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On irreducibility of Koopman representations corresponding to measure contracting actions

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Abstract. We introduce a notion of measure contracting actions and show that Koopman representations corresponding to ergodic measure contracting actions are irreducible. We also show that the actions of Higman–Thompson groups on intervals equipped with Lebesgue measure and the actions of weakly branch groups on the boundaries of rooted trees equipped with non-uniform Bernoulli measures are measure contracting. This gives a new point of view on irreducibility of the corresponding Koopman representations.

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1. Introduction

One of the most natural representations that one can associate to a measure class preserving action of a group G on a measure space (X, μ), where μ is a probabilty measure, is the Koopman representation κ of G in $L^2(X, \mu)$ defined by

$$(\kappa(g)f)(x) = \sqrt{\frac{\mathrm{d}\mu(g^{-1}(x))}{\mathrm{d}\mu(x)}}\, f(g^{-1}x).$$

This representation is important due to the fact that the spectral properties of κ reflect the measure-theoretic and dynamical properties of the action such as ergodicity and weak-mixing.

It is known that for an ergodic action operators $\kappa(g)$ together with operators of multiplication by functions from $L^\infty(X, \mu)$ generate the von Neumann algebra of all bounded operators on $L^2(X, \mu)$. A natural question is whether the operators $\kappa(g), g \in G$ generate the algebra of all bounded operator by themselves, that is whether κ is irreducible. Below are several examples of group actions with quasi-invariant measures for which the Koopman representation is known to be irreducible:

- actions of free non-commutative groups on their boundaries ([12] and [13]);
- actions of lattices of Lie-groups (or algebraic-groups) on their Poisson-Furstenberg boundaries ([7] and [3]);
- action of the fundamental group of a compact negatively curved manifold on its boundary endowed with the Paterson-Sullivan measure class ([2]);
- natural actions of Thompson's groups F and T on the unit segment ([14]);
- action of the group of compactly supported contactomorphisms of a contact manifold ([15]);
- actions of weakly branch groups on the boundaries of the corresponding rooted trees equipped with non-uniform Bernoulli measures ([11]).

However, the general question in what cases the Koopman representation is irreducible remains open.

In the present paper we introduce a notion of a *measure contracting action* and show that Koopman representations corresponding to ergodic measure contracting actions are irreducible. Using the above we show that Koopman representations corresponding to natural actions of Higman–Thompson groups are irreducible and reconsider the Koopman representations of weakly branch groups presented in [11]. We also notice that the measure contracting property applies to other interesting group actions. For example, Łukasz Garncarek pointed to the author that the results of the present paper can be used to prove irreducibility of the Koopman representation of the group of inner automorphims of a foliation.

Definition 1. Let G act on a probability space (X, μ) with a quasi-invariant measure μ. We will call this action *measure contracting* if for every measurable subset $A \subset X$ and any $M, \epsilon > 0$ there exists $g \in G$ such that

(1) $\mu(\mathrm{supp}(g) \setminus A) < \epsilon$;

(2) $\mu\left(\left\{x \in A\colon \sqrt{\frac{\mathrm{d}\mu(g(x))}{\mathrm{d}\mu(x)}} < M^{-1}\right\}\right) > \mu(A) - \epsilon.$

Here $\mathrm{supp}(g) = \{x \in X\colon gx \neq x\}$. One of the main results of the present paper is the following:

Theorem 1. *For any ergodic measure contracting action of a group G on a probability space (X, μ) the associated Koopman representation κ of G is irreducible.*

Apparently, the most famous group from the family of Higman–Thompson groups is the Thompson group $F_{2,1}$ consisting of all piecewise linear continuous transformations of the unit interval with singularities at the points $\{\frac{p}{2^q}\colon p, q \in \mathbb{N}\}$ and slopes in $\{2^q\colon q \in \mathbb{Z}\}$. This group satisfies a number of unusual properties and disproves several important conjectures in group theory. The group $F_{2,1}$ is infinite but finitely presented, is not elementary amenable, has exponential growth, and does not contain a subgroup isomorphic to the free group of rank 2. An important open question is whether the group $F_{2,1}$ is amenable. Further discussion of historical importance of Higman–Thompson groups and their various algebraic properties can be found in [6], [8], and [4]. Each of the groups $G_{n,r}$ and $F_{n,r}$ acts canonically on $[0,r]$. Denote by λ_r the Lebesgue probability measure on $[0,r]$. Using Theorem 1 we show the following:

