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Groups, Geometry, and Dynamics

Quasi-automorphisms of the infinite rooted 2-edge-coloured binary tree

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Abstract. We study the group QV, the self-maps of the infinite 2-edge coloured binary tree which preserve the edge and colour relations at cofinitely many locations. We introduce related groups QF, QT, $\tilde{Q}T$, and $\tilde{Q}V$, prove that QF, $\tilde{Q}T$, and $\tilde{Q}V$ are of type F_∞, and calculate finite presentations for them. We calculate the normal subgroup structure of all 5 groups, the Bieri–Neumann–Strebel–Renz invariants of QF, and discuss the relationship of all 5 groups with other generalisations of Thompson's groups.

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Contents

1. Introduction

R. Thompson's groups F, T, and V were originally used to, among other things, construct finitely presented groups with unsolvable word problem [33, 28]. Brown and Geoghegan proved F is FP_∞ [10], giving the first example of a torsion-free FP_∞ group with infinite cohomological dimension, and later Brown used an important new geometric technique to prove that F, T, and V are all of type F_∞ [9]. For more background on Thompson's groups, see [13].

Recently many variations on Thompson's groups have been studied, and using techniques similar to Brown's, many of these have been proven to be of type F_∞. These include Brin's groups sV [19], generalisations of these [26], and the braided Thompson's groups BV and BF [11] which were originally defined by Brin [8].

Let $\mathcal{T}_{2,c}$ denote the infinite binary 2-edge-coloured tree and let QV be the group of all bijections on the vertices of $\mathcal{T}_{2,c}$ which respect the edge and colour relations except for possibly at finitely many locations. The group QV was studied by Lehnert [25] who proved that QV is co-context-free, i.e. the co-word problem in QV is context-free. He also gave an embedding of V into QV. More recently, Bleak, Matucci, and Neunhöffer gave an embedding of QV into V [6]. The group QV was called $\mathrm{QAut}(\mathcal{T}_{2,c})$ by Bleak, Matucci, and Neunhöffer, and was called both $\mathrm{QAut}(\mathcal{T}_{2,c})$ and G by Lehnert.

We write QV instead of $\mathrm{QAut}(\mathcal{T}_{2,c})$ in order to follow the convention of naming varations of Thompson's groups using a prefix (e.g. BV for the braided Thompson's groups [12] and nV for the Brin–Thompson groups [7]). The Q may be thought of as standing for "quasi-automorphism".

There is a natural surjection $\pi\colon QV \to V$ (Lemma 2.1), and we denote by QF the preimage $\pi^{-1}(F)$ and by QT the preimage $\pi^{-1}(T)$. The group QF was studied by Lehnert, where it is called G^0 [25, Definition 2.9]. The surjection $\pi\colon QV \to V$ is not split (Proposition 2.9), although it does split when restricted to QF (Lemma 2.5). This situation is similar to that of the braided Thompson groups—BF splits as an extension of F by the infinite pure braid group P_∞, whereas BV is a non-split extension of V by P_∞ [36, §1.2]. There is no braided analogue of T.

The following theorem appeared in Lehnert's thesis, [25, Satz 2.14, 2.30], although his proof that QF is of type FP_∞ uses Brown's criterion, a different approach from that used here.

Theorems 3.6 and 4.10. *QF is F_∞ and has presentation*

$$QF = \left\langle \sigma, \alpha, \beta \;\middle|\; \begin{array}{l} \sigma^2,\ [\sigma, \sigma^{\alpha^2}],\ (\sigma\sigma^\alpha)^3,\ \sigma\sigma^\alpha\sigma\sigma^{\alpha\beta^{-1}\alpha^{-1}}, \\ [\alpha\beta^{-1},\ \alpha^{-1}\beta\alpha],\ [\alpha\beta^{-1}, \alpha^{-2}\beta\alpha^2], \\ [\nu, \sigma] \textit{ for all } \nu \in X. \end{array} \right\rangle$$

where

$$X = \left\{ \begin{array}{cc} \beta, \beta^\alpha, \alpha^3\beta^{-1}\alpha^{-2}, & \alpha^2\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-2}, \\ \alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}, & \alpha\beta\alpha^{-1}\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha\beta^{-1}\alpha^{-1} \end{array} \right\}.$$

We write $\mathcal{T}_{2,c} \cup \{\zeta\}$ for the disjoint union of $\mathcal{T}_{2,c}$ and a single vertex labelled ζ. We write $\widetilde{Q}V$ for the group of all bijections on the vertices of $\mathcal{T}_{2,c} \cup \{\zeta\}$ which respect the edge and colour relations except for possibly at finitely many locations. Again, there is a surjection $\pi\colon \widetilde{Q}V \to V$ and we write $\widetilde{Q}T$ for the preimage $\pi^{-1}(T)$. The groups $\widetilde{Q}V$ and $\widetilde{Q}T$ are easier to study than QV and QT because $\pi\colon \widetilde{Q}V \to V$ splits (Lemma 2.3).

In Section 3 we find finite presentations for $\widetilde{Q}V$ and $\widetilde{Q}T$ and in Section 4 we show $\widetilde{Q}V$, $\widetilde{Q}T$ and QF are of type F_∞.

Section 5 contains a complete classification of the normal subgroups of QF, QT, QV, $\widetilde{Q}T$, and $\widetilde{Q}V$. We find that the structure is very similar to that of F, T, and V.

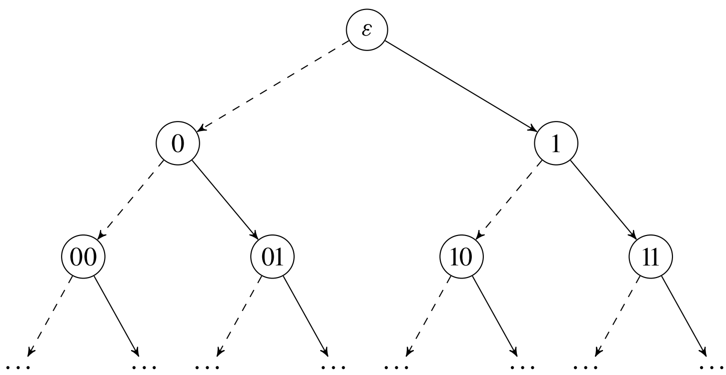

Figure 1. $\mathcal{T}_{2,c}$, the rooted 2-edge-coloured infinite binary tree.

We identify the vertices of $\mathcal{T}_{2,c}$ with the set $\{0,1\}^*$ of finite length words on the alphabet $\{0,1\}$, denoting the empty word by ε (see Figure 1), and hence view QV as acting on $\{0,1\}^*$. Let $\mathrm{Sym}(\{0,1\}^*)$ denote the finite support permutation group on $\{0,1\}^*$ and $\mathrm{Alt}(\{0,1\}^*)$ the finite support alternating group on $\{0,1\}^*$. Both these groups are normal in QV, QT, and QF. Let $Z = \{0,1\}^* \cup \{\zeta\}$, so that Z can be identified with the set of vertices of $\mathcal{T}_{2,c} \cup \{\zeta\}$, then $\mathrm{Sym}(Z)$ and $\mathrm{Alt}(Z)$ have the obvious definition and are normal in $\widetilde{Q}V$ and $\widetilde{Q}T$. Recall that $F/[F,F] \cong \mathbb{Z} \oplus \mathbb{Z}$, but since T and V are simple and non-abelian, $[T,T] \cong T$ and $[V,V] \cong V$ [13, Theorems 4.1, 5.8, 6.9].

Theorem 5.1 (normal subgroup structure). (1) *A non-trivial normal subgroup of* QF *is either* $\mathrm{Alt}(\{0,1\}^*)$, $\mathrm{Sym}(\{0,1\}^*)$, *or contains*

$$[QF, QF] = \mathrm{Alt}(\{0,1\}^*) \rtimes [F, F].$$

(2) *A proper non-trivial normal subgroup of* $\tilde{Q}T$ *is either* $\mathrm{Alt}(Z)$, $\mathrm{Sym}(Z)$, *or*

$$[\tilde{Q}T, \tilde{Q}T] = \mathrm{Alt}(Z) \rtimes T.$$

(3) *A proper non-trivial normal subgroup of* $\tilde{Q}V$ *is either* $\mathrm{Alt}(Z)$, $\mathrm{Sym}(Z)$, *or*

$$[\tilde{Q}V, \tilde{Q}V] = \mathrm{Alt}(Z) \rtimes V.$$

(4) *A proper non-trivial normal subgroup of* QT *is one of either* $\mathrm{Alt}(\{0,1\}^*)$, $\mathrm{Sym}(\{0,1\}^*)$, *or*

$$[QT, QT] = (\mathrm{Alt}(Z) \rtimes T) \cap QT.$$

Moreover, $(\mathrm{Alt}(Z) \rtimes T) \cap QT$ *is an extension of* T *by* $\mathrm{Sym}(\{0,1\}^*)$.

(5) *A proper non-trivial normal subgroup of* QV *is one of either* $\mathrm{Alt}(\{0,1\}^*)$, $\mathrm{Sym}(\{0,1\}^*)$, *or*

$$[QV, QV] = (\mathrm{Alt}(Z) \rtimes V) \cap QV.$$

Moreover, $(\mathrm{Alt}(Z) \rtimes V) \cap QV$ *is an extension of* V *by* $\mathrm{Sym}(\{0,1\}^*)$.

In Section 6 we calculate the Bieri–Neumann–Strebel–Renz invariants, or Sigma invariants, of QF (they are uninteresting for the groups QT, QV, $\tilde{Q}T$, and $\tilde{Q}V$ since they all have finite abelianisation). They have previously been computed for F by Bieri, Geoghegan, and Kochloukova [3], and for BF by Zaremsky [36, §3]. We also show that QF is an ascending HNN extension

$$QF \cong QF(1)*_{\theta,t},$$

where $QF \cong QF(1)$.

We find that $\Sigma^i(QF) \cong \Sigma^i(F)$ for all i. More precisely, we define $\pi^*\chi = \chi \circ \pi$ for any character χ of F and prove that π^* induces an isomorphism between the character spheres $S(F)$ and $S(QF)$.

Let $\chi_0 \colon F \to \mathbb{R}$ and $\chi_1 \colon F \to \mathbb{R}$ be the two characters given by $\chi_0(A) = -1$, $\chi_0(B) = 1$, and $\chi_0(A) = \chi_1(B) = 1$, where A and B are the standard generators of F.

Theorem 6.3 (the Bieri–Neumann–Strebel–Renz invariants of QF). *The character sphere* $S(QF)$ *is isomorphic to* S^1 *and for any ring* R,

(1) $\Sigma^1(QF, R) = \Sigma^1(QF) = S(QF) \setminus \{[\pi^*\chi_0], [\pi^*\chi_1]\}$;

(2) $\Sigma^i(QF, R) = \Sigma^i(QF) = S(QF) \setminus \{[a\pi^*\chi_0 + b\pi^*\chi_1] \colon a, b \geq 0\}$ *for all* $i \geq 2$.

In Section 7 we study the relationship of the groups studied in this note to previously studied families of groups which generalise Thompson's groups and are known to be of type F_∞. We show that the groups studied here cannot be proved to be of type F_∞ using Farley–Hughes' study of Groups of Local Similarities [16, Theorem 1.1], or Martínez-Pérez–Nucinkis' study of Automorphism Groups of Cantor Algebras [27]. We also show that one cannot use Thumann's study of Operad groups [34] to determine whether QV is of type F_∞ .

Remark 1.1 (notation). A tree diagram representing an element of Thompson's group V is a triple (L, R, f) where L and R are rooted subtrees of the infinite binary rooted tree with equal numbers of leaves, and f is a bijection from the leaves of L to the leaves of R. To represent an element of Thompson's group F we require only the first two elements of the tuple—the bijection is understood to preserve the left-to-right ordering of the leaves. For further background on tree diagrams see [13, §2].

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2. Introducing QF, QT, QV, $\tilde{Q}T$, and $\tilde{Q}V$

2.1. The group QV. As mentioned in the introduction the group QV is the group of all bijections on the vertices of $\mathcal{T}_{2,c}$, the infinite binary 2-edge-coloured tree, which respect the edge and colour relations except for possibly at finitely many locations. Given some $\tau \in QV$ and $a \in \{0,1\}^*$, let s_a be the maximal suffix such that $a = x_a \cdot s_a$ and $\tau a = y_a \cdot s_a$ for some $x_a, y_a \in \{0,1\}^*$ (here the operation $\cdot$ is concatenation of words). Bleak, Mattuci and Neunhöffer prove that the set

$$\{(x_a, y_a) : a \in \{0,1\}^*\}$$

is finite [6, Claim 8], and hence that the subset

$$M_\tau = \{(x, y) : \text{there are infinitely many } a \in \{0,1\}^* \text{ with } (x_a, y_a) = (x, y)\}$$

is finite also. Moreover, they show that the sets

$$\mathcal{L}_\tau = \{x : (x, y) \in M_\tau\}$$

and

$$\mathcal{R}_\tau = \{y : (x, y) \in M_\tau\}$$

both form finite complete anti-chains for the poset $\{0,1\}^*$, where $x \leq x'$ if x is a subword of x'. Write b_τ for the bijection

$$b_\tau : \mathcal{L}_\tau \longrightarrow \mathcal{R}_\tau.$$

Note that restricting τ to $\mathcal{L}_\tau$ does not necessarily give the bijection b_τ. This is because if σ is any finite permutation of $\{0,1\}^*$ then $b_{\sigma\circ\tau} = b_\tau$.

Let L and R be the binary trees with leaves $\mathcal{L}_\tau$ and $\mathcal{R}_\tau$ respectively, then (L, R, b_τ) is a tree-pair diagram for an element of V which we denote by v_τ.

In fact, v_τ can be viewed as the induced action on the space of ends of the simplicial tree $\mathcal{T}_{2,c}$, the set whose elements are enumerated by the right-infinite words $\{0,1\}^{\mathbb{N}}$. With its natural topology the space of ends is homeomorphic to a Cantor set with v_τ acting as a homeomorphism. Explicitly, the action of v_τ on $\{0,1\}^*$ is

$$\begin{aligned} v_\tau \colon \{0,1\}^{\mathbb{N}} &\longrightarrow \{0,1\}^{\mathbb{N}}, \\ x \cdot s &\longmapsto y \cdot s \text{ for } (x,y) \text{ in } M_\tau. \end{aligned}$$

This is well-defined since $\mathcal{L}_\tau$ is a complete anti-chain in the poset $\{0,1\}^*$.

Lemma 2.1. *The map $\pi\colon QV \to V$ sending $\tau \mapsto v_\tau$ is a group homomorphism.*

Proof. For any topological space X let $\mathrm{Ends}(X)$ denote the space of ends of X.

If $\tau \in QV$ then τ may not be a simplicial automorphism of $\mathcal{T}_{2,c}$, but there will always exist a rooted subtree $L \subseteq \mathcal{T}_{2,c}$ such that

$$\tau|_{\mathcal{T}_{2,c}\setminus L}\colon \mathcal{T}_{2,c} \setminus L \longrightarrow \mathcal{T}_{2,c} \setminus \tau(L)$$

is a simplicial map. Thus τ determines a map

$$\mathrm{Ends}(\tau)\colon \mathrm{Ends}(\mathcal{T}_{2,c}) \longrightarrow \mathrm{Ends}(\mathcal{T}_{2,c}).$$

This is well-defined—if we choose to remove a different rooted subtree L' instead then, since removing a compact set doesn't affect the ends of a space, we see that

$$\mathcal{T}_{2,c} \setminus (L \cup L') \longrightarrow \mathcal{T}_{2,c} \setminus \tau(L \cup L')$$

and hence

$$\mathcal{T}_{2,c} \setminus (L') \longrightarrow \mathcal{T}_{2,c} \setminus \tau(L')$$

induce the same map on $\mathrm{Ends}(\mathcal{T}_{2,c})$.

