Groups Geom. Dyn. 12 (2018), 703–719
DOI 10.4171/GGD/453

**Groups, Geometry, and Dynamics**


# The action of the mapping class group on the space of geodesic rays of a punctured hyperbolic surface

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**Abstract.** Let $\Sigma$ be a complete finite-area orientable hyperbolic surface with one cusp, and let $\mathcal{R}$ be the space of complete geodesic rays in $\Sigma$ emanating from the puncture. Then there is a natural action of the mapping class group of $\Sigma$ on $\mathcal{R}$. We show that this action is "almost everywhere" wandering.

**Mathematics Subject Classification (2010).** 20F65, 20F34, 20F67, 30F60, 57M50.

**Keywords.** Mapping class group, hyperbolic surface, geodesic ray, wandering domain.

## 1. Introduction

Let $\Sigma$ be a complete finite-area orientable hyperbolic surface with one cusp, and $\mathcal{R}$ the space of complete geodesic rays in $\Sigma$ emanating from the puncture. Then, there is a natural action of the (full) mapping class group $\mathrm{Map}(\Sigma)$ of $\Sigma$ on $\mathcal{R} \equiv S^1$, as we describe in Sections 2 and 3. (Here we are allowing orientation-reversing mapping classes.) The dynamics of the action of an element of $\mathrm{Map}(\Sigma)$ on $\mathcal{R}$ plays a key role in the Nielsen-Thurston theory for surface homeomorphisms. It also plays a crucial role in the variation of McShane's identity for punctured surface bundles with pseudo-Anosov monodromy, established by [3] and [1].

It is natural to ask what does the action of the whole group $\mathrm{Map}(\Sigma)$ (or its subgroups) look like. However, the authors could not find a reference which treats this natural question. On the other hand, there are various references which study the action of (subgroups of) the mapping class groups on the projective measured lamination spaces, which are homeomorphic to higher dimensional spheres (see for example, [8, 9, 11, 13]). In particular, such an action is minimal (cf. [6]) and moreover ergodic [8].

The purpose of this paper is to prove that the action of $\mathrm{Map}(\Sigma)$ on $\mathcal{R}$ is "almost everywhere" wandering (see Theorem 2.2 for the precise meaning). This forms a sharp contrast to the above result of [8].

[1] Makoto Sakuma was supported by JSPS Grants-in-Aid 15H03620.

While we restrict to the case where $\Sigma$ has one cusp, we note that one could generalise the statement when there is more than one. However, this would complicate the exposition somewhat. (See the remark at the end of Section 6.)

This paper is organised as follows. In section 2, we give a statement of the main result. In Section 3, we give a rigorous construction of the action of $\mathrm{Map}(\Sigma)$ on $\mathcal{R}$ by using the theory of the canonical boundary of a relatively hyperbolic group, and then state the main result of this paper. In Section 4, we give an account of the "loop-cutting" construction, and show how it gives rise to a sequence of elements of $\Gamma = \pi_1(\Sigma)$, and parabolic points in the boundary of $\tilde{\Sigma}$. These "derived sequences" are used in Section 5 to define the concept of a filling point. We show that the set, $F$, of filling points is an open subset of $C$ whose complement is "small", in particular $F$ has full measure (Proposition 5.1). In Section 6, we show that the image of $F$ in $\mathcal{R}$ is contained in the wandering domain of the action of $\mathrm{Map}(\Sigma)$ on $\mathcal{R}$ (Lemma 6.2). The proof of the main result is given by using this result. In Section 7, we give a list of notations used in this paper.

**Acknowledgements.** We would like to thank Katsuhiko Matsuzaki for his helpful comments on the first version of the paper. We would also like to thank the referees for their careful reading and valuable suggestions.

## 2. Statement of main result

Let $\Sigma = \mathbb{H}^2/\Gamma$ be a complete finite-area orientable hyperbolic surface with precisely one cusp, where $\Gamma = \pi_1(\Sigma)$. Let $\mathcal{R}$ be the space of complete geodesic rays in $\Sigma$ emanating from the puncture. Then $\mathcal{R}$ is identified with a horocycle, $\tau$, in the cusp. In fact, a point of $\tau$ determines a geodesic ray in $\Sigma$ emanating from the puncture, or more precisely, a bi-infinite geodesic path with its positive end going out the cusp and meeting $\tau$ in the given point. Any mapping class $\psi$ of $\Sigma$ maps each geodesic ray to another path which can be "straightened out" to another geodesic ray, and hence determines another point of $\tau$. This gives rise to an action of $\mathrm{Map}(\Sigma)$ on $\mathcal{R} \equiv \tau$. This will be discussed more formally in Section 3, where we give a description of the space $\mathcal{R}$ as a topological circle constructed independent of a hyperbolic structure of $\Sigma$, and then give a rigorous construction of the action of $\mathrm{Map}(\Sigma)$ on $\mathcal{R}$.

In order to state the main result, we prepare some terminology.

**Definition.** Let $G$ be a group acting by homeomorphism on a topological space $X$. An open subset, $U \subseteq X$, is said to be *wandering* if $gU \cap U = \emptyset$ for all $g \in G \setminus \{1\}$.

Note that this definition is stronger than the usual definition of wandering, where it is only assumed that the number of $g \in G$ such that $gU \cap U \neq \emptyset$ is finite.

The *wandering domain*, $W_G(X) \subseteq X$, is the union of all wandering open sets. Its complement, $W^c_G(X) = X \setminus W_G(X)$, is the *non-wandering set*.

The following easily verified fact is used in Section 6.

**Proposition 2.1.** *Let $H \lhd G$ be a normal subgroup. Then for the induced action of $G/H$ on $X/H$, we have $W_G(X)/H \subseteq W_{G/H}(X/H)$ with equality if $W_H(X) = X$.*

Note that any hyperbolic structure on $\Sigma$ induces a euclidean metric on $\mathcal{R}$ (via the horocycle $\tau$). If one changes the hyperbolic metric, the induced euclidean metrics on $\mathcal{R}$ are related by a quasisymmetry. However, they are completely singular with respect to each other (see [7, 16]). (That is, there is a set which has zero measure in one structure, but full measure in the other.) In general, this gives little control over how the Hausdorff dimension of a subset can change.

We say that a subset, $B \subseteq \mathcal{R}$ is *small* if it has Hausdorff dimension stricty less than 1 with respect to any hyperbolic structure on $\Sigma$. Now we can state our main theorem.

**Theorem 2.2.** *Let $\Sigma$ be a once-punctured closed orientable surface, with $\chi(\Sigma) < 0$, and consider the action of* $\mathrm{Map}(\Sigma)$ *on the circle $\mathcal{R}$. Then the non-wandering set in $\mathcal{R}$ with respect to the action of* $\mathrm{Map}(\Sigma)$ *is small.*

In particular, the non-wandering set has measure 0 with respect to any hyperbolic structure, and so has empty interior.

Given that two different hyperbolic structures give rise to quasisymmetically related metrics on $\mathcal{R}$, it is natural to ask if there is a more natural way to express this. For example, is there a property of (closed) subsets of $\mathcal{R}$, invariant under quasisymmetry and satisfied by the non-wandering set, which implies Hausdorff dimension less than 1 (or measure 0)?

