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A characterization of relatively hyperbolic groups via bounded cohomology

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Abstract. It was proved by Mineyev and Yaman that, if (Γ, Γ') is a relatively hyperbolic pair, the comparison map

$$H_b^k(\Gamma, \Gamma'; V) \longrightarrow H^k(\Gamma, \Gamma'; V)$$

is surjective for every $k \geq 2$, and any bounded Γ-module V. By exploiting results of Groves and Manning, we give another proof of this result. Moreover, we prove the opposite implication under weaker hypotheses than the ones required by Mineyev and Yaman.

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Contents

1. Introduction

The relations between bounded cohomology and geometric group theory have been proved to be fruitful on several occasions. For instance, the second bounded cohomology with real coefficients of most hyperbolic groups has uncountable dimension ([11]). This results generalizes an analogous fact for free non-abelian groups (see [9], or [29] for a simpler proof) and was in turn extended by considering groups acting properly discontinuously on Gromov hyperbolic spaces ([12]). The proper discontinuity condition was weakened in order to include other interesting classes of group actions on Gromov hyperbolic spaces where a Brook's type argument could be applied. For example, the WPD (weakly properly discontinuous) property and the the acylindricity condition were introduced by Bestvina and Fujiwara ([3]) and Bowditch ([7]) respectively in order to study actions of mapping class groups on curve complexes. The second bounded cohomology for more complicated coefficients of (most) acylindrically hyperbolic groups was shown to be infinite-dimensional in [19] and [2].

Two other cases somehow opposite to each other are the characterization of amenability in terms of the vanishing of bounded cohomology ([20]) and the characterization of Gromov hyperbolicity of groups in term of the surjectivity of the comparison map in higher degrees ([25]). The last two examples could be exploited to prove that the simplicial volume of connected closed (and aspherical of dimension at least 2) oriented manifolds with amenable (Gromov hyperbolic) fundamental group vanishes (is nonzero).

In the present paper we will consider a generalization of Mineyev's result to the relative setting. The absolute case was considered by Mineyev in [24] and [25]. He proved that, if Γ is hyperbolic, the comparison map $H_b^k(\Gamma, V) \to H^k(\Gamma, V)$ is surjective for every $k \geq 2$ and every bounded Γ-module V. Viceversa, if Γ is finitely presented and the comparison map $H_b^2(\Gamma, V) \to H^2(\Gamma, V)$ is surjective for every bounded Γ-module V, then Γ is hyperbolic (actually, it was proven by Gromov and Rips that hyperbolic groups are finitely presented: see [15, Corollary 2.2.A] or [10, Théorème 2.2]).

In this work we consider a relative version of the results of [24] and [25] which holds for group-pairs, i.e. pairs (Γ, Γ') where Γ is a group and Γ' is a finite family of subgroups of Γ. The following is our main result (see Section 2 for the definitions of the terms involved).

Theorem 1.1. *Let* (Γ, Γ') *be a group-pair.*

(*a*) *If* (Γ, Γ') *is relatively hyperbolic the comparison map*

$$H_b^k(\Gamma, \Gamma'; V) \longrightarrow H^k(\Gamma, \Gamma'; V)$$

is surjective for every bounded Γ*-module* V *and* $k \geq 2$.

(*b*) *Conversely, if* (Γ, Γ') *is a finitely presented group-pair such that* Γ *is finitely generated and the comparison map is surjective in degree* 2 *for any bounded* Γ*-module* V*, then* (Γ, Γ') *is relatively hyperbolic.*

Roughly speaking, the group-pair (Γ, Γ') is finitely presented if there is a presentation for Γ in the alphabet $\bigsqcup_{i\in I} \Gamma_i \sqcup \mathcal{A}$ – where $\mathcal{A} \subset \Gamma$ is a finite set – such that only finitely many relations involve elements of $\mathcal{A}$. See Definition 8.7 for more details.

Mineyev and Yaman proved in [26] a similar theorem. In particular, they proved (a), while the opposite implication was proved only under stronger hypotheses than (b) above (see [26, Theorem 59]).

In an article of Groves and Manning ([17]) written shortly thereafter, several useful results are proved which seem to provide an alternative strategy to prove (a) of Theorem 1.1. Indeed, quoting from [17, p. 4]:

> "In particular, in [16], we define a homological bicombing on the coned-off Cayley graph of a relatively relatively hyperbolic group (using the bicombing from this paper in an essential way) in order to investigate relative bounded cohomology and relatively hyperbolic groups, in analogy with [24] and [25]."

The article [16] was referred to as "in preparation," and has never appeared. It was our aim to provide such a proof. We take a small detour from the strategy outlined in the quotation above, since we will use the *cusped-graph* defined in [17] instead of the coned-off Cayley graph.

In [26] a weaker version of (b) is also considered. However, such implication was proved under additional *finiteness* hypotheses about the action of Γ on a graph or complex, which seem to be far more restricting than the finite presentability in the absolute case. By making use of recent results in a paper of Martinez-Pedroza [22], we will be able to prove this implication with a proof similar to the one in [26], but without mentioning Γ-actions in the statement.

In Section 7 we give two applications. The first one is a straightforward consequence of Theorem 1.1 (a) and was already proved in [26]: if the topological pair (X, Y) is a classifying-pair for (Γ, Γ'), then the Gromov norm on $H_k(X, Y)$ – which in general is merely a semi-norm – is actually a norm, for $k \geq 2$. This implies in particular interesting non-vanishing results for some classes of compact manifolds with boundary. The second application easily follows from our Rips complex construction, and can be obtained in the same way from an analogous construction in [26, Section 2.9]. It states that, for a hyperbolic pair (Γ, Γ'), there is $n \in \mathbb{N}$ such that, for any Γ-module V, the relative (non-bounded) cohomology of (Γ, Γ') with coefficients in V vanishes in dimensions at least n.

The plan of the paper is as follows. In sections 2 and 3 we recall some definitions and results from [26] and [17] (some technicalities pertaining to Section 2 are addressed later in the first addendum). In sections 3, 4, and 5 we introduce a Rips complex construction as our main tool, and prove some filling-inequalities of its simplicial chain complex, which will allow us to prove Theorem 1.1 (a) in Section 6. In the following section we give the applications already mentioned. In Section 8 we recall some results in [22] and prove Theorem 1.1 (b). In the second addendum we show that the definitions of relative bounded cohomology given in [26] and [5] respectively are isometric.

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2. Preliminaries

Several definitions and results in this section are taken from [26].

Given a set S, let $\mathbb{R}S$ be the vector space with basis S. Then S induces a natural ℓ^1-norm $\|\cdot\|$ on $\mathbb{R}S$

$$\Big\| \sum_{s\in S} \lambda_s s \Big\| := \sum_{s\in S} |\lambda_s|$$

(where almost all coefficients λ_s are null). We denote by $C_*(S)$ the complex defined by

$$C_k(S) = \begin{cases} \{0\} & \text{if } k \leq -1, \\ \mathbb{R}S^{k+1} & \text{if } k \geq 0, \end{cases}$$

with boundary operator given by

$$\partial_k(s_0, \dots, s_k) := \sum_{j=0}^{k} (-1)^k (s_0, \dots, \hat{s}_j, \dots, s_k).$$

Notice that ∂_k is a bounded linear operator for every k. If Γ is a group acting on S, then Γ also acts diagonally on $C_k(S)$ via isometries, and ∂_k is Γ-equivariant with respect to this action. The complex $C_*(S)$ admits an exact augmentation given by

$$C_0(S) \longrightarrow \mathbb{R}, \quad \sum_i \lambda_i s_i \longmapsto \sum_i \lambda_i.$$

The following definition of relative bounded cohomology is taken from [26] and is modelled on the analogous one for the non-bounded version in [4]. Our notation is slightly different from that of [26].

Definition 2.1. A Γ-module is a real vector space equipped with a linear Γ-action. A Γ-module P is *projective* if, given Γ-equivariant maps $\varphi\colon V \to W$ and $f\colon P \to W$, with φ surjective, there exists a Γ-equivariant map $\tilde{f}\colon P \to V$ making the following diagram commute

$$\begin{array}{ccc} & & V \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow \varphi \\ P & \xrightarrow{\ f\ } & W. \end{array} \tag{1}$$

Given a module M, a Γ-*resolution* for M is an exact Γ-complex

$$\cdots \longrightarrow E_k \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0.$$

A Γ-*projective resolution of* M is a Γ-resolution where all the E_i are Γ-projective.

The following lemma, (similar to [26, Lemma 52]) will be useful.

Lemma 2.2. *Let P be a Γ-module generated as a vector space by a basis S. Suppose that the action of Γ on P is such that, for every $s \in S$ and $\gamma \in \Gamma$, there is $t \in S$ such that $\gamma s = \pm t$. Moreover, suppose that $|\mathrm{Stab}_\Gamma(s)| < \infty$ for every $s \in S$. Then P is a Γ-projective module.*

Proof. Let $\varphi\colon V \to W$ and $f\colon P \to W$ be Γ-equivariant maps, and suppose that φ is surjective. If $a \in P$, let $\mathrm{Stab}^-(a) := \{\gamma \in \Gamma\colon \gamma a = -a\}$. Notice that $|\mathrm{Stab}^-(s)|$ is null or equals $|\mathrm{Stab}\,(s)|$, hence in particular it is finite, if $s \in S$. Fix $s \in S$ and $b \in V$ such that $f(s) = \varphi(b)$. Put

$$\tilde{f}(\pm\alpha s) := \pm \frac{\sum_{\gamma\in\mathrm{Stab}\,(s)} \gamma\alpha b - \sum_{\gamma\in\mathrm{Stab}^-(s)} \gamma\alpha b}{|\mathrm{Stab}\,(s) \cup \mathrm{Stab}^-(s)|} \quad \text{for all } \alpha \in \Gamma.$$

The definition above gives rise to a well defined $\mathbb{R}$-linear and Γ-equivariant map $\mathbb{R}\Gamma s \to V$. Since P is a direct sum of spaces of type $\mathbb{R}\Gamma$, $s \in S$, we obtain a Γ-equivariant map $\tilde{f}\colon P \to V$. Finally, it is easy to see that $\varphi \circ \tilde{f} = f$. □

In particular, if Γ acts freely on S, then $C_*(S) \to \mathbb{R} \to 0$ is a Γ-projective resolution of the trivial Γ-module $\mathbb{R}$.

We also have a normed version of projectivity.

Definition 2.3. Let Γ be a discrete group. A *bounded* Γ*-module* V is an $\mathbb{R}$-normed space equipped with a (left) Γ-action of equibounded automorphisms, i.e. there exists $L > 0$ such that

$$\|\gamma \cdot v\| \le L\|v\| \quad \text{for all } v \in V,\ \gamma \in \Gamma.$$

A *bounded* Γ*-complex* is a complex of bounded Γ-modules with Γ-equivariant bounded boundary operators.

Definition 2.4. A map $\varphi\colon V \to W$ between normed spaces is *undistorted* if there exists $K > 0$ such that, for every $w \in W$ in the image of φ, there exists $v \in V$ such that

$$\varphi(v) = w, \quad \|v\| \le K\|w\|.$$

Definition 2.5. A Γ-module P is *b-projective* if, given any surjective undistorted bounded Γ-map $\varphi\colon V \to W$ and any bounded Γ-map $f\colon P \to W$, there exists a bounded Γ-map $\tilde{f}\colon P \to V$ making the following diagram commute

$$\begin{array}{ccc} & & V \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \varphi \\ P & \xrightarrow{\ f\ } & W. \end{array} \tag{2}$$

Given a module M, a *bounded* Γ*-resolution* for M is an exact bounded Γ-complex

$$\cdots \longrightarrow E_k \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0.$$

A *b-projective resolution of* M is a bounded Γ-resolution of M where all the E_i are b-projective and all maps are undistorted.

Given (bounded) Γ-modules V and W, we denote by $\mathrm{Hom}_{(b)}(V, W)$ the space of all (bounded) $\mathbb{R}$-linear homomorphisms from V to W, and we denote by $\mathrm{Hom}^{\Gamma}_{(b)}(V, W)$ the subspace of $\mathrm{Hom}_{(b)}(V, W)$ whose elements are Γ-equivariant.

The following lemma is a simple exercise in homological algebra:

Lemma 2.6. *Given two* (b-)*projective* Γ*-resolutions* E_M *and* E'_M *of the same module* M, *there exists a* (*bounded*) *chain* Γ*-map* $\varphi_*\colon E_M \to E'_M$ *which extends the identity on* M. *This map is unique up to* (*bounded*) Γ*-homotopy.*

Dually, if V *is any* (*bounded*) Γ*-module and* $\varphi_1, \varphi_2\colon E_M \to E'_M$ *are as above, there is a* (*bounded*) Γ*-homotopy between* φ_1^* *and* $\varphi_2^*\colon \mathrm{Hom}^*_{(b)}(E'_M, V) \to \mathrm{Hom}^*_{(b)}(E_M, V)$.

Notice that, for every Γ-set S, the space $\mathbb{R}S$ is a bounded Γ-module and $C_*(S)$ is a bounded Γ-complex, whose augmentation is a Γ-projective and Γ-b-projective resolution of $\mathbb{R}$, if Γ acts on S as in Lemma 2.2.

Definition 2.7. Let Γ be a group, and let $\Gamma' := \{\Gamma_i\}_{i\in I}$ be a finite non-empty parametrized family of subgroups (this means that we allow repetitions among the Γ_i). We call such (Γ, Γ') a *group-pair*.

Definition 2.8. Given a group-pair (Γ, Γ'), let $I\Gamma$ be the Γ-set $\bigsqcup_{i\in I} \Gamma \sim \Gamma \times I$ (where Γ acts on $I\Gamma$ by left translation of each copy of Γ). We consider the complex

$$\mathrm{St} = \mathrm{St}_*(I\Gamma) := C_*(\Gamma \times I).$$

Let St' be the Γ-subcomplex of St with basis given by the tuples $(x_0, \dots, x_k) \in (\Gamma \times I)^{k+1}$ for which there exists $i \in I$ such that $x_j \in \Gamma \times \{i\}$ for all $0 \le j \le k$ and $x_j \in x_0\Gamma_i$ for every $1 \le j \le k$. Finally, let $\mathrm{St}^{\mathrm{rel}}_* := \mathrm{St}_* \,/\, \mathrm{St}'_*$ be the quotient Γ-complex. If V is a (bounded) Γ-module, the (*bounded*) *cohomology of the group-pair* (Γ, Γ') *with coefficients in* V is the cohomology of the cocomplex

$$\mathrm{St}^{\mathrm{rel}\,*}_{(b)}(\Gamma, \Gamma'; V) := \mathrm{Hom}^{\Gamma}_{(b)}(\mathrm{St}^{\mathrm{rel}}_*, V),$$

and it is denoted by $H^*_{(b)}(\Gamma, \Gamma'; V)$.

The complex $\mathrm{St}^{\mathrm{rel}}_*(\Gamma, \Gamma')$ is provided with a natural norm, hence we can equip $\mathrm{St}^{\mathrm{rel}\,*}_{(b)}(\Gamma, \Gamma'; V)$ with the corresponding ℓ^∞ norm, which descends to a semi-norm on $H^*_{(b)}(\Gamma, \Gamma'; V)$.

By Lemma 2.2 it is easily seen that $\mathrm{St}^{\mathrm{rel}}_*(\Gamma, \Gamma')$ induces a Γ-projective resolution of the Γ-module $\Delta := \ker(\mathbb{R}(\Gamma/\Gamma') \to \mathbb{R})$. Moreover, Lemma 2.2 could be easily adapted to the normed setting, proving that $\mathrm{St}^{\mathrm{rel}}_k(\Gamma, \Gamma')$ is b-projective for all $k \ge 2$. Mineyev and Yaman also proved that the boundaries of the complex $\mathrm{St}^{\mathrm{rel}}_* \to \Delta \to 0$ are undistorted, hence the resolution St^{rel}_* is b-projective (see [26, Section 8.3]). It follows from Lemma 2.6 that the relative (bounded) cohomology of (Γ, Γ') is computed by any Γ-equivariant (b-)projective resolution of Δ up to canonical (bilipschitz) isomorphism. Even if we don't actually use the fact that $\mathrm{St}^{\mathrm{rel}}_*$ provides a b-Γ-projective resolution of Δ, we will use the following result (proven in [26, Section 10]). For completeness we provide a proof of it in Addendum 8.1.