Theorem 2. *Let G be a group from the Higman–Thompson families $\{F_{n,r}\}$, $\{G_{n,r}\}$. Then the Koopman representation of G corresponding to the canonical action of G on $([0,r],\lambda_r)$ is irreducible.*

In [14] Garncarek proved irreducibility of a more general class of Koopman type representations (with Radon-Nikodym derivative twisted by a cocycle) of the Thompson groups $F = F_{1,2}$ and T. In fact, Garncarek's methods can be adapted to Koopman representations of Higman–Thompson groups. However, in the present paper we use a different approach to prove irreducibility of these representations.

In [10] the author of the present paper jointly with Medynets showed that Higman–Thompson groups have only discrete set of finite type factor-representations using the notion of a *compressible action*. We notice that the notion of measure contracting actions we introduce in the present paper is loosely related to the notion of compressible actions.

The second class of representations we consider is Koopman representations of weakly branch groups. A group acting on a rooted tree T is called weakly branch if it acts transitively on each level of the tree and for every vertex v of T it has a nontrivial element g supported on the subtree T_v emerging from v (see e.g. [1] and [16]). Weakly branch groups posses interesting and often unusual properties. The class of weakly branch groups contains groups of intermediate growth, amenable but not elementary amenable groups, groups with finite commutator width etc.. Weakly branch groups also play important role in studies in holomorphic dynamics (see [18]) and in the theory of fractals (see [17]).

For a d-regular rooted tree T its boundary ∂T can be identified with a space of sequences $\{x_j\}_{j\in\mathbb{N}}$ where $x_j \in \{1,\dots,d\}$. For a collection of positive real numbers $p = \{p_1,\dots,p_d\}$ with $p_1 + \cdots + p_d = 1$ let μ_p be the corresponding Bernoulli measure on ∂T. In [11] the authors showed the following:

Theorem 3. *Let G be a subexponentially bounded weakly branch group acting on a regular rooted tree and p be as above such that p_i are pairwise distinct. Then the Koopman representation associated to the action of G on $(\partial T,\mu_p)$ is irreducible.*

Here subexponentially bounded group means a group consisting of subexponentially bounded (in the sense similar to polynomial boundedness of Sidki [19]) automorphisms of T. In the present paper using the results of [11] we show that actions of subexponentially bounded weakly branch groups on $(\partial T, \mu_p)$ (with p_i pairwise distinct) are measure contracting. This gives a different view on the proof of Theorem 3 presented in [11].

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2. Proof of Theorem 1

For a measure space (X, μ) denote by $\mathcal{L}(X, \mu)$ the von Neumann algebra generated by operators of multiplication by functions from $L^\infty(X, \mu)$ on $L^2(X, \mu)$. Let $\mathcal{B}(X, \mu)$ be the algebra of all bounded linear operators on $L^2(X, \mu)$. For a set of operators $\mathcal{S} \subset \mathcal{B}(X, \mu)$ denote by

$$\mathcal{S}' = \{A \in \mathcal{B}(X, \mu) \colon AB = BA \text{ for all } B \in \mathcal{S}\}$$

the commutant of $\mathcal{S}$. The following result is folklore.

Theorem 4. *Let group G act ergodically by measure class preserving transformations on a standard Borel space (X, μ). Then the von Neumann algebra $\widetilde{\mathcal{M}}_\kappa$ generated by operators from $\mathcal{M}_\kappa$ and $\mathcal{L}^\infty(X, \mu)$ coincides with $\mathcal{B}(X, \mu)$.*

Proof. By von Neumann bicommutant theorem (see e.g. [5], Theorem 2.4.11) it is sufficient to show that the commutant $\widetilde{\mathcal{M}}'_\kappa$ consists only from scalar operators. Let $A \in \widetilde{\mathcal{M}}'_\kappa$. Since $\mathcal{L}(X, \mu)$ is a maximal abelian subalgebra of $\mathcal{B}(X, \mu)$ (see e.g. [9], Lemma 8.5.1) we obtain that $A \in \mathcal{L}(X, \mu)$. That is, A is the operator of multiplication by a function $m \in L^\infty(X, \mu)$. Since A commute with $\kappa(g)$ for all $g \in G$ the function m is G-invariant ($m(gx) = m(x)$ for all $g \in G$ for almost all $x \in X$). By ergodicity, m is constant almost everywhere. Therefore, operator A is scalar. This finishes the proof. □

