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On self-similarity of wreath products of abelian groups

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Abstract. We prove that in a self-similar wreath product of abelian groups $G = B \operatorname{wr} X$, if X is torsion-free then B is torsion of finite exponent. Therefore, in particular, the group $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ cannot be self-similar. Furthermore, we prove that if L is a self-similar abelian group then $L^{\omega} \operatorname{wr} C_2$ is also self-similar.

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1. Introduction

A group G is self-similar provided for some finite positive integer m, the group has a faithful representation on an infinite regular one-rooted m-tree $\mathcal{T}_m$ such that the representation is state-closed and is transitive on the tree's first level. If a group G does not admit such a representation for any m then we say G is not self-similar. In determining that a group is not self-similar we will use the language of virtual endomorphisms of groups. More precisely, a group G is not self-similar if and only if for any subgroup H of G of finite index and any homomorphism $f: H \to G$ there exists a non-trivial subgroup K of H which is normal in G and is f-invariant (in the sense $K^f \leq K$).

Which groups admit faithful self-similar representation is an on going topic of investigation. The first in depth study of this question was undertaken in [2] and then in book form in [3]. Faithful self-similar representations are known for many individual finitely generated groups ranging from the torsion groups of Grigorchuk and Gupta–Sidki to free groups. Such representations have been also studied for the family of abelian groups [4], of finitely generated nilpotent groups [7], as well as for arithmetic groups [9]. See [8] for further references.

One class which has received attention in recent years is that of wreath products of abelian groups $G = B \operatorname{wr} X$, such as the classical lamplighter group [1] in which B is cyclic of order 2 and X is infinite cyclic. The more general class $G = C_p \operatorname{wr} X$ where C_p is cyclic of prime order p and X is free abelian of rank $d \geq 1$ was the subject of [8] where self-similar groups of this type are constructed for every finite rank d.

We show in this paper that the properly of B being torsion is necessary to guarantee self-similarity of G. More precisely, we prove the following result.

Theorem 1. *Let $G = B \operatorname{wr} X$ be a self-similar wreath product of abelian groups. If X is torsion free then B is a torsion group of finite exponent. In particular, $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ cannot be self-similar.*

Observe that though $G = \mathbb{Z} \operatorname{wr} \mathbb{Z}$ is not self-similar, it has a faithful finite-state representation on the binary tree [5].

Next we produce a novel embedding of self-similar abelian groups into self-similar wreath products having higher cardinality.

Theorem 2. *Let L be a self-similar abelian group and L^ω an infinite countable direct sum of copies of L. Then $L^\omega \operatorname{wr} C_2$ is also self-similar.*

2. Preliminaries

We recall a number of notions of groups acting on trees and of virtual endomorphisms of groups from [4].

2.1. State-closed groups. 1. Automorphisms of one-rooted regular trees $\mathcal{T}(Y)$ indexed by finite sequences from a finite set Y of size $m \geq 2$, have a natural interpretation as automata on the alphabet Y, and with states which are again automorphisms of the tree. A subgroup G of the group of automorphisms $\mathcal{A}(Y)$ of the tree is said to have degree m. Moreover, G is *state-closed* of degree m provided the states of its elements are themselves elements of the same group.

2. Given an automorphism group G of the tree, v a vertex of the tree and l a level of the tree, we let $\mathrm{Fix}_G(v)$ denote the subgroup of G formed by its elements which fix v and let $\mathrm{Stab}_G(l)$ denote the subgroup of G formed by elements which fix all v of level l. Also, let $P(G)$ denote the permutation group induced by G on the first level of the tree. We say G is *transitive* provided $P(G)$ is transitive.

3. A group G is said to be *self-similar* provided it is a state-closed and transitive subgroup of $\mathcal{A}(Y)$ for some finite set Y.

2.2. Virtual endomorphisms. 1. Let G be a group with a subgroup H of finite index m. A homomorphism $f: H \to G$ is called a *virtual endomorphism* of G and (G, H, f) is called a *similarity triple*; if G is fixed then (H, f) is called a *similarity pair*.

2. Let G be a transitive state-closed subgroup of $\mathcal{A}(Y)$ where $Y = \{1, 2, \dots, m\}$. Then the index $[G: \mathrm{Fix}_G(1)] = m$ and the projection on the 1st coordinate of $\mathrm{Fix}_G(1)$ produces a subgroup of G; that is, $\pi_1: \mathrm{Fix}_G(1) \to G$ is a virtual endomorphism of G.