Proposition 2.9 ([26, the relative cone]). *Fix $y \in I\Gamma$. There is a (non-$\mathbb{R}$-linear) map*

$$[y, \cdot]_{\mathrm{rel}} \colon \mathrm{St}^{\mathrm{rel}}_1 \longrightarrow \mathrm{St}^{\mathrm{rel}}_2$$

*called the **relative cone**, such that $\|[y, b]_{\mathrm{rel}}\| \le 3\|b\|$ for all $b \in \mathrm{St}^{\mathrm{rel}}_1$ and $\partial[y, z] = z$ for any cycle $z \in \mathrm{St}^{\mathrm{rel}}_1$ with respect to the augmentation map $\mathrm{St}^{\mathrm{rel}}_1 \to \Delta$.*

In, we get the following corollary.

Corollary 2.10 ([26, equation (29), p. 38]). *Fix* $y \in I\Gamma$. *Let* $\beta \in \mathrm{St}_2^{\mathrm{rel}}(\Gamma, \Gamma; V)$. *Then* $\beta - [y, \partial\beta]_{\mathrm{rel}} \in \mathrm{St}_2^{\mathrm{rel}}(\Gamma, \Gamma; V)$ *is a cycle, and therefore also a boundary by the exactness of* $\mathrm{St}_*^{\mathrm{rel}}$. *Hence, if* $\alpha \in \mathrm{St}_b^2(\Gamma, \Gamma'; V)$ *is a cocycle, we have*

$$\langle \alpha, \beta \rangle = \langle \alpha, [y, \partial\beta]_{\mathrm{rel}} \rangle \tag{3}$$

Remark 2.11. A more general notion of relative bounded cohomology for pairs of *groupoids* is developed in [5]. By unravelling the definition of relative bounded cohomology given in [5, Definition 3.5.1 and 3.5.12], it is possible to see that those definitions are isometrically isomorphic. We refer the reader to Proposition 8.18 in Addendum 8.2 for a proof of this fact.

3. Hyperbolic group-pairs and cusped-graph construction

Given a graph G, we denote by $d := d_G$ the *graph-metric* on G. This is the path-metric on G induced by giving length 1 to every edge in G. Now, let Y be a simplicial complex, with 1-skeleton $Y^{(1)} = G$. Given a vertex $v_0 \in Y^{(0)}$ and a number $R \geq 0$, we define the ball $B_R(v_0)$ with radius R centered in v_0 as the full subgraph of Y whose vertex set is $\{v \in Y^{(0)} = G^{(0)} \colon d_G(v, v_0) \leq R\}$. Notice that this definition is slightly in contrast with the usual notion of balls in metric spaces, since we do not equip the whole Y with a metric if $\dim Y \geq 2$ and, even if $Y = G$, there could be a point p in the middle of an edge e such that $p \in B_R(v_0)$, but $d_G(p, v_0) > R$. More generally, if $A \subseteq Y^{(0)}$ and $r \in \mathbb{N}$, we denote by $\mathcal{N}_r(A)$ the full subcomplex of Y whose vertex set is $\{v \in Y^{(0)} \colon d_G(v, A) \leq r\}$.

Let $S \not\ni 1$ be a symmetric finite generating set of a group Γ, and consider the associated *simplicial Cayley graph* $G^{\mathrm{simp}}(\Gamma, S)$. This is the simplicial graph (i.e. no double edges allowed) whose vertex set is Γ, and with a single edge connecting γ_1 with γ_2 in Γ if and only if $\gamma_1\gamma_2^{-1} \in S$. In Section 8 we will consider a non-simplicial version of that graph.

There are many equivalent definitions of relative hyperbolicity for a group-pair (Γ, Γ'). We choose the one introduced in [17, p. 21, Definition 3.12; p. 25, Theorem 3.25(5)] which is based on the following *cusped-graph* construction. In particular, we will restrict our attention to the case when Γ is finitely generated and Γ' is a finite family of finitely generated subgroups of Γ. A (*combinatorial*) *horoball* $\mathcal{H} = \mathcal{H}(G)$ on a graph G is the graph whose vertex set is parametrized by $G^{(0)} \times \mathbb{N}$, and with the following edges:

- the full subgraph of $\mathcal{H}$ whose vertex set is $G^{(0)} \times \{0\}$ is a copy of G;
- there is a single edge between (g, n) and $(g, n+1)$, for every $(g, n) \in G^{(0)} \times \mathbb{N}$;
- there is a single edge between (g, n) and (h, n) if and only $d_G(g, h) \leq 2^n$.

Definition 3.1 (cusped-graph). Let $(\Gamma, \Gamma' = \{\Gamma_i\}_{i \in I})$ be a group-pair of finitely generated groups, and consider a symmetric finite generating set $S \not\ni 1$ of Γ such that $S \cap \Gamma_i$ is a finite generating set of Γ_i for every i (i.e. S is *compatible*). For every $i \in I$ and left coset $g\Gamma_i$ of Γ_i in Γ we consider the combinatorial horoball on the subgraph $g\,G^{\mathrm{simp}}(\Gamma_i, S \cap \Gamma_i)$ of $G^{\mathrm{simp}}(\Gamma, S)$. We glue those horoballs to $G^{\mathrm{simp}}(\Gamma, S)$ in the obvious way (see [17, p. 18] for more details). We obtain in this way the *cusped-graph* X.

We denote by the triple $(g, i, n) \in \Gamma \times I \times \mathbb{N}$ a vertex of the cusped-graph. Notice that $(g, i, 0)$ and $(g, j, 0)$ denote the same vertex for all $i, j \in I$. We call the parameter n in (g, i, n) the *height* of the vertex (g, i, n). Given a natural number n and a horoball $\mathcal{H}$, the n-horoball associated with $\mathcal{H}$ is the full subgraph $\mathcal{H}_n$ of X whose vertices are the ones contained in $\mathcal{H}$ with height at least n.

We will need the following result from [17].

Proposition 3.2. [17, Lemma 3.26] *If the cusped-graph X constructed in Definition 3.1 is δ-hyperbolic and $C > \delta$, then the C-horoballs are convex in X.*

Remark 3.3. From now on we fix some constant $C > \delta$, $C \geq 1$.

Remark 3.4. Notice that, by our definition, a cusped-graph is necessarily simplicial. Groves and Manning explicitely allow multiple edges in their definition of cusped-graph. We avoid double edges because we want to consider a cusped-graph as contained in every *Rips complex* over it (see the next section). By Remark 6.4, we can apply all relevant results of [17] also in our setting.

Definition 3.5 ([17, Definition 3.12; Theorem 3.25(5)]). Let (Γ, Γ') be a group-pair of finitely generated groups. The pair (Γ, Γ') is (*relatively*) *hyperbolic* if the cusped-graph of (Γ, Γ') is a Gromov hyperbolic metric space (with the graph metric).

4. Rips complexes on cusped graphs

Definition 4.1. Given a graph G and a parameter $1 \leq \kappa \in \mathbb{N}$, the *Rips complex* $\mathcal{R}_\kappa(G)$ *on* G is the simplicial complex with the same 0-skeleton as G, and an n-dimensional simplex for every set of $n+1$ vertices whose diameter (with respect to the metric of G) is at most κ.

Notice that, since $k \geq 1$, G is naturally a subcomplex of $\mathcal{R}_\kappa(G)$. We need the following fundamental result about Rips complexes over Gromov hyperbolic graphs.

Lemma 4.2. *Let G be a δ-hyperbolic graph. Then $\mathcal{R}_\kappa(G)$ is contractible for every $\kappa \geq 4\delta + 6$.*

By considering the proof of Lemma 4.2 given in [8, Proposition 3.23], it is possible to derive a more precise version of this lemma (see Corollary 4.8).

Notation 4.3. Let G be a graph, and let $\mathcal{R} = \mathcal{R}_\kappa(G)$ be a Rips complex over G. Then G and $\mathcal{R}$ induce two metrics d_G and $d_{\mathcal{R}}$ on $G^{(0)} = \mathcal{R}^{(0)}$. For $R \geq 0$ and a vertex v_0, we denote the full subcomplex of $\mathcal{R}$ whose vertex set is $\{x \in G^{(0)}: d_G(x, v_0) \leq R\}$ by $B^G_R(v_0)$, and refer to it as a G-ball.

Given a Rips complex $\mathcal{R}_\kappa(G)$ over G, we have, for every $l \in \mathbb{N}$ and every vertex v, the equality

$$B^G_{l\kappa}(v) = B_l(v). \tag{4}$$

Definition 4.4. Given a topological space Z and two subspaces W_1 and W_2, we say that there is a homotopy from W_1 to W_2 if the inclusion $W_1 \hookrightarrow Z$ is homotopic to a map $f: W_1 \to Z$ whose image is W_2.

A (geometric) simplex in a simplicial complex Z is determined by the set of its vertices. If $x_0, \dots, x_n$ are non-necessarily distinct vertices in Z, we denote by $[x_0, \dots, x_n]$ the corresponding simplex (if there is one). Notice that the dimension of $[x_0, \dots, x_n]$ could be less than n.

Definition 4.5. If Z is a simplicial complex, W_1 and W_2 are subcomplexes of Z, and $w_1 \in W_1$ and $w_2 \in W_2$ are vertices, we say that W_2 is obtained from W_1 by *pushing* w_1 toward w_2 if the following conditions hold:

(1) for every set of vertices $\{x_0, \dots, x_n\} \in Z^{(0)} \setminus \{w_1\}$, $[x_0, \dots, x_n, w_1]$ is a simplex in W_1 if and only if $[x_0, \dots, x_n, w_2]$ is a simplex in W_2;

(2) in that case, $[x_0, \dots, x_n, w_1, w_2]$ is a simplex in Z.

Notice that it follows that $W_2^{(0)} = (W_1^{(0)} \setminus \{w_1\}) \cup \{w_2\}$. Under the conditions of Definition 4.5, there is an obvious simplicial homotopy from W_1 to W_2.

Lemma 4.6 ([8, Proposition 3.23]). *Let G and $\mathcal{R}_\kappa = \mathcal{R}_\kappa(G)$ be as in Lemma 4.2. Let K be a compact subcomplex of $\mathcal{R}_\kappa$, and let $v_0 \in \mathcal{R}_\kappa^{(0)}$ be a vertex. Then, it is possible to inductively homotope the complex K into a sequence of subcomplexes $K_0 = K, K_1, \dots, K_m = \{v_0\}$ in such a way that*

(1) *there is a sequence of vertices $x_i \in K_i^{(0)}$ such that*

$$d_G(v_0, x_i) = \max\{d_G(v_0, y): y \in K_i^{(0)}\},$$

(2) *K_{i+1} is obtained from K_i by pushing x_i toward some vertex y_i such that $d_G(v_0, y_i) < d_G(v_0, x_i)$.*

Corollary 4.7. *Let G be a δ-hyperbolic locally compact graph and let $\kappa \geq 4\delta + 6$. Then every G-ball $B_R^G(v_0) \subseteq \mathcal{R}_\kappa(G)$ is a contractible topological space.*

Proof. In the notations of Lemma 4.6 simply note that, by point (2), the K_i are contained in $B_R^G(v_0)$. □

Given a Rips complex $\mathcal{R}_\kappa(X)$ over some cusped space X, an (n-)horoball of $\mathcal{R}_\kappa(X)$ is the full subcomplex of $\mathcal{R}_\kappa(X)$ having the same vertices of an (n-)horoball of X. Recall that we have fixed a constant $C > \delta$ (Remark 3.3).

Corollary 4.8. *Let X be the cusped space of a relatively hyperbolic group-pair (Γ, Γ') (with respect to some finite generating set S as described above) and let δ be a hyperbolicity constant of X, which we can assume to be an integer. Then, for $\kappa \geq 4\delta + 6$, the Rips complex $\mathcal{R} = \mathcal{R}_\kappa(X)$ is contractible, with contractible C-horoballs. Moreover, the balls of $\mathcal{R}_\kappa(X)$ are also contractible.*

Proof. The last assertion follows from Corollary 4.3 and equation (4). Now, let K be a compact subcomplex contained in some C-horoball $\mathcal{H}_C$ (recall that $\mathcal{H}_C$ is convex). Let $v_L = (g, i, n)$ be the lowest vertex of K, and let $D := \max\{d_X(v_L, v) : v \in K^{(0)}\}$. A geodesic segment between points w_1 and w_2 in a horoball can be given by two upward segments which start from w_1 and w_2, reach a common height, and are joined by a vertical path of length 1 or 2. From this, it is easy to see that K is contained in the X-ball $B_{D+1}^X(g, i, n + D)$.

Put $r := D + 1$ and $v_0 := (g, i, n + D)$. Then, the X-ball $B_r^X(v_0)$ contains K and is contained in $\mathcal{H}_{C-1}$.

With the notation as in Lemma 4.6, consider the sequence of compact sets $K_1, \ldots, K_m$ which collapses to the point v_0. Those K_i are contained in $B_r^X(v_0) \subseteq \mathcal{H}_{C-1}$. We now prove that the K_i are actually contained in $\mathcal{H}_C$. Indeed, $K_1 \subseteq \mathcal{H}_C$ by hypothesis. Suppose by induction that K_i contains no vertices of height $C - 1$, and suppose that the vertex $w_{i+1} \in K_{i+1} \setminus K_i$ has height $C - 1$. Let $w_i \in K_i$ be a vertex such that $d_X(w_{i+1}, v_0) < d_X(w_i, v_0)$. Then we get a contradiction, because $d_X(w_{i+1}, v_0) \geq \text{height}(v_0) - (C - 1) \geq r$, and $d_X(w_i, v_0) \leq r$ because $K_i \subseteq B_r^X(v_0)$.

Hence K is contractible in $\mathcal{H}_C$. By the arbitrariness of the compact subcomplex K, it follows that all homotopy groups of $\mathcal{H}_C$ are trivial and the conclusion follows from Whitehead's Theorem. □

Notice that, in order to prove that C-horoballs are contractible, we have actually proved the following more precise statement.

Proposition 4.9. *Every compact complex K in some C-horoball $\mathcal{H}_C$ is contained in a contractible space $B_r^X(v_0) \cap \mathcal{H}_C$, for some $r > 0$, whose diameter in $\mathcal{R}_\kappa$ is linearly bounded by the diameter of K.*

5. Filling inequalities on $\mathcal{R}_\kappa(X)$

If Y is a CW-complex, by $C_*(Y)$ we mean the real cellular chain complex of Y, i.e. the complex $H_*(Y^{(*)}, Y^{(*-1)})$ with real coefficients. We denote by $Z_k(Y)$ the subspace of cycles of $C_k(Y)$. There will be no confusion with the notation of Section 2. Notice that, if Y is a simplicial complex, the cellular chain complex $\cdots \to C_2(Y) \to C_1(Y) \to C_0(Y) \to \mathbb{R} \to 0$ is identifiable with the simplicial chain complex of oriented simplices. This is the chain complex whose k-th module is the real vector space generated by tuples $(y_0, \ldots, y_n)$ up to the identification

$$(y_0, \ldots, y_i, \ldots, y_j, \ldots, y_n) = -(y_0, \ldots, y_j, \ldots, y_i, \ldots, y_n)$$

(see [27, Chapter 1, paragraph 5] for more details).

We see a simplicial chain $c \in C_k(Y)$ as a finitely supported map from the set of n-dimensional oriented simplices of Y to $\mathbb{R}$, and we define the support $\operatorname{Supp}(c)$ of c as the set of unoriented n-dimensional simplices Δ of Y such that $c(\sigma) \neq 0$, where σ is one of the two oriented simplices over Δ. By $\operatorname{maxh} c$ ($\operatorname{minh} c$) we mean the height of the highest (lowest) vertex of simplices in $\operatorname{Supp}(c)$. We denote by $\operatorname{Supp}^{(0)}(c)$ the set of vertices that belong to some simplex in $\operatorname{Supp}(c)$. If A is a subset of Y and $c = \sum_i \lambda_i \sigma_i$ is a simplicial k-chain, we define the restriction of c to A as the chain

$$c_{|A} := \sum_{i:\sigma_i^{(0)} \subseteq A} \lambda_i \sigma_i .$$

5.1. A local lemma.

From now on we assume that $\mathcal{R}_\kappa = \mathcal{R}_\kappa(X)$ satisfies the hypotheses of Corollary 4.8. Recall that C is a fixed constant greater than δ.