Proof of Theorem. First, for every measurable subset $A \subset X$ fix a sequence of elements g_m^A such that

1) $\mu(\mathrm{supp}(g_m^A) \setminus A) < \frac{1}{m}$,
2) $\mu\left(\left\{x \in A \colon \sqrt{\frac{\mathrm{d}\mu(g_m^A(x))}{\mathrm{d}\mu(x)}} < \frac{1}{m}\right\}\right) > \mu(A) - \frac{1}{m}$

for every $m \in \mathbb{N}$. For a subset $B \subset X$ denote by P^B the orthogonal projection onto the subspace

$$\mathcal{H}^B = \{\eta \in L^2(X, \mu) \colon \mathrm{supp}(\eta) \subset X \setminus B\}.$$

Let us show that for every measurable subset $A \subset X$ one has

$$w - \lim_{m\to\infty} \pi(g_m^A) = P^A, \tag{2.1}$$

where $w - \lim$ stands for the limit in the weak operator topology.

Fix a measurable subset $A \subset X$. Introduce the sets

$$A_m = \left\{x \in A: \sqrt{\frac{\mathrm{d}\mu(g_m^A(x))}{\mathrm{d}\mu(x)}} < \frac{1}{m}\right\}, \quad B_m = \mathrm{supp}(g_m^A) \setminus A, \quad C_m = B_m \cup (A \setminus A_m).$$

Notice that

$$\mu(A \setminus A_m) < \frac{1}{m} \quad \text{and} \quad \mu(B_m) < \frac{1}{m}.$$

To prove (2.1) it is sufficient to show that for any $\eta_1, \eta_2 \in L^2(X, \mu)$ one has

$$(\pi(g_m^A)\eta_1, \eta_2) \longrightarrow (P^A\eta_1, \eta_2)$$

when $m \to \infty$. Since the subspace of essentially bounded functions $L^2(X, \mu) \cap L^\infty(X, \mu)$ is dense in $L^2(X, \mu)$ we can assume that η_1 and η_2 are essentially bounded. Let $M_i = \|\eta_i\|_\infty, i = 1, 2$. We have

$$\begin{aligned}
&|(\pi((g_m^A)^{-1})\eta_1, \eta_2) - (P^A\eta_1, \eta_2)| \\
&= \left| \int_X \sqrt{\frac{\mathrm{d}\mu(g_m^A(x))}{\mathrm{d}\mu(x)}} \eta_1(g_m^A x)\overline{\eta_2(x)}\mathrm{d}x - \int_{X\setminus A} \eta_1(x)\overline{\eta_2(x)}\mathrm{d}x \right| \\
&\leqslant \frac{1}{m}\left| \int_{A_m} \eta_1(g_m^A x)\overline{\eta_2(x)}\mathrm{d}x \right| + \left| \int_{C_m} \sqrt{\frac{\mathrm{d}\mu(g_m^A(x))}{\mathrm{d}\mu(x)}} \eta_1(g_m^A x)\overline{\eta_2(x)}\mathrm{d}x \right| \\
&\quad + \left| \int_{B_m} \eta_1(x)\overline{\eta_2(x)}\mathrm{d}x \right| \\
&\leqslant \frac{1}{m} M_1 M_2 + |(\pi((g_m^A)^{-1})\eta_1, P^{X\setminus C_m}\eta_2)| + \frac{1}{m} M_1 M_2 \longrightarrow 0
\end{aligned}$$

when $m \to \infty$, since $\|\pi((g_m^A)^{-1})\eta_1\| = \|\eta_1\|$ (in the norm on $L^2(X, \mu)$) and $P^{C_m/X}\eta_2 \to 0$ when $m \to \infty$. Observe that for every $g \in G$ one has $(\pi(g^{-1})\eta_1, \eta_2) = \overline{(\pi(g)\eta_2, \eta_1)}$. This finishes the proof of (2.1). In particular, we obtain that $P^A \in \mathcal{M}_\kappa$ for every measurable subset $A \subset X$.