3. Let G be a group with a subgroup H of finite index m and a homomorphism $f: H \to G$. If $U \leq H$ and $U^f \leq U$ then U is called *f-invariant.* The largest subgroup K of H, which is normal in G and is f-invariant is called the f-core(H). If the f-core(H) is trivial then f and the triple (G, H, f) are called *simple.*

4. Given a triple (G, H, f) and a right transversal $L = \{x_1, x_2, \dots, x_m\}$ of H in G, the permutational representation $\pi: G \to \mathrm{Perm}(1, 2, \dots, m)$ is $g^\pi: i \to j$ which is induced from the right multiplication $Hx_i g = Hx_j$. Generalizing the Kalujnine-Krasner procedure [6] , we produce recursively a representation $\varphi: G \to \mathcal{A}(Y)$, defined by

$$g^\varphi = ((x_i g.(x_{(i)g^\pi})^{-1})^{f\varphi})_{1 \leq i \leq m} g^\pi,$$

seen as an element of an infinitely iterated wreath product of $\mathrm{Perm}(1, 2, \dots, m)$. The kernel of φ is precisely the f-core(H) and G^φ is state-closed and transitive and $H^\varphi = \mathrm{Fix}_{G^\varphi}(1)$.

Lemma 1. *A group G is self-similar if and only if there exists a simple similarity pair (H, f) for G.*

3. Proof of Theorem 1

We recall B, X are abelian groups, X is a torsion-free group and $G = B \,\mathrm{wr}\, X$. Denote the normal closure of B in G by $A = B^G$. Let (H, f) be the similarity pair with respect to which G is self-similar and let $[G : H] = m$. Define

$$A_0 = A \cap H, \quad L = (A_0)^f \cap A, \quad Y = X \cap (AH).$$

Note that if $x \in X$ is nontrivial then the centralizer $C_A(x)$ is trivial. We develop the proof in four lemmas.

Lemma 2. *Either B^m is trivial or $(A_0)^f \leq A$. In both cases $A \neq A_0$.*

Proof. We have $A^m \leq A_0$ and $X^m \leq H$. As A is normal abelian and X is abelian,

$$[A^m, X^m] \lhd G,$$

$$[A^m, X^m] \leq [A_0, X^m] \leq A_0.$$

Also,

$$f: [A^m, X^m] \longrightarrow [(A^m)^f, (X^m)^f] \leq (A_0)^f \cap G'$$
$$\leq (A_0)^f \cap A = L.$$

(1) If L is trivial then $[A^m, X^m] \leq \ker(f)$. Since f is simple, it follows that $\ker(f) = 1$ and $[A^m, X^m] = 1 = [B^m, X^m]$. As $X^m \neq 1$, we conclude $A^m = 1 = B^m$.

(2) If L is nontrivial then L is central in $M = A(A_0)^f = A(X \cap M)$ which implies $X \cap M = 1$ and $(A_0)^f \leq A$.

(3) If B is a torsion group then $\mathrm{tor}(G) = A$; clearly, $(A_0)^f \leq A$ and $A \neq A_0$. □

Let G be a counterexample; that is, B has infinite exponent. By the previous lemma $(A_0)^f \leq A$ and so we may use Proposition 1 of [8] to replace the simple similarity pair (H, f) by a simple pair $(\dot{H}, \dot{f})$ where $\dot{H} = A_0 Y$ $(Y \leq X)$ and $(Y)^{\dot{f}} \leq X$. In other words, we may assume $(Y)^f \leq X$.

Lemma 3. *If $z \in X$ is nontrivial and $x_1, \dots, x_t, z_1, \dots, z_l \in X$, then there exists an integer k such that*

$$z^k\{z_1, \dots, z_l\} \cap \{x_1, \dots, x_t\} = \emptyset.$$

Proof. Note that the set $\{k \in \mathbb{Z} \mid \{z^k z_j\} \cap \{x_1, \dots, x_t\} \neq \emptyset\}$ is finite, for each $j = 1, \dots, l$. Indeed, otherwise there exist $k_1 \neq k_2$ such that $z^{k_1 - k_2} = 1$, a contradiction. □

