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Groups, Geometry, and Dynamics

Some virtual limit groups

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Abstract. We show that certain graphs of free groups with cyclic edge groups have a finite index subgroup that is fully residually free.

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1. Introduction

1.1. Limit Groups. The group G is *residually free* if for each nontrivial g, there is a free quotient $G \to \overline{G}$ such that $\bar{g}$ is nontrivial. The group G is *fully residually free* if for each finite set $\{g_1, \ldots, g_r\}$ of nontrivial elements, there is a free quotient $G \to \overline{G}$ such that $\bar{g}_i$ is nontrivial for each i. Finitely generated fully residually free groups are known as *limit groups* because of Sela's reinterpretation [7]. B.Baumslag showed in [2] that when G is residually free, being fully residually free is equivalent to not containing $F_2 \times \mathbb{Z}$ and also equivalent to transitivity of the commutativity relation on $G - \{1_G\}$. In addition to free groups and free-abelian groups, most closed surface groups are fully residually free.

A *double* $F *_{w=w'} F'$ is an amalgamated product of free groups where there is an isomorphism $F \to F'$ with $w \mapsto w'$. For instance, an orientable genus 2 surface group $\langle a, b\rangle *_{[a,b]=[a',b']} \langle a', b'\rangle$ is a double, as is a non-orientable genus 4 surface group: $\langle a, b\rangle *_{aabb=a'a'b'b'} \langle a', b'\rangle$.

The following influential result about doubles was proven in [1]:

Theorem 1.1 (G.Baumslag). *The double $G = F *_{w=w'} F'$ of a free group along a maximal cyclic subgroup is residually free.*

G.Baumslag raised the question of when a free product of free groups amalgamating a cyclic subgroup is residually free. Such groups $F *_{u=v} F'$ are often called *cyclically pinched one-relator groups* in the combinatorial group theory literature.

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1.2. Non residually free examples. There are easy examples of cyclically pinched one-relator groups that are not fully residually free. In particular, $G = F *_{u^m = v^n} F'$ is not residually free when $m, n \geq 2$. Indeed, $[u, v] \neq 1_G$ but $[\bar{u}, \bar{v}] = 1_{\bar{G}}$ in any free quotient $G \to \bar{G}$. Moreover, when $m \geq 3, n \geq 2$, the group G contains $F_2 \times \mathbb{Z}$, and hence so does any finite index subgroup of G, and thus G has no finite index subgroup that is fully residually free by B.Baumslag's characterization.

It is trickier to see that residual freeness of G can fail when the amalgamated subgroup is maximal on one side. However, Lyndon showed that in a free group $(a^2b^2c^2 = 1) \implies (ab = ba)$ [6]. Thus the genus 3 non-orientable surface group $\langle a, b\rangle *_{a^2b^2=c^2} \langle c\rangle$ is not residually free. A non-residually free example that is an amalgam with maximal cyclic subgroup in each factor is the following group obtained in [5]: $\langle a, b\rangle *_{aba^{-1}b^{-1}a^4=cdc^{-1}d^{-1}} \langle c, d\rangle$.

While the non-orientable genus 2 and 3 surface groups are cyclically pinched one-relator groups that are not residually free, they certainly have finite index subgroups that are fully residually free. We are thus led to ask whether this generalizes.

1.3. Main Result. The aim of this note is to prove the following:

Theorem 1.2. *Let $G = A *_C B$ be a group that splits as the free product of two free groups amalgamating a cyclic subgroup. Suppose G does not contain $F_2 \times \mathbb{Z}$. Then G has a finite index subgroup G' that is fully residually free.*

Theorem 1.2 is a special case of a statement about a graph of groups where each edge group is generated by an element with the same translation length in each of its vertex groups. Such a statement is most readily expressed in terms of graphs of spaces as described in the following Theorem whose proof is described at the beginning of Section 3.