Observe that every function from $L^\infty(X, \mu)$ can be approximated arbitrarily well in L^2-norm by finite linear combinations of characteristic functions of open sets. This implies that for every $m \in L^\infty(X, \mu)$ the operator of multiplication by m

$$\mathcal{H} \longrightarrow \mathcal{H}, \quad f \longrightarrow mf$$

can be approximated arbitrary well in the strong operator topology by finite linear combinations of projections $P_A \in \mathcal{M}_\kappa$, and thus belongs to the von Neumann algebra $\mathcal{M}_\kappa$ generated by operators $\kappa(g), g \in G$. It follows that $\mathcal{M}_\kappa$ contains the algebra $\mathcal{L}(X, \mu)$. Using Theorem 4 we obtain that $\mathcal{M}_\kappa$ coincides with $\mathcal{B}(X, \mu)$. This finishes the proof of Theorem 1. □

Remark 1. Garncarek pointed to the author that the proof of Theorem 1 works to show irreducibility of a more general class of Koopman type representations. Namely, representations of the form:

$$(\kappa_\gamma(g)f)(x) = \gamma(g^{-1}, x)\sqrt{\frac{\mathrm{d}\mu(g^{-1}(x))}{\mathrm{d}\mu(x)}}f(g^{-1}x),$$

where $\gamma: G \times X \to \{z \in \mathbb{C}: |z| = 1\}$ is any cocycle.

3. Higman–Thompson groups

Let us briefly recall the definition of Higman–Thompson groups. For details we refer the reader to [6], [8], and [4].

Definition 2. Fix two positive integers n and r.

(1) The group $F_{n,r}$ consists of all orientation preserving piecewise linear homeomorphisms h of $[0, r]$ such that all singularities of h are in $\mathbb{Z}[1/n] = \left\{\frac{p}{n^k}: p, k \in \mathbb{N}\right\}$ and the derivative of h at any non-singular point is n^k for some $k \in \mathbb{Z}$.

(2) The group $G_{n,r}$ is the group of all right continuous piecewise linear bijections h of $[0, r)$ with finitely many discontinuities and singularities, all in $\mathbb{Z}[1/n]$, such that the derivative of h at any non-singular point is n^k for some $k \in \mathbb{Z}$ and h maps $\mathbb{Z}[1/n] \cap [0, r)$ to itself.

Proposition 1. *Let G be a group from the Higman–Thompson families $\{F_{n,r}\}$, $\{G_{n,r}\}$. Then the canonical action of G on $([0, r], \lambda_r)$ is ergodic.*

Proof. Since $F_{n,r} < G_{n,r}$ without loss of generality we can assume that $G = F_{n,r}$ for some n, r. Let A be a G-invariant measurable subset of $[0, r]$ such that $0 < \lambda_r(A) < 1$. Denote by Λ the set of segments $I \subset (0, r)$ of the form $I = \left[\frac{(n-1)p}{n^m}, \frac{(n-1)(p+1)}{n^m}\right]$, where $p, m \in \mathbb{N}$. Corollary A5.6 from [4] implies that for any $I_1, I_2 \in \Lambda$ there exists $g \in G$ which maps I_1 onto I_2. Replacing, if necessary, g on I_1 by the linear orientation preserving map sending I_1 onto I_2 we may assume that $g'(x)$ is constant on I_1. G-invariance of A implies that

$$\frac{\lambda_r(A \cap I_1)}{\lambda_r(I_1)} = \frac{\lambda_r(A \cap I_2)}{\lambda_r(I_2)}.$$

It follows that $\frac{\lambda_r(A\cap I)}{\lambda_r(I)}$ does not depend on $I \in \Lambda$. Since every measurable subset $B \subset [0,r]$ can be approximated by measure arbitrarily well by finite unions of segments from Λ we obtain that the ratio $\frac{\lambda_r(A\cap B)}{\lambda_r(B)}$ is the same for all measurable subset $B \subset [0,r]$ with $\lambda_r(B) > 0$. Taking $B = A$ and $B = [0,r]$ we arrive at a contradiction which finishes the proof. □

Proposition 2. *Let G be a group from the Higman–Thompson families $\{F_{n,r}\}$, $\{G_{n,r}\}$. Then the canonical action of G on $([0,r],\lambda_r)$ is measure contracting.*

Proof. Since $F_{n,r} < G_{n,r}$ it is sufficient to consider the case $G = F_{n,r}$. Introduce a sequence g_m of elements of G as follows:

$$g_m(x) = \begin{cases} x & \text{if } 0 \leqslant x < \frac{r}{n^{2m}}, \\ n^m x - \frac{r}{n^m} + \frac{r}{n^{2m}} & \text{if } \frac{r}{n^{2m}} \leqslant x < \frac{r}{n^m}, \\ r - \frac{r}{n^m} + \frac{x}{n^m} & \text{if } \frac{r}{n^m} \leqslant x \leqslant r. \end{cases} \tag{3.1}$$

Observe that $\lambda_r\left(\left\{x \in [0,r]\colon \sqrt{\frac{\mathrm{d}\lambda_r(g_m(x))}{\mathrm{d}\lambda_r(x)}} > n^{-m}\right\}\right) = n^{-m}$.

