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Infinitely supported Liouville measures of Schreier graphs

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Abstract. We provide equivalent conditions for Liouville property of actions of groups. As an application, we show that there is a Liouville measure for the action of the Thompson group F on dyadic rationals. This result should be compared with a recent result of Kaimanovich, where he shows that the action of the Thompson group F on dyadic rationals is not Liouville for all finitely supported measures. As another application we show that there is a Liouville measure for lamplighter actions. This gives more examples of non-amenable Liouville actions.

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1. Introduction

Let a countable group G act on a set X, denote by $G \curvearrowright X$, and let μ be a probability measure on G. We say that μ is non-degenerate if the semigroup its support generates is all of G. We denote by P_μ the transition matrix on X induced by μ, that is

$$P_\mu(x, y) = \sum_{g \in G} \mathbf{1}_{\{g \cdot x = y\}} \mu(g).$$

For simplicity of notations we write $\mu \cdot x = P_\mu(x, \cdot)$. A function $f: X \to \mathbb{R}$ is P_μ-harmonic if $f(x) = \sum_{y \in X} f(y) P_\mu(x, y)$. The pair (X, P_μ) is *Liouville* if all bounded P_μ-harmonic functions are constant. We say that the action $G \curvearrowright X$ is μ-Liouville if (X, P_μ) is Liouville. In this paper we consider the following property of an action:

Definition 1. We say an action $G \curvearrowright X$ is Liouville if there is a non-degenerate measure μ on G such that (X, P_μ) is Liouville.

Note that any P_μ-harmonic function $f: X \to \mathbb{R}$ can be lifted to be a μ-harmonic function $\tilde{f}: G \to \mathbb{R}$ by $\tilde{f}(g) = f(g \cdot o)$, where o is a reference point on X. The main motivation for our definition of a Liouville action is a recent approach to showing non-amenability suggested by Kaimanovich in [8]. In order to show that a group is not amenable it is sufficient to find an action which does not admit any Liouville measure. Indeed, this will insure that there is no Liouville measure on the group itself, thus, by a renowned result of Kaimanovich and Vershik [9, Theorem 4.3], the group is not amenable. The problem of amenability of Thompson's group F can be approached with this idea. In particular, Kaimanovich [8] showed that every non-degenerate finitely supported measure on Thompson group F is not Liouville for its standard action on dyadic rationals. In this paper we show that this action admits an infinitely supported Liouville measure.

In fact, our method is more general, we give a criterion for a measure μ to be Liouville for an action $G \curvearrowright X$. Further we give a sufficient condition for an action to be Liouville in the sense of Definition 1. As an application we show that certain Schreier graph of a wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr G$ with G non-amenable is non-amenable as a graph but μ-Liouville for some infinitely supported measure μ on the group. We note that examples of non-amenable graphs with Liouville measures were previously discovered in [2].

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2. Liouville measures on Schreier graphs

In [9, Theorem 4.3], Kaimanovich and Vershik proved that a countable group G is amenable if and only if there exists a non-degenerate measure μ on G such that (G, μ) is Liouville. The result was obtained independently by Rosenblatt in [13]. In other words G is amenable if and only if the action of G on its own Cayley graph by left multiplication is Liouville. Given an action $G \curvearrowright X$, following the idea in [9, Theorem 4.3], we can build a measure μ such that P_μ is Liouville if there is a sequence of probability measure on G that have good coupling properties on X. For the following lemma, the original proof in [9] carries over to the setting of Schreier graphs without any change. We include a proof here for the reader's convenience.

Lemma 2. *Suppose there exists an increasing sequence of finite subsets (K_n) exhausting X and a sequence (ϵ_n) decreasing to 0 such that for each n, there exists a probability measure ν_n of finite support on G such that for any $x, y \in K_n$ such that $y = s \cdot x$ for some $s \in S$, we have*

$$\|\nu_n \cdot x - \nu_n \cdot y\|_1 < \epsilon_n.$$

Then there exists a non-degenerate probability measure μ on G such that (X, P_μ) is Liouville.

Proof. Following the proof of Theorem 4.3 in [9], the measure μ is obtained as a convex combination of a subsequence of (ν_n),

$$\mu = \sum_{j=0}^{\infty} c_j \zeta_j, \ \zeta_j = \nu_{n_j}.$$

To make μ non-degenerate, we take ν_0 to be uniform on $S \cup S^{-1}$.