Lemma 5.1 (Local lemma). *For every $i \geq 0$, there are non-decreasing functions*

$$R\colon \mathbb{N} \longrightarrow \mathbb{N} \quad \textit{and} \quad M_{\mathrm{loc}}\colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$$

such that, for every $D \in \mathbb{N}_{\geq 1}$, $v_0 \in \mathcal{R}_\kappa^{(0)}$ and $z \in Z_i(\mathcal{R}_\kappa)$ such that $\operatorname{Supp} z \subseteq B_D(v_0)$*, there is $a \in C_{i+1}(\mathcal{R}_\kappa)$ such that*

(1) $\partial a = z$;

(2) $\operatorname{Supp} a \subseteq B_{R(D)}(v_0)$;

(3) $\|a\| \leq M_{\mathrm{loc}}(D, \mathrm{maxh}(z))\|z\|$;

(4) *if z is contained in some C-horoball, then a is contained in the same C-horoball* (C *as in Remark* 3.3).

Proof. Fix integers h, D and $j \in I$. Let $c_1, \dots, c_n$ be the collection of the i-dimensional simplices contained in $B_D((1, j, h))$. Let $z_1, \dots, z_m$ be a basis of the subspace of cycles in $\langle c_1, \dots, c_n \rangle_{\mathbb{R}}$, which extends bases of the spaces of cycles contained in the C-horoballs. We choose $a_1, \dots, a_m$ so that $\partial a_1 = z_1, \dots, \partial a_m = z_m$. If z_k is not contained in any C-horoball, the chain a_k may be chosen in $B_D((1, j, h))$, since this is contractible by Corollary 4.8. Otherwise, if z_k is contained in some C-horoball, we take a_k in the subcomplex $B_r^X(v_0)$ contained in that horoball, as described in Proposition 4.9.

We extend the map $z_k \mapsto a_k$ by linearity, obtaining a linear map $\theta^{h,j,D}$ between normed spaces, where the first one is finite dimensional. Therefore $\theta^{h,j,D}$ is bounded (with respect to any norm on the finite dimensional vector space).

Let now z be a cycle in $C_i(\mathcal{R}_\kappa)$ with $\mathrm{diam}(\mathrm{Supp}\, z) \leq D$, and $\mathrm{maxh}(z) \leq H$. Up to Γ-action, we may suppose that z contains a vertex of the form $(1, j, h)$ for some $h \leq H$, and $j \in I$. It follows that $\mathrm{Supp}\, z \subseteq B_D((1, j, h))$. Then we set $a := \theta^{h,j,D}(z)$. Since (h, j) is an element of the finite set $\{1, 2, \dots, H\} \times I$, we may bound the norm of a uniformly, and put $M_{\mathrm{loc}}(D, H) := \max\{\|\theta^{h,j,D}\| : h \leq H, j \in I\}$. □

5.2. Finite sets of geodesic segments in hyperbolic spaces and filling inequalities. The results we are going to present are inspired by the well-known fact that geodesics in hyperbolic spaces can be approximated by embedded trees (see [13, Chapter 2]). The idea is that a set of n geodesic segments *resembles* a simplicial tree where all pairs of edges having a point in common diverge very rapidly from that point. In other words, the vertices of the tree are the only points near which two edges may be close to each other. Moreover, this tree is finite, and the number of vertices and edges depends only on n. Hence, we can split this tree into a set of balls of fixed diameter and a set of subedges that are very far from each other.

Let $k \geq 1$ and let z be a k-dimensional cycle. If $\mathrm{Supp}\, z$ is contained in the L-neighborhood of a set of n geodesic segments, we will be able to express it as a sum of *edge-cycles* and *vertex-cycles*, that we can fill using the Local Lemma 5.1 and Corollary 5.3 respectively. Therefore we will be able to fill z with some control on its norm, as described in Theorem 5.6. Some of the methods of this section are inspired by the proof of [23, Lemma 5.9].

Let $[0, |\gamma|] \ni t \mapsto \gamma(t)$ be an arc-length parametrization of a geodesic segment γ (where $|\gamma|$ is the length of γ) in some metric space W. Let $x = \gamma(t)$, for some $t \in [0, |\gamma|]$, and let $s \in \mathbb{R}$. By "$\gamma(x + s)$" we mean the point $\gamma(t + s)$, if this is defined. Otherwise, if $t + s > |\gamma|$ ($t + s < 0$) we set $\gamma(x + s) := \gamma(|\gamma|)$ ($\gamma(x + s) := \gamma(0)$). If $t < r$ and $y = \gamma(r)$, by $\gamma_{|[x,y]}$ we mean the restriction of γ to the interval $[t, r]$ (or its image in W).

Lemma 5.2. *Let $i \geq 1$. Then there are functions $R\colon \mathbb{N} \to \mathbb{N}$, $D\colon \mathbb{N} \to \mathbb{N}$ and $L\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ which satisfy the following properties: let $z \in Z_i(\mathcal{R}_\kappa)$ be such that* $\operatorname{Supp} z \subseteq \mathcal{N}_S(\gamma)$, *for some geodesic segment γ and $S \in \mathbb{N}$. Then, for $R = R(S)$ and $D = D(S)$ there is an expression*

$$z = \sum_k z_k$$

where the z_k are cycles such that

(1) $\operatorname{Supp} z_k \subseteq B_R(x_k)$, *where* $x_k := \gamma(kD + D/2)$;

(2) $\sum_k \|z_k\| \leq L(S, \operatorname{maxh}(z))\|z\|$;

(3) *if* $\operatorname{Supp} z \subseteq \mathcal{H}_C$ *for a C-horoball $\mathcal{H}_C$, then the same is true for every z_k.*

Proof. Take $D \geq 2S + 3$. Let $y_k := \gamma(kD)$. We put

$$\bar{z}_k := z_{|B_{(k+1)D}(y_0)} - z_{|B_{kD}(y_0)}.$$

In other words, $\bar{z}_k$ is the restriction of z to the set of simplices contained in $B_{(k+1)D}(y_0)$ that are not contained in $B_{kD}(y_0)$. It follows immediately that $z = \sum_k \bar{z}_k$. Let us put

$$R := D/2 + 2S, \quad r := S + 1.$$

Notice that $D > 2S + 2 = 2r$. We have

$$\operatorname{Supp} \bar{z}_k \subseteq \mathcal{N}_S(\gamma) \cap (B_{(k+1)D}(y_0) \setminus B_{kD-1}(y_0)) \subseteq B_{D/2+2S}(x_k) = B_R(x_k). \quad (5)$$

In fact, let v be a vertex in $\mathcal{N}_S(\gamma) \cap (B_{(k+1)D}(y_0) \setminus B_{kD-1}(y_0))$. Let $x \in \gamma$ be such that $d(v,x) \leq S$. Notice that $x \in B_{(k+1)D+S}(y_0) \setminus B_{kD-1-S}(y_0)$, i.e. $kD - S \leq d(y_0, x) \leq (k+1)D + S$. Hence $d(x, x_k) \leq D/2 + S$, and $d(v, x_k) \leq d(v,x) + d(x,x_k) \leq D/2 + 2S$, whence the second inclusion in (5) follows. It follows from (5) that

$$\|\bar{z}_k\| \leq \|z_{|B_R(x_k)}\|. \quad (6)$$

Now, from the first inclusion of (5) we get

$$\begin{aligned}
&\operatorname{Supp}^{(0)}(\partial \bar{z}_k)\\
&\quad \subseteq \mathcal{N}_S(\gamma) \cap (\{x \in \mathcal{R}_\kappa^{(0)}(X)\colon kD \leq d(y_0,x) \leq kD + 1\}\\
&\qquad\qquad \sqcup \{x \in \mathcal{R}_\kappa^{(0)}(X)\colon (k+1)D - 1 \leq d(y_0,x) \leq (k+1)D\})\\
&\quad \subseteq B_{S+1}(\gamma(kD)) \sqcup B_{S+1}(\gamma((k+1)D))\\
&\quad = B_r(y_k) \sqcup B_r(y_{k+1}).
\end{aligned}$$

Therefore, since the last two subcomplexes are disjoint, we can put

$$\partial \bar{z}_k = b'_k + b_k \quad \operatorname{Supp} b'_k \subseteq B_r(y_k), \quad \operatorname{Supp} b_k \subseteq B_r(y_{k+1}).$$

Notice that $\|b'_k\| + \|b_k\| = \|b'_k + b_k\| \leq (i+1)\|\bar{z}_k\| \leq (i+1)\|z_{|B_R(x_k)}\|$. We have

$$0 = \partial z = \sum_k \partial \bar{z}_k = \sum_k b_k + b'_k = \sum_k b_k + b'_{k+1}.$$

By looking at supports, we note that it follows that $b_k = -b'_{k+1}$. Since $b'_k + b_k$ is a cycle (in the augmented simplicial chain complex of $\mathcal{R}_\kappa$) and b_k and b'_k have disjoint supports, it follows that b_k and b'_k are cycles too, *if* their dimension is at least 1. The same is true if the $b_k^{(\prime)}$ are 0-dimensional. Indeed, it is easy to see that $b'_0 = 0$, hence $b_0 = b'_1$ is a cycle. By induction, if b'_k is a cycle, it follows that $b_k = b'_{k+1}$ is a cycle too. Hence all the $b_k^{(\prime)}$ are cycles.

We fill b_k and b'_k by a'_k and a_k using the local lemma, and we also require that $a'_k = -a_{k-1}$. Since b_k and b'_k have diameter bounded by $2r$, by the local lemma we have a function $L(S,\cdot) := M_{\mathrm{loc}}(2r,\cdot) = M_{\mathrm{loc}}(2(S+1),\cdot)$ such that $\|a_k\| \leq L(S, \operatorname{maxh}(b_k))\|b_k\|$. If $H = \operatorname{maxh}(z)$, then

$$\|a_k\| \leq L(S, \operatorname{maxh}(b_k))\|b_k\| \leq L(S,H)(i+1)\|z_{|B_R(x_k)}\|. \tag{7}$$

Hence also

$$\|a'_k\| = \|a_{k-1}\| \leq L(S,H)(i+1)\|z_{|B_R(x_{k-1})}\|. \tag{8}$$

We put

$$z_k := \bar{z}_k - a'_k - a_k.$$

By (6), (7) and (8), there is a function $L' \colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that

$$\|z_k\| \leq L'(S,H)\|z_{|B_{R+D}(x_k)}\|.$$

We have $\partial z_k = b_k + b'_k - b'_k - b_k = 0$, and

$$\sum_k z_k = \sum_k \bar{z}_k - a'_k - a_k = \sum_k \bar{z}_k - \sum_k \bar{a}'_k + a_k = z - \sum_k \bar{a}'_k + a_{k-1} = z.$$

Finally, since the balls $B_{R+D}(x_k)$ and $B_{R+D}(x_{k+5})$ have disjoint supports (because $4S < 2D \implies 2R = D + 4S \leq 3D \implies 2(R+D) \leq 5D$), we have

$$\sum_k \|z_{|B_{R+D}(x_k)}\| = \sum_{j=0}^{4} \Big\| \sum_{k=j \mod 5} z_{|B_{R+D}(x_k)} \Big\| \leq 5\|z\|.$$

Therefore, Condition (2) in the statement holds with $L(S,H) = 5L'(S,H)$. Finally, (3) follows from the local lemma. $\square$

Corollary 5.3. *For every $i \in \mathbb{N}$ there are functions*

$$S' \colon \mathbb{N} \longrightarrow \mathbb{N} \quad \textit{and} \quad M_{\text{thin}} \colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$$

such that, for every geodesic segment γ and every cycle $z \in Z_i(\mathcal{R}_\kappa)$ with $\operatorname{Supp} z \subseteq \mathcal{N}_S(\gamma)$ for some $S \geq 0$, there is a filling a of z with $\operatorname{Supp} a \subseteq \mathcal{N}_{S'}(\gamma)$ and such that

$$\|a\| \leq M_{\text{thin}}(S, \operatorname{maxh}(z))\|z\|.$$

Moreover, we may impose that a is contained in a C-horoball $\mathcal{H}_C$, if the same is true for z (C is as in Remark 3.3).

Proof. Split z as the sum of the cycles z_k be as in the previous lemma. Now, let $R = R(S)$ as in the previous lemma. If $\operatorname{Supp} z \cap B_R(x_k) \neq \emptyset$, we have $\operatorname{maxh}(z_k) \leq \operatorname{maxh}(B_R(x_k))$; otherwise it is clear from the construction of z_k that $z_k = 0$. In any case we have $\operatorname{maxh}(z_k) \leq \operatorname{maxh}(\operatorname{Supp}(z)) + 2R$. Moreover, by (1) of the previous lemma, $\max \operatorname{diam}(z_k) \leq 2R$. Fill z_k with a_k as in the local lemma, and put $a = \sum_k a_k$. Let $M_{\text{loc}} \colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be as in the local lemma. Hence

$$\begin{aligned}
\|a\| &\leq \sum_k \|a_k\| \\
&\leq M_{\text{loc}}(\operatorname{maxdiam}(z_k), \operatorname{maxh}(z) + 2R) \sum_k \|z_k\| \\
&\leq M_{\text{loc}}(2R, \operatorname{maxh}(z) + 2R)\ L(S, \operatorname{maxh}(z))\|z\|.
\end{aligned}$$

So we can put $M_{\text{thin}}(S, h) := M_{\text{loc}}(2R, h + 2R)L(S, h)$. □

The next lemma holds for every δ-hyperbolic space X.

Lemma 5.4. *Let $\alpha_1, \ldots, \alpha_n$ be n geodesic segments. Then, for every $S \in \mathbb{N}$, there exist constants $R = R(S, n)$, $p = p(n)$, $q = q(n)$, points $x_1, \ldots, x_p \in X$ and geodesic segments $\gamma_1, \ldots, \gamma_q$ such that*

$$\bigcup_{k=1}^{n} \alpha_k \subseteq \bigcup_{i=1}^{p} B_R(x_i) \cup \bigsqcup_{j=1}^{q} \mathcal{N}_{2\delta}(\gamma_j)$$

where the γ_j are S-far from each other.

Proof. We prove the statement by induction on n. The case $n = 1$ is obvious. Suppose that the statement is proved for $n - 1$ segments, and put

$$p = p(n-1), \quad q = q(n-1).$$

Hence we have balls $B_R(x_1), \ldots, B_R(x_p)$ and geodesic segments $\gamma_1, \ldots, \gamma_q$ associated with $\alpha_1, \ldots, \alpha_{n-1}$ as in the statement. We fix an orientation on α_n and for every $1 \leq j \leq q$ such that $d(\gamma_j, \alpha_n) \leq S$ (here d denotes the distance between sets) we denote by x_j (resp. y_j) the first (resp. the last) point on α_n such that $d(x_j, \gamma_j) \leq S$, $d(y_j, \gamma_j) \leq S$. By hyperbolicity, it is easy to see that

$$\alpha_{n|[x_j+S+\delta, y_j-S-\delta]} \subseteq \mathcal{N}_{2\delta}(\gamma_j)$$

(some of these intervals may be empty). Since the γ_j are $(2S + 6\delta + 1)$-far from each other, we claim that, up to reindexing, we have

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_k \leq y_k$$

for $k \leq q$. Indeed, $x_j \leq y_j$ by definition. Moreover, the points between x_j and y_j are $(S + 3\delta)$-close to γ_j. Since there cannot be points in X that are $(S + 3\delta)$-close to two different γ_j, we have $[x_j, y_j] \cap [x_k, y_k] = \emptyset$, for $j \neq k$, whence the claim.