Lemma 4. *If $x \in X$ is nontrivial, then $(x^m)^f$ is nontrivial.*

Proof. Suppose that there exists a nontrivial $x \in X$ such that $x^m \in \ker(f)$. Then for each $a \in A$ and each $u \in X$ we have

$$\begin{aligned}(a^{-mu} a^{mux^m})^f &= (a^{-mu})^f (a^{mux^m})^f \\ &= (a^{-mu})^f ((a^{mu})^f)^{(x^m)^f} \\ &= (a^{-mu})^f (a^{mu})^f \\ &= 1.\end{aligned}$$

Since $A^{m(x^m - 1)} \leq \ker(f)$ and is normal in G, we have a contradiction. □

Lemma 5. *The subgroup A^m is f-invariant.*

Proof. Let $a \in A$. Consider $T = \{c_1, \dots, c_r\}$, a transversal of A_0 in A, where r is a divisor of m. Since A^m is a subgroup of A_0 and $A = \bigoplus_{x \in X} B^x$, there exist $x_1, \dots, x_t$ such that

$$\langle (c_i^m)^f \mid i = 1, \dots, r \rangle \leq B^{x_1} \oplus \cdots \oplus B^{x_t}$$

and $z_1, \dots, z_l \in X$ such that

$$\langle (a^m)^f \rangle \leq B^{z_1} \oplus \cdots \oplus B^{z_l}.$$

Since $[G : H] = m$, it follows that $X^m \leq Y$. Fix a nontrivial $x \in X$ and let $z = (x^m)^f$.

For each integer k, define $i_k \in \{1, \dots, r\}$ such that

$$a^{x^{mk}} c_{i_k}^{-1} \in A_0.$$

Then

$$((a^{x^{mk}} c_{i_k}^{-1})^m)^f = ((a^{x^{mk}} c_{i_k}^{-1})^f)^m \in A^m,$$

but $(a^{x^{mk}} c_{i_k}^{-1})^m = a^{mx^{mk}} c_{i_k}^{-m}$, thus

$$((a^{x^{mk}} c_{i_k}^{-1})^m)^f = (a^{mx^{mk}})^f (c_{i_k}^{-m})^f = (a^{mf})^{z^k} c_{i_k}^{-mf}.$$

By Lemma 4, $z \neq 1$. There exists by Lemma 3 an integer k' such that

$$\{z^{k'} z_1, \dots, z^{k'} z_l\} \cap \{x_1, \dots, x_t\} = \emptyset,$$

and so,

$$(B^{z^{k'} z_1} \oplus \cdots \oplus B^{z^{k'} z_l}) \cap (B^{x_1} \oplus \cdots \oplus B^{x_t}) = 1.$$

It follows that

$$(a^{mf})^{z^{k'}} c_{i_k}^{-mf} \in A^m \cap [(B^{z^{k'} z_1} \oplus \cdots \oplus B^{z^{k'} z_l}) \oplus (B^{x_1} \oplus \cdots \oplus B^{x_t})];$$

But as

$$A^m = \bigoplus_{x \in X} B^{mx}$$

we conclude, $(a^{mf})^{z^{k'}} \in B^{mz^{k'} z_1} \oplus \cdots \oplus B^{mz^{k'} z_l} \leq A^m$ and $a^{mf} \in A^m$. Hence, $(A^m)^f \leq A^m$. □

With this last lemma, the proof of Theorem 1 is finished.

4. Proof of Theorem 2

Let L be a self-similar abelian group with respect to a simple triple (L, M, ϕ); then ϕ is a monomorphism. Define $B = \sum_{i \geq 1} L_i$, a direct sum of groups where $L_i = L$ for each i . Let X be cyclic group of order 2 and $G = B \text{ wr } X$, the wreath product of B by X. Denote the normal closure of B in G by A; then,

$$A = B^X = \Big(L_1 \oplus \sum_{i \geq 2} L_i\Big) \times B$$
$$G = A \bullet X.$$

Define the subgroup of G

$$H = \Big(M \oplus \sum_{i \geq 2} L_i\Big) \times B;$$

an element of H has the form

$$\beta = (\beta_1, \beta_2)$$

where

$$\beta_i = (\beta_{ij})_{j\geq 1}, \quad \beta_{ij} \in L,$$
$$\beta_1 = (\beta_{1j})_{j\geq 1}, \quad \beta_{11} \in M.$$

We note that $[G : H]$ is finite; indeed,

$$[A : H] = [L : M] \quad \text{and} \quad [G : H] = 2[L : M].$$

Define the maps

$$\phi_1' \colon M \oplus \Big(\sum_{i\geq 2} L_i\Big) \longrightarrow B, \quad \phi_2' \colon B \longrightarrow B,$$

where for $\beta = (\beta_1, \beta_2) = ((\beta_{1j}), (\beta_{2j}))_{j\geq 1}, \beta_{11} \in M$,

$$\phi_1' \colon \beta_1 \longmapsto (\beta_{11}^{\phi}\beta_{12}, \beta_{13}, \dots), \quad \phi_2' \colon \beta_2 \longmapsto (\beta_{22}, \beta_{21}, \beta_{23}, \dots).$$

Since L is abelian, ϕ_1' is a homomorphism and clearly ϕ_2' is a homomorphism as well.