First note that to show P_μ is Liouville, it suffices to show for any $x, y \in X$ connected by an edge, $y = s \cdot x$ for some $s \in S$, we have

$$\liminf_{m\to\infty} \|\mu^{(m)} \cdot x - \mu^{(m)} \cdot y\|_1 = 0,$$

since this implies that for any bounded P_μ-harmonic function h, $h(x) = h(y)$ for neighboring points, thus the function must be constant.

Since each ν_n is assumed to be of finite support on G, let $B(e, r_n)$ be a ball large enough on G such that $\mathrm{supp}(\nu_n) \subset B(e, R_n)$. Let r_n be the largest ball such that $B(o, r_n) \subseteq K_n$, we have $r_n \to \infty$ since (K_n) exhausts X. Fix a sequence of weights (c_j), select (n_j) inductively as follows: let m_j be the smallest integer such that $(c_0 + \ldots + c_{j-1})^{m_j} \le 1/j$, take n_j to be the least integer such that

$$m_j R_{n_{j-1}} \le r_{n_j} \quad \text{and} \quad m_j R_{n_{j-1}} \epsilon_{n_j} \le 1/j.$$

For m-th convolution power of μ,

$$\mu^{(m)} = \sum_k c_{k_1} \ldots c_{k_m} \zeta_{k_m} * \ldots * \zeta_{k_1}.$$

Consider two parts, $\mu_1^{(m)}$ consists these terms with $\max_{1\le i\le m} k_i < j$, and $\mu_2^{(m)} = \mu^{(m)} - \mu_1^{(m)}$. For $m = m_j$, the total mass of the first part is

$$\|\mu_1^{(m_j)}\|_1 = (c_0 + \ldots + c_{j-1})^{m_j}.$$

For each term in the second part, let $i = i(k)$ be the lowest index such that $k_i \ge j$. Starting at two neighboring points x, y, consider the distribution induced by $\zeta_{k_i} * \ldots * \zeta_{k_1}$ on these two points. Since for $\ell < i(k)$, $k_\ell < j$, it follows that the support of $\zeta_{k_{i-1}} * \ldots * \zeta_{k_1} \cdot x$ and $\zeta_{k_{i-1}} * \ldots * \zeta_{k_1} \cdot y$ are contained in the ball $B_X(o, d(o,x) + 1 + (m_j - 1) R_{n_{j-1}-1})$. By the choice of the (ζ_n), we have for $k_i \ge j$,

$$\|\zeta_{k_i} * \zeta_{k_{i-1}} * \ldots * \zeta_{k_1} \cdot x - \zeta_{k_i} * \zeta_{k_{i-1}} * \ldots * \zeta_{k_1} \cdot y\|_1 \le 2(m_j - 1) R_{n_{j-1}-1} \epsilon_{n_j}.$$

Combine the two parts, we have

$$\|\mu^{(m_j)} \cdot x - \mu^{(m_j)} \cdot y\|_1 \le (c_1 + \ldots + c_{j-1})^{m_j} + 2(m-1) R_{\ell_m - 1} \epsilon_{\ell_m} \le 3/j. \qquad \square$$

Remark 3. By the general results on 0-2 laws for Markov chains the condition in Lemma 2 is also necessary for existence of a Liouville measure μ. Suppose μ is non-denegerate and $\mu(e) \geq \frac{1}{2}$. By [7, Theorem 2.3], if the action of G on X is transitive and (X, P_μ) is Liouville, then for any two points $x, y \in X$,

$$\lim_{n\to\infty} \|\mu^{(n)} \cdot x - \mu^{(n)} \cdot y\|_1 = 0.$$

Therefore if we have a measure μ such that P_μ is Liouville, then finite-support approximations to convolution powers of μ provide the sequence that satisfies the assumptions of Lemma 2.

3. Applications to the Thompson group *F*

In this section we show that the action of the Thompson group F on dyadic rationals is Liouville in the sense of Definition 1. Thompson group F is the group of all strictly increasing piecewise linear homeomorphisms of $[0, 1]$ to itself, with finitely many dyadic rational numbers as breakpoints. The group operation is composition of homeomorphisms. Basic properties of F can be found in the survey [3]. In particular, F is finitely presented. We follow notations in [14] and represent a point in the interval $[0, 1]$ by its binary expansion. The action of F extends to X^ω, which consists of all infinite words over $\{0, 1\}$. The standard choice of generators of F are

$$x_0\colon \begin{cases} 0w \longmapsto 00w, \\ 10w \longmapsto w, \\ 11w \longmapsto 1w. \end{cases} \qquad x_1\colon \begin{cases} 0w \longmapsto 0w, \\ 10w \longmapsto 100w, \\ 110w \longmapsto 101w, \\ 111w \longmapsto 11w. \end{cases}$$