If σ is some other element of QV then choosing a large enough finite rooted subtree L, both maps in the composition

$$\mathcal{T}_{2,c} \setminus L \stackrel{\tau}{\longmapsto} \mathcal{T}_{2,c} \setminus \tau(L) \stackrel{\sigma}{\longmapsto} \mathcal{T}_{2,c} \setminus \sigma \circ \tau(L)$$

are simplicial maps. Now use that $\mathrm{Ends}(-)$ is a functor. □

The kernel of π are those permutations of $\{0,1\}^*$ which are the identity on all but finitely many points, the finite support permutation group $\mathrm{Sym}(\{0,1\}^*)$. Thus there is a group extension

$$1 \longrightarrow \mathrm{Sym}(\{0,1\}^*) \longrightarrow QV \longrightarrow V \longrightarrow 1.$$

This extension does not split, see Proposition 2.9.

2.2. The group $\widetilde{Q}V$. Recall that $\mathcal{T}_{2,c} \cup \{\zeta\}$ denotes the 2-edge-coloured tree $\mathcal{T}_{2,c}$ together with an isolated vertex ζ, so the vertex set of $\mathcal{T}_{2,c} \cup \{\zeta\}$ is $Z = \{0,1\}^* \cup \{\zeta\}$. There is a map π from $\widetilde{Q}V$ to V defined as previously, whose kernel is $\mathrm{Sym}(Z)$.

Let $\leq_{\mathrm{lex}}$ be the total order on $\{0,1\}^*$ defined by following rule: given a word $x, y \in \{0,1\}$ let $\tilde{x}, \tilde{y} \in \{0, \frac{1}{2}, 1\}^{\mathbb{N}}$ be the words obtained by concatenating x and y respectively with the infinite word containing only the symbol $\frac{1}{2}$; now we say that $x \leq_{\mathrm{lex}} y$ if and only if $\tilde{x}$ is smaller that $\tilde{y}$ in the lexicographical order where $0 \lneq \frac{1}{2} \lneq 1$. For example, $00 \leq_{\mathrm{lex}} 0$ but $01 \geq_{\mathrm{lex}} 0$. We extend the $\leq_{\mathrm{lex}}$ order to Z so that ζ is strictly larger than all elements of $\{0,1\}^*$.

Remark 2.2. If T is any finite binary tree, there is an order preserving bijection

$$b_T\colon \mathrm{nodes}(T) \cup \{\zeta\} \longrightarrow \mathrm{leaves}(T).$$

Moreover, adding a caret to T at any leaf doesn't affect the bijection on the remaining leaves.

Lemma 2.3. *The short exact sequence*

$$1 \longrightarrow \mathrm{Sym}(Z) \longrightarrow \widetilde{Q}V \xrightarrow{\pi} V \longrightarrow 1$$

is split.

Using the computer algebra package GAP Bleak, Matucci, and Neunhöffer provide the same splitting [6, Lemma 15].

Proof. Let (L, R, f) be a tree diagram for an element v of V, we define the element $\iota(v)$ of $\widetilde{Q}V$ by

$$\iota(v)\colon \{0,1\}^* \cup \{\zeta\} \longrightarrow \{0,1\}^* \cup \{\zeta\},$$
$$x \longmapsto \begin{cases} b_R^{-1} \circ f \circ b_L(x) & \text{if } x \in \mathrm{nodes}(L) \cup \{\zeta\}, \\ f(x) & \text{if } x \in \mathrm{leaves}(L), \end{cases}$$

and extend $\iota(v)$ onto all $\{0,1\}^*$ by having $\iota(v)$ preserve all edge and colour relations below $\mathrm{leaves}(L)$.

If (L', R', f') is an expansion of (L, R, f) then (L', R', f') determines the same element $\iota(v)$ (use Remark 2.2). Since any two tree diagrams representing the same element of V have a common expansion this proves $\iota(v)$ is independent of choice of tree diagram.

We claim that $\iota\colon v \mapsto \iota(v)$ is a group homomorphism. Let v and w be two elements of V represented by tree diagrams (L, R, f) and (R, S, g) (by performing expansions we may assume any two elements of V are represented by tree pair diagrams in this form). Thus the composition of the two elements of V has tree pair diagram $(L, S, g \circ f)$. Now one verifies that for $x \in \text{nodes}(L) \cup \{\zeta\}$,

$$\begin{aligned}\iota(v) \circ \iota(w)(x) &= b_S^{-1} \circ g \circ b_R \circ b_R^{-1} \circ f \circ b_L(x) \\ &= b_S^{-1} \circ g \circ f \circ b_L(x) \\ &= \iota(vw)(x).\end{aligned}$$

Similarly, for $x \in \text{leaves}(L)$, $\iota(v) \circ \iota(w)$ and $\iota(vw)$ agree. □

Remark 2.4. For an explicit description of this splitting for the generators of V, let A, B, C and D denote the generators of V as denoted by A, B, C and π_0 in [13, Figures 16 and 17]. Then $\iota(A)$, $\iota(B)$, $\iota(C)$, and $\iota(D)$ are shown in Figures 2–5.

Hereafter we define $\alpha = \iota(A)$, $\beta = \iota(B)$, $\gamma = \iota(C)$, and $\delta = \iota(D)$.

2.3. The group QF. We define the group QF to be the preimage $\pi^{-1}(F)$ of Thompson's group F under $\pi\colon QV \to V$.

Lemma 2.5. *The short exact sequence*

$$1 \longrightarrow \text{Sym}(\{0,1\}^*) \longrightarrow QF \xrightarrow{\pi} F \longrightarrow 1$$

is split.

Proof. Restrict the splitting map ι defined in Lemma 2.3 to QF. Since elements (L, R) of F always map the right-hand leaf of L to the right-hand leaf of R, we see that $\iota(f)$ fixes ζ for all $f \in F$. □

Remark 2.6. If (L, R) is a tree-diagram for an element f of F, where L has nodes $(x_1, \dots, x_n)$ and R has nodes $(y_1, \dots, y_n)$ then, assuming $x_i \leq_{\text{lex}} x_{i+1}$ and $y_i \leq_{\text{lex}} y_{i+1}$ for all $1 \leq i \leq n-1$, $\iota(f)$ maps $x_i \mapsto y_i$. The action on all other elements of $\{0,1\}^*$ is determined by L and R—if l_i is the i$^{\text{th}}$ leaf of L and r_i is the i^{th} leaf of R then $\iota(f)$ takes the full subtree with root l_i to the full subtree with root r_i.

In particular, for all $f \in F$, the image $\iota(f)$ preserves the $\leq_{\text{lex}}$ order on $\{0,1\}^*$.

2.4. The group QT. We define the group QT to be the preimage $\pi^{-1}(T)$ of Thompson's group T under $\pi\colon QV \to V$.

Restricting the map π to QT gives a short exact sequence

$$1 \longrightarrow \text{Sym}(\{0,1\}^*) \longrightarrow QT \xrightarrow{\pi} T \longrightarrow 1.$$

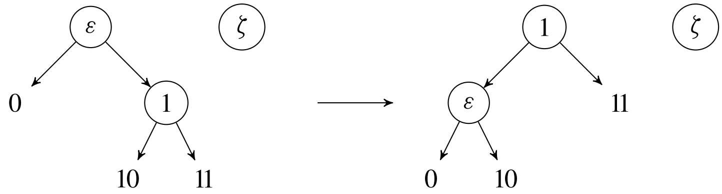

Figure 2. Element $\alpha = \iota(A)$.

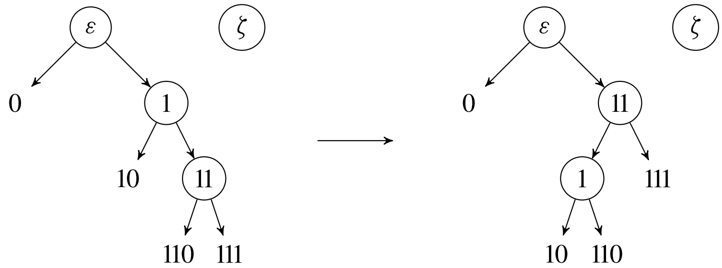

Figure 3. Element $\beta = \iota(B)$.

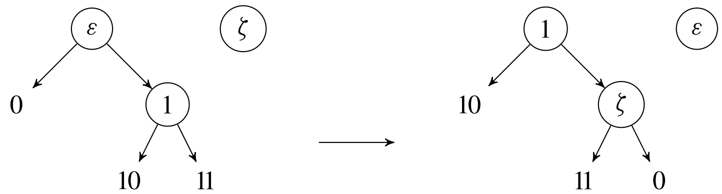

Figure 4. Element $\gamma = \iota(C)$.

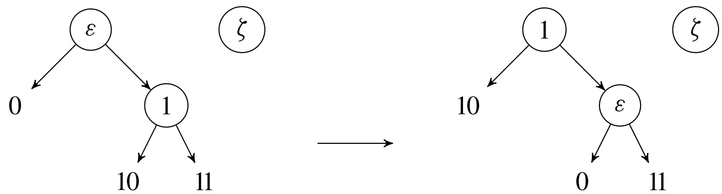

Figure 5. Element $\delta = \iota(D)$.

In [25, Lemma 2.10], Lehnert proves that the splitting $\iota\colon F \to QF$ does not extend to an embedding of T into QV. Examining the proof, he in fact shows the stronger statement below. Recall that A and B denote the standard generators of F.

Lemma 2.7. [25, Lemma 2.10] *Let j be an embedding of F into QV such that $j(A) = \iota(A)$, then j does not extend to an embedding of T into QV.*

Lemma 2.8. *Let f be an element of F and let $j\colon F \to QF$ be a splitting of $\pi\colon QF \to F$, then the actions of $\iota(f)$ and $j(f)$ on $\{0,1\}^*$ agree on cofinitely many points.*

Proof. Recall from Section 2.1 that given $\tau \in QV$, for all but finitely many $a \in \{0,1\}^*$ we have $a = x_a \cdot s_a$ for some $x_a \in \mathcal{L}_\tau$ and $\tau(a) = b_\tau(x_a) \cdot s_a$. Since the tree pair diagram for $\pi(\tau)$ is (L, R, b_τ), b_τ is determined completely by $\pi(\tau)$. We conclude that $\iota(f)$ and $j(f)$ agree on all but finitely many elements of $\{0,1\}^*$, since by assumption $\pi \circ \iota(f) = \pi \circ j(f)$. □

Proposition 2.9. *The group extension*

$$1 \longrightarrow \mathrm{Sym}(\{0,1\}^*) \longrightarrow QT \xrightarrow{\pi} T \longrightarrow 1$$

is not split.

Proof. Let $j\colon T \to QT$ be a splitting of π, by Lemma 2.8 we have that $j(A)$ and $\iota(A)$ agree on cofinitely many points. Hence, after possibly extending via adding carets, we can find a tree-pair diagram (L, R, f) and positive integer k such that in L every leaf under 0 has length exactly k, and every leaf under 1 has length exactly $k+1$ and similarly in R every leaf under 0 has length exactly $k+1$, and every leaf under 1 has length exactly k.

Following the argument of the proof of Lemma 2.7, we see that this is incompatible with every possible choice of $j(C)$. □

3. Finite presentations

In this section we compute finite presentations for the groups QF, $\widetilde{Q}T$, and $\widetilde{Q}V$. We use a method for finding a finite presentation for a semi-direct product $N \rtimes Q$, whereby we find a presentation for N such that Q acts by permutations and with finitely many orbits on both the generating set and the relating set. Then as long as Q is finitely presented, and the Q-stabilisers of the generating set are finitely generated, one can write down a finite presentation for $N \rtimes Q$ [26, Lemma A.1]. This technique predates the referenced paper of Martínez-Pérez, Matucci, and Nucinkis, but we are not aware of anywhere else where the proof is given.

We apply this method to QF, $\widetilde{Q}T$ and $\widetilde{Q}V$ using their decompositions as semi-direct products with kernel an infinite symmetric group and quotient F, T, and V respectively.

Let $N = \langle S_N | R_N \rangle$ and $Q = \langle S_Q | R_Q \rangle$ be groups such that Q acts on S_N by permutations and the action induces an action on R_N with $Q(R_N) = R_N$. Let S_0 be a set of representatives for the Q-orbits in S_N and R_0 a set of representatives for the Q-orbits in R_N.

For any set X, we write X^* for the finite length words of X and we write $q^{-1}sq$ to denote the action of $q \in S_Q$ on $s \in S_0$. Let $r \in R_0$, then r is a word in S_N^* and hence can be expressed as a word in $(S_0 \cup S_Q)^*$ using the conjugation notation just defined. Denote the set of these words, one for each $r \in R_0$, by $\widehat{R}_0$.

Lemma 3.1 ([26, Lemma A.1]). *With the notation above,*

$$N \rtimes Q \cong \langle S_0, S_Q | \widehat{R}_0, R_Q, [x, s] \textit{ for all } s \in S_0 \textit{ and all } x \in X_s \rangle,$$

where X_s is a generating set for $\mathrm{Stab}_Q(s)$ *and the semi-direct product is built using the given action of Q on N.*

3.1. A finite presentation for QF

Lemma 3.2. *F acts transitively on*

$$\Sigma_n = \left\{(x_1, \ldots, x_n) \in \prod_1^n \{0,1\}^* : x_1 \lneq_{\mathrm{lex}} x_2 \lneq_{\mathrm{lex}} \cdots \lneq_{\mathrm{lex}} x_n\right\}$$

for all $n \in \mathbb{N}$, where the action is via the splitting $\iota: F \to QF$, the inclusion $QF \to QV$, and the usual action of QV on $\{0,1\}^*$.

Proof. For $i \geq 1$, let 0^i be the following element of $\{0,1\}^*$

$$0^i = \overbrace{0 \cdots 0}^{i\text{-times}},$$

and let $0^0 = \varepsilon$. For $1 \leq i \leq n$, let $y_i = 0^{n-i}$.

For any $(x_1, \ldots, x_n) \in \Sigma_n$, we build an element γ of F such that

$$\iota(\gamma): (y_1, y_2, \ldots, y_n) \longmapsto (x_1, \ldots, x_n)$$

Let R be the smallest rooted subtree of $\mathcal{T}_{2,c}$ containing all the x_i as nodes and let L_0 be the smallest subtree of $\mathcal{T}_{2,c}$ containing all the y_i as nodes.

For any two nodes $z_1 \lneq_{\mathrm{lex}} z_2$ in a finite rooted subtree T of $\mathcal{T}_{2,c}$ we define

$$d_T(z_1, z_2) = |\{z \in \mathrm{nodes}(T): z_1 \lneq_{\mathrm{lex}} z \lneq_{\mathrm{lex}} z_2\}|,$$

$$d_T(z_1, *) = |\{z \in \mathrm{nodes}(T): z_1 \lneq_{\mathrm{lex}} z\}|,$$

$$d_T(*, z_2) = |\{z \in \mathrm{nodes}(T): z \lneq_{\mathrm{lex}} z_2\}|.$$

Thus $d_{L_0}(*, y_0) = d_{L_0}(y_i, y_{i+1}) = d_{L_0}(y_n, *) = 0$ for all $0 \leq i \leq n-1$.

Expand L_0 to L by adding carets until $d_L(y_i, y_{i+1}) = d_R(x_i, x_{i+1})$ for all i. This is possible because adding a caret at a leaf between y_i and y_{i+1} increases $d_L(y_i, y_{i+1})$ by 1. Now use the description of Remark 2.6 to check that the element (L, R) maps $(y_1, \ldots, y_n)$ onto $(x_1, \ldots, x_n)$. □

Lemma 3.3. (1) *The F-stabiliser of any element in Σ_n is isomorphic to the group $\prod_1^{n+1} F$.*

(2) *The F-stabiliser of $(0, \varepsilon) \in \Sigma_2$ has generating set*

$$\mathrm{Stab}_{\Sigma_2}((0,\varepsilon)) = \left\langle \begin{matrix} \beta, \beta^\alpha, \alpha^3\beta^{-1}\alpha^{-2}, & \alpha^2\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-2}, \\ \alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}, & \alpha\beta\alpha^{-1}\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha\beta^{-1}\alpha^{-1} \end{matrix} \right\rangle.$$

We use the method in [1, §1] to translate from tree diagrams to words in the generators α and β.