Theorem 1.3. *Let X be a graph of spaces where each vertex space X_v is a graph, and each edge space X_e is a graph homeomorphic to a circle, and all attaching maps are combinatorial immersions. Suppose $\pi_1 X$ does not contain $F_2 \times \mathbb{Z}$. Then $\pi_1 X$ has a finite index subgroup that is fully residually free.*

1.4. Problems. I expect the following more general result can be obtained:

Conjecture 1.4. *Let $G = A *_C B$ be an amalgamated product where A, B are fully residually free, and C is cyclic. Suppose G does not contain $F_2 \times \mathbb{Z}$. Then G is virtually fully residually free.*

On the other hand, the following appears to be less straightforward:

Problem 1.5. Let $G = F*_{Z_1^t=Z_2}$ be an HNN extension of a free group with a cyclic edge group. Suppose G is word-hyperbolic [relative to virtually $\mathbb{Z}^2$ subgroups]. Does G have a finite index subgroup that is fully residually free?

Both of these problems touch on the possibility that being virtually fully residually free is equivalent to having a finite index subgroup that is both hyperbolic relative to abelian subgroups and has a hierarchy with abelian edge groups.

1.5. Strategy of the proof of Theorem 1.3. The idea of the proof is to find a finite index subgroup G' of G, and a quotient $\rho\colon G' \to F$ to a free group such that each vertex group in the induced splitting of G' maps injectively to F. We use ρ to see that G' is a limit group, by precomposing ρ with Dehn twists at the edge groups of G', to ensure that any finite set of hyperbolic elements injects. Note that elliptic elements automatically inject because of the first property of ρ.

2. virtual subproduct

Let X be a graph of spaces with underlying graph Γ. Suppose each vertex space X_v is a graph, and that for each edge e of Γ attached along vertices $\iota(e), \tau(e)$, the edge space of $X_e \times [0,1]$ is a graph and the attaching maps $X_e \times \{0\} \to X_{\iota(e)}$ and $X_e \times \{1\} \to X_{\tau(e)}$ are combinatorial immersions with respect to the original graph structures. The map $X \to \Gamma$ sends each X_v to v, and projects each $X_e \times (0,1)$ to the open edge e.

Note that X is a square complex where in addition to the 0-cells and 1-cells in each vertex space X_v, there are 1-cells and 2-cells arising from each product $X_e \times [0,1]$. We regard the edges projecting to vertices of Γ as *vertical* and the edges projecting to edges of Γ as *horizontal*. Accordingly X is a $\mathcal{VH}$-complex in the sense that the attaching map of each 2-cell is of the form $abcd$ where a, c are vertical edges and b, d are horizontal edges. Finally, X is nonpositively curved since the combinatorial immersion condition on the attaching maps ensures that the link of each 0-cell is a simplicial bipartite graph.

The following is a strong form of virtual specialness that holds in our setting:

Theorem 2.1. *Let X be as in Theorem 1.3. There is a finite cover $\widehat{X} \to X$ such that $\widehat{X} \subset V \times H$ is a subcomplex of a product of two graphs.*

Proof. In [8, Thm 5.1] it was shown that $\pi_1 X$ is subgroup separable since it splits as a balanced finite graph of free groups with cyclic edge groups. In [9, Lem 5.7+Lem 16.2] it was shown that if X is a compact nonpositively curved $\mathcal{VH}$-complex, and the edge groups of the splitting of $\pi_1 X$ are separable, then X has a finite cover $\widehat{X}$ that is isomorphic to a subcomplex of a product of two graphs. $\square$

Combined with Theorem 2.1, the following Lemma could be used to complete the proof of Theorem 1.3.

Lemma 2.2. *Let G split as a graph of groups, where each vertex group is free, and each edge group is infinite cyclic. Suppose G does not contain $F_2 \times \mathbb{Z}$ or $\langle a, b \mid aabb \rangle$. Suppose there is a free quotient $\rho: G \to F$ that is injective on each vertex group. Then G is fully residually free.*

A weak form of Lemma 2.2 asserting that G is virtually fully residually free, is actually a consequence of Theorem 1.3. Indeed, the homomorphism ρ shows how to metrize the vertex groups so that each edge group has the same length on each side.

In the next section we will focus on a special case of Lemma 3.3 which is proved using a geometric argument together with Dehn twists. That geometric argument can also be used to prove Lemma 2.2, and the main issues are explained in Remark 3.4. Lemma 2.2 can also be proven using Dehn twists together with standard combinatorial group theory ideas about free cancellation in free groups.

3. Proof that G' is fully residually free

Proof of Theorem 1.3. By Theorem 2.1 there is a finite cover $\widehat{X} \to X$ such that $\widehat{X} \subset V \times H$ is a subcomplex of a product of two graphs. Moreover, under $\widehat{X} \to H$, the preimage of each edge of H is the union of open cylinders.