## 3. Actions

In this section, we give a more formal account of the action of $\mathrm{Map}(\Sigma)$ on $\mathcal{R} \equiv \tau$.

Choose a representative, $f$, of $\psi \in \mathrm{Map}(\Sigma)$, so that its lift $\tilde{f}$ to the universal cover $\mathbb{H}^2$ is a quasi-isometry. Then $\tilde{f}$ extends to a self-homeomorphism of the closed disc $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. For a geodesic ray $\nu \in \mathcal{R}$, let $\tilde{\nu}$ be the closure in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ of a lift of $\nu$ to $\mathbb{H}^2$. Then $\tilde{f}(\tilde{\nu})$ is an arc properly embedded in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$, and its endpoints determine a geodesic in $\mathbb{H}^2$, which project to another geodesic ray $\nu' \in \mathcal{R}$. Thus, we obtain an action of $\psi$ on $\mathcal{R}$, by setting $\psi\nu = \nu'$. However, one needs to verify that this action does not depend on the choice of a representative $f$ of $\psi$.

In the following, we settle this issue, by using the canonical boundary of a relatively hyperbolic group described in [4]. Though we are really interested here only in the case where the group is the fundamental group of a once-punctured closed orientable surface, and the peripheral structure is interpreted in the usual way (as the conjugacy class of the fundamental group of a neighbourhood of the puncture), we give a discussion in a general setting.

Let $\Gamma$ be a non-elementary relatively hyperbolic group with a given peripheral structure $\mathcal{P}$, which is a conjugacy invariant collection of infinite subgroups of $\Gamma$. By [4, Definition 1], $\Gamma$ admits a properly discontinuous isometric action on a path-metric space, $X$, with the following properties.

(1) $X$ is proper (i.e., complete and locally compact) and Gromov hyperbolic,

(2) every point of the boundary of $X$ is either a conical limit point or a bounded parabolic point,

(3) the peripheral subgroups, i.e., the elements of $\mathcal{P}$, are precisely the maximal parabolic subgroups of $\Gamma$, and

(4) every peripheral subgroup is finitely generated.

It was proven in [4, Theorem 9.4] that the Gromov boundary $\partial X$ is uniquely determined by $(\Gamma, \mathcal{P})$, (even though the quasi-isometry class of the space $X$ satisfying the above conditions is not uniquely determined). Thus the boundary $\partial\Gamma = \partial(\Gamma, \mathcal{P})$ is defined to be $\partial X$. By identifying $\Gamma$ with an orbit in $X$, we obtain a natural topology on the disjoint union $\Gamma \cup \partial\Gamma$ which is compact Hausdorff, with $\Gamma$ discrete and $\partial\Gamma$ closed.

The action of $\Gamma$ on itself by left multiplication extends to an action on $\Gamma \cup \partial\Gamma$ by homeomorphism. This gives us a geometrically finite convergence action of $\Gamma$ on $\partial\Gamma$. Let $\mathrm{Aut}(\Gamma, \mathcal{P})$ be the subgroup of the automorphism group, $\mathrm{Aut}(\Gamma)$, of $\Gamma$ which respects the peripheral structure $\mathcal{P}$. This contains the inner automorphism group, $\mathrm{Inn}(\Gamma)$. Now, by the naturality of $\partial\Gamma$ ([4, Theorem 9.4]), the action of $\mathrm{Aut}(\Gamma, \mathcal{P})$ on $\Gamma$ also extends to an action on $\Gamma \cup \partial\Gamma$, which is $\Gamma$-equivariant, i.e., $\phi \cdot (g \cdot x) = \phi(g) \cdot (\phi \cdot x)$ for every $\phi \in \mathrm{Aut}(\Gamma, \mathcal{P})$, $g \in \Gamma$ and $x \in \Gamma \cup \partial\Gamma$. (In order to avoid confusion, we use $\cdot$ to denote group actions, only in this place.) Under the natural epimorphism $\Gamma \to \mathrm{Inn}(\Gamma)$, this gives rise to the same action on $\partial\Gamma$ as that induced by left multiplication. The centre of $\Gamma$ is always finite, and for simplicity, we assume it to be trivial. In this case, we can identify $\Gamma$ with $\mathrm{Inn}(\Gamma)$.

Suppose that $p \in \partial\Gamma$ is a parabolic point. Its stabiliser, $Z = Z(\Gamma, p)$, in $\Gamma$ is a peripheral subgroup. Now $Z$ acts properly discontinuously cocompactly on $\partial\Gamma \setminus \{p\}$, so the quotient $T = (\partial\Gamma \setminus \{p\})/Z$ is compact Hausdorff (cf. [4, Section 6]). Let $A = A(\Gamma, \mathcal{P}, p)$ be the stabiliser of $p$ in $\mathrm{Aut}(\Gamma, \mathcal{P})$. Then $Z$ is a normal subgroup of $A$, and we get an action of $M = A/Z$ on $T$. If there is only one conjugacy class of peripheral subgroups, then the orbit $\Gamma p$ is $\mathrm{Aut}(\Gamma, \mathcal{P})$-invariant, and it follows that the group $A$ maps isomorphically onto $\mathrm{Out}(\Gamma, \mathcal{P}) = \mathrm{Aut}(\Gamma, \mathcal{P}) / \mathrm{Inn}(\Gamma)$, so in this case we can naturally identify the group $M$ with $\mathrm{Out}(\Gamma, \mathcal{P})$.

Suppose now that $\Sigma$ is a once-punctured closed orientable surface, with negative Euler characteristic $\chi(\Sigma)$. We write $\Sigma = D/\Gamma$, where $D = \widetilde{\Sigma}$, the universal cover, and $\Gamma \cong \pi_1(\Sigma)$. Let $\mathcal{P}$ be the peripheral structure of $\Gamma$ arising from the cusp of $\Sigma$, namely $\mathcal{P}$ consists of the conjugacy class of the fundamental group of a neighbourhood of the end of $\Sigma$. Then $(\Gamma, \mathcal{P})$ is a relatively hyperbolic group, because if we fix a complete hyperbolic structure on $\Sigma$ then $D$ is identified with $\mathbb{H}^2$ and the isometric action of $\Gamma$ on $D = \mathbb{H}^2$ satisfies conditions (1)–(4) in the above. Now $D$ admits a natural compactification to a closed disc, $D \cup C$, where $C$ is the dynamically defined circle at infinity. We can identify $C$ with $\partial\Gamma$. In fact, if $x$ is any point of $D$, then identifying $\Gamma$ with the orbit $\Gamma x$, we get an identification of $\Gamma \cup \partial\Gamma$ with $\Gamma x \cup C \subseteq D \cup C$. As above we get an action of $\mathrm{Aut}(\Gamma, \mathcal{P})$ on $C$. If $p \in \partial C$ is parabolic, then its stabiliser $Z$ in $\Gamma$ is isomorphic to the infinite cyclic group $\mathbb{Z}$, and we get an action of $\mathrm{Out}(\Gamma, \mathcal{P})$ on the circle $T = (C \setminus \{p\})/Z$. Since $\mathrm{Out}(\Gamma, \mathcal{P})$ is identified with the (full) mapping class group $\mathrm{Map}(\Sigma)$ of $\Sigma$, we obtain a well defined action of $\mathrm{Map}(\Sigma)$ on the circle $T$.