For any segment $I \subset [0,r]$ of the form

$$I = \left[r\frac{p}{n^m}, r\frac{p+1}{n^m}\right], \tag{3.2}$$

where $m \in \mathbb{N}$ and $p \in \mathbb{Z}_+$, let J_I be the unique increasing affine map sending $[0,r]$ onto I. Introduce elements $g_m^I \in F_{n,r}$ by

$$g_m^I(x) = \begin{cases} J_I g_m J_I^{-1}(x) & \text{if } x \in I, \\ x & \text{otherwise.} \end{cases}$$

We will call a set $A \subset [0,1]$ admissible if it is a finite union of segments of the form (3.2). For an admissible set $A \subset [0,r]$ and any sufficiently large $m \in \mathbb{N}$ fix a partition $A = I_1 \cup I_2 \cup \cdots \cup I_k$, where each of I_j is of the form (3.2) and I_j intersect I_l for $l \neq j$ by at most one point, and set $g_m^A = g_m^{I_1} g_m^{I_2} \cdots g_m^{I_k}$. Clearly, one has

$$\operatorname{supp}(g_m^A) \subset A \quad \text{and} \quad \lambda_r\left(\left\{x \in A\colon \sqrt{\frac{\mathrm{d}\lambda_r(g_m(x))}{\mathrm{d}\lambda_r(x)}} > n^{-m}\right\}\right) = n^{-m}\lambda_r(A).$$

Since any measurable subset of $[0,r]$ can be approximated by measure arbitrarily well by admissible sets the latter implies that the action of G on $[0,r]$ is measure contracting. □

As a Corollary of Propositions 1 and 2, and Theorem 1 we obtain Theorem 2.

4. Weakly branch groups.

First, let us give a brief introduction to weakly branch groups. See e.g. [1] and [16] for details.

A rooted tree is a tree T with vertex set divided into levels V_n, $n \in \mathbb{Z}_+$, such that V_0 consists of one vertex v_0 (called the root of T), the edges are only between consecutive levels, and each vertex from V_n, $n \geqslant 1$ (we consider infinite trees), is connected by an edge to exactly one vertex from V_{n-1} (and several vertexes from V_{n+1}). A rooted tree is called spherically homogeneous if each vertex from V_n connected to the same number d_n of vertexes from V_{n+1}. T is called $d-$regular, if $d_n = d$ is the same for all levels. For any vertex v removing the edge connecting v to a vertex of the previous level splits T into two connected components. Denote by T_v the component containing v. The automorphism group $\mathrm{Aut}(T)$ consists of all automorphisms of T (as a graph) preserving the root.

Definition 3. Let T be a spherically homogeneous tree and $G < \mathrm{Aut}(T)$. Rigid stabilizer of a vertex v is the subgroup $\mathrm{rist}_v(G)$ consisting of element g acting trivially outside of T_v. Rigid stabilizer of level n is

$$\mathrm{rist}_n(G) = \prod_{v \in V_n} \mathrm{rist}_v(G).$$

G is called *branch* if it is transitive on each level and $\mathrm{rist}_n(G)$ is a subgroup of finite index in G for all n. G is called *weakly branch* if it is transitive on each level V_n of T and $\mathrm{rist}_v(G)$ is nontrivial for each v.