Define the homomorphism

$$f \colon \Big(M \oplus \sum_{i\geq 2} L_i\Big) \times B \longrightarrow A$$

by

$$f \colon (\beta_1, \beta_2) \longmapsto ((\beta_1)^{\phi_1'}, (\beta_2)^{\phi_2'}).$$

Suppose by contradiction that K is a nontrivial subgroup of H, normal in G and f-invariant and let $\kappa = (\kappa_1, \kappa_2)$ be a nontrivial element of K. Since X permutes transitively the indices of κ_i, we conclude $\kappa_{i1} \in M$ for $i = 1, 2$. Let s_i (call it degree) be the maximum index of the nontrivial entries of κ_i; if $\kappa_i = 0$ then write $s_i = 0$. Choose κ with minimum $s_1 + s_2$; we may assume s_1 be minimum among those $s_i \neq 0$. Since

$$\kappa_1 = (\kappa_{1j})_{j\geq 1}, \quad \kappa_{11} \in M,$$
$$(\kappa_1)^{\phi_1'} = (\kappa_{11}^{\phi}\kappa_{12}, \kappa_{13}, \dots),$$

we conclude $(\kappa_1)^{\phi_1'}$ has smaller degree and therefore

$$\kappa_1 = (\kappa_{11}, e, e, e, \dots) \quad \text{or} \quad (\kappa_{11}, \kappa_{11}^{-\phi}, e, e, \dots).$$

Suppose $\kappa_1 = (\kappa_{11}, e, e, e, \dots)$. As, $\kappa = (\kappa_1, \kappa_2) \in K$, we have $\kappa_{11} \in M$ and therefore

$$\begin{aligned} \kappa^f &= ((\kappa_1)^{\phi_1'}, (\kappa_2)^{\phi_2'}) \in K, \\ (\kappa_1)^{\phi'} &= ((\kappa_{11})^{\phi}, e, e, e, \dots), \\ (\kappa_{11})^{\phi} &\in M; \end{aligned}$$

$$\begin{aligned} \kappa^{f^2} &= ((\kappa_1)^{(\phi_1')^2}, \kappa_2), \\ (\kappa_1)^{(\phi_1')^2} &= ((\kappa_{11})^{\phi^2}, e, e, e, \dots), \\ (\kappa_{11})^{\phi^2} &\in M; \end{aligned}$$

etc. By simplicity of ϕ, this alternative is out. That is,

$$\kappa_1 = (\kappa_{11}, \kappa_{11}^{-\phi}, e, e, \dots), \quad \kappa_{11} \in M.$$

Therefore

$$\begin{aligned} \kappa &= (\kappa_1, \kappa_2), \\ \kappa^x &= (\kappa_2, \kappa_1), \quad \kappa^{xf} = ((\kappa_2)^{\phi_1'}, (\kappa_{11}^{-\phi}, \kappa_{11}, e, e, \dots)), \\ \kappa^{xfx} &= ((\kappa_{11}^{-\phi}, \kappa_{11}, e, e, \dots), (\kappa_2)^{\phi_1'}) \end{aligned}$$

are elements of K and so, $\kappa_{11}^{\phi} \in M$. Furthermore,

$$\begin{aligned} \kappa^{xfxf} &= ((\kappa_{11}^{-\phi^2} \kappa_{11}, e, e, \dots), (\kappa_2)^{\phi_1'\phi_2'}), \\ \kappa_{11}^{-\phi^2} \kappa_{11} &\in M; \end{aligned}$$

successive applications of f to κ^{xfx} produces $\kappa_{11}^{\phi^i} \in M$. Therefore, $\langle \kappa_{11}^{\phi^i} \mid i \geq 0 \rangle$ is a ϕ-invariant subgroup of M; a contradiction is reached.

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