non-degeneracy for all words, we must use the fact that the contraction strength of g grows (technically the number $s(g)$ diminishes) exponentially with its length, so that the (Hausdorff) distance between $A_g^{\geq}$ and $A_{g_{i_1}^{\sigma_1}}^{\geq}$ is in fact a sum of exponentially diminishing increments and remains bounded. To take this into account, we prove by induction a series of slightly more complicated statements.

Proof. The proof is exactly the same as proof of Proposition 5.2 in [22], *mutatis mutandis*.

Given the importance of this point, let us briefly recap the strategy of this proof. Let us fix $C \geq 1$, a positive constant $s_{9.2}(C) \leq 1$ to be determined in the course of the proof, and a family $g_1, \ldots, g_k$ satisfying the hypotheses (H1), (H2), and (H3). We show by induction on l that whenever we take a nonempty cyclically reduced word $g = g_{i_1}^{\sigma_1} \cdots g_{i_l}^{\sigma_l}$, we have the following properties:

(i) the map g is of type X_0;

(ii) $$\begin{cases} \alpha^{\mathrm{Haus}}(A_g^{\geq}, A_{g_{i_1}^{\sigma_1}}^{\geq}) \lesssim_C 2(1 - 2^{-(l-1)}) s_{9.2}(C), \\ \alpha^{\mathrm{Haus}}(A_g^{\leq}, A_{g_{i_l}^{\sigma_l}}^{\leq}) \lesssim_C 2(1 - 2^{-(l-1)}) s_{9.2}(C); \end{cases}$$

(iii) $s(g) \leq 2^{-(l-1)} s_{9.2}(C)$;

(iv) $$\left\| M(g) - \sum_{m=1}^{l} M(g_{i_m}^{\sigma_m}) \right\| \leq (l-1) k_{8.1}(2C);$$

(v) If $h = g_{i'_1}^{\sigma'_1} \cdots g_{i'_{l'}}^{\sigma'_{l'}}$ is another nonempty cyclically reduced word of length $l' \leq l$ such that gh (or equivalently hg) is still cyclically reduced, the pair (g, h) is $2C$-non-degenerate.

The proposition then follows from the properties (i), (iv) and (v). For the actual proof of these five statements, we refer the reader to the proof of Proposition 5.2 in [22]. □

10. Construction of the group

Here we prove the Main Theorem. We closely follow Section 6 from [22], with only two substantial differences:

- while in the case of the adjoint representation, existence of a $-w_0$-invariant vector in V_0^t was automatic, here we must postulate it explicitly (Assumption 10.1);
- where we originally relied on Lemma 7.2 in [5], we now need the more general Lemma 4.3.a in [6].

In the next-to-last paragraph of the proof, we have also made more explicit the relationship between $s_{\mathrm{Main}}(C)$ and $s_{9.2}(C)$.

Let us recall the outline of the proof. We begin by showing (Lemma 10.3) that if we take a group generated by a family of C-non-degenerate, sufficiently contracting maps of type X_0 with suitable Margulis invariants, it satisfies all of the conclusions of the Main Theorem, except Zariski-density. We then exhibit such a group that is also Zariski-dense (and thus prove the Main Theorem).

The idea is to ensure that the Margulis invariants of all elements of the group remain close to some half-line. Obviously if $-w_0$ maps every element of V_0^t to its opposite, Proposition 8.1 (i) makes this impossible. So we now exclude this case:

Assumption 10.1. The representation ρ is such that the action of w_0 on V_0^t is not trivial.

This is precisely condition (i) from the Main Theorem. More precisely, V_0^t is the set of all vectors that satisfy (i)(a), and what we say here is that some of them also satisfy (i)(b).

Example 10.2. (1) Consider $G = \mathrm{SO}^+(p,q)$ acting on $\mathbb{R}^{p+q}$ (with $p \geq q$); we have already seen that the only case when $V_0^t \neq 0$ is when $p - q = 1$ (see Example 4.22.1). So let $p = n + 1$, $q = n$; then we may show that

$$w_0|_{V_0^t} = (-1)^n \operatorname{Id} \tag{10.1}$$

(this is essentially the content of Lemma 3.1 in [3] or of Proposition 2.7 in [21]). So $G = \mathrm{SO}^+(n+1, n)$ satisfies this assumption if and only if n is odd.