Proof. Recall that $y_i = 0^{n-i}$ for $0 \le i \le n-1$ and $y^n = \varepsilon$. For any $x \in \{0,1\}^*$, let $x \cdot \{0,1\}^*$ denote all finite words beginning with x. We calculate the stabiliser of $(y_1, \ldots, y_n) \in \Sigma_n$, since F acts transitively on Σ_n this determines the stabilisers of all elements of Σ_n up to conjugacy.

We claim the stabiliser of $(y_1, \ldots, y_n)$ is exactly those elements of F for which $\iota(F)$ preserves setwise $y_0 \cdot \{0,1\}^*$ and $y_i \cdot 1 \cdot \{0,1\}^*$ for all $1 \le i \le n$.

If $f \in F$ and $\iota(f)$ maps some $x \in y_0 \cdot \{0,1\}^*$ to an element in $y_i \cdot 1 \cdot \{0,1\}^*$ for some $1 \le i \le n$ then, since $\iota(f)$ preserves the $\lneqq_{\mathrm{lex}}$ ordering, it is impossible for $\iota(f)$ to map y_1 to y_1 as required. Similarly, again because $\iota(f)$ respects the $\lneqq_{\mathrm{lex}}$ ordering, any $\iota(f)$ setwise preserving these sets will necessarily map y_1 to y_1. Repeating this argument for elements in $y_j \cdot 1 \cdot \{0,1\}^*$ completes the proof of (1).

Rephrasing the above, the stabiliser of $(0, \varepsilon)$ is exactly those elements of F which preserve setwise the subtrees under 00, 01, and 1. Those elements preserving, for example the subtree under 00, and acting trivially elsewhere give a copy of F on that interval. Since we have three such intervals, the stabiliser is $F \times F \times F$.

Generators of the elements setwise preserving the subtree under 1 and acting trivially elsewhere are $\{\beta, \beta^\alpha\}$ where $\alpha = \iota(A)$ and $\beta = \iota(B)$. Generators of the elements setwise preserving the subtree under 00 and acting trivially elsewhere are $\alpha^3\beta^{-1}\alpha^{-2}$ and $\alpha^2\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-2}$. Generators of the elements setwise preserving the subtree under 01 and acting trivially elsewhere are $\alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}$ and $\alpha\beta\alpha^{-1}\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha\beta^{-1}\alpha^{-1}$. □

Lemma 3.4. *The finite support symmetric group on* Sym($\{0,1\}^*$) *has presentation*

$$\mathrm{Sym}(\{0,1\}^*) = \left\langle \sigma_{x,y},\quad \begin{matrix} x,y\in\{0,1\}^*, \\ x \lneq_{\mathrm{lex}} y \end{matrix} \;\middle|\; \begin{matrix} \sigma_{x,y}^2, & (x,y)\in\Sigma_2, \\ [\sigma_{x,y},\sigma_{z,w}], & (x,y,z,w)\in\Sigma_4, \\ (\sigma_{x,y}\sigma_{y,z})^3, & (x,y,z)\in\Sigma_3, \\ \sigma_{x,y}\sigma_{y,z}\sigma_{x,y}\sigma_{x,z}, & (x,y,z)\in\Sigma_3 \end{matrix} \right\rangle,$$

where $\sigma_{x,y}$ denotes the transposition of x and y.

Proof. Let $x_1,\dots,x_n$ be some collection of elements of $\{0,1\}^*$ with $x_i \lneq_{\mathrm{lex}} x_{i+1}$ for all $1\le i\le n-1$. The finite symmetric group on $\{x_1,\dots,x_n\}$ has a well-known presentation

$$\mathrm{Sym}(\{x_1,\dots,x_n\}) = \left\langle \sigma_{x_1,x_2},\dots,\sigma_{x_{n-1},x_n} \;\middle|\; \begin{matrix} \sigma_{x_i,x_{i+1}}^2,\ (\sigma_{x_i,x_{i+1}}\sigma_{x_{i+1},x_{i+2}})^3, & \\ [\sigma_{x_i,x_{i+1}},\sigma_{x_j,x_{j+1}}], & |i-j|\ge 2 \end{matrix} \right\rangle,$$

where σ_{x_i,x_j} denotes the transposition of x_i and x_j. We expand this presentation using Tietze transformations. First add new generators $\sigma_{x_i,x_{i+2}}$ for each suitable $1\le i\le n-2$, each such generator can be added with the relator

$$\sigma_{x_i,x_{i+1}}\sigma_{x_{i+1},x_{i+2}}\sigma_{x_i,x_{i+1}}\sigma_{x_i,x_{i+2}}.$$

Next add new generators $\sigma_{x_i,x_{i+3}}$, each can be added with the relator

$$\sigma_{x_i,x_{i+2}}\sigma_{x_{i+2},x_{i+3}}\sigma_{x_i,x_{i+2}}\sigma_{x_i,x_{i+3}}$$

and so on adding $\sigma_{x_i,x_{i+j}}$ for all i and increasing j, until all the required generators are present. Finally we can add all relators

$$\sigma_{x_i,x_j}\sigma_{x_j,x_k}\sigma_{x_i,x_j}\sigma_{x_i,x_k},\qquad \text{for all } i\lneq j\lneq k,$$

which are not already present. We've obtained the presentation

$$\mathrm{Sym}(\{x_1,\dots,x_n\}) = \left\langle \sigma_{x_i,x_j},\quad i\lneq j \;\middle|\; \begin{matrix} \sigma_{x_i,x_j}^2, & i\lneq j, \\ (\sigma_{x_i,x_j}\sigma_{x_j,x_k})^3, & i\lneq j\lneq k, \\ [\sigma_{x_i,x_j},\sigma_{x_k,x_l}], & i\lneq j\lneq k\lneq l, \\ \sigma_{x_i,x_j}\sigma_{x_j,x_k}\sigma_{x_i,x_j}\sigma_{x_i,x_k}, & i\lneq j\lneq k \end{matrix} \right\rangle.$$

It is now easy to check that this can be extended to the entire finite support symmetric group Sym($\{0,1\}^*$). □

Proposition 3.5 ([13, §3]). *Thompson's group F has presentation*

$$F = \langle \alpha,\beta \mid [\alpha\beta^{-1},\alpha^{-1}\beta\alpha],[\alpha\beta^{-1},\alpha^{-2}\beta\alpha^2]\rangle.$$

Theorem 3.6. *QF has presentation*

$$QF = \left\langle \sigma, \alpha, \beta \;\middle|\; \begin{array}{l} \sigma^2,\ [\sigma, \sigma^{\alpha^2}],\ (\sigma\sigma^{\alpha})^3,\ \sigma\sigma^{\alpha}\sigma\sigma^{\alpha\beta^{-1}\alpha^{-1}}, \\ [\alpha\beta^{-1},\ \alpha^{-1}\beta\alpha],\ [\alpha\beta^{-1}, \alpha^{-2}\beta\alpha^2], \\ [\nu, \sigma] \quad \text{for all } \nu \in X \end{array} \right\rangle.$$

where

$$X = \left\{ \begin{array}{cc} \beta, \beta^{\alpha}, \alpha^3\beta^{-1}\alpha^{-2}, & \alpha^2\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-2}, \\ \alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}, & \alpha\beta\alpha^{-1}\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha\beta^{-1}\alpha^{-1} \end{array} \right\}$$

and $\sigma = \sigma_{0,\varepsilon}$.

Proof. We apply Lemma 3.1 to the description of QF as the semi-direct product $\mathrm{Sym}(\{0,1\}^*) \rtimes F$ (Lemma 2.5), using the presentation of $\mathrm{Sym}(\{0,1\}^*)$ given in Lemma 3.4, and the presentation of F given in Proposition 3.5.

Let $f \in F$, then necessarily $f(x) \lneq_{\mathrm{lex}} f(y)$ (see Remark 2.6) and $f^{-1}\sigma_{x,y}f = \sigma_{f(x),f(y)}$. Hence by Lemma 3.2, F acts transitively on the generating set of $\mathrm{Sym}(\{0,1\}^*)$ and acts with three F-orbits on the relating set.

Let $\sigma = \sigma_{0,\varepsilon}$, this is a representative of the generating set. We take the representatives of the relating set,

$$\{\sigma^2,\ [\sigma, \sigma_{1,11}],\ (\sigma\sigma_{\varepsilon,1})^3,\ \sigma\sigma_{\varepsilon,1}\sigma\sigma_{0,1}\}.$$

Let α and β be the generators of F from Proposition 3.5. We can express the transpositions appearing in the relating set as

$$\sigma_{1,11} = \sigma^{\alpha^2},$$

$$\sigma_{\varepsilon,1} = \sigma^{\alpha},$$

$$\sigma_{0,1} = \sigma^{\alpha\beta^{-1}\alpha^{-1}}.$$

Since the stabiliser of $\sigma_{x,y}$ for some $x \lneq_{\mathrm{lex}} y$ is equal to the stabiliser of the pair (x, y) in Σ_2, Lemma 3.3 gives that the stabiliser of the F-action on the generators of $\mathrm{Sym}(\{0,1\}^*)$ is a copy of $F \times F \times F$, generated by the set

$$X = \left\{ \begin{array}{cc} \beta, \beta^{\alpha}, & \alpha^3\beta^{-1}\alpha^{-2}, \alpha^2\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-2}, \\ \alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}, & \alpha\beta\alpha^{-1}\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha\beta^{-1}\alpha^{-1} \end{array} \right\}.$$

In summary, with the notation of Lemma 3.1, $X_\sigma = X$ is as above, $S_0 = \{\sigma\}$, $S_Q = \{\alpha, \beta\}$,

$$\widehat{R}_0 = \{\sigma^2,\ [\sigma, \sigma^{\alpha^2}],\ (\sigma\sigma^{\alpha})^3,\ \sigma\sigma^{\alpha}\sigma\sigma^{\alpha\beta^{-1}\alpha^{-1}}\},$$

and

$$R_Q = \{[\alpha\beta^{-1},\ \alpha^{-1}\beta\alpha],\ [\alpha\beta^{-1}, \alpha^{-2}\beta\alpha^2]\}. \qquad \square$$

Corollary 3.7. $QF/[QF, QF] \cong \mathbb{Z} \oplus \mathbb{Z} \oplus C_2$.

Proof. Abelianising the relators we find that the only surviving relation is σ^2. □

3.2. A finite presentation for $\widetilde{Q}T$. Recall that $\widetilde{Q}T$ denotes the preimage of T under $\pi\colon \widetilde{Q}V \to V$. In this section we compute a finite presentation for $\widetilde{Q}T$.

Let $(x_1, \ldots, x_n) \in \prod_1^n Z$. Then we say $[x_1, \ldots, x_n]_{\text{lex}}$ if and only if $X_1, \ldots, x_n$ are cyclically ordered in the lex-order. In particular, $[x, y, z]_{\text{lex}}$ if and only if $x \leq_{\text{lex}} y \leq_{\text{lex}} z$ or $y \leq_{\text{lex}} z \leq_{\text{lex}} x$ or $z \leq_{\text{lex}} x \leq_{\text{lex}} y$.

The next lemma is an analogue of Lemma 3.2.

Lemma 3.8. *Let $Z = \{0, 1\}^* \cup \{\zeta\}$. T acts transitively on*

$$\Lambda_n = \left\{(x_1, \ldots, x_n) \in \prod_1^n Z\colon [x_1, x_2, \ldots, x_n]_{\text{lex}},\ x_i \neq x_j \text{ for all } i \neq j\right\}$$

for all $n \in \mathbb{N}$, where the action is via the splitting $\iota\colon T \to \widetilde{Q}T$, the inclusion $\widetilde{Q}T \to \widetilde{Q}V$, and the usual action of $\widetilde{Q}V$ on Z.

Proof. As in Lemma 3.2, let $y_i = 0^{n-i}$ for all $1 \leq i \leq n$. We build an element γ of T such that $\iota(\gamma)$ maps an arbitrary element $(x_1, \ldots, x_n) \in \Lambda_n$ onto $(y_1, \ldots, y_n)$.

Let L be the smallest rooted subtree of $\mathcal{T}_{2,c}$ such that L contains all the x_i as nodes. Let f be the cyclic permutation of the leaves of L such that the element t of T represented by (L, L, f) satisfies $\iota(t)x_i \leq \iota(t)x_{i+1}$ for all i.

If $\iota(t)x_n \neq \zeta$ then, via Lemma 3.2, there exists $f \in F$ such that $\iota(ft)x_i = y_i$ for all i.

Assume $\iota(t)x_n = \zeta$. Let L' be formed from L by adding a caret on the left-hand-most leaf, and let f' be the cyclic permutation of the leaves of L' which sends each leaf to it's immediate left-hand neighbour. Let t' be the element of T represented by (L', L', f'), then $\iota(t't)x_i \lneq_{\text{lex}} \iota(t't)x_{i+1}$, and $\iota(t't)x_n \neq \zeta$. Once again we may use Lemma 3.2. □

Lemma 3.9. (1) *The T-stabiliser of any element in Λ_n is isomorphic to $\prod_1^n F$.*

(2) *The T-stabiliser of $(\varepsilon, \zeta) \in \Lambda_2$ has generating set*

$$\operatorname{Stab}((\varepsilon, \zeta)) = \langle \beta, \beta^\alpha, \alpha^2\beta^{-1}\alpha^{-1}, \alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}\rangle.$$

Proof. The T-stabiliser of $(y_1, \ldots, y_{n-1}, \zeta)$ is exactly the F-stabiliser of $(y_1, \ldots, y_{n-1})$, which is isomorphic to $\prod_1^n F$ by Lemma 3.3, this proves part (1).

Finally, the stabiliser of the subtree with root 1 is $\langle \beta, \beta^\alpha\rangle$ and that of the subtree with root 0 is $\langle \alpha^2\beta^{-1}\alpha^{-1}, \alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}\rangle$. □

The proof of the next lemma is identical to that of Lemma 3.4.

Lemma 3.10. *The finite support symmetric group on* Sym(Z) *has presentation*

$$\mathrm{Sym}(Z) = \left\langle \sigma_{x,y}, \quad x, y \in \Lambda_2 \;\middle|\; \begin{array}{ll} \sigma_{x,y}^2, & (x,y) \in \Lambda_2, \\ [\sigma_{x,y}, \sigma_{z,w}], & (x,y,z,w) \in \Lambda_4, \\ (\sigma_{x,y}\sigma_{y,w})^3, & (x,y,z) \in \Lambda_3, \\ \sigma_{x,y}\sigma_{y,z}\sigma_{x,y}\sigma_{x,z}, & (x,y,z) \in \Lambda_3 \end{array} \right\rangle.$$

Proposition 3.11 ([13, §5]). *Thompson's group T has presentation*

$$T = \left\langle \langle\alpha, \beta, \gamma \;\middle|\; \begin{array}{l} [\alpha\beta^{-1}, \alpha^{-1}\beta\alpha], [\alpha\beta^{-1}, \alpha^{-2}\beta\alpha^2], \\ \gamma^{-1}\beta\alpha^{-1}\gamma\beta, \alpha^{-1}\beta^{-1}\alpha\beta^{-1}\gamma^{-1}\alpha\beta\alpha^{-2}\gamma\beta^2, \\ \alpha^{-1}\gamma^{-1}(\alpha^{-1}\gamma\beta)^2, \gamma^3 \end{array} \right\rangle.$$

Theorem 3.12. *$\tilde{Q}T$ has presentation*

$$\tilde{Q}T = \left\langle \sigma, \alpha, \beta, \gamma \;\middle|\; \begin{array}{l} \sigma^2, [\sigma, \sigma^{\alpha^2}], (\sigma\sigma^{\alpha})^3, \sigma\sigma^{\alpha}\sigma\sigma^{\alpha\beta^{-1}\alpha^{-1}}, \\ [\alpha\beta^{-1}, \alpha^{-1}\beta\alpha], [\alpha\beta^{-1}, \alpha^{-2}\beta\alpha^2], \\ \gamma^{-1}\beta\alpha^{-1}\gamma\beta, \alpha^{-1}\beta^{-1}\alpha\beta^{-1}\gamma^{-1}\alpha\beta\alpha^{-2}\gamma\beta^2, \\ \alpha^{-1}\gamma^{-1}(\alpha^{-1}\gamma\beta)^2, \gamma^3, \\ [\nu, \sigma] \quad \textit{for all } \nu \in X. \end{array} \right\rangle,$$

where

$$X = \{\beta, \beta^{\alpha}, \alpha^2\beta^{-1}\alpha^{-1}, \alpha\beta^2\alpha^{-1}\beta^{-1}\alpha\beta^{-1}\alpha^{-1}\}$$

and $\sigma = \sigma_{0,\varepsilon}$.

