

## Games orbits play and obstructions to Borel reducibility

Martino Lupini and Aristotelis Panagiotopoulos<sup>1</sup>

**Abstract.** We introduce a new, game-theoretic approach to anti-classification results for orbit equivalence relations. Within this framework, we give a short conceptual proof of Hjorth’s turbulence theorem. We also introduce a new dynamical criterion providing an obstruction to classification by orbits of CLI groups. We apply this criterion to the relation of equality of countable sets of reals, and the relations of unitary conjugacy of unitary and selfadjoint operators on the separable infinite-dimensional Hilbert space.

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### 1. Introduction

Borel complexity theory provides a framework for studying the relative complexity of classification problems in mathematics. In this context, a classification problem is regarded as an equivalence relation on a standard Borel space and *Borel reductions* provide the main notion of comparison between equivalence relations. Two distinguished classes of equivalence relations that are often used as benchmarks to measure the complexity of other equivalence relations are equality on a Polish space and isomorphism of countable structures. An equivalence relation  $E$  is called *concretely classifiable* if  $E$  is Borel reducible to equality on some Polish space  $Y$ . More generously, an equivalence relation is called *classifiable by countable structures* if it is Borel reducible to the relation  $\cong_{\mathcal{L}}$  of isomorphism within the class of countably infinite structures in some first order language  $\mathcal{L}$ .

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Many naturally arising equivalence relations are given as orbit equivalence relations  $E_G^X$  associated with a continuous action of a Polish group  $G$  on a Polish space  $X$ . For instance, the relation  $\cong_{\mathcal{L}}$  is induced by the canonical logic action of the group  $S_\infty$  of permutations of  $\omega$  on the space  $\text{Mod}(\mathcal{L})$  of  $\mathcal{L}$ -structures with universe  $\omega$ . Similarly, the equality relation on a Polish space is induced by the action of the trivial group.

In the process of determining the exact complexity of an equivalence relation one often has to establish both positive and negative results. For the later, it is important to isolate criteria that imply nonclassifiability by certain invariants. The well-known criterion of *generic ergodicity* (having meager orbits and a dense orbit) provides a dynamical condition on a continuous action of a Polish group on a Polish space ensuring that the corresponding orbit equivalence relation is not concretely classifiable. Hjorth's notion of *turbulent action* is a strengthening of generic ergodicity, ensuring that the associated orbit equivalence relation is not classifiable by countable structures. Both these results can actually be seen as addressing the following general problem.

**Problem 1.1.** Given a class of Polish groups  $\mathcal{C}$ , which dynamical conditions on a Polish  $G$ -space  $X$  ensure that the corresponding orbit equivalence relation is not Borel reducible to  $E_H^Y$  for some Borel action of a Polish group  $H$  in  $\mathcal{C}$  on a Polish space  $Y$ ?

Indeed, generic ergodicity provides such a criterion for the class  $\mathcal{C}$  of *compact Polish groups* [11, Proposition 6.1.10]. Hjorth's turbulence theorem [15] addresses this problem in the case when  $\mathcal{C}$  is the class of *non-Archimedean Polish groups*. Turbulence has played a key role in Borel complexity theory in the last two decades and it is to this day essentially the only known method to prove unclassifiability by countable structures; see [13, 7, 8, 27, 25, 24, 1, 2, 33, 34, 35, 12, 37, 18, 20, 23, 11, 15, 22, 6]. There has been so far little progress into obtaining similar criteria for other interesting classes of Polish groups.

The purpose of this paper is two-fold. Our first goal is to introduce a game-theoretic approach to Problem 1.1. This approach consists in endowing the space  $X/G$  of orbits of a Polish  $G$ -space  $X$  with different graph structures, and then showing that a Baire measurable  $(E_G^X, E_H^Y)$ -homomorphism  $f: X \rightarrow Y$  induces a graph homomorphism  $X/G \rightarrow Y/G$  after restricting to an invariant dense  $G_\delta$  set. This perspective allows us to give a short conceptual proof of Hjorth's turbulence theorem, avoiding the substantial amount of bookkeeping of Hjorth's original argument [15]; see also [11, Chapter 10].

The second goal of this paper is to use the above-mentioned game-theoretic approach to address Problem 1.1 for the class of *CLI groups*. Recall that a CLI group is a Polish group that admits a *compatible complete left-invariant metric*. Every locally compact group, as well as every solvable Polish group—in particular, every abelian Polish group—is CLI [17, Corollary 3.7]. This class of groups has

been considered in several papers so far. For instance, [3, Corollary 5.C.6] settled the topological Vaught conjecture for CLI groups. It is also proved in [3, Theorem 5.B.2] that CLI groups satisfy an analog of the Glimm-Effros dichotomy. In [10, Theorem 1.1] it is shown that the non-Archimedean CLI groups are precisely the automorphism groups of countable structures whose Scott sentence does not have an uncountable model. The class of CLI groups has been further studied in [28], where it is shown that it forms a coanalytic non-Borel subset of the class of Polish groups.