(2) If G is any semisimple real Lie group acting on $V = \mathfrak{g}$ (its Lie algebra) by the adjoint representation, then $\mathfrak{g}_0^t$ contains the Cartan subspace $\mathfrak{a}$ (see Example 4.22.2), on which w_0 obviously acts nontrivially unless $\mathfrak{a}$ is itself trivial. So this assumption is satisfied whenever G is noncompact.

Thanks to Assumption 10.1, we choose once and for all some nonzero vector $M_C \in V_0^t$ that is a fixed point of $-w_0$ (which is possible since w_0 is an involution). This requirement still leaves us free to prescribe the norm of this vector; let us additionally assume that $\|M_C\| = 2k_{8.1}(2C)$.

Lemma 10.3. *Take any family* $g_1, \dots, g_k \in G \ltimes V$ *satisfying the hypotheses* (H1), (H2), *and* (H3) *from Proposition* 9.2, *and also the following additional condition.*

(H4) *For every* i, $M(g_i) = M_C$.

Then the group generated by $g_1, \dots, g_k$ *is free* (*with* $g_1, \dots, g_k$ *being a basis*) *and acts properly discontinuously on the affine space* V_{Aff}.

Proof. The proof is exactly the same as the proof of Lemma 6.1 in [22], *mutatis mutandis*.

The (orthogonal) projection

$$\hat{\pi}_{\mathfrak{z}} \colon \hat{\mathfrak{g}} \longrightarrow \mathfrak{z} \oplus \mathbb{R}^0 \qquad \text{parallel to } \mathfrak{d} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

now becomes the (orthogonal) projection

$$\hat{\pi}_t \colon A \longrightarrow V_0^t \oplus \mathbb{R}^0 \quad \text{parallel to } V_0^r \oplus V_0^> \oplus V_0^<.$$

(Let us just explicitly restate the proof that the group is free, as it is very short. The group is free simply because any nonempty reduced word on the $g_i^{\pm 1}$ is conjugate to some cyclically reduced word, which, by Proposition 9.2, is of type X_0 and in particular different from the identity.) ☐

Proof of Main Theorem. First note that all the assumptions we have made on ρ in the course of the paper were legitimate, in the sense that they follow from the hypotheses of the Main Theorem.

- Assumption 10.1 is just the condition (i).
- Assumption 4.23 is the weaker condition (i)(a).
- Assumption 3.2 is an even weaker condition that follows from (i)(a) (see Remark 3.4). and
- Assumption 3.10 is just the condition (ii).

Once again, we use the same strategy as in the proof of the Main Theorem of [22]. We find a positive constant $C \geq 1$ and a family of maps $g_1, \dots, g_k$ in $G \ltimes V$ (with $k \geq 2$) that satisfy the conditions (H1) through (H4) and whose linear parts generate a Zariski-dense subgroup of G, then we apply Lemma 10.3. We proceed in several stages.

- We begin by using a result of Benoist: we apply Lemma 4.3.a in [6] to
  - $\Gamma = G$;
  - $t = k + 1$;
  - $\Omega_1 = \dots = \Omega_k = \mathfrak{a}_{\rho, X_0} \cap \mathfrak{a}^{++}$.

  This gives us, for any $k \geq 2$, a family of maps $\gamma_1, \dots, \gamma_k \in G$ (which we shall see as elements of $G \ltimes V$, by identifying G with the stabiliser of p_0), such that

(i) every γ_i is of type X_0 (this is (H1));

(ii) for any two indices i, i' and signs σ, σ' such that $(i', \sigma') \neq (i, -\sigma)$, the spaces $V^{\geq}_{\gamma_i^\sigma}$ and $V^{\leq}_{\gamma_{i'}^{\sigma'}}$ are transverse;

(iii) any single γ_i generates a Zariski-connected group;

(iv) all of the γ_i generate together a Zariski-dense subgroup of G.