The segments of type $\alpha_{n|[y_j+S+\delta, x_{j+1}-S-\delta]}$ (where by y_0 and x_{k+1} we mean the left and right extreme of α_n respectively) are S-far from all the γ_j and $(2S+2\delta)$-far from each other. Adding to the γ_j the segments of type $\alpha_{n|[y_j+S+\delta,x_{j+1}-S-\delta]}$ and to the $B_R(x_i)$ the balls of type $B_{S+\delta}(x_j)$, $B_{S+\delta}(y_j)$ we complete the inductive step. □

We need in Theorem 5.6 a stronger version of the lemma above in order to deal with 1-dimensional cycles. In the notation of Lemma 5.4, we say that two distinct balls B_1 and B_2 are *linked* if there is a γ_j such that $d(\gamma_j, B_1) \leq S$, $d(\gamma_j, B_2) \leq S$ and, if $v_1, v_2 \in \gamma_j$ are such that $d(v_1, B_1) \leq S$, $d(v_2, B_2) \leq S$, there is no point $v_3 \in \gamma_j$ between v_1 and v_2 such that $d(v_3, B_3) \leq S$, for some ball B_3 distinct from B_1 and B_2. We thus get a graph structure on the balls of Lemma 5.4.

For any $r \in \mathbb{N}$, we call *r-cycle* a sequence $\{B_u\}_{u\in\mathbb{Z}/r\mathbb{Z}}$ of r distinct balls, with B_u linked to B_{u+1} for all $u \in \mathbb{Z}/r\mathbb{Z}$. In the following lemma we prove that, if the balls are sufficiently far apart, there are no r-cycles for $r \geq 3$. Hence the graph is a forest, i.e. a graph which is a disjoint union of trees. Notice that, in the conditions of Lemma 5.5, for any pair of balls B_1 and B_2, there is at most one γ_j such that $d(\gamma_j, B_1) \leq S$ and $d(\gamma_j, B_2) \leq S$.

Lemma 5.5. *Suppose that we have an inclusion*

$$\bigcup_{k=1}^{n} \alpha_k \subseteq \bigcup_{u=1}^{p} B_R(x_u) \cup \bigsqcup_{j=1}^{q} \mathcal{N}_{2\delta}(\gamma_j)$$

where the α_k and γ_j are geodesic segments, and the γ_j are S-far apart, for some $S > (p + 6)\delta$.

Moreover, suppose that the x_u are $2p(R + S)$-far apart. Then there are no r-cycles, for any $r \geq 3$.

Proof. Up to reindexing, we may suppose that the balls $B_R(x_1)$, .., $B_R(x_r)$ constitute an r-cycle. Put $B_u := B_R(x_u)$. We slightly abuse notation by identifying the natural numbers 1, .., r with the corresponding elements of $\mathbb{Z}/r\mathbb{Z}$. Let l_u be the minimal subsegment of some γ_j such that the ends of l_u are S-close to B_u and B_{u+1} respectively.

In the following, we denote by $[x_u, x_{u+1}]'$ the subsegment of $[x_u, x_{u+1}]$ which is outside the balls $B_{R+S}(x_u)$ and $B_{R+S}(x_{u+1})$. By δ-hyperbolicity, the Hausdorff distance between l_u and $[x_u, x_{u+1}]'$ is at most 4δ. For a geodesic r-agon in a δ-hyperbolic space, any edge is contained in the $(r-2)\delta$-neighborhood of the union of the other edges. Hence $[x_u, x_{u+1}] \subset \bigcup_{k\neq u} \mathcal{N}_{(r-2)\delta}([x_k, x_{k+1}])$, therefore

$$[x_u, x_{u+1}]' \subset \bigcup_{k\neq u} \mathcal{N}_{(r-2)\delta}([x_k, x_{k+1}]') \cup \bigcup_{k\neq u} B_{R+S+(r-2)\delta}(x_k)$$

Since the length of $[x_u, x_{u+1}]'$ is at least

$$2p(R+S) - 2(R+S) > (r-1)(R+S+(r-2)\delta),$$

it follows that the $r-1$ balls $B_{R+S+(r-2)\delta}(x_k)$ can't cover all of $[x_u, x_{u+1}]'$. Hence there is some $k \neq u$ such that $d([x_u, x_{u+1}]', [x_k, x_{k+1}]') \leq (r-2)\delta$. Since the Hausdorff distance between l_u and $[x_u, x_{u+1}]'$ (l_k and $[x_k, x_{k+1}]'$) is at most 4δ, it follows that the distance between l_u and l_k is less than $(r+6)\delta \leq (p+6)\delta < S$, whence the contradiction. □

We now consider the problem of filling cycles whose supports are close to geodesic segments.

Theorem 5.6. *Let n, i, $L \in \mathbb{N}$, $i \geq 1$, and let $C \in \mathbb{N}$ be as in Remark 3.3. Then there exists $L' = L'(n, i, L) \in \mathbb{N}$ such that, for every cycle $z \in Z_i(\mathcal{R}_\kappa)$ and every family of geodesic segments $\alpha_1, \ldots, \alpha_n$ such that* $\operatorname{Supp} z \subseteq \mathcal{N}_L(\alpha_1 \cup \ldots \cup \alpha_n)$, *there exists $a \in C_{i+1}(\mathcal{R}_\kappa)$ with $\partial a = z$ and*

$$\operatorname{Supp} a \subset \mathcal{N}_{L'}(\operatorname{Supp} z). \tag{9}$$

In particular, up to increasing L', we have $\operatorname{Supp} a \subseteq \mathcal{N}_{L'}(\alpha_1 \cup \ldots \cup \alpha_n)$, *and* $\operatorname{maxh}(a) \leq \operatorname{maxh}(z) + L'$. *Moreover, there exists a function $M = M(n, i, L)\colon \mathbb{N} \to \mathbb{N}$ such that*

$$\|a\| \leq M(\operatorname{maxh}(z))\|z\|. \tag{10}$$

Finally, we can require that, if z is contained in some C-horoball, a is contained in the same C-horoball.

Proof. Fix $S \in \mathbb{N}$ such that $S \geq 2L + 4\delta + 1$, $S > (p+6)\delta$. Let $R = R(S,n)$, $p = p(n)$ and $q = q(n)$ be as in Lemma 5.4, in such a way that for some vertices x_u and geodesic segments γ_j which are S-far from each other

$$\bigcup_k \alpha_k \subseteq \bigcup_{u=1}^{p} B_R(x_u) \cup \bigsqcup_{j=1}^{q} \mathcal{N}_{2\delta}(\gamma_j). \tag{11}$$

Let z be a cycle whose support is contained in the L-neighborhood of the α_u. Hence

$$\operatorname{Supp} z \subseteq \bigcup_{u=1}^{p} B_{R+L}(x_u) \cup \bigsqcup_{j=1}^{q} \mathcal{N}_{2\delta+L}(\gamma_j). \tag{12}$$

The fact that the $\mathcal{N}_{2\delta+L}(\gamma_j)$ are pairwise disjoint is a consequence of our requirements on S.

By suitably choosing a subset I of $\{1,\dots,p\}$, we get that there exists $R+S+1 \leq R' \leq (2p+1)^p(R+S+1)$ such that the x_u, $u \in I$, are $2p(R'+S)$ far apart, and $\bigcup_{u=1}^{p} B_{R+S+1}(x_u) \subseteq \bigsqcup_{u\in I} B_{R'}(x_u)$. Indeed, the case $p=1$ is trivial. Otherwise, if two balls $B_{R+S+1}(x_{u_1})$ and $B_{R+S+1}(x_{u_2})$ are not $2p(R+S)$ far apart, we consider the balls $B_{(2p+1)(R+S+1)}(x_u)$, for all $u \neq u_2$. We have that $B_{R+S+1}(x_{u_1}) \cup B_{R+S+1}(x_{u_2}) \subseteq B_{2p(R+S+1)}(x_{u_1})$. Then we continue by reverse induction on p.

We have

$$\operatorname{Supp} z \subseteq \bigcup_{u=1}^{p} B_{R+L+1}(x_u) \cup \bigsqcup_{j=1}^{q} \mathcal{N}_{2\delta+L}(\gamma_j) \subseteq \bigsqcup_{u\in I} B_{R'}(x_u) \cup \bigsqcup_{j=1}^{q} \mathcal{N}_{2\delta+L}(\gamma_j). \tag{13}$$

Put

$$z' := z_{|\bigsqcup_{j=1}^{N} \mathcal{N}_{2\delta+L}(\gamma_j)}.$$

There is a unique expression

$$z' = \sum_{j=1}^{q} z_{\gamma_j} \quad \text{where } \operatorname{Supp} z_{\gamma_j} \subseteq \mathcal{N}_{2\delta+L}(\gamma_j).$$

By (13) and the definition of z' and $\bigcup_{u=1}^{p} B_{R+L+1}(x_u) \subseteq \bigsqcup_{u\in I} B_{R'}(x_u)$ we get

$$\operatorname{Supp} \partial z_{\gamma_j} \subseteq \bigsqcup_{u\in I} B_{R'}(x_u).$$

Hence we can put

$$\partial z_{\gamma_j} = \sum_{u\in I} b_j^u \quad \operatorname{Supp} b_j^u \subseteq B_{R'}(x_u) \tag{14}$$

(this expression being unique).

Suppose that the dimension i is at least 2. Then, by (14) and the disjointness of the $B_{R'}(x_u)$, $u \in I$, the b_j^u must all be cycles.

The same is true if $i = 1$. Fix j and u such that $d(B_{R'}(x_u), \gamma_j) \leq S$. Consider the set T_u^j of ball-indices $\hat{u}$ for which there is a sequence

$$B_{R'}(x_u) = B_{R'}(x_{u_1}), \quad B_{R'}(x_{u_2}), \quad \ldots, \quad B_{R'}(x_{u_s}) = B_{R'}(x_{\hat{u}})$$

such that $B_{R'}(x_{u_i})$ is linked to $B_{R'}(x_{u_{i+1}})$, for $1 \leq i \leq s-1$, and every such link is realized by some $\gamma_{\hat{j}} \neq \gamma_j$. Moreover, consider the set $T_u^{j\,\prime}$ of indices $\hat{j} \neq j$ for which $d(\gamma_{\hat{j}}, B_{R'}(x_{\hat{u}})) \leq S$ for some $\hat{u} \in T_u^j$.

If $u \neq u'$ are such that $d(B_{R'}(x_u), \gamma_j) \leq S$ and $d(B_{R'}(x_{u'}), \gamma_j) \leq S$, then $T_u^j \cap T_{u'}^j = \emptyset$ and $T_u^{j\,\prime} \cap T_{u'}^{j\,\prime} = \emptyset$. Otherwise there would be r-cycles, for some $r \geq 3$.

Consider the chain $t_u^j := \sum_{\hat{u} \in T_u^j} (z - z')_{|B_{R'}(x_{\hat{u}})} + \sum_{\hat{j} \in T_u^{j\,\prime}} z_{\gamma_{\hat{j}}}$. By the disjointness above, $\mathrm{Supp}(t_u^j) \cap \mathrm{Supp}(t_{u'}^j) = \emptyset$. Since $z_{\gamma_j} + \sum_{u : d(B_{R'}(x_u), \gamma_j) \leq S} t_u^j$ is a cycle and $\mathrm{Supp}(t_u^j) \cap \mathrm{Supp}(z_{\gamma_j}) \subset B_{R'}(x_u)$ for any such u, we have that the 0-chain $b_j^u = \partial z_{\gamma_j \, | B_{R'}(x_u)} = -\partial t_u^j$ is a cycle, whence the claim for dimension $i = 1$ too.

Let a_j^u, $u \in I$, be such that $\partial a_j^u = b_j^u$ as in the local lemma. By definition of z',

$$\mathrm{Supp}(z - z') \subseteq \bigsqcup_{u \in I} B_{R'}(x_u).$$

For $u \in I$, let z_u be the restriction of $z - z'$ to $B_{R'}(x_u)$. Then

$$z - z' = \sum_{u \in I} z_u$$

and

$$\begin{aligned} 0 &= \partial z \\ &= \partial(z - z') + \partial z' \\ &= \sum_{u \in I} \left(\partial z_u + \sum_{j=1}^{q} b_j^u \right) \\ &= \sum_{u \in I} \partial \left(z_u + \sum_{j=1}^{q} a_j^u \right) \\ &\Longrightarrow \partial \left(z_u + \sum_{j=1}^{q} a_j^u \right) = 0 \quad \text{for all } u \in I, \end{aligned}$$

the implication being true because the $B_{R'}(x_u)$ are disjoint. The chain $\bar{z}_{\gamma_j} = z_{\gamma_j} - \sum_{u\in I} a_j^u$ is a cycle (by (14)), and $\overline{z_u} = z_u + \sum_{j=1}^q a_j^u$, for $u \in I$, is also a cycle by the equality above. By summing, we get

$$\begin{aligned}\sum_{j=1}^q \bar{z}_{\gamma_j} + \sum_{u\in I} \bar{z}_u &= \sum_{j=1}^q \Big(z_{\gamma_j} - \sum_{u\in I} a_j^u\Big) + \sum_{u\in I}\Big(z_u + \sum_{j=1}^q a_j^u\Big)\\ &= \sum_{j=1}^q z_{\gamma_j} + \sum_{u\in I} z_u - \sum_{j=1}^q\sum_{u\in I} a_j^u + \sum_{u\in I}\sum_{j=1}^q a_j^u\\ &= \sum_{j=1}^q z_{\gamma_j} + \sum_{u\in I} z_u\\ &= z.\end{aligned}$$

For all u, $\operatorname{Supp} z_u \cup \bigcup_j \operatorname{Supp} b_u^j \subseteq B_{R'}(x_u)$. By the local lemma, there is a constant $R'' = R''(R')$ such that $\bigcup_j \operatorname{Supp} a_u^j \subseteq B_{R''}(x_u)$, hence $\operatorname{Supp} \bar{z}_u \subseteq B_{R''}(x_u)$ too. Analogously, we have $\operatorname{Supp} z_{\gamma_j} \cup \bigcup_u b_j^u \subseteq \mathcal{N}_{2\delta+L}(\gamma_j)$, hence also $\operatorname{Supp} \bar{z}_{\gamma_j} \subseteq \operatorname{Supp} z_{\gamma_j} \cup \bigcup_u a_j^u \subseteq \mathcal{N}_{S'}(\gamma_j)$, where we can put $S' = \max\{2\delta + L,\ R''\}$.

We fill the $\bar{z}_u$ and the $\bar{z}_{\gamma_j}$ by a_u and a_{γ_j} as in the local lemma and Corollary 5.3 respectively, and put

$$a := \sum_{u\in I} a_u + \sum_{j=1}^q a_{\gamma_j}.$$

By the local lemma again, the filling a_u of $\bar{z}_u$ has support contained in some $B_{R'''}(x_u)$, where R''' only depends on R''. Finally, by Lemma 5.3, we get that $\operatorname{Supp} a_{\gamma_j} \subseteq \mathcal{N}_{S''}(\gamma_j)$, for some S'' which only depends on S'. Hence condition (9) is easily verified, and we can put $L' = \max\{S'',\ R'''\}$.

In order to check the condition about the horoballs note that, if z is contained in some C-horoball, then all the z_{γ_j} and z_u are contained in the same C-horoball. Hence, by (4) in the local lemma and (3) in Corollary 5.3, the same is true for the a_u and the a_{γ_j}.

We are finally left to prove (10). Let

$$K := \max\{M_{\text{thin}}(S'', \operatorname{maxh}(z) + S''),\ M_{\text{loc}}(2R'', \operatorname{maxh}(z))\},$$

where M_{thin} is the function of Corollary 5.3 and $M_{\text{loc}}\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ is the function of point (3) of the local lemma. Then

$$\|a\| \le \sum_{u\in I}\|a_u\| + \sum_{j=1}^q \|a_{\gamma_j}\| \le K\Big(\sum_{u\in I}\|\bar{z}_u\| + \sum_{j=1}^q \|\bar{z}_{\gamma_j}\|\Big),$$

$$\sum_u \|\bar{z}_u\| \le \sum_u \|z_u\| + \sum_{uj}\|a_j^u\|, \qquad \sum_j \|\bar{z}_{\gamma_j}\| \le \sum_j \|z_{\gamma_j}\| + \sum_{uj}\|a_j^u\|.$$

By disjointness of the supports of the z_u and the $\bar{z}_{\gamma_j}$ we get

$$\sum_u \|z_u\| = \Big\| \sum_u z_u \Big\| \leq \|z\|$$

$$\sum_j \|\bar{z}_{\gamma_j}\| = \Big\| \sum_j \bar{z}_{\gamma_j} \Big\| \leq \|z\|.$$

Now, from the construction of the b^u_j, we get $\operatorname{maxh}(b^u_j) \leq \operatorname{maxh}(z)$. Since the a^u_j fill the b^u_j as in the local lemma, we get

$$\sum_{uj} \|a^u_j\| \leq M_{\mathrm{loc}}(R', \operatorname{maxh}(z)) \sum_{ju} \|b^u_j\|,$$

because the b^u_j are contained in balls of radius R'. □

6. Proof of part (*a*) of Theorem 1.1

Let (Γ, Γ') be a group-pair. Let St_* and St'_* be as in Definition 2.8. The augmented complexes $\mathrm{St}^+_* := \mathrm{St}_* \to \mathbb{R} \to 0$ and $\mathrm{St}'^+_* := \mathrm{St}_* \to \mathbb{R}(\Gamma/\Gamma') \to 0$ are Γ-projective resolutions of $\mathbb{R}$ and $\mathbb{R}(\Gamma/\Gamma')$ (see Definition 2.1). In general, by a *map between resolutions* of the same Γ-module M we mean a chain Γ-map that extends the identity of M.