Proof. The proof is similar to that of Theorem 3.6, except using Lemma 3.10 and Proposition 3.11. □

Question 3.13. Is QT finitely presented?

Corollary 3.14. $\tilde{Q}T/[\tilde{Q}T, \tilde{Q}T] \cong C_2$.

Proof. Abelianising the relators kills the generators α, β, and γ, leaving only the generator σ and the relation σ^2. □

In Corollary 5.3 we show that the abelianisation of QT is also isomorphic to the cyclic group of order 2.

3.3. A finite presentation for $\widetilde{Q}V$. In this section we compute a finite presentation for $\widetilde{Q}V$. Recall that $Z = \{0,1\}^* \cup \{\zeta\}$. We also define

$$\begin{aligned} \beta_n &= \alpha^{-(n-1)}\beta\alpha^{n-1} && \text{for all } n \geq 1, \\ \gamma_n &= \alpha^{-(n-1)}\gamma\beta^{n-1} && \text{for all } n \geq 1, \\ \delta_1 &= \gamma_2^{-1}\delta\gamma_2, \\ \delta_n &= \alpha^{-(n-1)}\delta_1\alpha^{n-1} && \text{for all } n \geq 2. \end{aligned}$$

These definitions will allow us to express the presentation of $\widetilde{Q}V$ in a simpler form. The same definitions appear in [13, p.13,p.16], where they are called X_n, C_n, and π_n respectively.

Lemma 3.15. *V acts transitively on*

$$\Delta_n = \Big\{(x_1,\dots,x_n) \in \prod_1^n Z\colon x_i \neq x_j \textit{ for all } i \neq j\Big\},$$

for all $n \in \mathbb{N}$, where the action is via the splitting $\iota\colon V \to \widetilde{Q}V$ and the usual action of $\widetilde{Q}V$ on Z.

Proof. Let $(x_1,\dots,x_n) \in \Delta_n$ and let L be the smallest subtree of $\mathcal{T}_{2,c}$ containing all the x_i as nodes. Choose a bijection f on the leaves of L such that the element γ of V represented by (L, L, f) has $\iota(f)x_i \leq_{\text{lex}} \iota(f)x_{i+1}$. Now use Lemma 3.8. □

Lemma 3.16. *The V-stabiliser of $(\varepsilon,\zeta) \in \Delta_2$ has generating set*

$$\operatorname{Stab}((\varepsilon,\zeta)) = \langle \alpha', \beta', \gamma', \delta', \lambda, \mu\rangle,$$

where

$$\begin{aligned} \alpha' &= (\alpha^2)(\beta^{-1}\gamma^{-1}\alpha\delta\alpha^{-1}\gamma\beta)(\alpha^{-1}\beta^{-1}), \\ \beta' &= (\alpha\beta_2^2)(\beta_3^{-1}\beta_2^{-1}\alpha^{-1}), \\ \gamma' &= (\alpha\beta_2)(\delta\delta_2\delta_1\delta)(\beta_2^{-1}\alpha^{-1}), \\ \delta' &= (\alpha\beta_2)(\delta_1\delta_0\delta_1)(\beta_2^{-1}\alpha^{-1}), \\ \lambda &= \alpha^2\beta^{-1}\alpha^{-1}, \\ \mu &= \beta. \end{aligned}$$

See Figures 6–11 for representatives of the elements α', β', γ', δ', λ, and μ as tree diagrams.

Between them, the elements α', β', γ', and δ' generate the subgroup V' of V which fixes the subtrees under 01 *and* 11.

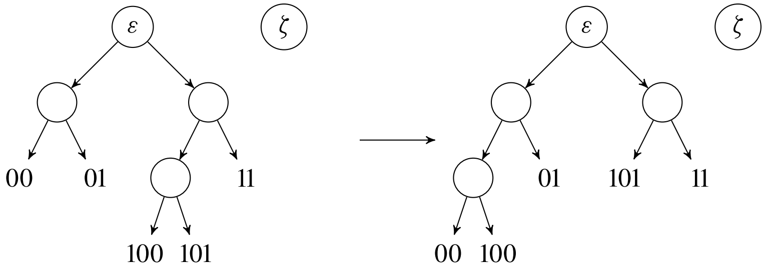

Figure 6. Element $\alpha' = (\alpha^2)(\beta^{-1}\gamma^{-1}\alpha\delta\alpha^{-1}\gamma\beta)(\alpha^{-1}\beta^{-1})$ from Lemma 3.16.

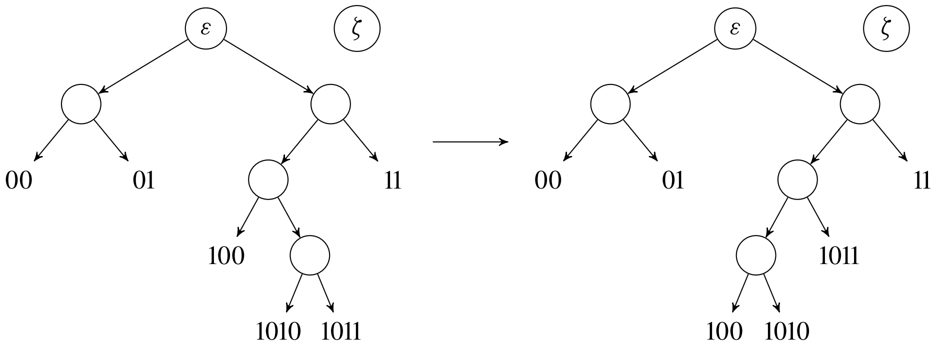

Figure 7. Element $\beta' = (\alpha\beta_2^2)(\beta_3^{-1}\beta_2^{-1}\alpha^{-1})$ from Lemma 3.16.

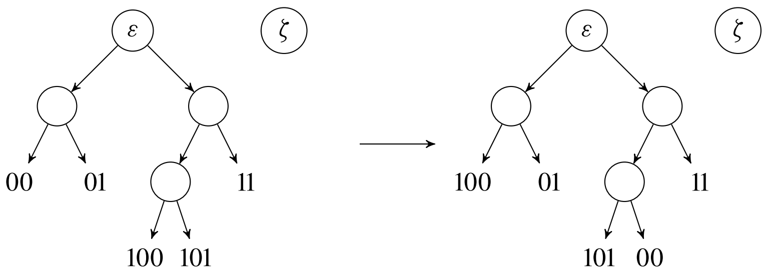

Figure 8. Element $\gamma' = (\alpha\beta_2)(\delta\delta_2\delta_1\delta)(\beta_2^{-1}\alpha^{-1})$ from Lemma 3.16.

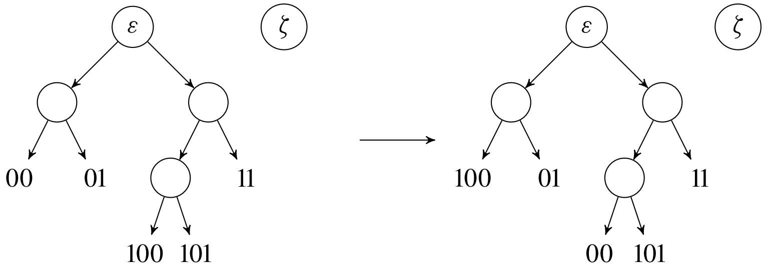

Figure 9. Element $\delta' = (\alpha\beta_2)(\delta_1\delta_0\delta_1)(\beta_2^{-1}\alpha^{-1})$ from Lemma 3.16.

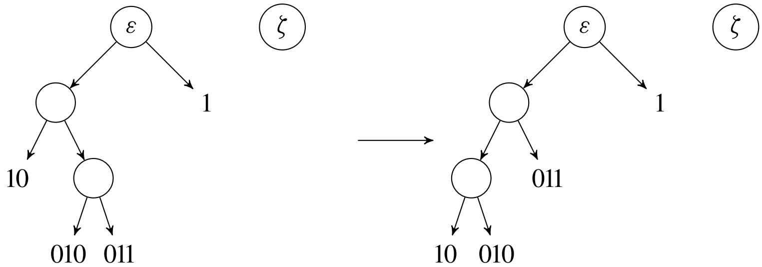

Figure 10. Element $\lambda = \alpha^2\beta^{-1}\alpha^{-1}$ from Lemma 3.16.

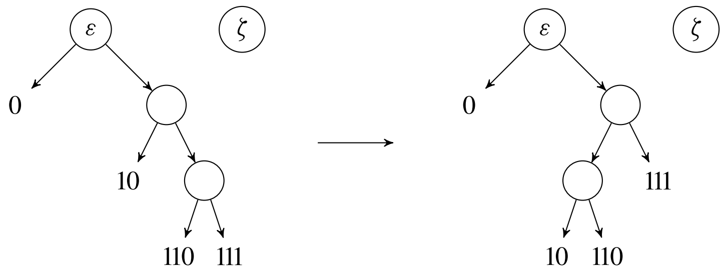

Figure 11. Element $\mu = \beta$ from Lemma 3.16.

The method of finding expressions in the generators $\alpha, \beta, \gamma, \delta$ is adapted from that in [1, §1], and the applet [24] was used for checking these calculations.

Proof. Let T be a finite rooted subtree of $\mathcal{T}_{2,c}$ with at least two leaves and let S be the rooted subtree with leaves 00, 01, 10, and 11. We denote by T' the subtree obtained by taking the subtree of T with root 0 and glueing 0 to the node 00 of S, similarly we take the subtree of T with root 1 and glue it to the leaf 10 of S.

If $f\colon \text{leaves}(L) \to \text{leaves}(R)$ is a bijection between the leaves of finite rooted subtrees of $\mathcal{T}_{2,c}$ then we write $f'\colon \text{leaves}(L') \to \text{leaves}(R')$ for the corresponding bijection (which fixes the leaves 01 and 11).

The homomorphism

$$\varphi\colon V \longrightarrow V,$$
$$(L, R, f) \longmapsto (L', R', f')$$

is an injection whose image is V'.

Let $\alpha' = \varphi(A)$, $\beta' = \varphi(B)$, $\gamma' = \varphi(C)$, and $\delta' = \varphi(D)$ and let λ and μ be the elements shown in Figures 10 and 11. By construction, α', β', γ', and δ' generate V'.

The elements λ, μ, and the group V' stabilise ε and ζ, we claim that they also generate $\text{Stab}_V((\varepsilon, \zeta))$. Let $\tau \in V$ be an element of the stabiliser represented by (L, R, f), via Lemma 3.17 we may assume that τ fixes the subtrees under 01 and 11, and thus is an element of V'.

Finally, we calculate α', β', γ', and δ'. Each is a product of three elements of V: the first is an element of F which maps the tree L to a right-vine (a tree formed from the trivial tree by adding carets to the right-hand most leaf only) , the second is the necessary permutation of the leaves on the right vine, and the third is the element of F which maps the right-vine to the tree R. The outcome of these calculations is as shown in the statement of the lemma. Note that the word length of these elements may not be minimal. □

Let T be a rooted subtree of $\mathcal{T}_{2,c}$ and recall the definition of

$$b_T\colon \text{nodes}(T) \cup \{\zeta\} \longrightarrow \text{leaves}(T)$$

from Section 2.1. We define $l_\varepsilon(T)$ to be the word length in $\{0, 1\}^*$ of the leaf $b_T(\varepsilon)$ and define $l_\zeta(T)$ to be the word length of the leaf $b_T(\zeta)$.

Lemma 3.17. *Let $v \in V$ satisfy $\iota(v)(\zeta) = \zeta$ and $\iota(v)(\varepsilon) = \varepsilon$. There exist non-negative integers a, b, c, and d such that $\iota(\lambda^{-a}\mu^{-b}v\lambda^c\mu^d)$ fixes the subtrees under* 01 *and* 11.

Proof. Note that $\iota(\lambda)(\varepsilon) = \iota(\mu)(\varepsilon) = \varepsilon$ and $\iota(\lambda)(\zeta) = \iota(\mu)(\zeta) = \zeta$. To prove the lemma, it suffices to find non-negative integers a, b, c, and d and tree diagram representative (L, R, f) for $\lambda^{-a}\mu^{-b}v\lambda^{c}\mu^{d}$ such that

$$l_\varepsilon(L) = l_\varepsilon(R) = l_\zeta(L) = l_\zeta(R) = 2.$$

Consider the element λ^n for some non-negative integer n. This element has tree diagram representative (L, R, f), where $l_\varepsilon(L) = n + 2$, $l_\varepsilon(R) = 2$, $l_\zeta(L) = 1$, and $l_\zeta(R) = 1$. Similarly consider the element μ^n for some non-negative integer n. This element has tree diagram representative (L, R, f), where $l_\zeta(L) = n + 2$, $l_\zeta(R) = 2$, $l_\varepsilon(L) = 1$, and $l_\varepsilon(R) = 1$.

Let $v \in V$ be such that $\iota(v)(\zeta) = \zeta$ and $\iota(v)(\varepsilon) = \varepsilon$ and let (L, R, f) be a tree diagram representative for v. If either $l_\zeta(L)$, $l_\zeta(R)$, $l_\varepsilon(L)$, or $l_\varepsilon(R)$ are strictly less than 2 than expand L and R by adding carets until they are equal or greater than 2.

Let $a = l_\varepsilon(L) - 2$, $b = l_\zeta(L) - 2$, $c = l_\varepsilon(R) - 2$, and $d = l_\zeta(R) - 2$. Then, one calculates that $\lambda^{-a}\mu^{-b}v\lambda^{c}\mu^{d}$ has tree diagram representative (L, R, f) with

$$l_\varepsilon(L) = l_\varepsilon(R) = l_\zeta(L) = l_\zeta(R) = 2,$$

as required. □

Modifying Lemma 3.10 slightly we obtain the following.