We now return to the setting in the beginning of this section, where $\Sigma = \mathbb{H}^2/\Gamma$ is endowed with a complete hyperbolic structure. Then we can identify the (dynamically defined) circle $T$ with the horocycle, $\tau$, in the cusp, which in turn is identified with the space of geodesic rays, $\mathcal{R}$. This gives an action of $\mathrm{Map}(\Sigma)$ on $\mathcal{R}$. Since the action of $\Gamma$ on $\mathbb{H}^2$ satisfies the conditions (1)–(4) in the above (i.e., [4, Definition 1]), we see that, for each mapping class $\psi$ of $\Sigma$, its action on $\mathcal{R}$, defined via the "straightening process" presented at the beginning of this section, is identical with the action which is dynamically constructed in the above, independently from the hyperbolic structure. Thus the problem raised at the beginning of this section is settled.

## 4. The loop-cutting construction

Let $\Sigma = \mathbb{H}^2/\Gamma$ be a complete finite-area orientable hyperbolic surface with precisely one cusp.

The construction described in this section aims to associate combinatorial data to a ray, $\nu$, emanating from the cusp of $\Sigma$. In particular, we will construct a (finite or infinite) sequence of properly embedded arcs, $\lambda_i$, with both endpoints at the cusp. In Section 6, we will show that if these arcs eventually "fill" $\Sigma$, then the ray corresponds to an element of the wandering domain. Since this situation is "generic", the main result will then follow.

The basic idea behind the construction is as follows. If the ray, $\nu$, is embedded in $\Sigma$ (properly or not), we immediately stop with the empty sequence. Otherwise (and indeed "generically") we consider the first point at which $\nu$ crosses itself. This determines an embedded essential loop, $\nu_1$, based at this intersection point.

In other words, we can write $\nu$ as a concatenation, $\nu_0 \cup \nu_1 \cup \nu_2$, where $\nu_0$ and $\nu_2$ are respectively initial and final segments of $\nu$. We now cut out the loop $\nu_1$. That is, we take the piecewise geodesic paths, $\nu_0 \cup \nu_1 \cup (-\nu_0)$ and $\nu_0 \cup \nu_2$, and straighten them out to geodesics, to give us respectively, an arc, $\lambda_1$, based at the cusp, and a new ray, $\nu'$, emanating from the cusp. We now start again with $\nu'$ in place of $\nu$, and iterate the procedure. This gives a (generically infinite) sequence, $\lambda_1, \lambda_2, \lambda_3, \ldots$, of such arcs.

While this may be intuitively clearer in $\Sigma$, it is best expressed formally in terms of the universal cover, $\tilde{\Sigma}$. This entails fixing some parabolic point, $p$. Then any other ideal point, $x$, gives rise to a sequence, $g_i$, of elements of $\Gamma = \pi_1(\Sigma)$. We should think of the geodesic from $p$ to $x$ as projecting to $\nu$, while the elements $g_i$ correspond to the loops in $\Sigma$, which we have cut out in the above process.

Here is a formal account. Recall that $D = \tilde{\Sigma}$, is identified with the hyperbolic plane $\mathbb{H}^2$. Write $C$ for the ideal boundary of $D$, which we consider equipped with a preferred orientation. Thus $\Gamma$ acts on $C$ as a geometrically finite convergence group. Let $\Pi \subseteq C$ be the set of parabolic points of $\Gamma$. Given $p \in \Pi$, let $\theta(p)$ be the generator of $\mathrm{stab}_\Gamma(p)$ which acts on $C \setminus \{p\}$ as a translation in the positive direction. Given distinct $x, y \in C$, let $[x, y] \subseteq D \cup C$ denote the oriented geodesic from $x$ to $y$. If $g \in \Gamma$ is hyperbolic, write $a(g)$, $b(g)$ respectively, for its attracting and repelling fixed points; $\alpha(g) = [b(g), a(g)]$ for its axis; and $\lambda(g)$ for the oriented closed geodesic in $\Sigma$ corresponding to $g$, i.e., the image of $\alpha(g) \cap D$ in $\Sigma$. If $x, y \in C$ are distinct, then $[x, y] \cap D$ projects to an oriented bi-infinite geodesic path, $\lambda(x, y)$, in $\Sigma$. If $x, y \in \Pi$, then this is a proper geodesic path, with a finite number, $\nu(x, y)$, of self-intersections. Let $\Delta = \{(p, q) \in \Pi^2 \mid \nu(p, q) = 0\}$, i.e., $\Delta$ consists of pairs $(p, q)$ of parabolic points such that $\lambda(p, q)$ is a proper geodesic arc. (By an *arc*, we mean an embedded path.) Given $p \in \Pi$, write $\Pi(p) = \{q \in \Pi \mid (p, q) \in \Delta\}$.

Pick an element $(p, q) \in \Delta$. Then the proper arc $\lambda(p, q)$ intersects a sufficiently small horocycle, $\tau$, in precisely two points. Let $\tilde{\tau} \subseteq D$ be the horocircle centred at $p$ which is a connected component of the inverse image of $\tau$. Put $s_0 = [p, q] \cap \tilde{\tau}$ and $s_{2i} = (\theta(p))^i s_0$ for $i \in \mathbb{Z}$. Then $\{s_{2i}\}_{i \in \mathbb{Z}}$ is the inverse image in $\tilde{\tau}$ of the point of $\lambda(p, q) \cap \tau$ from which the ray $\lambda(p, q)$ emanates into the non-cuspidal part of $\Sigma$. For each $i \in \mathbb{Z}$, there is a unique point in the open interval of $\tilde{\tau}$ bounded by $s_{2i}$ and $s_{2(i+1)}$ which projects to the point of $\lambda(p, q) \cap \tau$ from which the ray $\lambda(p, q)$ enters the cusp. We denote the point by $s_{2i+1}$. Then $\{s_i\}_{i \in \mathbb{Z}}$ is the inverse image of the two points in $\tilde{\tau}$, located in this order with respect to the preferred orientation of $\tilde{\tau}$ (the orientation determined by the preferred orientation of $C$), and we have $\theta(p)s_i = s_{i+2}$ for every $i \in \mathbb{Z}$. Note that, for each $s_i$, there is a unique lift of $\lambda(p, q)$ which passes through $s_i$, and in particular, there is a unique element $g(p, q) \in \Gamma$ such that $g(p, q)^{-1}[p, q] \cap \tilde{\tau} = s_1$. Then $g(p, q)p = q$ and $g(p, q)^{-1}[p, q]$ is the closure of the lift of $\lambda(p, q)$ with endpoint $p$ which is closest to $[p, q]$, among the lifts of $\lambda(p, q)$ with endpoint $p$, with respect to the preferred orientation of $\tilde{\tau}$. (See Figure 1.)