Let G a group finitely generated group equipped with a finite generating set S. Suppose G acts on X. Given a point $o \in X$, the (oriented) Schreier graph $\Gamma(G, S, x)$ is defined as the labelled graph: the vertex set is $\mathrm{Orb}(x) = \{g \cdot x \colon g \in G\}$, and there is an edge labelled with $s \in S$ from x to y if and only if $s \cdot x = y$. In [8] and [11], the authors show that the Schreier graph of the action of F (equipped with generating set $S = \{x_0, x_1\}$) on the orbit of $x = 1/2 = 1000\ldots$ is not Liouville with respect to a strictly non-degenerate probability measure with finite support. Here we show that there are non-degenerate probability measures with infinite support on F that make this graph Liouville. In fact, one can choose the measures to be symmetric.

The orbital Schreier graph of $1/2$ under the action of F was described by Savchuk [14], see Figure 1. There are two parts in the graph: the skeleton that corresponds to the binary tree and another is rays attached to every node of the tree. These rays imitate a half ray in Cayley graph of $\mathbb{Z}$, and we will call them hairs. In the oriented Schreier graph there are two types of orientation on hairs, towards the tree or away from the tree.

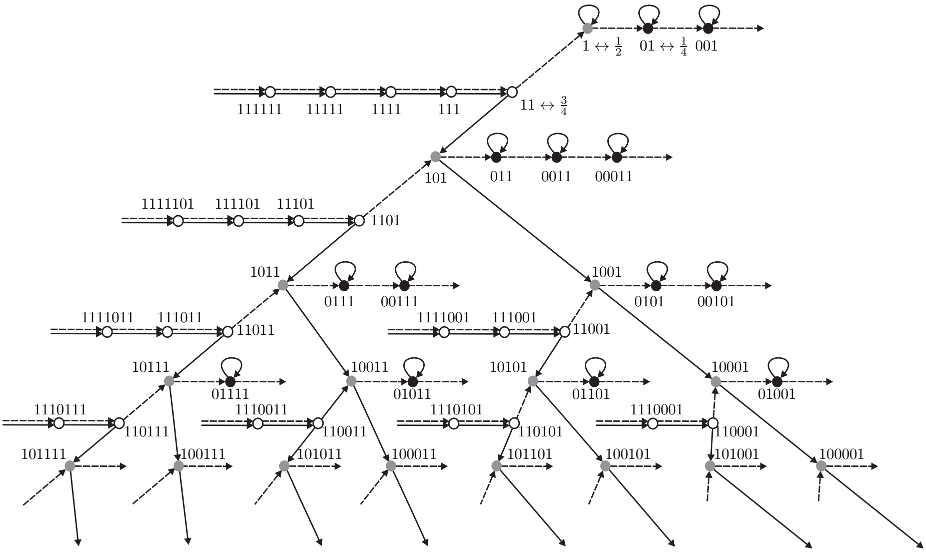

Figure 3.1. Schreier graph of the action of F on the orbit of $1/2 = 1000\ldots$. Figure from Savchuk [14].

Theorem 4. *There is a non-degenerate symmetric probability measure μ on Thompson group F such that the action on* Orb(1/2) *is μ-Liouville.*

Proof. We will show that for every finite subset K of the orbit of $1/2$ and for every $\epsilon > 0$, there is a finite set E in F such that

$$\left| \bigcap_{x \in K} \{gx : g \in E\} \right| \geq (1-\epsilon)|E|. \tag{3.1}$$

By Lemma 2, we have to find a measure that approximates any finite subset in Orb(1/2). The main property that we need is that there is a subset of the orbital Schreier graph of $1/2$ (the hairs as explained as above), which can be enumerated by natural numbers, such that one of the generators g_0 acts on it as $x \mapsto x+1$ and another g_1 fixes every point of it. We will refer to this set as $\mathbb{N}$.