A fundamental tool in the study of dichotomies for orbit equivalence relations from [3] is the notion of left embeddability, or  $l$ -embeddability, for points in a Polish  $G$ -space. We work here with the right variant of left embeddability which we call *Becker embeddability*. We prove that a Baire-measurable homomorphism between orbit equivalence relations necessarily preserves Becker embeddability on an invariant dense  $G_\delta$  set. From this we extract in Theorem 2.9 a dynamical condition which answers Problem 1.1 for the class of CLI groups. We then apply it to show that the Friedman–Stanley jump of equality  $=^+$  is not Borel reducible to the orbit equivalence relation induced by a Borel action of a CLI group. The only proof of this fact that we are aware of relies on meta-mathematical reasoning and involves the theory of pinned equivalence relations; see [19]. A natural reduction from this relation to the relations of unitary equivalence of bounded unitary or selfadjoint operators on an infinite-dimensional Hilbert space, shows that the latter relations are also not classifiable by the orbits of a CLI group actions. We note that it is still an open question if an action of the unitary group can induce an orbit equivalence relation which is universal for orbit equivalence relations induced by Polish group actions. Our results show that the complexity of possible orbit equivalence relations of  $\mathcal{U}(\mathcal{H})$ -actions is not bounded from above by the complexity of orbit equivalence relations induced by continuous CLI group actions.

We conclude by discussing how all the results of the present paper admit natural generalizations from Polish group actions to Polish groupoids. Turbulence theory for Polish groupoids has been developed in [12]. Applications of this more general framework to classification problems in operator algebras have also been presented in [12].

Besides this introduction, the present paper is divided into three sections. In Section 2 we present the results about Becker-embeddability and CLI groups. In Section 3 we present the short and conceptual proof of Hjorth’s turbulence theorem mentioned above. Finally in Section 4 we recall the fundamental notions about Polish groupoids, and explain how the main results of this paper can be adapted to this more general setting. In the rest of the paper, we will use the category quantifier  $\forall^* x \in U$  for the statement “for a comeager set of  $x \in U$ ”; see [11, Section 3.2].

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## 2. Nonreducibility to CLI group actions

**2.1. The Becker-embedding game.** Recall that a CLI group is a Polish group that admits a compatible *complete left-invariant metric*. It is easy to see that a Polish group is CLI if and only if it admits a compatible complete right-invariant metric; see [3, 3.A.2. Proposition]. Throughout this section, we will let  $G$  be a Polish group, and  $X$  be a Polish  $G$ -space. The main goal of this section is to provide a dynamical criterion for a Polish  $G$ -space ensuring that the corresponding orbit equivalence relation is not Borel reducible to the orbit equivalence relation induced by a Borel action of a CLI group. A characterization of CLI groups in terms of tameness of the corresponding orbit equivalence relations has been obtained in [36].

The notion of l-embeddability for points of  $X$  has been introduced in [3, Definition 3.D.1]. Recall that a sequence  $(g_n)$  in  $G$  is Cauchy with respect to some left-invariant metric on  $G$  if and only if it is Cauchy with respect to every left-invariant metric on  $G$  [3, Proposition 3.B.1]. If this holds, we say that  $(g_n)$  is *left Cauchy*. We define when  $(g_n)$  is right Cauchy in a similar way, by replacing left-invariant metrics with right-invariant metrics.

**Definition 2.1.** Fix  $x, y \in X$ . We say that  $x$  is *l-embeddable* into  $y$  if there exists a left Cauchy sequence  $(h_n)_{n \in \omega}$  in  $G$  such that  $h_n x \rightarrow y$ . We say that  $x$  is *r-embeddable* into  $y$  if there exists a right Cauchy sequence  $(h_n)_{n \in \omega}$  in  $G$  such that  $h_n y \rightarrow x$ .

In the rest of the paper, we will focus on the notion of r-embeddability. Similar results can be proved for l-embeddability. It is easy to see as in the proof of [3, Proposition 3.D.4] that the relation of r-embeddability is a preorder. Furthermore, if  $x$  is r-embeddable into  $y$ ,  $x'$  belongs to the  $G$ -orbit of  $x$ , and  $y'$  belongs to the  $G$ -orbit of  $y$ , then  $x'$  is r-embeddable into  $y'$ .

We now consider a natural game between two players, and show that it captures the notion of r-embeddability from Definition 2.1. A natural variation of the same game captures the notion of l-embeddability.

**Definition 2.2.** Suppose that  $X$  is a Polish  $G$ -space, and  $x, y \in X$ . We consider the *Becker-embedding game*  $\text{Emb}(x, y)$  played between two players as follows. Set  $U_0 = X$  and  $V_0 = G$ .

- (1) In the first turn, Player I plays an open neighborhood  $U_1$  of  $x$ , and an open neighborhood  $V_1$  of the identity of  $G$ . Player II replies with an element  $g_0$  in  $V_0$ .
- (2) In the second turn, Player I then plays an open neighborhood  $U_2$  of  $x$ , and an open neighborhood  $V_2$  of the identity of  $G$ , and Player II replies with an element  $g_1$  in  $V_1$ .
- (n) At the  $n$ -th turn, Player I plays an open neighborhood  $U_n$  of  $x$ , and an open neighborhood  $V_n$  of the identity of  $G$ , and Player II responds with an element  $g_{n-1}$  in  $V_{n-1}$ .