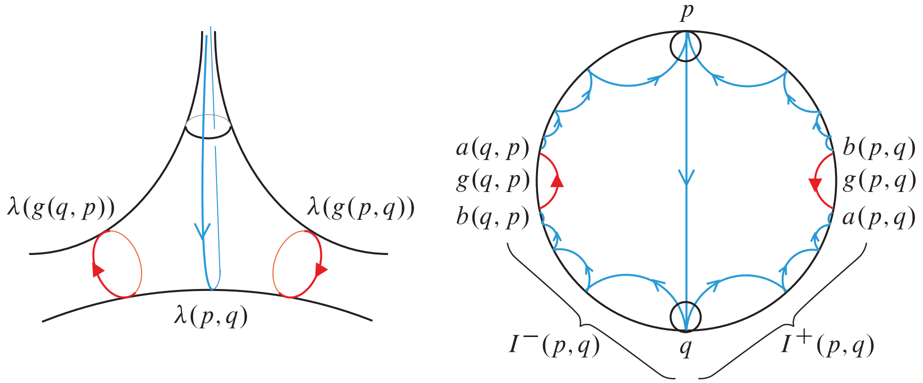


Figure 1. In the right figure, the two red arcs with thick arrows represent the axes $\alpha(g(p,q))$ and $\alpha(g(q,p))$ of the hyperbolic transformations $g(p,q)$ and $g(q,p)$ respectively. The blue arcs with thin arrows represent the oriented geodesic $[p,q]$ and its images by the infinite cyclic groups $\langle g(p,q)\rangle$ and $\langle g(q,p)\rangle$. The three intersection points of the blue arcs and the horocircle $\tilde{\tau}$ centred at $p$ are $s_{-1}$, $s_0$ and $s_1$, from left to right.

In the quotient surface $\Sigma$, the oriented closed geodesic $\lambda(g(p,q))$ is homotopic to the simple oriented loop obtained by shortcutting the oriented arc $\lambda(p,q)$ by the horocyclic arc which is the image of the subarc of $\tilde{\tau}$ bounded by $s_0$ and $s_1$. Thus $\lambda(g(p,q))$ is a simple closed geodesic disjoint from the proper geodesic arc $\lambda(p,q)$. In particular, $[p,\theta(p)q]\cap\alpha(g(p,q))=\emptyset$. In fact, the map $[(p,q)\mapsto g(p,q)]\colon\Delta\to\Gamma$ is characterised by the following properties: for all $(p,q)\in\Delta$, we have $g(p,q)p=q$, $g(q,p)g(p,q)=\theta(p)$, and $[p,\theta(p)q]\cap\alpha(g(p,q))=\emptyset$.

Write $a(p,q)=a(g(p,q))$ and $b(p,q)=b(g(p,q))$. Then the points $p$, $a(q,p)$, $b(q,p)$, $q$, $a(p,q)$, $b(p,q)$ occur in this order around $C$. Let $I^+(p,q)=(q,a(p,q))$, $I^-(p,q)=(b(q,p),q)$ and $I(p,q)=(b(q,p),a(p,q))$ be open intervals in $C$. Thus $I(p,q)=I^-(p,q)\cup\{q\}\cup I^+(p,q)$, $I(p,q)\cap\theta(p)^nI(p,q)=\emptyset$ for all $n\neq 0$, and $I(p,q)\cap\theta(p)^nI(q,p)=\emptyset$ for all $n$.

In the quotient surface $\Sigma$, the oriented simple closed geodesics $\lambda(g(p,q))$ and $\lambda(g(q,p))$ cut off a punctured annulus containing the geodesic arc $\lambda(p,q)$, in which the simple geodesic rays $\lambda(p,a(p,q))$ and $\lambda(p,b(q,p))$ emanating from the puncture spiral to $\lambda(g(p,q))$ and $\lambda(g(q,p))$, respectively. Thus, each of $I^{\pm}(p,q)$ projects homeomorphically onto a *gap* in the horocircle $\tau$, in the sense of [12, p.610]. In fact, each of $I^{\pm}(p,q)$ is a maximal connected subset of $C\setminus\{p\}$ consisting of points $x$ such that the geodesic ray $\lambda(p,x)$ is non-simple. Moreover, if $\lambda(p,x)$ is non-simple, then $x$ is contained in $I^{\pm}(p,q)$ for some $q\in\Pi(p)$ (see [12, 14]).

Write $\mathcal{I}(p)=\{I(p,q)\mid q\in\Pi(p)\}$. Then we obtain the following as a consequence of [12, Corollary 5] and [2] (see also [14, Section 5]).

**Theorem 4.1.** *The elements of $\mathcal{I}(p)$ are pairwise disjoint. The complement, $C \setminus \bigcup \mathcal{I}(p)$, is a Cantor set of Hausdorff dimension* 0.

Here, of course, the Hausdorff dimension is taken with respect to the euclidean metric on the horocycle, $\tau$. Up to a scale factor, this is the same as the euclidean metric in the upper-half-space model with $p$ at $\infty$. (Note that we could equally well use the circular metric on the boundary, $C$, induced by the Poincaré model, since all the transition functions are Möbius, and in particular, smooth.)

Write $R(p) = \{p\} \cup \Pi(p) \cup \big(C \setminus \bigcup \mathcal{I}(p)\big) \subseteq C$. This is a closed set, whose complementary components are precisely the intervals $I^{\pm}(p,q)$ for $q \in \Pi(p)$. Thus the set $R(p)$ is characterised by the following property: a point $x \in C \setminus \{p\}$ belongs to $R(p)$ if and only if the geodesic ray $\lambda(p,x)$ in $\Sigma$ is simple.

In the following, we define, for each $x \in C \setminus \{p\}$ with $p \in \Pi$, the derived sequences $(g_i)_i$, $(p_i)_i$ and $(\epsilon_i)_i$ which are finite or infinite sequences in $\Gamma$, the set of parabolic points of $\Gamma$ and $\{+,-\}$, respectively, by applying the loop cutting construction to the geodesic ray $\lambda(p,x)$. To this end, we introduce a few notations.

For $p \in \Pi$, we define maps $\epsilon(p)$, $\mathbf{q}(p)$ and $\mathbf{g}(p)$ from $C \setminus R(p)$ to $\{+,-\}$, $\Pi(p)$ and $\Gamma$, respectively, by the following rule. If $x \in C \setminus R(p)$, then $x \in I^{\epsilon}(p,q)$ for some unique $\epsilon = \pm$ and $q \in \Pi(p)$. Define $\epsilon(p)(x) = \epsilon$, $\mathbf{q}(p)(x) = q$, and $\mathbf{g}(p)(x) = g(p,q)$ or $g(q,p)^{-1}$ according to whether $\epsilon = +$ or $-$. Note that the definition is symmetric under simultaneously reversing the orientation on $C$ and swapping $+$ with $-$.