A comment about item (i): we actually get not only that every γ_i is of type X_0, but also that every γ_i is $\mathbb{R}$-regular.

A comment about item (ii): since we have taken Benoist's Γ to be the whole group G, we have $\theta = \Pi$, so that Y_Γ is the complete flag variety G/P^+. Benoist's conclusion can then be restated by saying that the pair of cosets

$$(\phi_g P^+,\ \phi_h P^-)$$

(where ϕ_g and ϕ_h are respective canonizing maps of g and h as defined by us in Proposition 4.16) is in the open G-orbit of $G/P^+ \times G/P^-$. Once again it is actually stronger than our conclusion, which is equivalent to saying that the pair of cosets

$$(\phi_g P^+_{X_0},\ \phi_h P^-_{X_0})$$

is in the open G-orbit of $G/P^+_{X_0} \times G/P^-_{X_0}$.

- Clearly every pair of transverse spaces is C-non-degenerate for some finite C; and here we have a finite number of such pairs. Hence if we choose some suitable value of C (which we fix for the rest of this proof), the hypothesis (H2) becomes a direct consequence of the condition (ii) above.
- From condition (iii) (Zariski-connectedness), it follows that any algebraic group containing some power γ_i^N of some generator must actually contain the generator γ_i itself. This allows us to replace every γ_i by some power γ_i^N without sacrificing condition (iv) (Zariski-density). Clearly, conditions (i), (ii) and (iii) are then preserved as well. If we choose N large enough, we may suppose (thanks to Remark 5.7) that the numbers $s(\gamma_i^{\pm 1})$ are as small as we wish: this gives us (H3). In fact, we shall suppose that for every i, we have $s(\gamma_i^{\pm 1}) \leq s_{\mathrm{Main}}(C)$ for an even smaller constant $s_{\mathrm{Main}}(C)$, to be specified soon.
- To satisfy (H4), we replace the maps γ_i by the maps

$$g_i := \tau_{\phi_i^{-1}(M_C)} \circ \gamma_i \tag{10.2}$$

(for $1 \leq i \leq k$), where ϕ_i is a canonizing map for γ_i.

We need to check that this does not break the first three conditions. Indeed, for every i, we have $\gamma_i = \ell(g_i)$; even better, since the translation vector $\phi_i^{-1}(M_C)$ lies in the subspace $V^{=}_{\gamma_i}$ stable by γ_i, obviously the translation commutes with γ_i, hence g_i has the same geometry as γ_i (meaning that $A^{\geqq}_{g_i} = A^{\geqq}_{\gamma_i} = V^{\geqq}_{\gamma_i} \oplus \mathbb{R}p_0$ and $A^{\leqq}_{g_i} = A^{\leqq}_{\gamma_i} = V^{\leqq}_{\gamma_i} \oplus \mathbb{R}p_0$). Hence the g_i still satisfy the hypotheses (H1) and (H2), but now we have $M(g_i) = M_C$ (this is (H4)). As for contraction strength, we have, by Lemma 5.8:

$$s(g_i) \lesssim_C s(\gamma_i)\|\tau_{M_C}\| \leq s_{\mathrm{Main}}(C)\|\tau_{M_C}\|, \tag{10.3}$$

and similarly for g_i^{-1}. Recall that $\|M_C\| = 2k_{8.1}(2C)$, hence $\|\tau_{M_C}\|$ depends only on C: in fact it is equal to the norm of the 2-by-2 matrix $\left(\begin{smallmatrix} 1 & \|M_C\| \\ 0 & 1 \end{smallmatrix}\right)$. It follows that if we choose

$$s_{\mathrm{Main}}(C) \leq s_{9.2}(C) \begin{Vmatrix} 1 & 2k_{8.1}(2C) \\ 0 & 1 \end{Vmatrix}^{-1}, \tag{10.4}$$

then the hypothesis (H3) is satisfied.

We conclude that the group generated by the elements $g_1, \dots, g_k$ acts properly discontinuously (by Lemma 10.3), is free (by the same result), nonabelian (since $k \geq 2$), and has linear part Zariski-dense in G. ☐

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