Let $\{g_1, \dots, g_r\}$ be a set of nontrivial elements of $\pi_1 \widehat{X}$. By Lemma 3.3, there exists a homeomorphism $\phi: \widehat{X} \to \widehat{X}$ such that for sufficiently large m, the conjugacy class of each $\phi_*^m(g_i)$ is represented by a closed geodesic that is the concatenation of vertical and diagonal geodesics. By Lemma 3.2, the projection $\rho: \widehat{X} \to V$ maps each such closed geodesic to a closed immersed path in V. Thus $\rho_* \circ \phi_*^m: \pi_1 \widehat{X} \to \pi_1 V$ provides a homomorphism to a free group that is nontrivial on each g_i. □

Definition 3.1 (cylindrical subspaces). Two closed edge spaces of $\widetilde{X}$ are *equivalent* if one lies in a finite neighborhood of the other. A *cylindrical subspace* $\widetilde{\Lambda} \subset \widetilde{X}$ is the union of all edge spaces in an equivalence class. A *cylindrical subspace* $\Lambda \subset \widehat{X}$ is the image of a cylindrical subspace of $\widetilde{X}$.

Consider the quotient map $\widehat{X} \to \widehat{\Gamma}$ to the underlying graph $\widehat{\Gamma}$ of the induced splitting of $\widehat{X}$ as a graph of spaces. We can assume that $\widehat{X} \to H$ is surjective, and there is thus a surjection $H \to \widehat{\Gamma}$, however this map might not be an isomorphism.

Each cylindrical space $\Lambda_j \subset \widehat{X}$ equals $c_j \times \widehat{\Gamma}_j$ for some subgraph $\widehat{\Gamma}_j \subset \widehat{\Gamma}$ and some embedded circle $c_j \subset V$. We have $\chi(\widehat{\Gamma}_j) \geq 0$, i.e. $\widehat{\Gamma}_j$ is either a tree or is unicyclic, for otherwise $\mathbb{Z} \times F_2 \subset \pi_1 X$. Each edge of $\widehat{\Gamma}$ lies in a unique $\widehat{\Gamma}_j$, and $\widehat{\Gamma} = \cup_j \widehat{\Gamma}_j$.

Every nontrivial conjugacy class in the fundamental group of a compact non-positively curved complex is represented by a closed geodesic [4].

A *vertical* geodesic in $\widehat{X}$ is a nontrivial closed geodesic that projects to a singleton under the map $\widehat{X} \to H$. A *cylindrical* geodesic in $\widehat{X}$ is a nontrivial closed geodesic that is not vertical and lies in a cylindrical subspace of $\widehat{X}$. The cylindrical geodesic is *horizontal* if it projects to a singleton under $\rho\colon \widehat{X} \to V$, and it is *diagonal* if it is not horizontal. Any closed geodesic γ in $\widehat{X}$ is the concatenation of maximal vertical geodesics and cylindrical geodesics.

Lemma 3.2 (immersion). *Let $\gamma \to \widehat{X}$ be a nontrivial closed geodesic that is the concatenation of verticals and diagonals. Then $\rho \circ \gamma$ is an immersed path in V.*

Proof. The map $\rho\colon \widehat{X} \to V$ obviously projects a vertical geodesic γ_v and a diagonal geodesic γ_d to an immersed path in V. The map ρ preserves the immersion property at the concatenation between vertical and diagonal geodesics. That is, concatenations $\rho(\gamma_v)\rho(\gamma_d)$ and $\rho(\gamma_d)\rho(\gamma_v)$ are immersed paths in V. See Figure 1. Indeed, in $\widetilde{X}$, the flat strip carrying the initial part of γ_d intersects the terminal edge of γ_v at a single vertex. For a concatenation between diagonal geodesics $\gamma_d\gamma'_d$, the map ρ projects this concatenation to a local geodesic in V, since in the case where the concatenation point lies in the interior of the attaching line intersection, γ_d and γ'_d have the same nonzero slope and ρ looks locally like projection of $\mathbb{R}^2 \to \mathbb{R}^1$. And in the case where the concatenation point is an endpoint of the attaching line intersections, then either $\rho(\gamma_d), \rho(\gamma'_d)$ are disjoint rays not in the intersection, or one is a ray in the intersection and the other is a ray outside the intersection. □