It should be noted that if $x \in C \setminus R(p)$, then, in the quotient surface $\Sigma$, the geodesic ray $\lambda(\mathbf{q}(p)(x), x) = \lambda(q,x)$ is obtained from the non-simple geodesic ray $\lambda(p,x)$ by cutting a loop, homotopic to $\lambda(\mathbf{g}(p)(x)) = \lambda(g(p,q))$, and straightening the resulting piecewise geodesic (see Figure 2). (In the quotient, we are allowing ourselves to cut out any peripheral loops that occur at the beginning.) In particular, if $x \in \Pi \setminus R(p)$, then both $\lambda(p,x)$ and $\lambda(\mathbf{q}(p)(x), x)$ are proper geodesic paths in $\Sigma$, and their self-intersection numbers satisfy the inequality $\nu(p,x) > \nu(\mathbf{q}(p)(x), x)$.

We now fix, once and for all, some $p \in \Pi$. (Since the construction is equivariant with respect to the action of $\Gamma$, the choice does not ultimately matter. We will always get the same picture on projecting to $\Sigma$.)

By repeatedly applying the maps above, we associate for a given $x \in C$, a sequence $(g_i)_i$ in $\Gamma$, $(p_i)_i$ in $\Pi$, and $(\epsilon_i)_i$ in $\{+,-\}$ as follows.

**Step 0.** Set $p_0 = p$. (Thus, $p_0$ is independent of $x \in C$.)

**Step 1.** If $x \in R(p_0)$, we stop with the 1-element sequence $p_0$, and define $(g_i)_i$ and $(\epsilon_i)_i$ to be the empty sequence. If $x \notin R(p_0)$, set $g_1 = \mathbf{g}(p_0)(x)$, $p_1 = g_1 p_0$, $\epsilon_1 = \epsilon(p_0)(x)$, and continue to the next step. (The sequences $(g_i)_i$ and $(\epsilon_i)_i$ begin with index $i = 1$.)

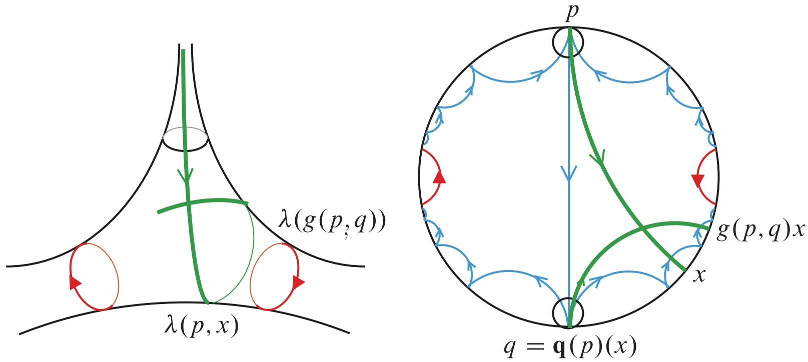


Figure 2. In the figure, we assume $\epsilon(p)(x) = +$ and so $\mathbf{g}(p)(x) = g(p,q)$.

**Step 2.** If $x \in R(p_1)$, we stop with the 1-element sequences $g_1$ and $\epsilon_1$ and 2-element sequence $p_0, p_1$. If $x \notin R(p_1)$, set $g_2 = \mathbf{g}(p_1)(x)$, $p_2 = g_2 p_1$ and $\epsilon_2 = \epsilon(p_1)(x)$.

We continue this process, forever or until we stop.

We call the resulting sequences $(g_i)_i$, $(p_i)_i$ and $(\epsilon_i)_i$ the *derived sequences* for $x$. More specifically, we call $(g_i)_i$ and $(p_i)_i$ the *derived* $\Gamma$*-sequence* and the *derived* $\Pi$*-sequence* for $x$, respectively.

**Lemma 4.2.** *Let $x \in C$, and let $(g_i)_i$, $(p_i)_i$ and $(\epsilon_i)_i$ be the derived sequences for $x$. Then the following hold.*

(1) *The sequences $(p_i)_i$ and $(\epsilon_i)_i$ are determined by the sequence $(g_i)_i$ by the following rule: $p_i = h_i p_0$ where $h_i = g_i g_{i-1} \cdots g_1$, and $\epsilon_i = +$ or $-$ according to whether $g_i = g(p_{i-1}, p_i)$ or $g(p_{i-1}, p_i)^{-1}$.*

(2) *A point $y \in C$ has the derived $\Gamma$-sequence beginning with $g_1, g_2, \dots, g_n$ for some $n \geq 1$, if and only if $y \in \bigcap_{i=1}^{n} I^{\epsilon}(p_{i-1}, p_i)$.*

(3) *Set $R = \bigcup_{p \in \Pi} R(p)$. If $x \notin R$, then the derived $\Gamma$-sequence $(g_i)_i$ is infinite.*

(4) *If $x \in \Pi$, then the derived $\Gamma$-sequence $(g_i)_i$ is finite.*

*Proof.* (1), (2), and (3) follow directly from the definition of the derived sequences. To prove (4), let $x$ be a point in $\Pi$. If $x \in R(p)$, then $(g_i)_i$ is the empty sequence. So we may assume $x \in \Pi \setminus R(p)$. Then by repeatedly using the observation made prior to the construction of the derived sequences, we see that the self-intersection number $\nu(p_i, x)$ of the proper geodesic path $\lambda(p_i, x)$ is strictly decreasing. Hence $\nu(p_n, x) = 0$ for some $n$. This means that $x \in R(p_n)$ and so the derived sequences terminate at $n$. $\square$

The following is an immediate consequence of Lemma 4.2(2).

**Corollary 4.3.** *Suppose that $x \in C$ has derived $\Gamma$-sequence beginning with $g_1, \ldots, g_n$ for some $n \geq 1$. Then there is an open set, $U \subseteq C$, containing $x$, such that if $y \in U$, then $g_1, \ldots, g_n$ is also an initial segment of the derived $\Gamma$-sequence for $y$.*

Recall from Section 3 that $A(\Gamma, \mathcal{P}, p)$ denotes the subgroup of $\mathrm{Aut}(\Gamma)$ preserving $\Pi$ setwise and fixing $p \in \Pi$.

**Lemma 4.4.** *Let $\phi$ be an element of $A = A(\Gamma, \mathcal{P}, p)$ with $p = p_0$. Then the following holds for every point $x \in C$. If $(g_i)_i$, $(p_i)_i$ and $(\epsilon_i)_i$ are the derived sequences for $x$, then the derived sequences for $\phi x$ are $(\phi(g_i))_i$, $(\phi p_i)_i$ and $(\deg(\phi)\epsilon_i)_i$.*

Here $\deg(\phi)$ is the degree of the mapping class of $\Sigma$ corresponding to $\phi$: thus $\deg(\phi)$ is $+1$ or $-1$ according to whether $\phi$ is induced by an orientation-preserving homeomorphism or by an orientation-reversing homeomorphism.

*Proof.* This can be proved through induction, by using the fact that the following hold for each $\phi \in A$.

(1) $\phi(R(p)) = R(p)$.