The following homological lemma helps us to outline the strategy we intend to pursue in order to prove Theorem 1.1 (a).

Lemma 6.1 (Homological lemma). *Let $\varphi_i\colon \mathrm{St}_* \to \mathrm{St}_*$, $i = 1, 2$ be chain Γ-maps which satisfy the following hypotheses*:

(1) *φ_i extends to a map between resolutions $\varphi^+_i\colon \mathrm{St}^+_* \to \mathrm{St}^+_*$;*

(2) *φ_i restricts to a map $\varphi'_i\colon \mathrm{St}'_* \to \mathrm{St}'_*$;*

(3) *φ'_i extends to a map between resolutions $\varphi'^+_i\colon \mathrm{St}'^+_* \to \mathrm{St}'^+_*$.*

Then there is a Γ-equivariant homotopy T between φ^+_1 and φ^+_2 that restricts to a homotopy between φ'^+_1 and φ'^+_2 (in St'_*). Given a Γ-module V, the dual maps φ^1 and φ^2 of φ_1 and φ_2 induce homotopically equivalent maps on the complex $\mathrm{Hom}^\Gamma(\mathrm{St}_* \,/\, \mathrm{St}'_*, V) =: \mathrm{St}^{\mathrm{rel}\,*}(\Gamma, \Gamma'; V)$, *for every Γ-module V.*

We will apply the homological lemma to the diagram

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{St}_2(\Gamma) & \xrightarrow{\psi_2} & C_2(\mathcal{R}_\kappa) & \xrightarrow{\varphi_2} & \mathrm{St}_2(\Gamma) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{St}_1(\Gamma) & \xrightarrow{\psi_1} & C_1(\mathcal{R}_\kappa) & \xrightarrow{\varphi_1} & \mathrm{St}_1(\Gamma) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{St}_0(\Gamma) & \xrightarrow{\psi_0} & C_0(\mathcal{R}_\kappa) & \xrightarrow{\varphi_0} & \mathrm{St}_0(\Gamma) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R} & \xrightarrow{\mathrm{Id}} & \mathbb{R} & \xrightarrow{\mathrm{Id}} & \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0,
\end{array} \tag{15}$$

where $\mathcal{R}_\kappa = \mathcal{R}_\kappa(X)$ and $\kappa \geq 4\delta + 6$ as in Corollary 4.8.

We wish to prove that the composition $\varphi_* \circ \psi_*$ satisfies the hypotheses of the homological lemma, and that $\psi^n \circ \varphi^n(f) = f \circ \varphi_n \circ \psi_n$ is a bounded cocycle for every $n \geq 2$ and for every cocycle $f \in \mathrm{Hom}^\Gamma(\mathrm{St}_n; V)$. This will prove the surjectivity of the comparison map since, by Lemma 6.1, for any given cocycle f, the cocycle $f \circ \varphi_n \circ \psi_n$ is cobordant to f and bounded.

In order to fulfill conditions (1), (2), (3) of the homological lemma it is sufficient to find Γ-equivariant chain maps φ_* and ψ_* such that ψ_* maps simplices in St' into simplices in the corresponding C-horoballs of $\mathcal{R}_\kappa$, and vice versa for ψ_*.

We now define φ_*. If $n \geq 1$ we set $\varphi_0(g, i, n) := (g, i)$. Otherwise, we define $\varphi_0(g, 0) := \frac{1}{|I|} \sum_{i \in I} (g, i)$. For $m \geq 1$ and an m-dimensional simplex $[x_0, \ldots, x_m]$ of $\mathcal{R}_\kappa(X)$ we set

$$\varphi_m([x_0, \ldots, x_m]) := \frac{1}{(1+m)!} \sum_{\pi \in S_{m+1}} \varepsilon(\pi)(\varphi_0(x_{\pi(0)}), \ldots, \varphi_0(x_{\pi(m)})), \tag{16}$$

where S_{m+1} is the group of permutations of $\{0, \ldots, m\}$, and $\varepsilon(\pi) = \pm 1$ is the sign of π. The apparently cumbersome definition of the map φ_* follows from the fact that in $C_*(\mathcal{R}_\kappa)$ we have oriented simplices, whereas in St_* we have *ordered* ones, and that the action of Γ on $\mathcal{R}_\kappa(X)$ may map a simplex to itself, changing the order of the vertices.

Much more effort will be needed for the definition of ψ_*, to which the rest of this section is dedicated. The fundamental tool that we will use is the bicombing defined in [17].

Definition 6.2 ([24, Section 3]). Given a group Γ acting on a graph G through simplicial automorphisms, a *homological bicombing* is a function

$$q\colon G^{(0)} \times G^{(0)} \longrightarrow C_1(G)$$

such that $\partial q(a,b) = b - a$ for all $(a,b) \in G^{(0)} \times G^{(0)}$. We say that q is *antisymmetric* if $q(a,b) = -q(b,a)$ for all $a, b \in G^{(0)}$, and Γ-*equivariant* if $\gamma q(a,b) = q(\gamma a, \gamma b)$ for all $\gamma \in \Gamma$ and $a, b \in G^{(0)}$. Moreover, q is *quasi-geodesic* if there is a constant $D > 0$ such that, for all $a, b \in G^{(0)}$:

(1) $\|q(a,b)\| \leq D d(a,b)$;

(2) $\operatorname{Supp} q(a,b) \subseteq \mathcal{N}_D([a,b])$.

We note that, if G is a hyperbolic graph, the precise choice of a geodesic $[a,b]$ between a and b is, up to increasing the constant D, irrelevant.

Note that, by the antisymmetricity requirement in the definition of homological bicombing, we can extend q to a function on 1-dimensional oriented simplices of the graph.

The homological bicombing Q in the following theorem is based on the bicombing constructed by Mineyev in [24]. The relevant properties of Q are described in [17, Section 5] and [17, Theorem 6.10].

Theorem 6.3. *If* (Γ, Γ') *is a relatively hyperbolic pair, there is a bicombing* Q *on the associated cusped space* X *such that*

(1) Q *is quasi-geodesic*;

(2) Q *is* Γ*-equivariant*;

(3) Q *is antisymmetric*;

(4) *there is* $K > 0$ *such that, for all* $a, b, c \in X^{(0)}$*, there are* 1*-dimensional cycles* $z = z(a,b,c)$ *and* $w = w(a,b,c)$ *such that*

- $Q(\partial(a,b,c)) = z + w$;
- $\operatorname{minh}(w) \geq C > \delta$;
- $\|z\| \leq K$;
- $\operatorname{maxh}(z) \leq K$;
- *for all* $\gamma \in \Gamma$,

$$z(\gamma a, \gamma b, \gamma c) = \gamma z(a,b,c)$$

  and

$$w(\gamma a, \gamma b, \gamma c) = \gamma w(a,b,c);$$

- z *and* w *are contained in the* K*-neighborhood of* $[a,b] \cup [b,c] \cup [c,a]$.

Remark 6.4. Groves and Manning allow multiple edges in their definition of cusped graph (as already noted in Remark 3.4). However, it is easy to see that, if $\bar{X}$ is the simplicial graph obtained by identifying edges of X with the same endpoints, the obvious bicombing induced by Q on $\bar{X}$ satisfies all of the properties of Theorem 6.3. See also [17, Remark 6.12]

We want to find a decomposition $\{\psi_k = z_k + w_k\}_{k\geq 2}\colon \mathrm{St}_*(\Gamma) \to C_*(\mathcal{R}_\kappa)$ of ψ_k into Γ-equivariant chain maps $\{z_k\}_{k\geq 2}$ and $\{w_k\}_{k\geq 2}$ such that

(A) $\|z_k(\Delta)\|$ is uniformly bounded independently on the simplex Δ in St_k;

(B) $\mathrm{maxh}(z_k(\Delta))$ is uniformly bounded independently of $\Delta \in \mathrm{St}_k$;

(C) $\mathrm{minh}(w_k(\Delta)) \geq C$ for every $\Delta \in \mathrm{St}_k$;

(D) z_* and w_* map elements in the basis of St' into C-horoballs.

We now show how the conclusion follows from the existence of a map ψ_* satisfying the four conditions above, and then we construct such a ψ_*. It is easy to see that, if an i-dimensional simplex s of $\mathcal{R}_\kappa(X)$ is not contained in a single C-horoball, it must satisfy $\mathrm{maxh}(s) \leq 2\kappa + 2$. For $i \geq 2$, let $f\colon \mathrm{St}_i \,/\, \mathrm{St}'_i \to V$ be a Γ-equivariant map, that we see as a map defined on St_i which is null on St'_i. Then $f \circ \varphi_i\colon C_i(\mathcal{R}_\kappa(X)) \to V$ is a bounded map: in fact,

$$\begin{aligned}
&\sup\{\|f \circ \varphi_i(s)\|\colon s \text{ is an } i\text{-dimensional simplex in } \mathcal{R}_\kappa(X)\}\\
&\quad = \sup\{\|f \circ \varphi_i(s)\|\colon \mathrm{maxh}(s) \leq 2\kappa + 2\}\\
&\quad < \infty,
\end{aligned}$$

because, up to the Γ-action, there is only a finite number of simplices s with $\mathrm{maxh}(s) \leq 2\kappa + 2$. Moreover, $f \circ \varphi_i \circ \psi_i$ is also bounded since, for every simplex $\Delta \in \mathrm{St}_k$,

$$\|f \circ \varphi_i \circ \psi_i(\Delta)\| = \|f \circ \varphi_i \circ z_i(\Delta)\|,$$

and z_i is a bounded map.

We now construct ψ_*, inductively verifying that it satisfies conditions (A), …, (D) above. Recall that, by our hypotheses, X is a subcomplex of $\mathcal{R}_\kappa(X)$. Let Q be the bicombing of Theorem 6.3. Since Q is quasi-geodesic and C-horoballs are convex, it follows that $Q(a,b)$ is completely contained in a C-horoball $\mathcal{H}_C$ if a and b lie in $\mathcal{H}_L$, for L sufficiently large. Therefore for such an L we set

$$\psi_0(g,i) := (g,i,L)$$

$$\psi_1((g,i),(h,j)) := Q(\psi_0(g,i), \psi_0(h,j)) \in C_1(X) \subset C_1(\mathcal{R}_\kappa(X)).$$

In order to simplify our notation, we denote by Δ^m a generic m-dimensional simplex in St. If $\Delta^2 = (p_0, p_1, p_2)$, we write

$$\psi_1(\partial\Delta^2) = z(\Delta^2) + w(\Delta^2),$$

where $z(\Delta^2) := z(\psi_0(p_0), \psi_0(p_1), \psi_0(p_2))$ as in the notation of Theorem 6.3, and $w(\Delta^2) = \psi_1(\partial\Delta^2) - z(\Delta^2)$.

Notice that the cycles $z(\Delta^2)$ fulfill the conditions of Theorem 5.6 for a uniform constant L and with $\mathrm{maxh}(z(\Delta^2))$ uniformly bounded. Therefore we can fill $z(\Delta^2)$ with a chain $z_2(\Delta^2)$, where $\mathrm{maxh}(z_2(\Delta^2))$ and its norm $\|z_2(\Delta^2)\|$ are uniformly bounded (i.e. independently of Δ^2), and moreover $\mathrm{Supp}(z_2(\Delta^2))$ is contained in some C-horoball, if the same is true for $\mathrm{Supp}(z(\Delta^2))$. We extend z and z_2 by linearity. In what follows, all fillings are required to satisfy the conditions of Theorem 5.6. We have

$$z(\partial\Delta^3) + w(\partial\Delta^3) = 0$$

hence $-z(\partial\Delta^3) = w(\partial\Delta^3)$ is a 1-dimensional cycle with bounded norm and minimum height at least C. Hence $\mathrm{Supp}(w(\partial\Delta^3))$ is contained in the union of some C-horoballs. Since the C-horoballs of X are disjoint complexes and because of (4) of Lemma 5.1, we have that $w(\partial\Delta^3)_{|\mathcal{H}_C}$ is a cycle for every C-horoball $\mathcal{H}_C$.

Let $\omega_2(\Delta^3)$ be a filling of $w(\partial\Delta^3)$ as in Theorem 5.6, i.e.

$$\partial\omega_2(\Delta^3) = -z(\partial\Delta^3) = w(\partial\Delta^3).$$

Applying Theorem 5.6 we find a chain z_3 such that

$$\partial(z_3(\Delta^3)) = z_2(\partial(\Delta^3)) + \omega_2(\Delta^3).$$

Fix $m \geq 4$, and suppose by induction that, for any m-symplex in St_m

$$\partial(z_m(\Delta^m)) = z_{m-1}(\partial\Delta^m) + \omega_{m-1}(\Delta^m),$$

with z_m, z_{m-1} and ω_{m-1} of uniformly bounded maximum height and ℓ^1-norm, and such that $\mathrm{minh}(\omega_{m-1}) \geq C$. Moreover, suppose that the geometric conditions of Theorem 5.6 for z_m, z_{m-1} and ω_{m-1} are also satisfied, where n and L in the statement of Theorem 5.6 that only depends on the dimension m. Then

$$\partial(z_m(\partial\Delta^{m+1})) = z_{m-1}(\partial(\partial\Delta^{m+1})) + \omega_{m-1}(\Delta^{m+1}) = \omega_{m-1}(\partial\Delta^{m+1}),$$

hence we can find a filling $\omega_m(\Delta^{m+1})$ of the cycle $-\omega_{m-1}(\partial\Delta^{m+1})$. Finally, we define z_{m+1} in such a way that

$$\partial(z_{m+1}(p_0, \ldots, p_{m+1})) = z_m(\partial(p_0, \ldots, p_{m+1})) + \omega_m(p_0, \ldots, p_{m+1}).$$

All inductive conditions are satisfied.

Now we consider the construction of w_*. Similarly as before, by Theorem 6.3, $\mathrm{minh}(w(\Delta^2)) \geq C$. Hence $w(\Delta^2)_{|\mathcal{H}_C}$ is a cycle for every C-horoball $\mathcal{H}_C$. By the contractibility of the C-horoballs (Corollary 4.8), we can fill every $w(\Delta^2)_{|\mathcal{H}_C}$ in $\mathcal{H}_C$. Let $w_2(\Delta^2)$ be a filling of $w(\Delta^2)$ given by filling any $w(\Delta^2)_{|\mathcal{H}_C}$ in the same C-horoball. Note that we have defined ω_* in such a way that $\partial\omega_2(\Delta^3) = w(\partial\Delta^3)$, and $\partial\omega_{m+1}(\Delta^{m+2}) = -\omega_m(\partial\Delta^{m+2})$ for $m \geq 2$. We have that

$$\partial w_2(\partial\Delta^3) = w(\partial\Delta^3) = \partial\omega_2(\Delta^3).$$

Hence we can define $w_3(\Delta^3)$ in such a way that

$$\partial w_3(\Delta^3) = w_2(\partial\Delta^3) - \omega_2(\Delta^3).$$

Now, fix $m \geq 4$, and suppose by induction that

$$\partial w_m(\Delta^m) = w_{m-1}(\partial\Delta^m) - \omega_{m-1}(\Delta^m).$$

Then $\partial w_m(\partial\Delta^{m+1}) = -\omega_{m-1}(\partial\Delta^{m+1}) = \partial\omega_m(\Delta^{m+1})$, hence $w_m(\partial\Delta^{m+1}) - \omega_m(\Delta^{m+1})$ is a cycle, which we can fill by $w_{m+1}(\Delta^{m+1})$.