Lemma 3.18. *The finite support symmetric group on* $\mathrm{Sym}(Z)$ *has presentation*

$$\mathrm{Sym}(Z) = \left\langle \sigma_{x,y}, \quad x, y \in \Delta_2 \;\middle|\; \begin{array}{ll} \sigma_{x,y} = \sigma_{y,x}, & (x,y) \in \Delta_2, \\ \sigma_{x,y}^2, & (x,y) \in \Delta_2, \\ [\sigma_{x,y}, \sigma_{z,w}], & (x,y,z,w) \in \Delta_4, \\ (\sigma_{x,y}\sigma_{y,w})^3, & (x,y,z) \in \Delta_3, \\ \sigma_{x,y}\sigma_{y,z}\sigma_{x,y}\sigma_{x,z}, & (x,y,z) \in \Delta_3 \end{array} \right\rangle.$$

Proposition 3.19 ([13, p.18]). *Thompson's group V has presentation*

$$V = \left\langle \alpha, \beta, \gamma, \delta \;\middle|\; \begin{array}{l} [\alpha\beta^{-1}, \beta_2], [\alpha\beta^{-1}, \beta_3], \beta\gamma_2\gamma_1^{-1}, \beta\gamma_3(\gamma_2\beta_2)^{-1}, \gamma_2^2(\gamma_1\alpha)^{-1}, \\ \gamma_1^3, \delta_1^2, \delta_3\delta_1(\delta_1\delta_3)^{-1}, (\delta_2\delta_1)^3, \delta_1\beta_3(\beta_3\delta_1)^{-1}, \\ \beta\gamma_2\gamma_1(\gamma_1\beta_2)^{-1}, \beta\delta_3(\delta_2\beta)^{-1}, \gamma_3\delta_2(\delta_1\gamma_3)^{-1}, (\delta_1\gamma_2)^3 \end{array} \right\rangle.$$

Theorem 3.20. *$\widetilde{Q}V$ has presentation*

$$\widetilde{Q}V = \left\langle \sigma, \alpha, \beta, \gamma, \delta \;\middle|\; \begin{array}{l} \sigma\sigma^{\alpha\delta\alpha^{-1}}, \sigma^2, [\sigma, \sigma^{\alpha^2}], (\sigma\sigma^{\alpha})^3, \sigma\sigma^{\alpha}\sigma\sigma^{\alpha\beta^{-1}\alpha^{-1}}, \\ [\alpha\beta^{-1}, \beta_2], [\alpha\beta^{-1}, \beta_3], \beta\gamma_2\gamma_1^{-1}, \beta\gamma_3(\gamma_2\beta_2)^{-1}, \\ \gamma_2^2(\gamma_1\alpha)^{-1}, \gamma_1^3, \delta_1^2, \delta_3\delta_1(\delta_1\delta_3)^{-1}, (\delta_2\delta_1)^3, \delta_1\beta_3(\beta_3\delta_1)^{-1}, \\ \beta\gamma_2\gamma_1(\gamma_1\beta_2)^{-1}, \beta\delta_3(\delta_2\beta)^{-1}, \gamma_3\delta_2(\delta_1\gamma_3)^{-1}, (\delta_1\gamma_2)^3, \\ [\nu, \sigma] \quad \textit{for all } \nu \in X \end{array} \right\rangle.$$

where

$$X = \left\{ \begin{array}{l} (\alpha^2)(\beta^{-1}\gamma^{-1}\alpha\delta\alpha^{-1}\gamma\beta)(\alpha^{-1}\beta^{-1}), (\alpha\beta_2^2)(\beta_3^{-1}\beta_2^{-1}\alpha^{-1}), \\ (\alpha\beta_2)(\delta\delta_2\delta_1\delta)(\beta_2^{-1}\alpha^{-1}), (\alpha\beta_2)(\delta_1\delta_0\delta_1)(\beta_2^{-1}\alpha^{-1}), \alpha^2\beta^{-1}\alpha^{-1}, \beta \end{array} \right\}$$

and $\sigma = \sigma_{0,\varepsilon}$.

Proof. As in Theorem 3.6 and Theorem 3.12, except using the presentations from Lemma 3.18 and 3.19.

Explicitly,

$$S_Q = \{\alpha, \beta, \gamma, \delta\},$$

$$R_Q = \left\{ \begin{array}{l} [\alpha\beta^{-1}, \beta_2], [\alpha\beta^{-1}, \beta_3], \beta\gamma_2\gamma_1^{-1}, \beta\gamma_3(\gamma_2\beta_2)^{-1}, \gamma_2^2(\gamma_1\alpha)^{-1} \\ \gamma_1^3, \delta_1^2, \delta_3\delta_1(\delta_1\delta_3)^{-1}, (\delta_2\delta_1)^3, \delta_1\beta_3(\beta_3\delta_1)^{-1}, \\ \beta\gamma_2\gamma_1(\gamma_1\beta_2)^{-1}, \beta\delta_3(\delta_2\beta)^{-1}, \gamma_3\delta_2(\delta_1\gamma_3)^{-1}, (\delta_1\gamma_2)^3 \end{array} \right\},$$

$$S_0 = \{\sigma\} = \{\sigma_{0,\varepsilon}\},$$

$$\widehat{R}_0 = \{\sigma\sigma^{\alpha\delta\alpha^{-1}}, \sigma^2, [\sigma, \sigma^{\alpha^2}], (\sigma\sigma^{\alpha})^3, \sigma\sigma^{\alpha}\sigma\sigma^{\alpha\beta^{-1}\alpha^{-1}}\},$$

$$X_\sigma = \left\{ \begin{array}{l} (\alpha^2)(\beta^{-1}\gamma^{-1}\alpha\delta\alpha^{-1}\gamma\beta)(\alpha^{-1}\beta^{-1}), (\alpha\beta_2^2)(\beta_3^{-1}\beta_2^{-1}\alpha^{-1}), \\ (\alpha\beta_2)(\delta\delta_2\delta_1\delta)(\beta_2^{-1}\alpha^{-1}), (\alpha\beta_2)(\delta_1\delta_0\delta_1)(\beta_2^{-1}\alpha^{-1}), \alpha^2\beta^{-1}\alpha^{-1}, \beta \end{array} \right\}.$$

Compared to the calculation for $\widetilde{Q}T$, there is one new element of R_0, namely $\sigma\sigma^{\alpha\delta\alpha^{-1}}$ which corresponds to the relation $\{\sigma_{x,y} = \sigma_{y,x}$ for all $x, y \in \Delta_2\}$. □

Corollary 3.21. $\widetilde{Q}V/[\widetilde{Q}V, \widetilde{Q}V] \cong C_2$.

Proof. Abelianising the presentation leaves the generator σ and the relator σ^2. □

In Corollary 5.3 we show that the abelianisation of QV is also isomorphic to C_2.

Question 3.22 ([25, p.31]). Is QV finitely presented?

4. Type F_∞

In this section we show that the groups QF, $\widetilde{Q}T$ and $\widetilde{Q}V$ are of type F_∞. The idea of the proof for QF is to consider $\mathrm{Sym}(\{0,1\}^*)$ as a countably generated Coxeter group then to form the Davis complex $\mathcal{U}$ (a certain contractible CW-complex on which $\mathrm{Sym}(\{0,1\}^*)$ acts properly), and then extend this to an action of QF on $\mathcal{U}$ which has stabilisers of type F_∞. For $\widetilde{Q}T$ and $\widetilde{Q}V$ we substitute $\mathrm{Sym}(Z)$ for $\mathrm{Sym}(\{0,1\}^*)$.

Let (W, S) be a countably generated Coxeter group, so W is generated by a countable set of involutions S. We start by giving a quick overview of the construction of the Davis complex of a Coxeter group, for background see [14, §5, §7]. After this we show that if Q is a group acting by automorphisms on W such that $q(S) = S$ for all $q \in Q$ then there is an action of $W \rtimes Q$ on the Davis complex, where the semi-direct product is formed using the given action of Q on W. This is already well known, see for example [14, §9.1], so we only give an overview.

For any subset T of S we denote by W_T the subgroup of W generated by T. Recall that a *spherical subset* is a finite subset T of S for which W_T is finite. The group W_T is known as a *spherical subgroup*. Let $\mathcal{S}$ denote the poset of spherical subsets in (W, S) and $\mathcal{C}$ the poset of cosets of spherical subgroups, thus elements of $\mathcal{C}$ may be written as wW_T for $w \in W$ and $T \in \mathcal{S}$. The poset $\mathcal{C}$ admits a left action by W,

$$w' \cdot wW_T = (w'w)W_T.$$

The *Davis Complex* $\mathcal{U}$ is the geometric realisation of $\mathcal{C}$ and thus admits a left action by W as well. Note that the W-orbits of n-simplices in $\mathcal{U}$ are $(n+1)$-element subsets of S generating a finite subgroup of W [15, p.2].

Proposition 4.1. [15, p.3] *The Davis complex $\mathcal{U}$ is contractible.*

Consider a group Q acting by automorphisms on W such that every $q \in Q$ satisfies $q(S) = S$. We denote by G the semi-direct product $W \rtimes Q$ formed using this action.

The action of Q on S extends to an action of Q on $\mathcal{S}$, by setting

$$q\{s_1, \dots, s_n\} = \{qs_1, \dots, qs_n\}.$$

We use this to define a G-action on $\mathcal{C}$ by

$$(h, q) \cdot wW_T = hw^{q^{-1}} W_{qT}.$$

One checks that this is well-defined and preserves the poset structure. The G-action on $\mathcal{C}$ induces a G-action on $\mathcal{U}$.

The next lemma appears in [14, Propostion 9.1.9], as does the first part of Proposition 4.3.

Lemma 4.2. *The $W \rtimes Q$-isotropy subgroup of $wW_T \in \mathcal{C}$ is $(W_T)^{w^{-1}} \rtimes Q_T$, where Q_T is the Q-isotropy of $T \in \mathcal{S}$.*

Let $\mathcal{S}_n$ denote the set of unordered n-element subsets of S, equivalently the spherical subsets of size n. The set $\mathcal{S}_n$ admits a Q-action, the restriction of that on $\mathcal{S}$.

Proposition 4.3. *If, for all positive integers n, Q acts on $\mathcal{S}_n$ with finitely many Q-orbits and with stabilisers of type* F_∞*, then G is of type* F_∞.

Proof. Since the set of W-orbits of n-simplices in $\mathcal{U}$ is exactly the set $\mathcal{S}_{n+1}$, there are as many G-orbits of n-simplicies in $\mathcal{U}$ as Q-orbits in $\mathcal{S}_{n+1}$. Thus G acts cocompactly on $\mathcal{U}$.

By construction, the Q-isotropy of any point in $\mathcal{U}$ is the Q-isotropy of some spherical subgroup W_T. Thus assumption (2) implies that the Q-stabiliser of any point has type F_∞, so combining with Lemma 4.2 and [2, Proposition 2.7], the G-stabiliser of any point in $\mathcal{U}$ is F_∞. Theorem 4.4 below completes the proof. □

Theorem 4.4. [20, Theorem 7.3.1] *If there exists a contractible G-CW complex which is finite type mod G and has stabilisers of type* F_∞*, then G is of type* F_∞.

At this point we specialise to the Coxeter group $\mathrm{Sym}(Z)$. Let S be the set of unordered 2-element subsets of Z. This is the set of generators of $\mathrm{Sym}(Z)$ given in Lemma 3.10. Recall from Section 3.3 that Δ_n is the set of ordered n-element subsets of Z, so there is a surjective V-map $\Delta_2 \to S$, given by projection. Recall that $\mathcal{S}_n$ denotes the set of unordered n-element subsets of elements of S (note this is not equivalent to the set of unordered $2n$-element subsets of S). There is also a surjective V-map $\Delta_{2n} \to \mathcal{S}_n$.

Lemma 4.5. *The V-stabiliser of $\{(x_1, y_1), \ldots, (x_n, y_n)\} \in \mathcal{S}_n$ is of type F_∞ for all $\{(x_1, y_1), \ldots, (x_n, y_n)\} \in \mathcal{S}_n$.*

Proof. Since V acts transitively on $\mathcal{S}_n$ it is sufficient to check that the stabiliser $\mathrm{Stab}_V(\{(0^{2n-2}, 0^{2n-1}), \ldots, (\varepsilon, \zeta)\})$ is F_∞.

Let $\{(0^{2n-2}, 0^{2n-1}), \ldots, (\varepsilon, \zeta)\} \in \mathcal{S}_n$, and let A be the preimage of this element under the V-map $\Delta_{2n} \to \mathcal{S}_n$. There is an homomorphism $\pi\colon \mathrm{Stab}_V(A) \to \mathrm{Sym}_{2n}$ which records the permutation on the elements $\{0^{2n-2}, \ldots, 0, \varepsilon, \zeta\}$. The kernel of π is exactly $\mathrm{Stab}_V(\{(0^{2n-2}, 0^{2n-1}), \ldots, (\varepsilon, \zeta)\})$ which is F_∞ by Corollary 4.17 so combining with [2, Proposition 2.7] and the fact that Sym_n is finite and hence F_∞, we deduce that $\mathrm{Stab}_V(A)$ is F_∞.

Observing that $\mathrm{Stab}_V(A) = \mathrm{Stab}_V(\{(0^{2n-2}, 0^{2n-1}), \ldots, (\varepsilon, \zeta)\})$ completes the proof for the V-stabiliser. □

Theorem 4.6. *The group $\widetilde{Q}V$ is of type* F_∞.

Proof. This follows from Proposition 4.3. Since V acts transitively on Δ_{2n}, it acts transitively on $\mathcal{S}_n$ also. The stabilisers are of type F_∞ by Lemma 4.5. □

Next, we will prove the group $\widetilde{Q}T$ is F_∞, for which we require the following technical lemma.

Lemma 4.7. *T acts with finitely many orbits and stabilisers of type* F_∞ *on* Δ_n.

Proof. Let Cyc_n denote the subgroup of Sym_n generated by the cyclic permutation $(1, 2 \dots, n)$. Let R be a set of representatives of the cosets $\mathrm{Sym}_n / \mathrm{Cyc}_n$. For any $\sigma \in R$, we define a T-map

$$\begin{aligned} i_\sigma \colon \Lambda_n &\longrightarrow \Delta_n, \\ (x_1, \dots, x_n) &\longmapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}). \end{aligned}$$

Taking a product of these gives an T-map,

$$\coprod_{\sigma \in R} \Lambda_n \xrightarrow{\coprod i_\sigma} \Delta_n,$$

which one checks is a bijection. Since the T-stabiliser of any element of Λ_n is F_∞ (Lemma 3.9), the T-stabiliser of any element of Δ_n is also. □

Theorem 4.8. *The group $\widetilde{Q}T$ is of type* F_∞.

Proof. Again, we let S be the unordered 2-element subsets of Z. Using the argument of Lemma 4.5 we deduce from Lemma 4.7 that T acts with finitely many orbits and with stabilisers of type F_∞ on $\mathcal{S}_n$. The statement follows from Proposition 4.3. □

To prove that QF is of type F_∞, we will again use Proposition 4.3, but now we are considering the F-action on the symmetric group $\mathrm{Sym}(\{0,1\}^*)$ so we let S be the unordered 2-element subsets of $\{0,1\}^*$. Thus the correct analogue of Lemma 4.7 is the following.

Lemma 4.9. *F acts with finitely many orbits and stabilisers of type* F_∞ *on*

$$X_n = \Big\{ (x_1, \dots, x_n) \in \prod_1^n \{0,1\}^* \colon x_i \neq x_j \textit{ for all } i \neq j \Big\}.$$

Proof. The proof is similar to that of Lemma 4.7, we construct a bijection of F-sets

$$\coprod_{\sigma \in \mathrm{Sym}_n} \Sigma_n \xrightarrow{\coprod i_\sigma} X_n,$$

where i_σ is the map

$$\begin{aligned} i_\sigma \colon \Sigma_n &\longrightarrow X_n, \\ (x_1, \dots, x_n) &\longmapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}). \end{aligned}$$

Now, since the F-stabiliser of any element of Σ_n is F_∞ (Lemma 3.3), the F-stabiliser of any element of X_n is also. □

Theorem 4.10. *The group QF is of type* F_∞.

Proof. Using the argument of Lemma 4.5 we deduce from Lemma 4.9 that F acts with finitely many orbits and stabilisers of type F_∞ on $\mathcal{S}_n$. The statement now follows from Proposition 4.3. ☐

4.1. The group $\mathrm{Stab}_V((0^{n-2},\dots,0,\varepsilon,\zeta))$ is of type F_∞. In this section we study the action of V on Δ_n and prove in Corollary 4.17 that it acts with stabilisers of type F_∞.