(2) For any $q \in \Pi(p)$,

  (a) if $\phi$ is orientation-preserving, then $\phi(\theta(p)) = \theta(p)$, $\phi(I^{\epsilon}(p,q)) = I^{\epsilon}(p, \phi(q))$, $\phi(g(p,q)) = g(p, \phi q)$, and $\phi(g(q,p)) = g(\phi q, p)$;

  (b) if $\phi$ is orientation-reversing, then $\phi(\theta(p)) = \theta(p)^{-1}$, $\phi(I^{\epsilon}(p,q)) = I^{-\epsilon}(p, \phi(q))$, $\phi(g(p,q)) = g(\phi q, p)^{-1}$, and $\phi(g(q,p)) = g(p, \phi q)^{-1}$.

□

## 5. Filling arcs

Let $p = p_0$ be our chosen parabolic point. Let $x$ be a point in $C$ and $(p_i)_i$ the (finite or infinite) derived $\Pi$-sequence for $x$. Write $\lambda_i = \lambda(p_{i-1}, p_i)$ for the projection of $[p_{i-1}, p_i] \cap D$ to $\Sigma$. This is a proper geodesic arc in $\Sigma$. We call the sequence $(\lambda_i)_i$ the *derived sequence of arcs* for $x$. We say that $x$ is *filling* if the arcs $(\lambda_i)_i$ eventually fill $\Sigma$, namely, there is some $n$ such that $\Sigma \setminus \bigcup_{i=1}^{n} \lambda_i$ is a union of open discs. Let $F$ be the subset of $C$ consisting of points which are filling. In this section, we prove the following proposition.

**Proposition 5.1.** *The set $F$ is open in $C$, and its complement has Hausdorff dimension strictly less than 1. In particular, $F$ has full measure.*

We begin with some preparation. Let $\gamma$ be a simple closed geodesic in $\Sigma$, and let $X(\gamma)$ be the path-metric completion of the component of $\Sigma \setminus \gamma$ containing the cusp. Then we can identify $X(\gamma)$ as $(H(G) \cap D)/G$, where $G = G(\gamma)$ is a subgroup of $\Gamma$ containing $Z = \mathrm{stab}_\Gamma(p)$, and $H(G) \subseteq D \cup C$ is the convex hull of the limit set $\Lambda G \subseteq C$. In other words, $X(\gamma)$ is the "convex core" of the hyperbolic surface $\mathbb{H}^2/G$, where $G = G(\gamma) \cong \pi_1(X(\gamma))$ and $p \in \Lambda G$. To be more precise, pick a base point $\tilde{*}$ on a small horocircle $\tilde{\tau}$ centred at $p$, and identify $\Gamma$ with $\pi_1(\Sigma, *)$ by using the base point, where $*$ is the image of $\tilde{*}$ in $\Sigma$. Then $G = G(\gamma) = j_*(\pi_1(X(\gamma), *)) < \pi_1(\Sigma, *) = \Gamma$, where $j$ is the inclusion map.

Let $\delta$ be the closure of a component of $\partial H(G) \cap D$. This is a bi-infinite geodesic in $D \cup C$. Let $J \subseteq C$ be the component of $C \setminus \delta$ not containing $p$. Thus, $J$ is an open interval in $C$, which is a component of the discontinuity domain of $G$. Note in particular, that $J \cap Gp = \emptyset$.

**Lemma 5.2.** *Suppose $x \in J \setminus R(p)$, and let $g = \mathbf{g}(p)(x)$, $\epsilon = \epsilon(p)(x)$ and $q = \mathbf{q}(p)(x)$. Then, if $g \in G = G(\gamma)$, we have $J \subseteq I^\epsilon(p, q)$. In particular, $\mathbf{g}(p)(y) = g$ for every $y \in J$.*

*Proof.* To simplify notation we can assume (via the orientation-reversing symmetry of the construction) that $\epsilon = +$. Note that $q \in Gp \subseteq \Lambda G$, so $[p, q] \subseteq H(G)$. Also $\alpha(g(p, q)) \subseteq H(G)$ and $\delta \subseteq \partial H(G)$. It follows that $[p, q]$, $\alpha(g(p, q))$ and $\delta$ are pairwise disjoint. Thus, $J$ lies in a component of $Y := C \setminus \{p, q, a(p, q), b(p, q)\}$. Since $\epsilon = +$, the four points, $p, q, a(p, q), b(p, q)$ are located in $C$ in this cyclic order, and so $I^+(p, q) = (q, a(p, q))$ is a component of $Y$. Since $J$ and $I^+(p, q)$ share the point $x$, we obtain the first assertion that $J \subseteq I^\epsilon(p, q)$ with $\epsilon = +$. The second assertion follows from the first assertion and the definition of $\mathbf{g}(p)(y)$. □

**Lemma 5.3.** *Suppose that $x \in J$ and that the derived $\Gamma$-sequence $(g_i)_i$ for $x$ is infinite. Then there is some $i$ such that $g_i \notin G = G(\gamma)$.*

*Proof.* Suppose, for contradiction, that $g_i \in G$ for all $i$. It follows that $h_i = g_i g_{i-1} \cdots g_1 \in G$ for all $i$, and so $p_i = h_i p \in Gp \subseteq \Lambda G$ for all $i$. By Lemma 5.2, we have $\mathbf{g}(p)(y) = \mathbf{g}(p)(x) = g_1$ for all $y \in J$. (Here $(p_i)_i$ is the derived $\Pi$-sequence for $x$ and $p = p_0$.) Now, applying Lemma 5.2 with $p_1$ in place of $p$, we get that $\mathbf{g}(p_1)(y) = \mathbf{g}(p_1)(x) = g_2$. Continuing inductively we get that $\mathbf{g}(p_i)(y) = g_{i+1}$ for all $i$. In other words, the derived $\Gamma$-sequence for $y$ is identical to that for $x$, and so, in particular, it must be infinite. We now get a contradiction by applying Lemma 4.2(4) to any point $y \in \Pi \cap J$. □

If we take $B$ to be a standard horoball neighbourhood of the cusp, then $B \cap \gamma = \emptyset$ for all simple closed geodesics in $\Sigma$, and so we can identify $B$ with a neighbourhood of the cusp in any $X(\gamma)$.

**Lemma 5.4.** *There is some $\theta < 1$ such that for each simple closed geodesic, $\gamma$, the Hausdorff dimension of $\Lambda G(\gamma)$ is at most $\theta$.*

*Proof.* This is an immediate consequence of [5, Theorem 3.11] (see also [10, Theorem 1]) which refines the result of [15], on observing that the groups $G(\gamma)$ are uniformly "geometrically tight", as defined in that paper. Here, this amounts to saying that there is some fixed $r \geq 0$ (independent of $\gamma$) such that the convex core, $X(\gamma)$, is the union of $B$ and the $r$-neighbourhood of the geodesic boundary of the convex core. From the earlier discussion, we see that $r$ is bounded above by the diameter of $\Sigma \setminus B$, and so in particular, is independent of $\gamma$. □

Let $L \subseteq C$ be the union of the limit sets $\Lambda G$ as $G = G(\gamma)$ ranges over all subgroups of $\Gamma$ obtained from a simple closed geodesic $\gamma$ in $\Sigma$. Applying Lemma 5.4, we see that $L$ is a $\Gamma$-invariant subset of $C$ of Hausdorff dimension strictly less than 1. This is because it is a countable union of the limit sets $\Lambda G$ whose Hausdorff dimensions are uniformly bounded by a constant $\theta < 1$.