Let $K \subset$ Orb(1/2). Since F is strongly transitive (see [1]), we can find an element g in F that maps this set $\bar{K} = \{1, \ldots, |K|\}$ in $\mathbb{N}$. The action of g_0 on $\mathbb{N}$ is partially defined, but the action of all positive powers of g_0 is defined on $\mathbb{N}$. Moreover, it is straightforward that for every $\epsilon > 0$ there is a sufficiently large n such that

$$\left| \bigcap_{x \in \bar{K}} \{g_0^k x : 1 \leq k \leq n\} \right| \geq (1-\epsilon)n.$$

In particular, inequality 3.1 holds with $E = \{g_0^k g : 1 \leq k \leq n\}$.

In order to obtain a symmetric measure in Lemma 2, we can take

$$E = \{g^{-1} g_0^k g : n \leq k \leq n\},$$

where g maps the set K to $\{n+1, \dots, n+K\}$. □

4. Liouville non-amenable Schreier graphs and lamplighters

As another application of Lemma 2 we show that certain Schreier graphs of lamplighter group are non-amenable but μ-Liouville for some infinitely supported measure μ. We note that examples of non-amenable graphs with Liouville measures were previously discovered in [2]. There are many examples where a non-amenable group G admits an action on a set X such that the induced action of the lamplighter group $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$ on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ is not amenable. In fact, if the action of G on X is not amenable then the action of $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$ on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ is not amenable as well, see for example [5] and [6]. However, this action always admits Liouville measures as a consequence of the following lemma. We denote elements in a semi-direct product $G \ltimes_\varphi A$ by (a, g), where $g \in G$, $a \in A$. Consider the action of $G \ltimes A$ on A given by $(a, g) \cdot a' = a\varphi_g(a')$. In the lamplighter example, $A = \bigoplus_X \mathbb{Z}/2$ is abelian, thus amenable.

Lemma 5. *Consider a semi-direct product $G \ltimes_\varphi A$ with G, A countable groups and A amenable. Then there exists a non-degenerate probability measure μ on $G \ltimes_\varphi A$ such that the action of $G \ltimes_\varphi A$ on A is μ-Liouville.*

Proof. Fix a sequence of finite subsets (K_n) that exhaust A. Since A is amenable, we can find a sequence of measures (ν_n) on A such that

$$\sup_{x,y \in K_n} \|\nu_n \cdot x - \nu_n \cdot y\|_{\ell^1(A)} \leq \frac{1}{n}.$$

Regard ν_n as a measure supported on $\{(a, e_G) : a \in A\}$, then by Lemma 2 we can find a non-degenerate measure μ on $G \ltimes A$ such that P_μ is Liouville.

□

Remark 6. As pointed out to us by Kaimanovich, the proof of the previous Lemma actually shows the following. Let $G \curvearrowright X$ and suppose there is an amenable subgroup A of G that acts transitively on X, then the action of G on X is Liouville in the sense of Definition 1.

Such examples show that the notion of an amenable Schreier graph (sometimes referred to as an amenable action) is quite distinct from the our notion of a Liouville action. We further illustrate this point by considering actions with amenable stabilizers. If G is an amenable group, then every action of G is amenable and admits a Liouville measure (possibly infinitely supported); moreover, the stabilizers of the action are amenable. In the other direction, if a transitive action of G on X is amenable and the stabilizers are amenable, then G is amenable. One can ask whether having a Liouville action with amenable stabilizers can imply amenability.

Question 7. Let G act transitively on a set X and assume that this action is Liouville in the sense of Definition 1. In addition suppose that $Stab_G(x)$ is abelian for some (equivalently for all) x in X. Is G amenable?

Since the circulation of the first draft of this paper, the question above was discussed in the AIM workshop *Amenability of discrete groups*. Here we mention a counter-example to Question 7 due to Narataka Ozawa. The group $G = SL_2(\mathbb{R})$ can be written as $G = SO_2T$, where SO_2 is the rotation group of $\mathbb{R}^2$ (abelian), T is the group of upper triangular matrices. Then the action of G on G/SO_2 is Liouville because the solvable group T acts transitively on it, and the stabilizer is abelian. To have a discrete group one can take $SL_2(R)$ over certain ring R.

We end this note with the following question. For amenable actions, there are examples of subgroups $K < H < G$ such that the action of G on G/K is amenable, but the action of H on H/K is not amenable, see [12]. In other words amenability of an action does not pass to subgroups. In analogy to this property, we can ask whether the Liouville property of an action passes to subgroups.

Question 8. Let $K < H < G$ be subgroups of G. Suppose the action of G on G/K is Liouville in the sense of Definition 1. Does it follow that the action of H on H/K is Liouville?

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