Let T be a rooted subtree of $\mathcal{T}_{2,c}$ and recall the definition of

$$b_T\colon \mathrm{nodes}(T)\cup\{\zeta\}\longrightarrow \mathrm{leaves}(T)$$

from Section 2.1. For $x\in\{0,1\}^*$ we define $l_x(T)$ to be the word length in $\{0,1\}^*$ of the leaf $b_T(x)$. This definition has already been seen in Section 3.3, just before Lemma 3.17.

Let $v\in V$ satisfy $\iota(v)(x)=x$ and let (L,R,f) be a representative for v. We define

$$\widetilde{\chi}_x(v)=l_x(L)-l_x(R).$$

Lemma 4.11. *For any $x\in Z$, the map $\widetilde{\chi}_x$ is a group homomorphism*

$$\mathrm{Stab}_V(x)\longrightarrow\mathbb{Z}.$$

Proof. Since $\widetilde{\chi}_x$ is defined only on $\mathrm{Stab}_V(x)$, the values of $\widetilde{\chi}_x$ are invariant under adding carets. Thus, since any two tree diagram representatives of an element of V have a common expansion, values taken by $\widetilde{\chi}_x$ don't depend on the tree diagram representative. Let v,w be any two elements of $\mathrm{Stab}_V(x)$, let (L,R,f) be a tree diagram representative for v and let (R,S,g) be a tree diagram representative for w. Then,

$$\begin{aligned}\widetilde{\chi}_x(vw)&=l_x(L)-l_x(S)\\&=l_x(L)-l_x(R)+l_x(R)-l_x(S)\\&=\widetilde{\chi}_x(f)+\widetilde{\chi}_x(g).\end{aligned}$$ ☐

Let T_1 be the tree with 2 leaves and let T_n be obtained from T_{n-1} by adjoining carets to the first and second leaves (measuring from smallest to largest using $\le_{\mathrm{lex}}$). In particular, T_n has $2n$ leaves. See Figure 12 for a picture of T_3. Let L_n be the subgroup of V consisting of elements which fix the full subtrees with root the $2i^{\mathrm{th}}$-leaf of T_n for all $1\le i\le n$. For example, the group L_3 is the subgroup of V fixing the full subtrees under 001, 011, and 11. One checks that $L_n\in\mathrm{Stab}_V((0^{n-2},\dots,0,\varepsilon,\zeta))$.

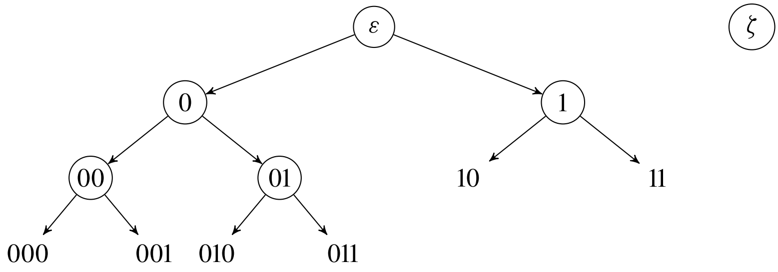

Figure 12. The tree T_3 from the proof of Lemma 4.12.

Lemma 4.12. *The group L_n is isomorphic to V for all $n \geq 2$.*

Proof. The group L_n is a copy of $V_{2,n}$, Thompson's group V but on the disjoint union of n-trees. These trees are exactly the full subtrees with roots the $(2i-1)^{\text{th}}$-leaves of T_n for each $1 \leq i \leq n$. There is a result of Higman that $V_{2,n}$ is isomorphic to V [22]. □

For $1 \leq i \leq n$, let $\lambda_{n,i}$ be the element of F with tree diagram representative $(L_{n,i}, R_{n,i})$ where $L_{n,i}$ is obtained from T_n by adjoining a caret to the $2i^{\text{th}}$-leaf, and $R_{n,i}$ is obtained from T_n by adjoining a caret to the $(2i-1)^{\text{th}}$-leaf of T_n. For example, $\lambda_{2,1}$ and $\lambda_{2,2}$ are the elements λ and μ of Section 3.3 and $\lambda_{3,2}$ is shown in Figure 13.

Lemma 4.13. *For all positive integers n, all integers i such that $1 \leq i \leq n-2$, and all integers j with $1 \leq j \leq n$,*

$$\begin{aligned}
\widetilde{\chi}_\zeta \colon L_n &\longmapsto 0,\\
\widetilde{\chi}_\varepsilon \colon L_n &\longmapsto 0,\\
\widetilde{\chi}_{0^i} \colon L_n &\longmapsto 0,\\
\widetilde{\chi}_\zeta \colon \lambda_{n,j} &\longmapsto \begin{cases} 1 & \textit{if } j = n,\\ 0 & \textit{else,}\end{cases}\\
\widetilde{\chi}_\varepsilon \colon \lambda_{n,j} &\longmapsto \begin{cases} 1 & \textit{if } j = n-1,\\ 0 & \textit{else,}\end{cases}\\
\widetilde{\chi}_{0^i} \colon \lambda_{n,j} &\longmapsto \begin{cases} 1 & \textit{if } j = n-1-i,\\ 0 & \textit{else.}\end{cases}
\end{aligned}$$

Lemma 4.14. *Let $v \in \mathrm{Stab}_V((0^{n-2}, \dots, \varepsilon, \zeta))$. For $1 \leq i \leq n$, there exist non-negative integers a_i and b_i such that the element $\lambda_{n,1}^{-a_1} \cdots \lambda_{n,n}^{-a_n} v \lambda_{n,n}^{b_1} \lambda_{n,1}^{b_n}$ is contained in L_n.*

Proof. In the case $n = 2$ this is Lemma 3.17, for $n > 2$ the proof is similar. □

Lemma 4.15. *The group* $\mathrm{Stab}_V((0^{n-2}, 0^{n-1}, \ldots, \varepsilon, \zeta))$ *is generated by* L_n *and* $\lambda_{n,i}$ *for* $i = 1, \ldots, n$.

Proof. This follows from Lemma 4.14. □

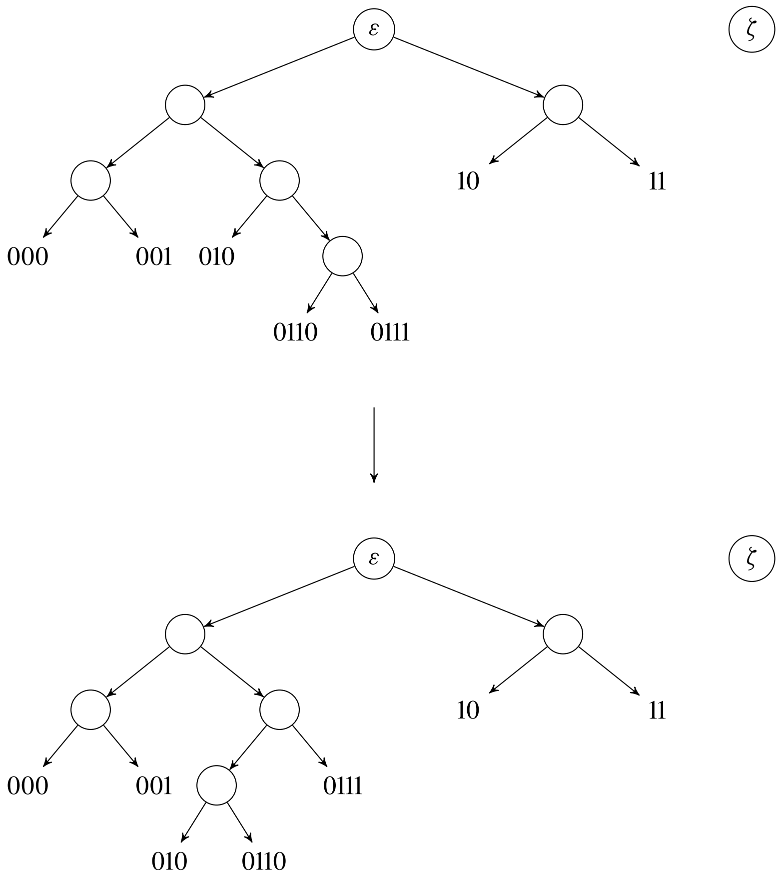

Figure 13. $\lambda_{3,2}$.

If G is a group, $\theta\colon G \to G$ an endomorphism, and $\tilde{\lambda}$ any symbol, then we denote by $G*_{\tilde{\lambda},\theta}$ the corresponding HNN-extension. Recall that an HNN-extension is said to be *ascending* if θ is injective.

Proposition 4.16. *We have*

$$\mathrm{Stab}_V((0^{n-2}, 0^{n-2}, \ldots, \varepsilon, \zeta)) \cong L_n *_{\tilde{\lambda}_1,\theta_1} *_{\tilde{\lambda}_2,\theta_2} \cdots *_{\tilde{\lambda}_n,\theta_n},$$

where the injective homomorphisms θ_i *correspond to conjugation by* $\lambda_{n,i}^{-1}$.

Proof. Let $\theta_i\colon L_n \to L_n$ be the injective homomorphism $v \mapsto \lambda_{n,i} v \lambda_{n,i}^{-1}$, and let

$$\Phi\colon (V' *_{\tilde{\mu},\theta_1}) *_{\tilde{\lambda}_i,\theta_2} \longrightarrow \mathrm{Stab}_V((0^{n-2}, 0^{n-1}, \dots, \varepsilon, \zeta)),$$

be the map specialising $\tilde{\lambda}_i$ to $\lambda_{n,i}^{-1}$. Since L_n together with the $\lambda_{n,i}$ generate $\mathrm{Stab}_V((0^{n-2}, \dots, \varepsilon, \zeta))$ (Lemma 4.15), the map Φ is a surjection. Let $v \in \operatorname{Ker}\Phi$, since the HNN-extensions are ascending, we can write

$$v = \lambda_1^{-a_1} \lambda_2^{-a_2} \cdots \lambda_n^{-a_n} w \lambda_n^{b_n} \cdots \lambda_1^{b_1},$$

for some $w \in L_n$ and some non-negative integers a_i and b_i for $1 \le i \le n$.

For any fixed i, the homomorphism

$$\begin{aligned} L_n *_{\tilde{\lambda}_1,\theta_1} *_{\tilde{\lambda}_2,\theta_2} \cdots *_{\tilde{\lambda}_n,\theta_n} &\longrightarrow \mathbb{Z}, \\ L_n &\longmapsto 0, \\ \tilde{\lambda}_j &\longmapsto \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else}, \end{cases} \end{aligned}$$

factors through $\mathrm{Stab}_V((0^{n-2}, \dots, \varepsilon, \zeta))$ using Lemma 4.13, thus $a_i = b_i$ for all i. In particular, $w \in \operatorname{Ker}\Phi$, thus $w = 1$, since Φ is the identity when restricted to L_n. □

Corollary 4.17. *Given* $(x_1, \dots, x_n) \in \Delta_n$, *the group* $\mathrm{Stab}_V((x_1, \dots, x_n))$ *is of type* F_∞.

Proof. Since V acts transitively on Δ_n it is sufficient to check that the stabiliser $\mathrm{Stab}_V((0^{n-2}, \dots, \varepsilon, \zeta))$ is F_∞. This follows since by Lemma 4.12 L_n is isomorphic to V and hence F_∞ and HNN-extensions of groups of type F_∞ are themselves F_∞ [2, Proposition 2.13(b)]. □

5. Normal subgroups

In this section we give a complete description of the normal subgroups of QF, QT, QV, $\tilde{Q}T$, and $\tilde{Q}V$.

Recall that $\mathrm{Sym}(\{0,1\}^*)$ has only one non-trivial proper normal subgroup, this is the finite support alternating group $\mathrm{Alt}(\{0,1\}^*)$, the subgroup of all permutations with even parity. Moreover, $\mathrm{Sym}(\{0,1\}^*)$ and $\mathrm{Alt}(\{0,1\}^*)$ are normal in the infinite support symmetric group on $\{0,1\}^*$, because conjugating any finite permutation by any bijection $\{0,1\}^* \to \{0,1\}^*$ preserves the cycle type of the finite permutation.

Recall also that $F/[F,F] \cong \mathbb{Z} \oplus \mathbb{Z}$ but, since T and V are simple and non-abelian, $[T,T] \cong T$ and $[V,V] \cong V$ [13, Theorems 4.1,5.8,6.9].

Theorem 5.1 (normal subgroups of QF). (1) *A non-trivial normal subgroup of* QF *is either* $\mathrm{Alt}(\{0,1\}^*)$, $\mathrm{Sym}(\{0,1\}^*)$, *or contains*

$$[QF, QF] = \mathrm{Alt}(\{0,1\}^*) \rtimes [F, F].$$

(2) *A proper non-trivial normal subgroup of* $\tilde{Q}T$ *is either* $\mathrm{Alt}(Z)$, $\mathrm{Sym}(Z)$, *or*

$$[\tilde{Q}T, \tilde{Q}T] = \mathrm{Alt}(Z) \rtimes T.$$

(3) *A proper non-trivial normal subgroup of* $\tilde{Q}V$ *is either* $\mathrm{Alt}(Z)$, $\mathrm{Sym}(Z)$, *or*

$$[\tilde{Q}V, \tilde{Q}V] = \mathrm{Alt}(Z) \rtimes V.$$

(4) *A proper non-trivial normal subgroup of* QT *is one of either* $\mathrm{Alt}(\{0,1\}^*)$, $\mathrm{Sym}(\{0,1\}^*)$, *or*

$$[QT, QT] = (\mathrm{Alt}(Z) \rtimes T) \cap QT.$$

Moreover, $(\mathrm{Alt}(Z) \rtimes T) \cap QT$ *is an extension of* T *by* $\mathrm{Sym}(\{0,1\}^*)$.

(5) *A proper non-trivial normal subgroup of* QV *is one of either* $\mathrm{Alt}(\{0,1\}^*)$, $\mathrm{Sym}(\{0,1\}^*)$, *or*

$$[QV, QV] = (\mathrm{Alt}(Z) \rtimes V) \cap QV.$$

Moreover, $(\mathrm{Alt}(Z) \rtimes V) \cap QV$ *is an extension of* V *by* $\mathrm{Sym}(\{0,1\}^*)$.

This theorem is similar to results for the braided Thompson groups F_{br} and V_{br}. The braided Thompson group F_{br} is a split extension with kernel the group P_{br}, a direct limit of finite pure braid groups, and quotient F. The braided Thompson group V_{br} is a non-split extension with kernel P_{br} and quotient V. Zaremsky shows that for any normal subgroup N of F_{br}, either $N \leq P_{\mathrm{br}}$ or $[F_{\mathrm{br}}, F_{\mathrm{br}}] \leq N$ [36, Theorem 2.1], and a proper normal subgroup N of V_{br} is necessarily contained in P_{br} [36, Corollary 2.8]. There is no braided version of Thompson's group T. We borrow some of the methods in this section from [36, §2].

The next proposition should be compared with [31, Theorem 5], that every element of the full permutation group on an infinite set is a commutator, so in particular the commutator subgroup of the full permutation group on $\{0,1\}^*$ is itself.

We write $\langle\langle H \rangle\rangle_G$ for the normal closure in G of a subgroup $H \leq G$, and for an element $g \in G$ we write $\langle\langle g \rangle\rangle_G$ for the normal subgroup generated by g.

Proposition 5.2. (1) $\langle\langle [\iota F, \iota F] \rangle\rangle_{QF} = [QF, QF] = \mathrm{Alt}(\{0,1\}^*) \rtimes [F, F]$.

(2) $\langle\langle [\iota F, \iota F] \rangle\rangle_{\tilde{Q}T} = [\tilde{Q}T, \tilde{Q}T] = \mathrm{Alt}(Z) \rtimes T$.

(3) $\langle\langle [\iota F, \iota F] \rangle\rangle_{\tilde{Q}V} = [\tilde{Q}V, \tilde{Q}V] = \mathrm{Alt}(Z) \rtimes V$.