This concludes the construction of ψ_*, whence the proof of Theorem 1.1 (a).

7. Applications

Let (X, A) be a topological pair. Let $S_*(X)$ be the singular complex of X with real coefficients. In other words, $S_k(X)$ is the real vector space whose basis is the set $C^0(\Delta^k, X)$ of singular k-dimensional simplices in X, and we take the usual boundary operator $\partial_k \colon S_k(X) \to S_{k-1}(X)$, for $k \geq 1$. The natural inclusion of complexes $S_*(A) \hookrightarrow S_*(X)$ allows us to define the relative singular complex $S_*(X, A) := S_*(X)/S_*(A)$. Dually, we define the relative singular cocomplex as

$$S^*(X, A) = \mathrm{Hom}(S_*(X, A), \mathbb{R}),$$

where $\mathrm{Hom}(S_*(X, A), \mathbb{R})$ denotes the set of real linear maps on $S_*(X, A)$. We put $S^*(X, \emptyset) =: S^*(X)$. We will often identify $S^*(X, A)$ with the subspace of $S^*(X)$ whose elements are null on $S_*(A)$. We put an ℓ^1-norm on $S_*(X, A)$ through the identification

$$S_*(X, A) \sim \mathbb{R}(C^0(\Delta^i, X) \setminus C^0(\Delta^i, A)).$$

Given a cochain $f \in S^*(X, A)$, the (possibly infinite) ℓ^∞-norm of f is

$$\|f\|_\infty := \sup\{|f(c)| \colon c \in S_*(X, A),\ \|c\| \leq 1\}.$$

We denote by $S_b^*(X, A)$ the subcocomplex of $S^*(X, A)$ whose elements have finite ℓ^∞-norm. Since the boundary operator $\partial_*: S_*(X, A) \to S_{*-1}(X, A)$ is bounded with respect to the ℓ^1-norms, its dual maps bounded cochains into bounded cochains (and is bounded with respect to the ℓ^∞-norm). Therefore $S_b^*(X, A)$ is indeed a cocomplex.

The following definition appeared for the first time in [14, Section 4.1].

Definition 7.1. Given a topological pair (X, A), the *relative bounded cohomology* $H_b^*(X, A)$ is the cohomology of the cocomplex $S_b^*(X, A)$.

Definition 7.2. Let $S_*(X, A)$ be the real singular chain complex of a topological pair. The norm on $S_*(X, A)$ descends to a natural semi-norm on homology, called *Gromov norm*: for every $\alpha \in H_*(X, A)$,

$$\|\alpha\| = \inf\{\|c\|: c \in S_*(X, A),\ [c] = \alpha\}.$$

If M is an n-dimensional oriented compact manifold with boundary, the *simplicial volume* of M is the Gromov norm of the fundamental class in $H_n(M, \partial M)$.

Definition 7.3. A topological pair (X, Y) is a *classifying space* for the group-pair $(\Gamma, \{\Gamma_i\}_{i\in I})$ if

(1) X is path-connected, and $Y = \bigsqcup_{i\in I} Y_i$ is a disjoint union of path-connected subspaces Y_i of X parametrized by I;

(2) there are basepoints $x \in X$ and $y_i \in Y_i$, and isomorphisms $\pi_1(X, x) \sim \Gamma$ and $\pi_1(Y_i, y_i) \sim \Gamma_i$;

(3) the Y_i are π_1-injective in X, and there are paths γ_i from x to y_i such that the induced injections
$$\pi_1(Y_i, y_i) \longrightarrow \pi_1(X, x)$$
correspond to the inclusions $\Gamma_i \hookrightarrow \Gamma$ under the isomorphisms above;

(4) X and Y are aspherical.

The following theorem applies in particular to negatively curved compact manifolds with totally geodesic boundary.

Theorem 7.4. *Let (X, Y) be a classifying space of a relatively hyperbolic pair (Γ, Γ'). Then the Gromov norm on $H_k(X, Y)$ is a norm for any $k \geq 2$.*

Proof. Let $H^*(\Gamma, \Gamma')$ be the relative cohomology of (Γ, Γ') as defined in [4] (the definition of Bieri and Eckmann is completely analogous to the one of Mineyev and Yaman, but without any reference on the norm). It is possible to define natural maps

$$H_b^*(X, Y) \longrightarrow H_b^*(\Gamma, \Gamma') \longrightarrow H^*(\Gamma, \Gamma') \longrightarrow H^*(X, Y) \tag{17}$$

such that the first map is an isometric isomorphism, the third one is an isomorphism, and the compositions of all maps in (17) is the comparison map from singular bounded cohomology to singular cohomology (the fact that the first map is an isometry also follows from weaker hypotheses; see [5, Theorem 5.3.11]). By hypothesis, the second map in (17) is surjective. Hence the conclusion follows from the following proposition ([26, Proposition 54]), which is the relative version of an observation by Gromov ([14, p. 17]) and could be generalized for any normed chain complex (see [21, Theorem 3.8]).

Proposition 7.5. *For any $z \in H_k(Y, Y'; \mathbb{R})$,*

$$\|z\| = \sup\Big(\Big\{\frac{1}{\|\beta\|_\infty} : \beta \in H_b^k(Y, Y'; \mathbb{R}) : \langle \beta, z\rangle = 1\Big\} \cup \{0\}\Big). \qquad \square$$

Now we consider our second application: a relatively hyperbolic group-pair has finite cohomological dimension. More precisely

Theorem 7.6. *Let (Γ, Γ') be a relatively hyperbolic pair. Then there is $n \in \mathbb{N}$ such that, for every $m > n$ and every bounded Γ-module V, $H^m(\Gamma, \Gamma'; V) = 0$.*

We note that this theorem admits a straightforward proof in the case of a torsion-free hyperbolic group Γ. Indeed, consider a contractible Rips complex Y over the Cayley-graph X of Γ. The complex Y is finite dimensional by the uniform local compactness of X. Since Y is contractible and Γ acts freely on it, the cohomology of Γ is isomorphic to the (simplicial) cohomology of Y, whence the conclusion.

Let $\mathcal{R}_\kappa := \mathcal{R}_\kappa(X)$ be the Rips complex associated to a cusped space X of the relatively hyperbolic pair (Γ, Γ'), as described in Corollary 4.8. Then

Lemma 7.7. *For every $C > 0$ there exists $n \in \mathbb{N}$ such that, for every $m \geq n$ and for every m-simplex Δ of $\mathcal{R}_\kappa(X)$, we have* $\mathrm{minh}(\Delta) > C$.

Proof. For m sufficiently large, every subset $A \subseteq \mathcal{R}_\kappa(X)^{(0)}$ of cardinality m and such that $\mathrm{minh}(A) \leq C$ has X-diameter greater than κ. This follows easily from the fact that, up to Γ-action, there are only finitely many such sets A. Therefore, by definition of Rips complex, the conclusion follows. $\square$

We can now prove Theorem 7.6.

Proof. Let V be a bounded Γ-module, and let f be a cochain in $\mathrm{Hom}^\Gamma(\mathrm{St}_k^{\mathrm{rel}}, V)$, which we can see as a Γ-equivariant map which is null on $\mathrm{St}'_k \subset \mathrm{St}_k$. By Lemma 7.7, $f \circ \varphi_k \circ \psi_k = 0$, for k sufficiently big and independent of f. If f is a cocycle, by Lemma 6.1, f is cohomologous to the null map, hence $H^k(\Gamma, \Gamma'; V) = 0$ by the arbitrariness of f. $\square$

8. Proof of part (*b*) of Theorem 1.1

In the following we will work in the category of *combinatorial cell complexes* (see [8, 8A.1]). We are particularly interested in the 2-skeleton of a combinatorial complex X. This is described as follows: $X^{(1)}$ is any graph, and the 2-cells are l-polygons e_λ, $l \geq 2$, such that the attaching map $\partial e_\lambda \to X^{(1)}$ is a loop whose restriction to any open cell of ∂e_λ (i.e., open edge or point) is a homeomorphism to some open cell of $X^{(1)}$.

The following characterization of relative hyperbolicity is due to Bowditch [6, Definition 2].

Definition 8.1. Let G be a graph. A circuit in G is a closed path that meets any vertex at most once. We say that G is *fine* if, for any edge, the set of circuits of any given length containing e is finite. A group Γ is *hyperbolic relative to a finite collection of subgroups* Γ' if Γ acts on a connected, fine, δ-hyperbolic graph G with finite edge stabilizers, finitely many orbits of edges, and Γ' is a set of representatives of distinct conjugacy classes of vertex stabilizers (such that each infinite stabilizer is represented).

Definition 8.2 ([22, Definition 1.2]). Let $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and let X be a combinatorial cell complex. The *homological Dehn function of* X *over* $\mathbb{K}$ is the map $FV_{X,\mathbb{K}} \colon \mathbb{N} \to \mathbb{R}$ defined by

$$FV_{X,\mathbb{K}}(k) := \sup\{\|\gamma\|_{f,\mathbb{K}} \colon \gamma \in Z_1(X,\mathbb{Z}),\ \|\gamma\| \leq k\}$$

where

$$\|\gamma\|_{f,\mathbb{K}} := \inf\{\|\mu\| \colon \mu \in C_2(X,\mathbb{K}), \partial\mu = \gamma\}.$$

By a result given in [25] (which generalizes [1, Theorem 3.3]) the linearity of $FV_{X,\mathbb{K}}$ is equivalent to the undistortedness of the boundary $\partial_2 \colon C_2(X,\mathbb{K}) \to C_1(X,\mathbb{K})$, if $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$. Indeed, given a cycle $z \in Z_1(X,\mathbb{K})$, we can express it as a sum of circuits $z = \sum_c a_c c$ in such a way that $\mathbb{K} \ni a_c \geq 0$ for all c, and $\|z\| = \sum_c a_c \|c\|$ (see [25, Theorem 6 (b)], with $T = \emptyset$). Suppose that $\|c\|_{f,\mathbb{K}} \leq K\|c\|$ for some constant $K \geq 0$ and any circuit c. Then

$$\|z\|_{f,\mathbb{K}} \leq \sum_c g(c)\|c\|_{f,\mathbb{K}} \leq K \sum_c g(c)\|c\| = K\|z\|.$$

Moreover, we have

Proposition 8.3. *Let X be a simply connected combinatorial cell complex. Then*

$$FV_{X,\mathbb{Q}} = FV_{X,\mathbb{R}}.$$

Proof. We have to prove that $FV_{X,\mathbb{Q}} \leq FV_{X,\mathbb{R}}$, since the opposite inequality is clear. Let $\gamma \in Z_1(X,\mathbb{Z})$, and let $a = \sum_i \lambda_i \sigma_i \in C_2(X,\mathbb{R})$ be such that $\partial a = \gamma$. We approximate the λ_i with rational coefficients λ_i', in such a way that, if $a' = \sum_i \lambda_i' \sigma_i$, then $\|\partial(a - a')\| \leq \varepsilon$. Let W be the normed subspace of $Z_1(X,\mathbb{Q})$ whose elements are $\mathbb{Q}$-linear combinations of faces of the σ_i. Let $\theta\colon W \to C_2(X,\mathbb{Q})$ be a $\mathbb{Q}$-linear map such that $\partial\theta(w) = w$ for all $w \in W$. Since W is finite-dimensional, θ is bounded. Moreover, $\partial(a - a') \in W$. Hence

$$\partial(a' + \theta\partial(a - a')) = \gamma$$

and

$$\|a' + \theta\partial(a - a')\| \leq \|a'\| + \|\theta\|_\infty \varepsilon$$

from which the conclusion follows immediately by the arbitrariness of ε. □

The following lemma is stated as such in [22], but is proven in [17, Theorem 2.30] with a different notation.

Lemma 8.4. [22, Theorem 3.4] *Let X be a simply connected complex such that there is a bound on the length of attaching maps of 2-cells. If $FV_{X,\mathbb{Q}}$ is bounded by a linear function, then the 1-skeleton of X is a hyperbolic graph.*

The following theorem is a slight modification of the "if part" of [22, Theorem 1.8]: we require the complex to be simply connected instead of 1-acyclic, and we write $FV_{X,\mathbb{Q}}$ instead of $FV_{X,\mathbb{Z}}$ in (2).

Theorem 8.5. *Let (Γ,Γ') be a group-pair. Then (Γ,Γ') is relatively hyperbolic if there is a simply connected combinatorial complex X such that*

(1) *Γ acts cocompactly on $X^{(2)}$;*

(2) *$FV_{X,\mathbb{Q}}(k) \leq Ck$ for every $k \in \mathbb{N}$;*

(3) *the stabilizers in Γ of edges are finite;*

(4) *Γ' is a set of representatives of (distinct) conjugacy classes of stabilizers of 0-cells such that each infinite stabilizer is represented. This means that there is an injection*

$$\Gamma' \longrightarrow \{[\mathrm{Stab}\,(v)]\colon v \in X^{(0)}\}, \quad \Gamma_i \longmapsto [\Gamma_i]$$

(where $[H]$ denotes the conjugacy class of a subgroup H of Γ) whose image contains all conjugacy classes of infinite stabilizers in Γ of vertices in $X^{(0)}$.

Proof. Points (1) and (2) imply the hyperbolicity of the graph by Lemma 8.4. Hence, in order to apply Bowditch's characterization of relative hyperbolicity it remains to prove that $X^{(1)}$ is fine.

Condition (1) in the statement implies that there is a bound on the number of edges on the boundary of 2-cells. Moreover, conditions (1) and (3) imply that any edge belongs to just a finite number of 2-cells (because edge-stabilizers act cocompactly on the 2-cells adjacent to the edge).

We conclude by means of the following lemma, which is proven in [22, Theorem 1.6 (2)]. □

Lemma 8.6. *Let X be a simply connected combinatorial cell complex such that each 1-cell is adjacent to finitely many 2-cells and there is a bound on the length of attaching maps of 2-cells. Suppose that there is $C \geq 0$ such that*

$$FV_{X,\mathbb{Q}}(k) \leq Ck \quad \text{for all } k \in \mathbb{N}.$$

Then $X^{(1)}$ is fine.

Definition 8.7 ([28, Definition 2.1]). Let (Γ, Γ') be a group-pair. We say that Γ is *finitely presented relative to* Γ' if

(1) Γ is generated by $\bigcup_{i\in I} \Gamma_i$ and a finite subset $\mathcal{A}$ of Γ;

(2) the kernel of the natural projection

$$F(\mathcal{A}) * (*_{i\in I}\Gamma_i) \longrightarrow \Gamma$$

is generated – as a normal subgroup of $F(\mathcal{A}) * (*_{i\in I}\Gamma_i)$ – by a finite set $\mathcal{R} \subseteq F(\mathcal{A}) * (*_{i\in I}\Gamma_i)$ of *relations*.

In this case, the datum of $\langle \mathcal{A}, \Gamma' | \mathcal{R}\rangle$ is a *finite presentation* of (Γ, Γ').

Notation 8.8. From now on, we will assume that $(\Gamma, \Gamma' = \{\Gamma_i\}_{i\in I=\{1,\dots,n\}})$ is a finitely presented group-pair, and that Γ is finitely generated. By a result in [28, Proposition 2.29], it follows that the groups in Γ' are finitely generated too.

Since there exist slightly different definitions of Cayley-graph in the literature, from now on we will rely on the following one. Let S be a (non-necessarily symmetric) generating set of a group Γ. The *Cayley graph $G = G(\Gamma, S)$ of* Γ *with respect to* S is the graph whose 0-skeleton is Γ and with an edge connecting x and xs labelled by (x, s), for any $(x, s) \in \Gamma \times S$.

Notice that Γ acts freely and isometrically on $G(\Gamma, S)$ by mapping the vertex x to γx and the edge (x, s) to the edge $(\gamma x, s)$.

Recall that a *compatible* generating set S of (Γ, Γ') is a generating set of Γ which restricts to a generating set for any group in Γ'.