The boundary ∂T of a d-regular rooted tree is homeomorphic to the space of infinite sequences $\{1, 2, \ldots, d\}^{\mathbb{N}}$ and hence is homeomorphic to a Cantor set. For a d-tuple $p = (p_1, \ldots, p_d)$ such that

$$p_i > 0 \quad \text{for all } i \quad \text{and} \quad p_1 + \cdots + p_d = 1 \tag{4.1}$$

define a measure ν_p on $\{1, 2, \ldots, d\}$ by

$$\nu_p(\{1\}) = p_1, \nu_p(\{2\}) = p_2, \ldots, \nu_p(\{d\}) = p_d.$$

Let $\mu_p = \nu_p^{\mathbb{N}}$ be the corresponding Bernoulli measure on ∂T. For each level V_n of a d-regular rooted tree an automorphism g of T can be presented in the form

$$g = \sigma \cdot (g_1, \ldots, g_{d^n}),$$

where $\sigma \in \mathrm{Sym}(V_n)$ is a permutation of the vertexes from V_n and g_i are the automorphisms induced by the action of g on the subtrees rooted at the vertexes from V_n.

Definition 4. We will call an element $g \in \operatorname{Aut}(T)$ subexponentially bounded (in the sense similar to polynomial boundedness of Sidki [19]) if the numbers $k_n(g)$ of restrictions g_i to the vertexes of level n not equal to identity satisfy

$$\lim_{n\to\infty} k_n(g)\gamma^n = 0 \quad \text{for any } 0 < \gamma < 1.$$

A group $G < \operatorname{Aut}(T)$ is subexponentially bounded if each $g \in G$ is subexponentially bounded.

In [11], Proposition 2 the authors showed that for any subexponentially bounded automorphism g and any p as in (4.1) the measure μ_p is quasi-invariant with respect to the action of g. For a subexponentially bounded weakly branch group G acting on a d-regular rooted tree and p be as in (4.1) denote by κ_p the Koopman representation corresponding to the action of G on $(\partial T, \mu_p)$. Let $\xi_A \in L^2(\partial T, \mu_p)$ stand for the characteristic function of a measurable subset $A \subset \partial T$. From [11] (Corollary 3 and Lemma 4) we deduce the following:

Proposition 3. *Let G be a subexponentially bounded weakly branch group acting on a d-regular rooted tree and p be as in (4.1) with pairwise distinct p_i. Then for every clopen set $A \subset \partial T$ there exists a sequence of elements $g_n \in G$ with* $\operatorname{supp}(g_n) \subset A$ *such that*

$$\lim_{n\to\infty} (\kappa_p(g_n)\xi_A, \xi_A) = 0. \tag{4.2}$$

Using results of [11] we show:

Proposition 4. *Let G be a subexponentially bounded weakly branch group acting on a d-regular rooted tree and $p_1, p_2, \ldots, p_d \in (0,1)$ such that p_i are pairwise distinct and $\sum_{i=1}^d p_i = 1$. Then the action of G on $(\partial T, \mu_p)$ is measure contracting.*

Proof. Let A be a measurable subset of $(\partial T, \mu_p)$. Since clopen sets approximate all measurable subsets of $(\partial T, \mu_p)$ by measure arbitrarily well to show that Definition 1 is satisfied we may assume that A is clopen. Let g_n be a sequence of elements from Proposition 3. Clearly, for large enough n condition (1) from Definition 1 is satisfied for g_n. Assume that for some $\epsilon, M > 0$ for all n condition (2) does not hold. Set

$$B_n = \left\{x \in A\colon \sqrt{\frac{\mathrm{d}\mu_p(g_n(x))}{\mathrm{d}\mu_p(x)}} > M^{-1}\right\}.$$

Then $\mu_p(B_n) \geqslant \epsilon$ for all n and we have

$$(\kappa_p(g_n)\xi_A, \xi_A) = (\kappa_p(g_n^{-1})\xi_A, \xi_A) \geqslant \int_{B_n} \sqrt{\tfrac{\mathrm{d}\mu_p(g_n(x))}{\mathrm{d}\mu_p(x)}}\mathrm{d}\mu_p(x) \geqslant M^{-1}\epsilon.$$

This contradicts to (4.2). It follows that for large enough n condition (2) from Definition 1 is also satisfied for g_n. This finishes the proof. □

References

[1] L. Bartholdi, R. I. Grigorchuk, and Z. Šunić, *Branch groups.* In M. Hazewinkel (ed.), *Handbook of algebra.* Vol. 3. Elsevier/North-Holland, Amsterdam, 2003, 989–1112. Zbl 1140.20306 MR 2035113

[2] U. Bader and R. Muchnik, Boundary unitary representations – irreducibility and rigidity. *J. Mod. Dyn.* **5** (2011), no. 1, 49–69. Zbl 1259.46051 MR 2787597