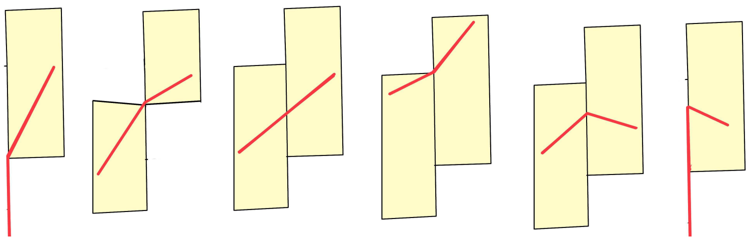

Figure 1. On the left are four possible concatenation scenarios. The two impossible scenarios on the right have a concatenation that is not a geodesic.

Lemma 3.3. *Let $\widehat{X}$ be a subcomplex of a product of two graphs $V \times H$. Suppose $\widehat{X} \cap (V \times h)$ is the union of open cylinders, for each open edge h of H.*

There is a homeomorphism $\phi\colon \widehat{X} \to \widehat{X}$ such that for each nontrivial $g \in \pi_1\widehat{X}$, for all sufficiently large m, the element $\phi_*^m(g)$ is represented by a closed geodesic that is the concatenation of vertical and diagonal geodesics.

Proof. We assign a *weight* $w(e)$ to each edge e of $\widehat{\Gamma}$ as follows: If $\widehat{\Gamma}_j$ is a tree with edges $\{e_1, \ldots, e_m\}$ we let $w(e_i) = 3^i$ for $1 \leq i \leq m$. If $\widehat{\Gamma}_j$ is a unicyclic graph containing a k-cycle $f_1 \cdots f_k$ together with m edges $e_1, \ldots, e_m$, then we again let $w(e_i) = 3^i$, and we let $w(f_\ell) = 3^{m+1}$ for $1 \leq \ell \leq k$. The key property of these weights is that there is no non-zero dependence relation of these weights with coefficients in $\{-1, 0, 1\}$ for $w(e_i)$ and with nonnegative integer coefficients for $w(f_\ell)$.

We direct the edges of $\widehat{\Gamma}$ so that for each cycle above, its edges are directed cyclically. For each directed edge e of $\widehat{\Gamma}$, we let ϕ_e be the degree w_e Dehn twist at the cylinder over e, so a horizontal edge $\hat{e}$ of $\widehat{X}$ projecting to e has the property that ρ projects $\phi_e(\hat{e})$ to $c_j^{w(e)}$ where e lies in Λ_j. Note that $\phi_e\phi_{e'} = \phi_{e'}\phi_e$ for edges e, e'. Let ϕ be the product of all ϕ_e as e ranges over the edges of Γ.

The impact of ϕ^m on any immersed cylinder $c_e \times \sigma \subset \Lambda_j \subset \widehat{X}$ is a nontrivial Dehn twist, since the path $\sigma \to \widehat{\Gamma}_j$ traverses all cycle-edges with the same orientation, and traverses each tree-edge at most once positively and at most once negatively, so its total contribution to the twist is its weight times $\{-1, 0, 1\}$.

We now show that $\phi^m(g)$ is represented by a closed geodesic with no horizontal subgeodesics for all $m \gg 0$. Let γ be a closed geodesic representing the conjugacy class of g. Let γ_m be the closed geodesic representing the conjugacy class of $\phi^m(g)$.

In the special case where γ is a single closed cylindrical subgeodesic, $\tilde{\gamma}$ lies in a torus within some cylindrical subspace of $\widehat{X}$. It is then clear that $\gamma_m = \phi_m(\gamma)$ is horizontal for at most one value of m.