Definition 8.9 ([17, relative Cayley complex]). Let $G := G(\Gamma, S)$ be the Cayley graph of Γ, with respect to some compatible generating set S. Consider the graph G^I constructed as follows:

(1) $(G^I)^{(0)} = G^{(0)} \times I$;

(2) for any $i \in \{1, \dots, n\} = I$, $G^i := G \times \{i\}$. For all $v \in G^{(0)}$ and $1 \leq i < n$ there is a single edge connecting (v, i) and $(v, i+1)$.

We call the edges contained in some G^i *horizontal*, and the other ones *vertical*.

By writing elements of $\mathcal{R}$ with the alphabet S, we can, for every i, (non-uniquely) associate them to loops in G^i based in 1. We add Γ-equivariantly 2-cells to those loops and their Γ-translates. Let $i \in I$, $i < n$. If e^i is an edge in G^i we have a corresponding edge e^{i+1} in G^{i+1}, and two vertical edges connecting the initial and final points of e^i and e^{i+1}. We add a rectangular 2-cell to this quadrilateral. We denote by $\mathrm{Cay}(\Gamma, \Gamma')$ the 2-dimensional combinatorial cell complex obtained in this way, and call it the *relative Cayley-complex of* (Γ, Γ') (*with respect to* S).

The group Γ naturally acts on $\mathrm{Cay}(\Gamma, \Gamma')$.

Definition 8.10. The 2*-dimensional quotient complex* $\widehat{X} = \widehat{X}(\Gamma, \Gamma')$ is the CW-complex obtained by collapsing to points the full subcomplexes of $\mathrm{Cay}(\Gamma, \Gamma')$ whose vertices are contained in the same left coset of $\Gamma_i \times \{i\}$, $i \in I$.

Remark 8.11. This means that, if Y_i is the full subcomplex of $\mathrm{Cay}(\Gamma, \Gamma')$ whose vertices correspond to $\Gamma_i \times \{i\}$, then all (left) Γ-translates of Y_i are collapsed to points. It is easily seen that $\widehat{X}$ could be given the structure of a combinatorial complex.

At the 0-dimensional level, we have a natural Γ-isomorphism $\Gamma/\Gamma' := \bigsqcup_{i \in I} \Gamma/\Gamma_i \to \widehat{X}^{(0)}$. We use it to label the vertices of $\widehat{X}^{(0)}$ by Γ/Γ'. Given a horizontal edge (x, s) in $G^i \subseteq \mathrm{Cay}(\Gamma, \Gamma')$, this is either collapsed to a point in $\widehat{X}$ if $s \in \Gamma_i$, or is left unchanged. Hence the horizontal edges of $\widehat{X}$ are naturally labelled by the set $\bigsqcup_{i \in I} \Gamma \times (S \setminus \Gamma_i)$. Notice that vertical edges are never collapsed.

The complex $\widehat{X}$ carries a natural Γ-action. The action on the 0-skeleton has already been described. A cell of dimension at least 1 in $\widehat{X}$ corresponds to exactly one cell of the same dimension in $\mathrm{Cay}(\Gamma, \Gamma')$, hence the action of Γ on $\widehat{X}$ is defined accordingly. Notice that, since the action of Γ on the Cayley complex is free, the same is true for the action of Γ on the 1-skeleton of $\widehat{X}$. In particular Condition (3) of Theorem 8.5 holds.

Proposition 8.12. *$\widehat{X}$ is simply connected.*

Proof. Let Y_i be the full subcomplex of $\mathrm{Cay}(\Gamma, \Gamma')$ whose vertices are labelled by $\Gamma_i \times \{i\}$. For all $i \in I$, we add Γ-equivariantly 2-cells to Y_i and to its Γ-translates, in order to make them simply connected. Let Z be the combinatorial complex thus obtained. This comples is homotopically equivalent to the complex Z' obtained by contracting the vertical edges to points and the 2-cells to edges. The complex Z' is simply connected by construction and Definition 8.7, hence Z is simply connected too.

The complex obtained by collapsing to points the simply connected full subcomplexes of Z containing Y_i and their Γ-translates is simply connected. Moreover, it is obviously homeomorphic to $\widehat{X}$. □

We add Γ-equivariantly higher dimensional cells to $\widehat{X}$ in order to make it a contractible combinatorial complex, that we also denote by $\widehat{X}$, and call it the *quotient complex*. Consider the exact cellular sequence

$$\cdots \longrightarrow C_1(\widehat{X}) \longrightarrow C_0(\widehat{X}) \longrightarrow \mathbb{R} \longrightarrow 0.$$

Recall that we have a Γ-isomorphism between the Γ-sets $\widehat{X}^{(0)}$ and Γ/Γ'. Therefore, if Δ is the kernel of the augmentation map $\mathbb{R}(\Gamma/\Gamma') \to \mathbb{R}$ we also have the exact sequence

$$C_*(\widehat{X}) \longrightarrow \Delta \longrightarrow 0. \tag{18}$$

By the following lemma, the sequence (18) provides a Γ-projective resolution of Δ (i.e., all the Γ-modules except Δ are Γ-projective).

Lemma 8.13. *Let X be a contractible CW-complex, and let Γ act on X through cellular homeomorphisms. Suppose that the stabilizers in Γ of 1-cells are finite. Then $C_k(X)$ is a Γ-projective module for every $k \geq 1$.*

Proof. For $k \geq 1$, the stabilizer of any k-dimensional cell is finite. By (arbitrarily) choosing an orientation for every k-cell of X, we get a basis of $C_k(X)$. Then we conclude by applying Lemma 2.2 to such a basis. □

We are now ready to prove Theorem 1.1 (b). Most of the proof follows almost verbatim [26, Theorem 57]. We note however that the existence of a combinatorial isoperimetric function required in the statement of Theorem 57 is never actually exploited in its proof.

Proof. We will prove that $\widehat{X}$ satisfies conditions (1), …, (4) of Theorem 8.5. Condition (4) is obvious. Since $\widehat{X}^{(2)}$ is a quotient of the relative Cayley complex, the Γ-action on it is obviously cocompact, whence (1). Condition (3) was already proved in Remark 8.11.

Now, let $V := (B_1(\widehat{X}); \|\cdot\|_f)$, where $B_1(\widehat{X}) \subset C_1(\widehat{X})$ is the set of boundaries and $\|\cdot\|_f$ is the *filling norm*

$$\|c\|_f := \inf\{\|a\| : a \in C_2(\widehat{X}),\ \partial a = c\}.$$

This is actually a norm (and not just a semi-norm) because, by Condition (1), the boundary map $\partial_2 : C_2(\widehat{X}) \to C_1(\widehat{X})$ is bounded, with respect to the ℓ^1-norms.

We have already seen that $\mathrm{St}_*^{\mathrm{rel}}(\Gamma, \Gamma')$ and $C_*(\widehat{X})$ provide Γ-projective resolutions of Δ. Hence by Lemma 2.6 there are, up to (non-bounded) Γ-homotopy, unique chain maps

$$\varphi_* : \mathrm{St}_*^{\mathrm{rel}}(\Gamma, \Gamma') \longrightarrow C_*(\widehat{X}), \quad \psi_* : C_*(\widehat{X}) \longrightarrow \mathrm{St}_*^{\mathrm{rel}}(\Gamma, \Gamma')$$

that extend the identity on Δ. Put

$$u := \partial_2 \colon C_2(\widehat{X}) \longrightarrow V.$$

The cochain u is a cocycle. Since $\psi^2 \circ \varphi^2$ induces the identity in ordinary cohomology, there is $v \in C^1(X, V)$ such that

$$u = \psi^2(\varphi^2(u)) + \delta v.$$

From the surjectivity hypothesis we get

$$\varphi^2(u) = u' + \delta v',$$

for some bounded cocycle $u' \in \mathrm{St}^2_{\mathrm{rel}}(\Gamma, \Gamma'; V)$ and $v' \in C^1_{\mathrm{rel}}(\Gamma, \Gamma')$. Let $b \in C_1(\widehat{X})$ be a cycle, and let $a \in C_2(\widehat{X})$ be a filling of b. Then

$$b = \partial a = \langle u, a\rangle = \langle (\psi^2 \circ \varphi^2)(u) + \delta v, a\rangle = \langle (\psi^2 \circ \varphi^2)(u), a\rangle + \langle v, b\rangle. \quad (19)$$

By Corollary 2.10 we have

$$\begin{aligned}
\langle (\psi^2 \circ \varphi^2)(u), a\rangle &= \langle \varphi^2(u), \psi_2(a)\rangle \\
&= \langle \varphi^2(u), [y, \partial(\psi_2(a))]_{\mathrm{rel}}\rangle \\
&= \langle \varphi^2(u), [y, \psi_1(b)]_{\mathrm{rel}}\rangle \\
&= \langle u' + \delta v', [y, \psi_1(b)]_{\mathrm{rel}}\rangle \\
&= \langle u', [y, \psi_1(b)]_{\mathrm{rel}}\rangle + \langle v', \partial[y, \psi_1(b)]_{\mathrm{rel}}\rangle \\
&= \langle u', [y, \psi_1(b)]_{\mathrm{rel}}\rangle + \langle v', \psi_1(b)\rangle \\
&= \langle u', [y, \psi_1(b)]_{\mathrm{rel}}\rangle + \langle \psi^1(v'), b\rangle.
\end{aligned}$$

Summarizing,

$$b = \langle u', [y, \psi_1(b)]_{\mathrm{rel}}\rangle + \langle \psi^1(v') + v, b\rangle.$$

Hence

$$\begin{aligned}
|b|_f &\le |\langle u', [y, \psi_1(b)]_{\mathrm{rel}}\rangle|_f + |\langle \psi^1(v') + v, b\rangle|_f \\
&\le |u'|_\infty \|[y, \psi_1(b)]_{\mathrm{rel}}\| + |\psi^1(v') + v|_\infty \|b\| \\
&\le 3|u'|_\infty \|\psi_1(b)\| + |\psi^1(v') + v|_\infty \|b\| \\
&\le (3|u'|_\infty |\psi_1|_\infty + |\psi^1(v') + v|_\infty)\|b\|.
\end{aligned}$$

Hence it remains to prove that $(3|u'|_\infty|\psi_1|_\infty + |\psi^1(v') + v|_\infty)$ is bounded.

The cocycle u' is bounded by definition. Moreover $\psi_1 \colon C_1(\widehat{X}) \to \mathrm{St}_1^{\mathrm{rel}}(\Gamma, \Gamma')$ and $\psi^1(v') + v \colon C_1(\widehat{X}) \to V$ are Γ-equivariant, hence also bounded by the cocompactness of the action of Γ over $\widehat{X}^{(2)}$.

It follows that $\partial \colon C_2(X, \mathbb{R}) \to C_1(X, \mathbb{R})$ is undistorted, hence $FV_{X,\mathbb{R}} = FV_{X,\mathbb{Q}}$ is linearly bounded. □

Remark 8.14. The proof of part (b) of Theorem 1.1 could be adapted, as in [26], by weakening the hypotheses in the statement by requiring the surjectivity only for *Banach* coefficients (that is, Banach spaces equipped with an isometric Γ-action).

8.1. Addendum I: the relative cone of [26]. We recall the definition and properties of the *relative cone* given in [26, 10.1,…, 10.5].

Consider the (non-linear) map $\Phi\colon \mathrm{St}_0(\Gamma) \to \mathrm{St}_1(\Gamma)$,

$$\Phi(c) := \frac{1}{\sum_{x\in I\Gamma} \alpha_x^+} \sum_{x,y\in I\Gamma} \alpha_x^- \alpha_y^+ [x, y],$$

where $c \in \mathrm{St}_0$ is written as $c = \sum_x \alpha_x^+ x - \sum_x \alpha_x^- x$ with all the α_x^+ and α_x^- non-negative and, for any $x \in I\Gamma$, $\alpha_x^+ = 0$ or $\alpha_x^- = 0$. The following fact is immediate.

Proposition 8.15. *For every* $c \in \mathrm{St}_0$*,* $\|\Phi(c)\| \leq \|c\|$*. If* c *is contained in the kernel of the map* $\mathrm{St}_0 \to \mathbb{R}$*, then*

$$\partial\Phi(c) = c. \tag{20}$$

Definition 8.16. [26, The absolute cone] Fix $y \in I\Gamma$ and $k \geq 0$. The *k-dimensional cone* (associated to y) is the map $[y, \cdot]\colon \mathrm{St}_k \to \mathrm{St}_{k+1}$ given by

$$[y, (\gamma_0, \dots, \gamma_k)] := (y, \gamma_0, \dots, \gamma_k) \quad \text{for all } \gamma_0, \dots, \gamma_k \in I\Gamma$$

and extended over the whole St_k by linearity.

It is trivialy seen that $[y, \cdot]$ is a linear map of norm 1 for every k. Moreover,

$$\partial[y, z] = z, \tag{21}$$

for any k-dimensional cycle z, $k \geq 0$.

Let $\mathrm{pr}_*\colon \mathrm{St}_* \to \mathrm{St}_*^{\mathrm{rel}}$ be the projection, and let $j_*\colon \mathrm{St}_*^{\mathrm{rel}} \to \mathrm{St}_*$ be the obvious right inverse of pr_*. This map has norm 1. For any left coset $s \in \Gamma/\Gamma'$ and $a \in \mathrm{St}_*$, let $\partial^s(a)$ be the restriction of ∂a to s.

Definition 8.17. [26, The relative cone] Fix $y \in I\Gamma$. The 1*-dimensional relative cone* (associated to y) is the (non-linear!) map

$$[y, \cdot]_{\mathrm{rel}}\colon \mathrm{St}_1^{\mathrm{rel}} \longrightarrow \mathrm{St}_2^{\mathrm{rel}},$$

$$[y, b]_{\mathrm{rel}} := \mathrm{pr}_2 \left[y, j(b) - \sum_{s\in\Gamma/\Gamma'} \Phi[\partial^s(j(b))] \right] \quad \text{for all } b \in \mathrm{St}_1^{\mathrm{rel}}(\Gamma).$$

We prove Proposition 2.9. Let $b \in \mathrm{St}_1^{\mathrm{rel}}$.

$$\begin{aligned}\|[y,b]_{\mathrm{rel}}\| &= \Big\| \mathrm{pr}\Big[y, j(b) - \sum_{s\in\Gamma/\Gamma'} \Phi[\partial^s(j(b))]\Big]\Big\| \\ &\le \Big\| \Big[y, j(b) - \sum_{s\in\Gamma/\Gamma'} \Phi[\partial^s(j(b))]\Big]\Big\| \\ &= \Big\| \Big[j(b) - \sum_{s\in\Gamma/\Gamma'} \Phi[\partial^s(j(b))]\Big]\Big\| \\ &\le \|j(b)\| + \sum_{s\in\Gamma/\Gamma'} \|\Phi[\partial^s(j(b))]\| \\ &\le \|b\| + \sum_{s\in\Gamma/\Gamma'} \|\partial^s(j(b))]\| \\ &\le \|b\| + \Big\| \sum_{s\in\Gamma/\Gamma'} \partial^s(j(b))]\Big\| \\ &= \|b\| + \|\partial(j(b))]\| \\ &\le \|b\| + 2\|j(b)\| \\ &\le \|b\| + 2\|b\| \\ &= 3\|b\|.\end{aligned}$$

Now, let $b \in \mathrm{St}_1^{\mathrm{rel}}$ be a cycle with respect to the augmentation map

$$\mathrm{St}_1^{\mathrm{rel}} \longrightarrow \Delta, \quad \mathrm{pr}_2(x,y) \longmapsto [y]-[x]$$

(where $[\cdot]$ refers to the class in Γ/Γ'). We prove that

$$\partial^{\mathrm{rel}}[y,b]_{\mathrm{rel}} = b. \tag{22}$$

Write

$$b = \sum_i \lambda_i[x_i, y_i]$$

(if $[x_i,y_i] \notin \Gamma'$ we will identify $[x_i,y_i]$ and $pr([x_i,y_i])$). By hypothesis,

$$\sum_i \lambda_i([y_i]-[x_i]) = 0 \in \Delta \subseteq \mathbb{R}(\Gamma/\Gamma').$$

Equivalently, for any $s \in \Gamma/\Gamma'$, $\sum_{y_i\in s}\lambda_i - \sum_{x_j\in s}\lambda_j = 0$. Hence, $\partial^s j(b) = \sum_{y_i\in s}\lambda_i y_i - \sum_{x_j\in s}\lambda_j x_j$ is a cycle with respect to the augmentation map $\mathrm{St}_0(\Gamma,\Gamma') \to \mathbb{R} \to 0$. Therefore we get

$$\partial \sum_{s\in\Gamma/\Gamma'} \Phi(\partial^s(j(b))) = \sum_{s\in\Gamma/\Gamma'} \partial\Phi(\partial^s(j(b))) = \sum_{s\in\Gamma/\Gamma'} \partial^s(j(b)),$$

because of 20. Moreover

$$\begin{aligned}\partial^{\rm rel}[y,b]_{\rm rel} &:= \partial^{\rm rel}\,{\rm pr}_2\Big[y, j(b) - \sum_{s\in\Gamma/\Gamma'}\Phi[\partial^s(j(b))]\Big]\\ &= {\rm pr}_2\,\partial\Big[y, j(b) - \sum_{s\in\Gamma/\Gamma'}\Phi[\partial^s(j(b))]\Big]\\ &= {\rm pr}_2\Big(j(b) - \sum_{s\in\Gamma/\Gamma'}\Phi[\partial^s(j(b))]\Big)\\ &= b\end{aligned}$$

because $\Phi[\partial^s(j(b))] \in {\rm St}'_2$ for every $s \in \Gamma/\Gamma'$.