(4) $\langle\langle [\iota F, \iota F] \rangle\rangle_{QT} = [QT, QT] = (\mathrm{Alt}(Z) \rtimes T) \cap QT$, *moreover this subgroup is an extension with kernel* $\mathrm{Sym}(\{0,1\}^*)$ *and quotient* T.

(5) $\langle\langle [\iota F, \iota F] \rangle\rangle_{QV} = [QV, QV] = (\mathrm{Alt}(Z) \rtimes V) \cap QV$, *moreover this subgroup is an extension with kernel* $\mathrm{Sym}(\{0,1\}^*)$ *and quotient* V.

Proof. For part (1), consider $[\alpha, \beta] \in [\iota F, \iota F]$ then for $\sigma_{1,11} \in \mathrm{Sym}(\{0,1\}^*)$, we have

$$\sigma_{1,11}[\alpha, \beta]^{-1}\sigma_{1,11}[\alpha, \beta] \in \langle\langle[\iota F, \iota F]\rangle\rangle.$$

Calculating explicitly,

$$\sigma_{1,11}[\alpha, \beta]^{-1}\sigma_{1,11}[\alpha, \beta] = \sigma_{1,11,1110}.$$

The smallest normal subgroup of $\mathrm{Sym}(\{0,1\}^*)$ containing a 3-cycle is $\mathrm{Alt}(\{0,1\}^*)$ so $\mathrm{Ker}\,\pi|_{\langle\langle[\iota F,\iota,F]\rangle\rangle} = \mathrm{Alt}(\{0,1\}^*)$. Combining this with the fact that $[F, F] \leq \pi(\langle\langle[\iota F, \iota F]\rangle\rangle)$ gives

$$\mathrm{Alt}(\{0,1\}^*) \rtimes [F, F] \leq \langle\langle[\iota F, \iota F]\rangle\rangle.$$

Since $\langle\langle[\iota F, \iota F]\rangle\rangle_{QF} \leq [QF, QF]$, we know that $\mathrm{Alt}(\{0,1\}^*) \rtimes [F, F] \leq [QF, QF]$ and we now show they are equal.

Consider the following map induced by π,

$$\pi' \colon QF/(\mathrm{Alt}(\{0,1\}^*) \rtimes [F, F]) \longrightarrow F/[F, F].$$

Applying π' to the quotient $[QF, QF]/(\mathrm{Alt}(\{0,1\}^*) \rtimes [F, F])$ gives the trivial group. Since the kernel of π' is contained in $\mathrm{Sym}(\{0,1\}^*)/\mathrm{Alt}(\{0,1\}^*) = C_2$, the quotient $[QF, QF]/(\mathrm{Alt}(\{0,1\}^*) \rtimes [F, F])$ is either trivial or C_2. However, if the quotient $[QF, QF]/(\mathrm{Alt}(\{0,1\}^*) \rtimes [F, F])$ was C_2 then $[QF, QF] = \mathrm{Sym}(\{0,1\}^*) \rtimes [F, F]$ which would contradict Corollary 3.7. Hence $[QF, QF] = \mathrm{Alt}(\{0,1\}^*) \rtimes [F, F]$. This completes the proof of (1).

For parts (2) and (3) substitute Corollary 3.14 or Corollary 3.21 for Corollary 3.7 and use that $[T, T] \cong T$ and $[V, V] \cong V$ [13, Theorems 5.8,6.9].

For part (5), by the argument at the beginning of this proof, $\mathrm{Alt}(\{0,1\})^*$ is contained in $\langle\langle[\iota F, \iota F]\rangle\rangle_{QV}$ and $\langle\langle[\iota F, \iota F]\rangle\rangle_{QV}$ cannot contain any finite cycle with odd parity, else $\langle\langle[\iota F, \iota F]\rangle\rangle_{\tilde{Q}V}$ would also, and we already know that it doesn't. Also, $\langle\langle[\iota F, \iota F]\rangle\rangle_{QV}$ projects onto a non-trivial normal subgroup of V, so it must project onto V. We conclude that $\langle\langle[\iota F, \iota F]\rangle\rangle_{QV}$ is an extension of V by $\mathrm{Alt}(\{0,1\}^*)$. Observe that,

$$\langle\langle[\iota F, \iota F]\rangle\rangle_{QV} \leq (\langle\langle[\iota F, \iota F]\rangle\rangle_{\tilde{Q}V}) \cap QV = (\mathrm{Alt}(Z) \rtimes V) \cap QV.$$

Since $(\mathrm{Alt}(Z) \rtimes V) \cap QV$ is also an extension of V by $\mathrm{Alt}(\{0,1\}^*)$, we necessarily have

$$\langle\langle[\iota F, \iota F]\rangle\rangle_{QV} = (\mathrm{Alt}(Z) \rtimes V) \cap QV.$$

The proof of (4) is similar to that of (5). □

Corollary 5.3. *The abelianisation of QF is isomorphic $\mathbb{Z} \oplus \mathbb{Z} \oplus C_2$ and the abeliansations of QT, QV, $\widetilde{Q}T$, and $\widetilde{Q}V$ are all isomorphic to C_2.*

Lemma 5.4. (1) *Let $\sigma \in \mathrm{Sym}(\{0,1\}^*)$ and $1 \neq f \in \iota F$ be arbitrary, then there exists $h \in \iota F$ such that $[h,\sigma] = 1$ but $[h, f] \neq 1$.*

(2) *Let $\sigma \in \mathrm{Sym}(Z)$ and $1 \neq v \in \iota V$ be arbitrary, then there exists $h \in \iota F$ such that $[h,\sigma] = 1$ by $[h, f] \neq 1$.*

Proof. Let T be the full sub-tree under some $x \in \{0,1\}^*$, where T is chosen to intersect the support of f but not the support of σ, this is possible since the support of f is infinite but the support of σ is finite. Thus any h with $\mathrm{supp}(h) \subseteq T$ satisfies $[h,\sigma] = 1$.

Let F' denote the copy of F acting on T, with the usual generators denoted α' and β'. If f restricts to an action on T then, since F' is centreless, there exists some $h \in F'$ which doesn't commute with f. If f doesn't restrict to an action on T then there is necessarily an orbit $\{x_i\}_{i\in\mathbb{Z}}$ which is neither contained in T nor its complement. Since α' acts non-trivially on all of T, one of the α' orbits intersects $\{x_i\}_{i\in\mathbb{Z}}$ but is not equal to $\{x_i\}_{i\in\mathbb{Z}}$, thus α' and h do not commute.

The proof of (2) is analagous. □

Lemma 5.5. *Let G be one of QF, QT, QV, $\widetilde{Q}T$, and $\widetilde{Q}V$, and let $f \in G$ such that $\pi(f) \neq 1$, then $\langle\langle f\rangle\rangle_G \cap \iota F \neq \{\mathrm{id}\}$.*

Proof. Let $\sigma f \in QF$, so that $\sigma \in \mathrm{Sym}(\{0,1\}^*)$ and $f \in \iota F$ with $\pi(f) \neq 1$. Let $h \in \iota F$ be as in Lemma 5.4(1), then

$$\sigma h f h^{-1} = h\sigma f h^{-1} \in \langle\langle \sigma f\rangle\rangle.$$

So,

$$[f,h] = (\sigma f)^{-1}\sigma h f h^{-1} \in \langle\langle \sigma f\rangle\rangle,$$

since $[f,h]$ is a non-trivial element of ιF, this is sufficient. This proves the statement for QF.

For $G = \widetilde{Q}T$ or $\widetilde{Q}V$, start with an element $\sigma t \in \widetilde{Q}T$ (respectively $\sigma v \in \widetilde{Q}V$), so that $\sigma \in \mathrm{Sym}(Z)$ and $t \in \iota T$ (resp. $t \in \iota V$). Let $h \in \iota F$ be as in Lemma 5.4(2), then the proof is as for QF.

For $G = QT$ or QV, start with $\sigma t \in QT$ (respectively $\sigma v \in QV$), so that $\sigma \in \mathrm{Sym}(\{0,1\}^*)$ and $t \in \iota T$ (resp. $t \in \iota V$). Let $h \in \iota F$ be as in Lemma 5.4(1), then the proof is as for QF. □

Proof of Theorem 5.1. Let N be a normal subgroup of QF with $N \not\leq \mathrm{Sym}(\{0,1\}^*)$, then using Lemma 5.5 N contains an element $f \in \iota F$. For any $g \in \iota F$, we have $g^{-1} f^{-1} g \in N$ and hence

$$[g, f] = g^{-1} f^{-1} g f \in N.$$

Without loss of generality, $f \in [\iota F, \iota F]$.

Since $[\iota F, \iota F]$ is simple [13, Theorem 4.5], we deduce that $[\iota F, \iota F] \leq N$, and thus $\langle\langle [\iota F, \iota F] \rangle\rangle \leq N$. Proposition 5.2(1) completes the proof.

For the other parts, start with an element f of either $\widetilde{Q}T$, $\widetilde{Q}V$, QT, or QV and then use the appropriate parts of Lemmas 5.5 and Proposition 5.2. $\square$

6. Bieri–Neumann–Strebel–Renz invariants

In this section we compute the Bieri–Neumann–Strebel–Renz invariants $\Sigma^i(QF)$ and $\Sigma^i(QF, R)$ for any commutative ring R. The invariant $\Sigma^1(G)$ was introduced by Bieri, Neumann, and Strebel in [4] and the higher invariants $\Sigma^i(QF)$ for $i \geq 2$ were introduced in [5]. In general, for any group G, there is a hierachy of invariants

$$\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots \supseteq \Sigma^i(G) \supseteq \cdots,$$

furthermore we set $\Sigma^\infty(G) = \bigcap_{i=1}^\infty \Sigma^i(G)$. There are also homological versions $\Sigma^i(G, R)$ for any commutative ring R fitting into a similar hierachy. Furthermore, $\Sigma^i(G) \subseteq \Sigma^i(G, R)$ for all i all commutative rings R and also $\Sigma^1(G, R) = \Sigma^1(G)$ for all commutative rings R.

Some of the interest in the Bieri–Neumann–Strebel–Renz invariants comes from the following theorem, which classifies the finiteness length of normal subgroups N of a group G which contain the commutator subgroup $[G, G]$.

Theorem 6.1 ([5, Theorem B], [32]). *Let G be a group of type* F_n (*respectively* FP_n *over* R) *and let N be a normal subgroup containing the commutator* $[G, G]$. *Then N is* F_n (*resp.* FP_n *over* R) *if and only if every non-zero character χ of G such that $\chi(N) = 0$ satisfies* $[\chi] \in \Sigma^n(G)$ (*resp.* $[\chi] \in \Sigma^n(G, R)$).

Recall that a *character* of a group G is a group homomorphism $G \to \mathbb{R}$, where $\mathbb{R}$ is viewed as a group under addition, and the *character sphere* $S(G)$ is the set of equivalence classes of non-zero characters modulo multiplication by a positive real number. Thus $S(G)$ is isomorphic to a sphere of dimension $n-1$ where n is the torsion-free rank of $G/[G, G]$.

Since $F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}$, the character sphere $S(F)$ is isomorphic to S^1. We denote by χ_0 and χ_1 the two characters of F given by $\chi_0(A) = -1$, $\chi_0(B) = 0$, $\chi_1(A) = 1$, and $\chi_1(B) = 1$, these two characters are linearly independent and hence do not represent antipodal points on $S(F)$.

Theorem 6.2 (the Bieri–Neumann–Strebel–Renz invariants of F [3]). *The character sphere $S(F)$ is isomorphic to S^1 and, for any commutative ring R,*

(1) $\Sigma^1(F,R) = \Sigma^1(F) = S(F) \setminus \{[\chi_0],[\chi_1]\}$;

(2) $\Sigma^i(F,R) = \Sigma^i(F) = S(F) \setminus \{[a\chi_0 + b\chi_1]: a,b \geq 0\}$ *for all* $i \geq 2$.

Since the abelianisation of QF also has torsion-free rank 2, the character sphere $S(QF)$ is again isomorphic to S^1. By pre-composing χ_0 and χ_1 with $\pi: QF \to F$ we obtain characters $\pi^*\chi_0$ and $\pi^*\chi_1$. Once again these are linearly independent. In this section we prove the following theorem.

Theorem 6.3 (the Bieri–Neumann–Strebel–Renz invariants of QF). *The character sphere $S(QF)$ is isomorphic to S^1 and, for any ring R,*

(1) $\Sigma^1(QF,R) = \Sigma^1(QF) = S(QF) \setminus \{[\pi^*\chi_0],[\pi^*\chi_1]\}$;

(2) $\Sigma^i(QF,R) = \Sigma^i(QF) = S(QF) \setminus \{[a\pi^*\chi_0 + b\pi^*\chi_1]: a,b \geq 0\}$ *for all* $i \geq 2$.

Remark 6.4 (ν-symmetry). For $x \in \{0,1\}^*$, let $\bar{x}$ be the element obtained by swapping 0 and 1 in the word x. The map $x \mapsto \bar{x}$ induces an automorphism $\nu: QV \to QV$, which projects to the automorphism of F denoted ν in [3, §1.4]. Furthermore, ν induces an automorphism ν^* of $\Sigma^i(QF)$ for all i, and one checks that $\nu^*[\chi_0] = [\chi_1]$. Following Bieri, Geoghegan, and Kochloukova we will call this the *ν-symmetry* of $\Sigma^i(QF)$. Recall that for any group G the invariants $\Sigma^i(G)$ and $\Sigma^i(G,R)$ are invariant under automorphisms of G.

Lemma 6.5 ([30, Corollary 2.8][29, Corollary 3.12]). *Let $\pi: G \twoheadrightarrow Q$ be an split surjection and χ a character of Q. Then if $[\pi^*\chi] \in \Sigma^i(G)$ (respectively $\Sigma^i(G,R)$) then $[\chi] \in \Sigma^i(Q)$ (resp. $\Sigma^i(Q,R)$).*

Theorem 6.6 ([3, Theorems 2.1(1), 2.3]). *Let H be a group of type F_∞ and let $G = H*_{\theta,t}$ be an ascending HNN-extension such that θ is not surjective.*

(1) *Let $\chi: G \to \mathbb{R}$ be the character given by $\chi(H) = 1$ and $\chi(t) = 1$, then $[\chi] \in \Sigma^\infty(G)$.*

(2) *Let $\chi: G \to \mathbb{R}$ be a character such that $\chi|_H \neq 0$ and $[\chi|_H] \in \Sigma^\infty(H)$ then $[\chi] \in \Sigma^\infty(G)$, for any $i \geq 0$.*

Recall that the characters $\pi^*\chi_i$ are completely determined by the values they take on α and β (defined in Remark 2.4) and satisfy $\pi^*\chi_i(\alpha) = \chi_i(A)$ and $\pi^*\chi_i(\beta) = \chi_i(B)$ for all i, where A and B are the standard generators of F.

We now show that QF can be written as an ascending HNN-extension similarly to those of F [10, Proposition 1.7] and BF [36, Lemma 1.4].

Let $QF(1)$ be the subgroup of QF which fixes all of $\{0,1\}^*$ except the subtree with root 1. It is easy to see that $QF(1)$ is isomorphic to QF. Using Theorem 3.6 we see that QF is generated by $\{\alpha, \beta, \sigma_{\varepsilon,1}\}$. This shows that $QF(1)$ is generated by β, β^α and $\sigma_{1,11} = \sigma_{\varepsilon,1}^\alpha$. Moreover, conjugating $QF(1)$ by α maps $QF(1)$ isomorphically onto the subgroup of QF which fixes all but the subtree with root 11.

Lemma 6.7. *The group QF can be written as an ascending HNN-extension $QF = QF(1)*_{\theta,t}$, where $QF(1) \cong QF$.*

The proof is similar to [36, Lemma 1.4].