Recall the set $R = \bigcup_{p \in \Pi} R(p)$ defined in Lemma 4.2(3). Then $R$ is also $\Gamma$-invariant and has Hausdorff dimension zero by Theorem 4.1.

**Lemma 5.5.** *If $x \in C \setminus (R \cup L)$, then $x$ is filling. Namely, $C \setminus (R \cup L) \subseteq F$.*

*Proof.* Suppose, for contradiction, that some $x \in C \setminus (R \cup L)$ is not filling. Then there must be some simple closed geodesic, $\gamma$, in $\Sigma$, which is disjoint from every $\lambda_i$, where $(\lambda_i)_i$ is the derived sequence of arcs for the point $x$. Consider the hyperbolic surface $X(\gamma)$ and its fundamental group $G = G(\gamma) \subseteq \Gamma$, as described at the beginning of this section. By hypothesis, $x \notin \Lambda G$, and so $x$ lies in some component, $J$, of the discontinuity domain of $G$. By Lemma 5.3, there must be some $i \in \mathbb{N}$ with $g_i \notin G$. Choose the minimal such $i$. Thus, $h_{i-1} \in G$ but $h_i \notin G$, where $h_i = g_i g_{i-1} \cdots g_1$. We have $p_{i-1} = h_{i-1} p \in \Pi \cap \Lambda G$ and $p_i = h_i p \in \Pi \setminus \Lambda G$. (The latter assertion can be seen as follows. If $p_i \in \Lambda G$ then $p_i$ is a parabolic fixed point of $G$. Since $X(\gamma)$ has a single cusp, there is an element $f \in G$ such that $p_i = f p_{i-1}$. Since $p_i = g_i p_{i-1}$, we have $f^{-1} g_i \in \mathrm{stab}_\Gamma(p_{i-1}) = \mathrm{stab}_G(p_{i-1})$. This implies $g_i \in fG = G$, a contradiction.) Therefore $[p_{i-1}, p_i]$ meets $\partial H(G)$, giving the contradiction that $\lambda_i$ crosses $\gamma$ in $\Sigma$. □

*Proof of Proposition 5.1.* By Lemma 5.5, we have $C \setminus F \subseteq R \cup L$. Since $R$ and $L$ both have Hausdorff dimension strictly less than 1, the same is true of $C \setminus F$. Thus, we have only to show that $F$ is open. Pick an element $x \in F$. Then there is some $n$ such that $\Sigma \setminus \bigcup_{i=1}^{n} \lambda_i$ is a union of open discs, where $(\lambda_i)_i$ is a derived sequence of arcs for $x$. By Corollary 4.3, there is an open neighbourhood $U$ of $x$ in $C$ such that every $y \in U$ shares the same initial derived $\Gamma$-sequence $g_1, \ldots, g_n$ with $x$. Thus, every $y \in U$ shares the same beginning derived sequence of arcs $(\lambda_i)_{i=1}^{n}$ with $x$. Hence every $y \in U$ is filling, i.e., $U \subseteq F$. □

## 6. Wandering

In this section, we complete the proof of Theorem 2.2. To this end, we need the following lemma which appears to be well known, though we were unable to find an explicit reference.

**Lemma 6.1.** *Let $\lambda_1, \dots, \lambda_n$ be a set of proper oriented arcs in $\Sigma$ which together fill $\Sigma$. Suppose that $\psi$ is a mapping class on $\Sigma$ fixing the proper homotopy class of each $\lambda_i$. Then $\psi$ is trivial.*

*Proof.* Fix any complete finite-area hyperbolic structure on $\Sigma$, and use it to identify $\widetilde{\Sigma}$ with $\mathbb{H}^2$. Construct a graph, $K$, as follows. The vertex set, $V(K)$, is the set of bi-infinite geodesics which are lifts of the arcs $\lambda_i$ for all $i$. Two arcs $\mu, \mu' \in V(K)$ are deemed adjacent in $K$ if either (1) they cross (that is, meet in $\mathbb{H}^2$), or (2) they have a common ideal point in $\partial\mathbb{H}^2$, and there is no other arc in $V(K)$ which separates $\mu$ and $\mu'$. One readily checks that $K$ is locally finite. Moreover, the statement that the arcs $\lambda_i$ fill $\Sigma$ is equivalent to the statement that $K$ is connected. Note that $\Gamma = \pi_1(\Sigma)$ acts on $K$ with finite quotient. Note also that $K$ can be defined formally in terms of ordered pairs of points in $S^1 \equiv \partial\mathbb{H}^2$ (that is corresponding to the endpoints of the geodesics, and where crossing is interpreted as linking of pairs). The action of $\Gamma$ on $K$ is then induced by the dynamically defined action of $\Gamma$ on $S^1$.

Now suppose that $\psi \in \mathrm{Map}(\Sigma)$. Lifting some representative of $\psi$ and extending to the ideal circle gives us a homomorphism of $S^1$, equivariant via the corresponding automorphism of $\Gamma$. Suppose that $\psi$ preserves each arc $\lambda_i$, as in the hypotheses. Then $\psi$ induces an automorphism, $f\colon K \to K$. Given some $\mu \in V(K)$, by choosing a suitable lift of $\psi$, we can assume that $f(\mu) = \mu$.

We claim that this implies that $f$ is the identity on $K$. To see this, first let $V_0 \subseteq V(K)$ be the set of vertices adjacent to $\mu$. This is permuted by $f$. Consider the order on $V_0$ defined as follows. Let $I_R$ and $I_L$, respectively, be the closed intervals of $S^1$ bounded by $\partial\mu$ which lies to the right and left of $\mu$. Orient each of $I_R$ and $I_L$ so that the initial/terminal points of $\mu$, respectively, are those of the oriented $I_R$ and $I_L$. Each $\nu \in V_0$ determines a unique pair $(x_R(\nu), x_L(\nu)) \in I_R \times I_L$ such that $x_R(\nu)$ and $x_L(\nu)$ are the endpoints of $\nu$. Now we define the order $\leq$ on $V_0$, by declaring that $\nu \leq \nu'$ if either (i) $x_R(\nu) \leq x_R(\nu')$ or (ii) $x_R(\nu) = x_R(\nu')$ and $x_L(\nu) \leq x_L(\nu')$. This order must be respected by $f$, because $f$ preserves the orders on $I_R$ and $I_L$. Since $V_0$ is finite, we see that $f|V_0$ is the identity. The claim now follows by induction, given that $K$ is connected.