8.2. Addendum II: coincidence between Mineyev–Yaman and Blank definitions of relative bounded cohomology. We prove that Blank's definition of relative bounded cohomology for pairs of groupoids, when restricted to group-pairs, coincides with the one of Mineyev and Yaman, up to isometry.

First we briefly sketch Blank's definition of relative bounded cohomology for groupoids. For more details, see [5, Chapter 3]. If G is a groupoid, we write "$g \in G$" if $g \in {\rm Hom}(e, f)$, i.e. if g is a morphism between two objects e and f of G. In that case we also put $s(g) = e$, $t(g) = b$. A *bounded G-module V* is a set of normed vector spaces $= \{V_e\}_{e\in{\rm obj}(G)}$ which carries a bounded groupoid G-action. This means that to any $g \in G$ an operator $\rho_g\colon V_{s(e)} \to V_{t(e)}$ is assigned whose norm is bounded independently of $g \in G$, and the composition rule $\rho_{g\circ h} = \rho_g \circ \rho_h$ is respected when defined (our definition of bounded groupoid module is slightly more general than that of normed G-module in [5, Chapter 3.3.1] in that we consider actions by uniformly bounded operators on normed spaces, instead of isometries on Banach spaces). If V and W are bounded G-modules, by ${\rm Hom}^b_G(V, W)$ we mean the space of bounded maps $(f_e\colon V_e \to W_e)_{e\in{\rm obj}(G)}$ such that $\rho_g \circ f_{s(g)} = f_{t(g)} \circ \rho_g$ and $\|f_e\| \le L$ for some L independent of $e \in {\rm obj}(G)$.

To G we associate the *Bar resolution* $\{\mathscr{C}_n(G)\}_{n\in\mathbb{N}}$ defined as follows. For $n \in \mathbb{N}$ put $\mathscr{C}_n(G) := \{\mathscr{C}_n(G)\}_{e\in{\rm obj}(G)}$, where $(\mathscr{C}_k(G))_e$ is the normed space generated by the $n+1$-tuples $(g_0,\dots,g_n)$ such that $s(g_0) = e$ and $s(g_j) = t(g_{j-1})$, for $1 \le j \le n$, with the corresponding ℓ^1-norm. The module $\mathscr{C}_n(G)$ is equipped with the G-action

$$g \longmapsto \rho_g\colon \mathscr{C}_{s(e)}(G) \longrightarrow \mathscr{C}_{t(e)}(G), \quad \rho_g(g_0, g_1,\dots,g_n) := (gg_0, g_1,\dots,g_n).$$

For $n \ge 1$ we define the boundary map $\mathscr{C}_n(G) \to \mathscr{C}_{n-1}(G)$ by the formula

$$\partial(g_0,\dots,g_n) := \sum_{j=0}^{n-1}(-1)^j(g_0,\dots,g_i\cdot g_{i+1}, g_n) + (-1)^n(g_0,\dots,g_{n-1}).$$

We also have an augmentation

$$\mathcal{C}_0(G) \longrightarrow \mathbb{R}G, \quad g \longmapsto t(g) \cdot 1,$$

where $\mathbb{R}G$ is the groupoid $\{\mathbb{R}_e\}_{e \in \mathrm{obj}(G)}$, where G acts on $\mathbb{R}G$ by mapping g to the map $\mathrm{Id}_{\mathbb{R}}: G_{s(e)} \to G_{t(e)}$ (see [5, Definition 3.2.4]). Notice that we have equipped $\mathcal{C}_k(G)$ with a structure of bounded G-module, and that the boundary maps are G-linear.

If (G, A) is a pair of groupoids (i.e. if A is a subgroupoid of G) we have an inclusion of complexes $\mathcal{C}_*(A) \hookrightarrow \mathcal{C}_*(G)$. The *relative bounded cohomology of* (G, A) *with coefficients in* V is then given by the cocomplex

$$\mathcal{C}_b^*(G, A; V) := \{f \in \mathrm{Hom}_G^b(\mathcal{C}_*(G), V) \colon f_{e|\mathcal{C}_*(A)_e} = 0 \text{ for all } e \in \mathrm{obj}(G)\}$$

and is denoted by $\mathcal{H}_b^*(G, A; V)$ (see [5, Definition 3.5.1(iii), (iv)]).

Let $(\Gamma, \Gamma' = \{\Gamma_i\}_{i \in I})$ be a group-pair. Let Γ_I be the groupoid with $\mathrm{obj}(G) = I$, and $\mathrm{Hom}(i, j) = G$, for all $i, j \in \mathrm{obj}(G)$. If V is a bounded Γ-module, then V_I denotes the bounded Γ_I-module $(V_i)_{i \in \mathrm{obj}(\Gamma_I)}$ with Γ_I-action given by $\rho_g(v) = gv$, where $v \in V_{s(g)}$ and $gv \in V_{t(g)}$. Let $\bigsqcup_{i \in I} \Gamma_i$ be the groupoid with $\mathrm{obj}\left(\bigsqcup_{i \in I} \Gamma_i\right) = I$ and $\mathrm{Hom}(i, j) = G$ if $i = j$, and $\mathrm{Hom}(i, j) = \emptyset$ otherwise (see [5, Definitions 3.1.10, 3.5.11, Examples 3.1.3(iii)]). The *relative bounded cohomology of the group-pair* (Γ, Γ') *with coefficients in* V is defined to be the relative bounded cohomology of the corresponding groupoid-pair $\left(\Gamma_I, \bigsqcup_{i \in I} \Gamma_i\right)$, i.e.

$$\mathcal{H}_b^*(\Gamma, \Gamma'; V) := \mathcal{H}_b^*\Big(\Gamma_I, \bigsqcup_{i \in I} \Gamma_i; V\Big)$$

(see [5, Definition 3.5.12]).

Proposition 8.18. *Let* (Γ, Γ') *be a group-pair, and let* V *be a bounded* Γ*-module. There is a natural isometric chain isomorphism* $\mathcal{C}_b^*(\Gamma, \Gamma'; V) \to \mathrm{St}_b^*(\Gamma, \Gamma'; V)$.

Proof. We see an element in $\mathrm{St}_{(b)}^{\mathrm{rel}\,k}(\Gamma, \Gamma'; V) := \mathrm{Hom}_{(b)}^{\Gamma}(\mathrm{St}_k^{\mathrm{rel}}, V)$ as a Γ-linear map $f: \mathbb{R}(\Gamma \times I)^{k+1} \to V$ which is null on St_k', i.e. on tuples $(x_0, \dots, x_n)$ for which there exists $i \in I$ such that $x_j \in \Gamma \times \{i\}$ for all $0 \le j \le k$ and $x_j \in x_0 \Gamma_i$ for every $1 \le j \le k$. If $i, j \in I$ and $g \in \Gamma$, we write $g^{i \to j}$ for the corresponding element in $\mathrm{Hom}(i, j)$, and g^i for the corresponding element in $\Gamma \times \{i\} \subset I\Gamma$.

Fix $\bar{\imath} \in I$ and consider the maps

$$\varphi^k: \mathcal{C}_b^k(\Gamma_I; V) \longrightarrow St^k(\Gamma, \Gamma'; V) \quad \psi^k: St_{\mathrm{rel}}^k(\Gamma, \Gamma'; V) \longrightarrow \mathcal{C}_b^k(\Gamma, \Gamma'; V)$$

defined as follows: if $f \in \mathcal{C}_b^k(\Gamma, \Gamma'; V)$ we set

$$\varphi^k(f)\ (g_0^{i_0}, \dots, g_n^{i_n}) := f(g_0^{i_0 \to \bar{\imath}}, (g_0^{-1} g_1)^{i_1 \to i_0}, \dots, (g_{n-1}^{-1} g_n)^{i_n \to i_{n-1}}).$$

If $h \in St^k_{\mathrm{rel}}(\Gamma, \Gamma'; V)$ we set

$$\psi^k(h)\,(g_0^{i_0 \to \bar{i}}, g_1^{i_1 \to i_0}, \dots, g_n^{i_n \to i_{n-1}}) = h(g_0^{i_0}, (g_0 g_1)^{i_1}, \dots (g_0 \cdots g_n)^{i_n}).$$

The computations that show that φ^* and ψ^* are mutually inverse chain maps are similar to the ones that prove that the bar-resolution and the homogeneous bar-resolution are isomorphic. Indeed, they resemble dual versions of the ones in [18, Chapter VI 13 (b)]. We simply note that those maps are well-defined, i.e. the restrictions of $\varphi^k(f)$ on St'_k and of $\psi^k(h)$ on $\mathbb{R}\left(\bigsqcup_i \Gamma_i\right)^{n+1}$ are null. Indeed, if $g_0^i, \dots, g_n^i \in (\Gamma_i \times \{i\})^{n+1}$, then $(g_0^{i \to \bar{i}}, (g_0^{-1} g_1)^{i \to i}, \dots, (g_n^{-1} g_n)^{i \to i})$ is an $(n+1)$-tuple of elements in a Γ-translate of $\Gamma_i \subset \mathrm{Hom}(i, i)$, and therefore f is null on it. Conversely, if $(g_0^{i \to i}, g_1^{i \to i}, \dots, g_n^{i \to i})$ is a tuple of elements in $\Gamma_i \subset \mathrm{Hom}(i, i)$, then $(g_0^i, (g_0 g_1)^i, \dots (g_0 \cdots g_n)^i) \in (\Gamma_i \times \{i\})^{n+1}$, hence h is null over it. □

References

[1] F. J. Allcock and S. M. Gersten, A homological characterization of hyperbolic groups. *Invent. Math.* **135** (1999), no. 3, 723–742. Zbl 0909.20032 MR 1669272

[2] M. Bestvina, K. Bromberg, and K. Fujiwara, Bounded cohomology with coefficients in uniformly convex Banach spaces. *Comment. Math. Helv.* **91** (2016), no. 2, 203–218. Zbl 06588943 MR 3493369

[3] M. Bestvina and K. Fujiwara, Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.* **6** (2002), 69–89. Zbl 1021.57001 MR 1914565

[4] R. Bieri and B. Eckmann, Relative homology and Poincaré duality for group pairs. *J. Pure Appl. Algebra* **13** (1978), no. 3, 277–319. Zbl 0392.20032 MR 0509165

[5] M. Blank, *Relative bounded cohomology for groupoids.* Ph.D. thesis. Universität Regensburg, Regensburg, 2014. http://epub.uni-regensburg.de/31298/1/thesis.pdf

[6] B. H. Bowditch, Relatively hyperbolic groups. Preprint 2011. http://www.warwick.ac.uk/~masgak/papers/rhg.pdf

[7] B. H. Bowditch, Tight geodesics in the curve complex. *Invent. Math.* **171** (2008), no. 2, 281–300. Zbl 1185.57011 MR 2367021

[8] M. R. Bridson and A. Haefliger, *Metric spaces of non-Positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999. Zbl 0988.53001 MR 1744486

[9] R. Brooks, Some remarks on bounded cohomology. In I. Kra and B. Maskit (eds.), *Riemann surfaces and related topics*: *Proceedings of the 1978 Stony Brook Conference.* (Stony Brook, N.Y., 1978.) Annals of Mathematics Studies, 97. Princeton University Press, Princeton, N.J., and University of Tokyo Press, Tokyo, 1981, 53–63. Zbl 0677.30016 MR 0624804

[10] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes.* Les groupes hyperboliques de Gromov. Lecture Notes in Mathematics, 1441. Springer-Verlag, Berlin, 1990. Zbl 0727.20018 MR 1075994

[11] D. B. A. Epstein and K. Fujiwara, The second bounded cohomology of word-hyperbolic groups. *Topology* **36** (1997), no. 6, 1275–1289. Zbl 0884.55005 MR 1452851

[12] K. Fujiwara, The second bounded cohomology of a group acting on a Gromov-hyperbolic space. *Proc. London Math. Soc.* (3) **76** (1998), no. 1, 70–94. Zbl 0891.20027 MR 1476898

[13] E. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov.* (Bern, 1988.) Progress in Mathematics, 83. Birkhäuser Boston, Boston, MA, 1990. Zbl 0731.20025 MR 1086648

[14] M. Gromov, Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.* **56** (1982), 5–99. Zbl 0516.53046 MR 0686042

[15] M. Gromov, Hyperbolic groups. In S. M. Gersten (ed.), *Essays in group theory.* Mathematical Sciences Research Institute Publications, 8. Springer-Verlag, New York, 1987, 75–263. Zbl 0634.20015 MR 0919829

[16] D. Groves and J. F. Manning, Relative bounded cohomology and relatively hyperbolic groups. In preparation.

[17] D. Groves and J. F. Manning, Dehn filling in relatively hyperbolic groups. *Israel J. Math.* **168** (2008), 317–429. Zbl 1211.20038 MR 2448064

[18] P. J. Hilton and U. Stammbach, *A course in homological algebra.* Graduate Texts in Mathematics, Vol. 4. Springer-Verlag, Berlin etc., 1971. Zbl 0238.18006 MR 0346025

[19] M. Hull and D. Osin, Induced quasicocycles on groups with hyperbolically embedded subgroups. *Algebr. Geom. Topol.* **13** (2013), no. 5, 2635–2665. Zbl 1297.20045 MR 3116299

[20] B. E. Johnson, *Cohomology in Banach algebras.* Memoirs of the American Mathematical Society, 127. American Mathematical Society, Providence, R.I., 1972. Zbl 0256.18014 MR 0374934

[21] C. Löh, *ℓ^1-homology and simplicial volume*. Ph.D. thesis. Westfälische Wilhelms-Universität Münster, Münster, 2007. http://www.mathematik.uni-regensburg.de/loeh/theses/thesis.pdf

[22] E. Martínez-Pedroza, A note on fine graphs and homological isoperimetric inequalities. *Canad. Math. Bull.* **59** (2016), no. 1, 170–181. Zbl 1360.20037 MR 3451908

[23] I. Mineyev, ℓ^∞-cohomology and metabolicity of negatively curved complexes. *Internat. J. Algebra Comput.* **9** (1999), no. 1, 51–77. Zbl 1161.58300 MR 1695305

[24] I. Mineyev, Straightening and bounded cohomology of hyperbolic groups. *Geom. Funct. Anal.* **11** (2001), no. 4, 807–839. Zbl 1013.20034 MR 1866802

[25] I. Mineyev, Bounded cohomology characterizes hyperbolic groups. *Q. J. Math.* **53** (2002), no. 1, 59–73. Zbl 1013.20048 MR 1887670

[26] I. Mineyev and A. Yaman, Relative hyperbolicity and bounded cohomology. Preprint, 2007. http://www.math.uiuc.edu/~mineyev/math/art/rel-hyp.pdf

[27] J. R. Munkres, *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, etc., 1984. Zbl 0673.55001 MR 0755006

[28] D. V. Osin, Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.* **179** (2006), no. 843. Zbl 1093.20025 MR 2182268

[29] P. Rolli, Quasi-morphisms on free groups. Preprint 2009. arXiv:0911.4234v2 [math.GR]

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