Proof. Let $\theta\colon QF(1) \to QF(1)$ be the monomorphism $\tau \mapsto \alpha^{-1}\tau\alpha$ and let $\psi\colon QF(1)*_{\theta,t} \to QF$ be the map given by setting t to be α. The map ψ is surjective since $\psi(\beta) = \beta$, $\psi(t) = \alpha$, and $\psi(\sigma_{1,11}^{t^{-1}}) = \sigma_{\varepsilon,1}$.

Let $g \in \operatorname{Ker}\psi$, since $QF(1)*_{\theta,t}$ is an ascending HNN-extension we can write g in normal form as $g = t^n h t^{-m}$ where $n, m \geq 0$ and $h \in QF(1)$. Let $\chi\colon QF(1)*_{\theta,t} \to \mathbb{Z}$ be the homomorphism sending t to 1 and sending $QF(1)$ to 0. The homomorphism χ factors through ψ—one can check there is a homomorphism $QF \to \mathbb{Z}$ sending α to 1 and $\beta, \sigma_{\varepsilon,1}$ to 0. Thus, if $g \in \operatorname{Ker}\psi$ then $g \in \operatorname{Ker}\chi$ and so $n = m$. In particular, $\alpha^n\psi(h)\alpha^{-n} = \mathrm{id}_{QF}$ so $h \in \operatorname{Ker}\psi$. However ψ is the identity on $QF(1)$, thus $h = 1$ and ψ in injective. ☐

Proof of Theorem 6.3. Lemma 6.5 implies that for all i, we have $\Sigma^i(QF, R) \subseteq \Sigma^i(F, R)$. In particular,

$$\Sigma^2(QF, R) \subseteq S(QF) \setminus \{[a\pi^*\chi_0 + b\pi^*\chi_1]\colon a, b \geq 0\}$$

The character which takes 1 on α and 0 elsewhere is exactly $-\pi^*\chi_0$ so using Theorem 6.6(1) we find that $[-\pi^*\chi_0] \in \Sigma^\infty(QF)$ and by Remark 6.4, $[-\pi^*\chi_1] \in \Sigma^\infty(QF)$ as well.

Next we claim that

$$\{[\chi] \in S(QF)\colon \chi(\beta) < 0\} \subseteq \Sigma^\infty(QF),$$

the argument is similar to that of [3, Corollary 2.4]. Let $\chi\colon QF \to \mathbb{R}$ be a character with $\chi(\beta) < 0$, then

$$\psi^*[\chi|_{QF(1)}](\alpha) = \chi(\beta) < 0,$$

$$\psi^*[\chi|_{QF(1)}](\beta) = \chi(\beta^\alpha) = \chi(\beta) < 0.$$

So $\psi^*[\chi|_{QF(1)}] = -[\chi_1]$ and thus $[\chi|_{QF(1)}] \in \Sigma^\infty(QF(1))$. Since $QF(1)$ is isomorphic to QF and hence of type F_∞ (Theorem 4.10), applying Theorem 6.6(2) completes the claim.

The above implies that $[a\chi_0 + b\chi_1] \in \Sigma^\infty(QF)$ if $b < 0$, so by ν symmetry $[a\chi_0 + b\chi_1] \in \Sigma^\infty(QF)$ if $a < 0$ also. Thus,

$$\{[a\pi^*\chi_0 + b\pi^*\chi_1] : a, b \geq 0\} \subseteq \Sigma^\infty(QF)^c$$

which completes the descriptions of $\Sigma^i(QF)$ and $\Sigma^i(QF, R)$ for all $i \geq 2$.

It remains to complete the description of $\Sigma^1(QF)$, we use Theorem 6.1 for this. Let $\chi \in S(QF) \setminus \{[\pi^*\chi_0], [\pi^*\chi_1]\}$ and consider the normal subgroup $N = \operatorname{Ker} \pi^*\chi$. We will show that N is finitely generated, so necessarily $\chi \in \Sigma^1$.

From Theorem 6.2 we know that $\pi(N)$ is finitely generated. Since $\operatorname{Ker} \pi|_N = \operatorname{Sym}(\{0,1\}^*)$ we have a group extension

$$1 \longrightarrow \operatorname{Sym}(\{0,1\}^*) \longrightarrow N \longrightarrow \pi(N) \longrightarrow 1.$$

Thus, using the method of Section 3 it is sufficient to find a generating set of $\operatorname{Sym}(\{0,1\}^*)$ on which $\pi(N)$ acts by permutations and with finitely many orbits, this is given by Lemma 6.8 below, since a generating set for $\operatorname{Sym}(\{0,1\}^*)$ is given by the set Σ_2, which we recall below. □

Recall from Section 3.1 that for all n,

$$\Sigma_n = \left\{(x_1, \ldots, x_n) \in \prod_1^n \{0,1\}^* : x_1 \lneq_{\text{lex}} x_2 \lneq_{\text{lex}} \cdots \lneq_{\text{lex}} x_n\right\},$$

and that Σ_n admits an action of F via the splitting $\iota: F \to QF$, the inclusion $QF \to QV$, and the usual action of QV on $\{0,1\}^*$.

Recall the definition of $\widetilde{\chi}_\zeta$ from Section 4.1. By comparing the values they take on α and β, one checks that $\pi^*\chi_1 = \widetilde{\chi}_\zeta$.

Lemma 6.8. *Let $\chi \in S(QF)$ and $N = \operatorname{Ker} \chi$ then $\pi(N)$ acts transitively on Σ_2.*

Proof. Let $(x_1, x_2) \in \Sigma_2$ and let (L, R) be a representative for an element f of F such that $\iota(f)(x_1) = 0$ and $\iota(f)(x_2) = \varepsilon$ (use, for example, Lemma 3.2). Applying Lemma 6.9 below with $a = -\chi_1(f)$ gives an element f_1, and applying the lemma a second time with $a = -\chi_0(f)$ gives a second element f_0.

We claim that $\nu(f_0)f_1 f$ is the required element of F. Note that since f_0 fixes ε and 1 by construction, the element $\nu(f_0)$ fixes $\bar{1} = 0$ and $\bar{\varepsilon} = \varepsilon$. Finally,

$$\chi_0(\nu(f_0)f_1 f) = \chi_0(\nu(f_0)) + \chi_0(f_1) + \chi_0(f) = \chi_1(f_0) + \chi_0(f) = 0,$$

and similarly one shows that $\chi_1(\nu(f_0)f_1 f) = 0$.

Let $\tau = \iota(\nu(f_0)f_1 f)$. Since $\chi_1(\nu(f_0)f_1 f) = \chi_0(\nu(f_0)f_1 f) = 0$, also $\pi^*\chi_1(\tau) = \pi^*\chi_0(\tau) = 0$, so the element τ is in the kernel of any character in $S(QF)$. □

Lemma 6.9. *For any integer a there exists an element $f \in F$ such that $\iota(f) \cdot \varepsilon = \varepsilon$, $\iota(f) \cdot 0 = 0$, $\iota(f) \cdot 1 = 1$, $\chi_0 \iota(f) = 0$, and $\chi_1 \iota(f) = a$.*

Proof. Assume for now that $a > 0$. Let T be the smallest tree containing ε, 1, and 11 as nodes. Let L be obtained from T by attaching a carets iteratively to the largest leaf of T (using the $\leq_{\text{lex}}$ ordering). Let R be obtained from T by attaching a carets iteratively to the second largest caret of T. Denote by f the element of F with tree diagram representative (L, R). For example when $a = 2$ we obtain the element of Figure 14. One checks that $\chi_1(f) = a$ and that $\iota(f)$ fixes ε, 0, and 1.

If $a < 0$ then perform the steps above for $-a$ and then replace f_0 by f_0^{-1}. □

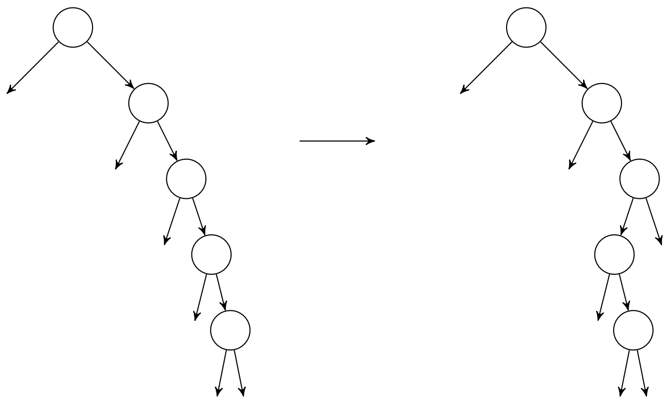

Figure 14. Element f from Lemma 6.9 when $a = 2$.

7. Relationships with other generalisations of Thompson's groups

In [34] Thumann studies the class of *Operad groups*, this class contains the generalised Thompson groups $F_{n,r}$ and $V_{n,r}$, the higher dimensional Thompson groups nV, and the braided Thompson groups BV. Moreover, the class of *diagram groups* appearing in [21] may be described as Operad groups, as may the class of *self-similarity groups* described by Hughes [23]—for a proof of these facts see [34, §3.5]. For each of these classes of groups there exist results showing that certain groups in the class are of type F_∞ [18, Theorem 1.3][16, Theorem 1.1][34, Theorem 4.3]. We show in Section 7.1 that while QV admits an easy description in all three of these classes, it cannot be proved to be of type F_∞ using these descriptions and the results listed above.

There is a generalisation of Thompson's groups due to Martínez-Pérez and Nucinkis, describing a family of automorphism groups of Cantor algebras [27]. We show in Proposition 7.4 that none of the groups studied in this paper appear in this family of groups.

Witzel and Zaremsky have introduced *Thompson groups for systems of groups* [35], this class contains the Thompson groups F and V and also the braided Thompson groups BF and BV. They study the finiteness lengths of some groups in this class. We do not know of a way to express any of the groups studied here as a group in this class and we are unsure of the relationship between Witzel–Zaremsky's class of Thompson groups for systems of groups and either Thumann's class of Operad groups or the class of groups of Martínez-Pérez and Nucinkis.

7.1. Operad groups. Following [34] we denote by

$$\mathrm{End}\colon \mathbf{MON} \longrightarrow \mathbf{OP}$$

the endomorphism functor taking a strict monoidal category $\mathcal{C}$ to the endomorphism operad $\mathrm{End}(\mathcal{C})$. Let $\mathcal{S}$ denote the left adjoint of End. Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$ we write $\pi_1(\mathcal{C}, X)$ for the fundamental group of $\mathcal{C}$ based at X. The *operad group* associated to an operad $\mathcal{O}$ and object $X \in \mathcal{S}(\mathcal{O})$ is the group $\pi_1(\mathcal{S}(\mathcal{O}), X)$.

Let $\mathcal{O}_{QV}$ be the operad with colours l and n (one should think of these as representing a leaf and a node of a finite subtree of $\mathcal{T}_{2,c}$) and with a single operation $\mathcal{O}(l, n, l; l) = \{\varphi\}$ (one should think of this operation as adding a caret).

Lemma 7.1. *There is an isomorphism*

$$QV \cong \pi_1(\mathcal{S}(\mathcal{O}_{QV}), l).$$

Proof. This is essentially [17, Example 4.4], where Farley and Hughes describe QV as a braided diagram group over the semi-group presentation

$$\mathcal{P} = \langle l, n \colon (l, lnl) \rangle.$$

One can then convert this to a description of QV as a group acting on a compact ultrametric space via a small similarity structure [17, Theorem 4.12] and then in turn to a description as an operad group [34, §3.5]. Following this method one obtains the operad $\mathcal{O}_{QV}$ given above. □

The next lemma demonstrates that one cannot use the result of Thumann [34, Theorem 4.3]) to show that QV is F_∞ because $\mathcal{O}_{QV}$ is not colour-tame, see [34, Definition 4.2]. Recall that a colour word is said to be *reduced* if no subword is in the domain of a non-identity operation.

Lemma 7.2. *$\mathcal{O}_{QV}$ is not colour-tame.*

Proof. The operad $\mathcal{O}_{QV}$ is planar and has a finite set of colours, however the colour word

$$\overbrace{n \cdot n \cdots n}^{i \text{ times}},$$

is reduced for all $i \geq 0$. □

Using [17, Theorem 4.12] to convert the description of QV as a braided diagram group over the semi-group presentation $\mathcal{P}$ to a description of QV as a self-similarity group gives the following. Let $\mathcal{T}_{\mathcal{P}}$ be the tree obtained from $\mathcal{T}_{2,c}$ by forgetting the colouring and adding to every node a single child. Let X be the ultrametric space $X = \mathrm{Ends}(\mathcal{T}_{\mathcal{P}})$ obtained by setting $d_X(p, p') = e^{-i}$ if p and p' are any two paths (without backtracking) which contain exactly i edges in common. Balls in X are all of the form

$$B_v = \{p \in \mathrm{Ends}(\mathcal{T}_{\mathcal{P}})\colon v \text{ lies on } p\},$$

for some vertex in v. If v and w are both leaves or both nodes then $\mathrm{Sim}_X(B_v, B_w)$ contains a single element and otherwise $\mathrm{Sim}_X(B_v, B_w)$ is empty.

The next lemma demonstrates that one cannot use the result of Farley and Hughes [16, Theorem 1.1] together with the description of QV as the self-similarity group associated to the ultrametric space X and the similarity structure Sim_X to prove that QV is F_∞. This is because [16, Theorem 1.1] requires that Sim_X is rich in simple contractions [16, Definition 5.11].

Lemma 7.3. Sim_X *is not rich in simple contractions.*

Proof. Given an arbitrary constant $k \in \mathbb{N}_{>0}$ it suffices to exhibit a pseudo-vertex v of height k such that for every pseudo vertex $w \subseteq v$, either $\|w\| = 1$ or there is no simple contraction of v at w.

Let $k \in \mathbb{N}_{>0}$ be an arbitrary constant and choose k leaves $n_1, \dots, n_k \in \mathrm{Ends}(\mathcal{T}_{\mathcal{P}})$ with no children. Each of these leaves is a ball B_{n_i} and so

$$v = \{[\mathrm{incl}_{B_{n_i}}, B_{n_i}]\colon i = 1, \dots, k\}$$

is a pseudo-vertex of height k. Since any simple expansion of any pseudo-vertex necessarily introduces balls containing leaves with children, there can be no simple contraction of v. □

7.2. Automorphisms of Cantor algebras. In [27], Martínez-Pérez and Nucinkis study certain automorphism groups $G_r(\Sigma)$, $T_r(\Sigma)$, and $F_r(\Sigma)$ of certain Cantor algebras.

Proposition 7.4. *None of QV, $\tilde{Q}V$, QT, $\tilde{Q}T$, and QF are isomorphic to either $G_r(\Sigma)$, $T_r(\Sigma)$, or $F_r(\Sigma)$ for any Σ and r.*

Proof. By [27, Theorems 4.3, 4.8], any group $G_r(\Sigma)$ or $T_r(\Sigma)$ has at most finitely many conjugacy classes of finite subgroups isomorphic to a given finite subgroup. This is false however for QV and QT as one can find infintely many non-conjugate subgroups isomorphic to the cyclic group of order 2: let $\{x_1, \ldots, x_n, \ldots\}$ and $\{y_1, \ldots, y_n, \ldots\}$ be two disjoint countably infinite subsets of $\{0, 1\}^*$ and let G_i be the subgroup of QF which tranposes x_j and y_j for all $j \leq i$. Clearly $G_i \cong C_2$ for all i, but the G_i are all pairwise non-conjugate (this is because conjugation preserves cycle type in infinite support permutation groups). Thus none of QV, $\tilde{Q}V$, QT, $\tilde{Q}T$, and QF may be isomorphic to $G_r(\Sigma)$ or $T_r(\Sigma)$.

Since $F_r(\Sigma)$ is always torsion-free [27, Remark 2.17], none of QV, $\tilde{Q}V$, QT, $\tilde{Q}T$, or QF may be isomorphic to $F_r(\Sigma)$. □

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