[3] M. E. B. Bekka and M. Cowling, Some irreducible unitary representations of $G(K)$ for a simple algebraic group G over an algebraic number field K. *Math. Z.* **241** (2002), no. 4, 731–741. Zbl 1022.22018 MR 1942238

[4] R. Bieri and R. Strebel, On groups of PL-homeomorphisms of the real line. Mathematical Surveys and Monographs, 215. American Mathematical Society, Providence, R.I., 2016. Zbl 1377.20002 MR 3560537

[5] O. Brattelli and W. Robinson, *Operator algebras and quantum statistical mechanics.* 1. C^*- and W^*-algebras, symmetry groups, decomposition of states. Texts and Monographs in Physics. Springer-Verlag, Berlin etc., 1979. Zbl 0421.46048 MR 0545651

[6] K. Brown, Finiteness properties of groups. *J. Pure Appl. Algebra* **44** (1987), no. 1-3, 45-75. Zbl 0613.20033 MR 0885095

[7] M. Cowling and T. Steger, The irreducibility of restrictions of unitary representations to lattices. *J. Reine Angew. Math.* **420** (1991), 85–98. Zbl 0760.22014 MR 1124567

[8] J. W. Cannon, J. W. Floyd, and W .R. Parry, Introductory notes on Richard Thompson's groups. *Enseign. Math.* (2) **42** (1996), no. 3-4, 215–256. Zbl 0880.20027 MR 1426438

[9] J. Dixmier, *Les C^*-algèbres et leurs représentations.* Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars, Paris, 1969. Zbl 0174.18601 MR 0246136

[10] A. Dudko and K. Medynets, Finite factor representations of Higman–Thompson groups. *Groups Geom. Dyn.* **8** (2014), no. 2, 375–389. Zbl 1328.20012 MR 3231220

[11] A. Dudko and R. Grigorchuk, On irreducibility and disjointness of Koopman and quasi-regular representations of weakly branch groups. In A. Katok, Y. Pesin, and F. R. Hertz (eds.), *Modern theory of dynamical systems.* Contemporary Mathematics, 692. American Mathematical Society, Providence, R.I., 2017, 51–66. Zbl 06859889 MR 3666066

[12] A. Figá-Talamanca and M. A. Picardello, *Harmonic analysis on free groups.* Lecture Notes in Pure and Applied Mathematics, 87. Marcel Dekker, New York, 1983. Zbl 0536.43001 MR 0710827

[13] A. Figá-Talamanca and T. Steger, *Harmonic analysis for anisotropic random walks on homogeneous trees.* Mem. Amer. Math. Soc. **110** (1994), no. 531. Zbl 0836.43019 MR 1219707

[14] Ł. Garncarek, Analogs of principal series representations for Thompson's groups F and T. *Indiana Univ. Math. J.* **61** (2012), no. 2, 619–626. Zbl 3043590 MR 3043590

[15] Ł. Garncarek, Irreducibility of some representations of the groups of symplectomorphisms and contactomorphisms. *Colloq. Math.* **134** (2014), no. 2, 287–296. Zbl 1301.57027 MR 3194413

[16] R. I. Grigorchuk, Some topics in the dynamics of group actions on rooted trees. *Tr. Mat. Inst. Steklova* **273** (2011), Sovremennye Problemy Matematiki, 72–191. In Russian. English translation, *Proc. Steklov Inst. Math.* **273** (2011), no. 1, 64–175. Zbl 1268.20027 MR 2893544

[17] R. Grigorchuk, V. Nekrashevych, and Z. Sunic, From self-similar groups to self-similar sets and spectra. In Ch. Bandt, K. Falconer and M. Zähle (eds.), *Fractal geometry and stochastics* V. (Tabarz, 2014.) Progress in Probability, 70. Birkhäuser/Springer, Cham, 2015, 175–207. Zbl 1359.37008 MR 3558157

[18] V. Nekrashevych, *Self-similar groups.* Mathematical Surveys and Monographs, 117. American Mathematical Society, Providence, R.I., 2005. Zbl 1087.20032 MR 2162164

[19] S. Sidki, Automorphisms of one-rooted trees: growth, circuit Structure, and acyclicity. *J. Math. Sci.* (*New York*) **100** (2000), no. 1, 1925–1943. Zbl 1069.20504 MR 1774362

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