It now follows that the lift of $\psi$ is the identity on the set of all endpoints of elements of $V(K)$. Since this set is dense in $S^1$, it follows that it is the identity on $S^1$, and we deduce that $\psi$ is the trivial mapping class as required. $\square$

Recall that $\mathrm{Map}(\Sigma)$ is identified with $M = A/Z$, where $A = A(\Gamma, \mathcal{P}, p)$ and $Z = Z(\Gamma, p)$, respectively, are the stabilisers of $p$ in $\mathrm{Aut}(\Gamma, \mathcal{P})$ and $\Gamma$. As described in Section 3, $A$ acts on $C \setminus \{p\}$, and $\mathrm{Map}(\Sigma) = M$ acts on the circle $T = (C \setminus \{p\})/Z$. The wandering domain $W_M(T)$ is equal to $W_A(C \setminus \{p\})/Z$, because $W_Z(C \setminus \{p\}) = C \setminus \{p\}$ (see Proposition 2.1).

Note that the set $F$ in Proposition 5.1 is actually an open set of $C \setminus \{p\}$. For this set $F$, we prove the following lemma.

**Lemma 6.2.** $F \subseteq W_A(C \setminus \{p\})$.

*Proof.* We want to show that any $x \in F$ has a wandering neighbourhood. By assumption, some initial segment, $\lambda_1, \ldots, \lambda_n$, of the derived sequence of arcs for $x$ fills $\Sigma$. By Corollary 4.3, there is an open neighbourhood, $U$, of $x$, such that for every $y \in U$, the initial segment of length $n$ of the derived sequence of arcs is identical with $\lambda_1, \ldots, \lambda_n$. Suppose that $U \cap \phi U \neq \emptyset$ for some non-trivial element $\phi$ of $\mathrm{Map}(\Sigma) = A/Z$. Pick a point $y \in U \cap \phi U$ and set $x = \phi^{-1} y \in U$. By assumption, the derived sequences of arcs for both $x$ and $y$ begin with $\lambda_1, \ldots, \lambda_n$. On the other hand, Lemma 4.4 implies that the derived sequence of arcs for $y = \phi x$ is equal to the image of that for $x$ by $\phi$. Hence we see that $\phi\lambda_i = \lambda_i$ for all $i = 1, \ldots, n$. It follows by Lemma 6.1 below, that $\phi$ is the trivial element of $\mathrm{Map}(\Sigma)$, a contradiction. □

*Proof of Theorem* 2.2. By Proposition 5.1, $F$ is an open set of $C \setminus \{p\}$ whose complement has Hausdorff dimension strictly less than 1. Since $W_A(C \setminus \{p\})$ contains $F$ by Lemma 6.2, its complement in $C \setminus \{p\}$ also has Hausdorff dimension strictly less than 1. Since $W_M(T) = W_A(C \setminus \{p\})/Z$, this implies that the non-wandering set, $T \setminus W_M(T)$, has Hausdorff dimension strictly less than 1. □

**Remark 6.3.** As noted in the introduction, one could allow $\Sigma$ to have more than one cusp. In this case, we get an action of the pure mapping class group of $\Sigma$, (i.e. that which preserves the cusps) on the horocycle about any cusp. The construction is similar. We take $\mathrm{Aut}(\Gamma, \mathcal{P})$ to preserve each conjugacy class of peripheral subgroup, and $\mathrm{Aut}(\Gamma, \mathcal{P}, p)$ to be the subgroup fixing a parabolic point $p$. We again get an $\mathrm{Aut}(\Gamma, \mathcal{P}, p)$ action on a horocircle about $p$, and this gives the action of the pure mapping class group alluded to. The statement is again, that the complement of the wandering domain has Hausdorff dimension less than 1. In the loop-cutting construction in Section 4, we need to allow $g(p,q)$, for some $(p,q) \in \Delta$, to be parabolic, so that $a(p,q) = b(p,q)$ is a parabolic point (not in $\Pi := \Gamma p$). The arguments of Section 4 then go through with some reinterpretation. Some further modification would be needed to Sections 5 and 6, though we will not elaborate on that here.

## 7. Notation

We summarise notations used in this paper.

- $\Sigma = \mathbb{H}^2/\Gamma$: a complete hyperbolic surface of finite area with precisely one puncture with $\Gamma = \pi_1(\Sigma)$.
- $D = \widetilde{\Sigma}$: the universal cover of $\Sigma$, which is identified with $\mathbb{H}^2$.
- $C$: the ideal boundary of $D$ which is equipped with a preferred orientation.
- $\Pi \subseteq C$: the set of parabolic points of $\Gamma$.
- $\theta(p)$: the generator of $\mathrm{stab}_\Gamma(p)$ which acts on $C \setminus \{p\}$ as a translation in the positive direction.
- $[x, y] \subseteq D \cup C$: the oriented geodesic from $x$ to $y$ for $x, y \in C$.
- $\lambda(x, y) \subseteq \Sigma$: the image of $[x, y] \cap D$ in $\Sigma$.
- $\nu(p, q)$: the self-intersection number of the proper geodesic path $\lambda(p, q)$, for $p, q \in \Pi$.
- $$\begin{aligned} \Delta &= \{(p,q) \in \Pi^2 \mid \nu(p,q) = 0\} \\ &= \{(p,q) \in \Pi^2 \mid \lambda(p,q) \text{ is a proper geodesic arc}\} \end{aligned}$$
- For $(p, q) \in \Delta$,
  - $g(p,q) \in \Gamma$: the element defined in Section 4;
  - $a(p,q)$, $b(p,q)$: the attracting and repelling fixed points of $g(p,q)$ respectively;
  - $\alpha(g(p,q)) = [b(p,q), a(p,q)]$: the axis of $g(p,q)$;
  - $\lambda(g(p,q))$: the oriented closed geodesic in $\Sigma$ corresponding to $g(p,q)$, i.e., the image of $\alpha(g(p,q)) \cap D$ in $\Sigma$;
  - $I(p,q) = I^+(p,q) \cup \{q\} \cup I^-(p,q) \subseteq C$: the open interval defined in Section 4.
- For $p \in \Pi$,
  - $\Pi(p) = \{q \in \Pi \mid (p,q) \in \Delta\}$;
  - $\mathfrak{I}(p) = \{I(p,q) \mid q \in \Pi(p)\}$;
  - $R(p) = \{p\} \cup \Pi(p) \cup \left(C \setminus \bigcup \mathfrak{I}(p)\right) \subseteq C$;
  - $\epsilon(p)$, $\mathbf{q}(p)$, $\mathbf{g}(p)$: the maps from $C \setminus R(p)$ to $\{+, -\}$, $\Pi(p)$ and $\Gamma$, respectively, defined in Section 4.
- $R = \bigcup_{p \in \Pi} R(p) \subseteq C$.
- $F \subseteq C$: the set of filling points.

- For a simple closed geodesic $\gamma$ in $\Sigma$,
  - $X(\gamma)$: the path-metric completion of the component of $\Sigma \setminus \gamma$ containing the cusp;
  - $G(\gamma) = \pi_1(X(\gamma)) < \pi_1(\Sigma) = \Gamma$.
- $L \subseteq C$: the union of the limit sets $\Lambda G$ as $G = G(\gamma)$ ranges over all subgroups of $\Gamma$ obtained from a simple closed geodesic $\gamma$ in $\Sigma$.

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Received September 15, 2016

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