Observe that $\phi^m(\tilde{\gamma})$ and $\tilde{\gamma}_m$ have the same projection in the Bass-Serre tree $\widetilde{\Gamma}$ of $\widetilde{X}$ associated to the graph of spaces $\widehat{X} \to \widehat{\Gamma}$. Since preimages of points are convex, we see that, like $\phi^m(\tilde{\gamma})$, each cylindrical subgeodesic of $\tilde{\gamma}_m$ maps injectively to T. Finally, each maximal cylindrical subgeodesic of $\tilde{\gamma}_m$ and of $\phi^m(\tilde{\gamma})$ lies in a product $\mathbb{R} \times \sigma$ where $\sigma \to \Lambda_j$ is an immersed path. We will show that the initial [terminal] transition point of corresponding cylindrical subgeodesics of γ_m and $\phi_m(\gamma)$ are within a uniformly bounded distance of each other. Indeed, if the transition is with a vertical subgeodesic then since it is ϕ-invariant, we see that the transition points are actually equal to the same vertex. And otherwise the initial [terminal] point lie in a vertical segment isomorphic to some component $c_j \cap c_k$ when the transition is between a Λ_j and Λ_k cylindrical subspace. Indeed, $\tilde{\gamma}_m$ must travel between these consecutive cylindrical spaces, and hence the transitions must occur between such components. Let E be the number of edges in V, so E exceeds the diameter of any $c_i \cap c_j$ with $i \neq j$. The result follows when m is sufficiently large to ensure that the vertical positions of the endpoints of each cylindrical subpath of $\phi_m(\tilde{\gamma})$ differ by $2E$. Indeed, the corresponding cylindrical subgeodesic in $\tilde{\gamma}_m$ cannot be horizontal since its endpoints must then have distinct vertical positions. However, this difference grows linearly with m because of the nontriviality of the Dehn twist within each cylindrical subspace. □

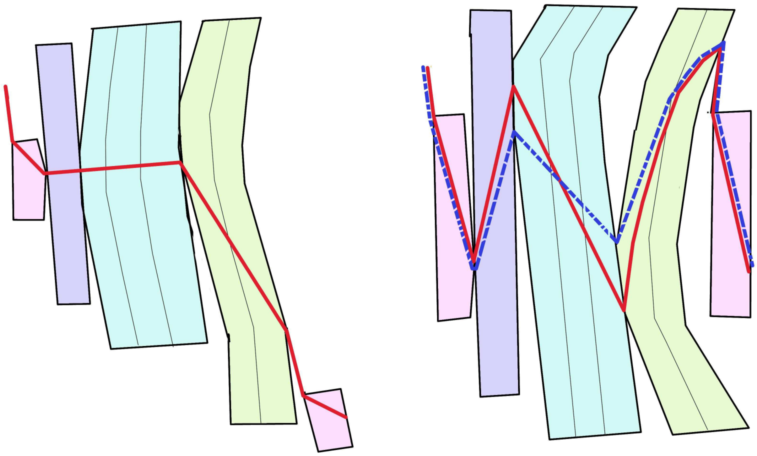

Figure 2. ϕ^m maps $\tilde{\gamma}$ to a path $\phi^m(\tilde{\gamma})$ that is closely fellow travelled by a geodesic $\tilde{\gamma}_m$ which thus cannot have a horizontal cylindrical subgeodesic.

Remark 3.4. The proof of Lemma 3.3 can be adjusted to handle the slightly more general scenario of Lemma 2.2. Indeed, the quotient $\rho\colon G \to F$ allows us to geometrize the situation, and the embeddedness of the edge spaces plays little role in the proof, and the constant E is replaced by an upper bound on the overlap between axes of incommensurable edge groups within vertex groups. The key point is that a cylindrical path σ will become diagonal after Dehn twisting, and this is protected by the algebraic exclusion. Indeed, the Dehn twisting is ineffective precisely when there is an immersion $C \times B \to \widehat{X}$ or $K \to \widehat{X}$, where C is a circle and B is a bouquet of two circles labeled b_1, b_2, and where K is obtained from a cylinder by quotienting each boundary circle by a nontrivial covering space automorphism (e.g. a Klein bottle). In each case, there is a cylindrical path that remains horizontal under any Dehn twist. In the case of $C \times B$ the path is the commutator $[b_1, b_2]$. In the case of K, there is a path of the form $h_1 h_2^{-1}$ where h_1, h_2 are horizontal edges in K that project to the same directed edge of the underlying graph $\widehat{\Gamma}$.

Remark 3.5. The referee proposes that there might be a relationship with Sela's notion of "strict resolution" – see [7, Def 5.9] or [3, Def 1.25]. It is suggested that ρ_* is strict with respect to the abelian decomposition induced from $\widehat{\Gamma}$ by "popping off" tori to make them into abelian vertex groups. The strictness of ρ_* implies that $\pi_1 \widehat{X}$ is a limit group. I hope the referee's suggestion might help a reader interested in continuing the direction of this work.

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