The game proceed in this way, producing a sequence  $(g_n)$  of elements of  $G$ , a sequence  $(U_n)$  of open neighborhoods of  $x$  in  $X$ , and a sequence  $(V_n)$  of open neighborhoods of the identity in  $G$ . Player II wins the game if for every  $n > 0$ ,  $g_{n-1} \dots g_0 y \in U_n$ . We say that  $x$  is *Becker embeddable* into  $y$ —and write  $x \preceq_B y$ —if Player II has a winning strategy for the game  $\text{Emb}(x, y)$ .

**Lemma 2.3.** *If Player II has a winning strategy for the Becker-embedding game as described in Definition 2.2, then it also has a winning strategy for the same game with the additional winning conditions that  $g_n$  belongs to some given comeager subset  $V_n^*$  of  $V_n$ , and  $g_{n-1} \dots g_0 y$  belongs to some given comeager subset  $X_0$  of  $X$ , provided that the set of  $g \in G$  such that  $gy \in X_0$  is comeager.*

*Proof.* Suppose that Player II has a winning strategy for the Becker-embedding game  $\text{Emb}(x, y)$ . The strategy consists of one function  $g_n(U_1, \dots, U_{n+1}, V_1, \dots, V_{n+1})$  for every  $n \geq 0$ , where  $U_0, \dots, U_{n+1}$  are open neighborhoods of  $x$  and  $V_1, \dots, V_n$  are open neighborhoods of the identity in  $G$ . Since the strategy is winning, one has that  $g_{n-1} \dots g_0 y \in U_n$  and  $g_n \in V_n$  for every  $n \geq 0$ .

Let us consider now a run of the game  $\text{Emb}(x, y)$ . Suppose that Player I at turn 1 plays open sets  $U_1, V_1$ . Consider an open neighborhood  $\tilde{V}_1$  of 1 such that  $\tilde{V}_1 \tilde{V}_1 \subseteq V_1$ . Let  $g'_0 := g_0(U_1, \tilde{V}_1)$  obtained by applying the given winning strategy of Player II. Using the assumption in the statement of the lemma we can find  $\varepsilon_0 \in G$  such that the following conditions are satisfied:

- $\varepsilon_0^{-1} \in \tilde{V}_1$ ;
- $\varepsilon_0 g'_0 \in V_0^*$ ;
- $\varepsilon_0 g'_0 y \in X_0 \cap U_1$ .

We can then define  $\tilde{g}_0 := \varepsilon_0 g'_0$ . Observe that  $\tilde{g}_0 \in V_0^*$  and  $\tilde{g}_0 y \in X_0 \cap U_1$ .

Suppose inductively that in the first  $n$  turns of the game Player I has played open sets  $U_1, V_1, \dots, U_n, V_n$ . Consider the elements  $g'_i := g_i(U_1, \dots, U_{i+1}, \tilde{V}_1, \dots, \tilde{V}_{i+1})$  of  $G$  for  $i \in \{0, 1, \dots, n-1\}$  produced by applying the original strategy for Player II, where  $\tilde{V}_i$  is an open neighborhood of the identity in  $G$  such that  $\tilde{V}_i \tilde{V}_i \subseteq V_i$  for  $i \in \{1, 2, \dots, n\}$ . We suppose furthermore that we have defined

$\varepsilon_0, \dots, \varepsilon_{n-1}, \tilde{g}_0, \dots, \tilde{g}_{n-1} \in G$  are such that the following conditions are satisfied for every  $i \in \{0, 1, \dots, n-1\}$ :

- $\varepsilon_i^{-1} \in \tilde{V}_{i+1}$ ;
- $\varepsilon_i g'_i \in V_i^*$ ;
- $\varepsilon_i g'_i g'_{i-1} \dots g'_0 y \in X_0 \cap U_{i+1}$ ;
- $\tilde{g}_i = \varepsilon_i g_i \varepsilon_{i-1}^{-1}$  where  $\varepsilon_{-1} = 1$ .

Suppose that Player I plays open sets  $U_{n+1}, V_{n+1}$  at turn  $n$ . Consider then  $g'_n := g_n(U_1, \dots, U_{n+1}, \tilde{V}_1, \dots, \tilde{V}_{n+1})$  obtained by applying the original winning strategy of Player II. Thus we have that  $g'_n \dots g'_0 y \in X_0 \cap U_{n+1}$  and that  $g'_n \varepsilon_{n-1}^{-1} \in \tilde{V}_n \tilde{V}_n \subseteq V_{n+1}$ . By applying the assumption in the statement of the lemma we can find  $\varepsilon_n \in G$  such that the following conditions are satisfied:

- $\varepsilon_n^{-1} \in \tilde{V}_{n+1}$ ;
- $\varepsilon_n g'_n \varepsilon_{n-1}^{-1} \in V_n^*$ ;
- $\varepsilon_n g'_n \dots g'_0 y \in X_0 \cap U_{n+1}$ .

We then set  $\tilde{g}_n := \varepsilon_n g'_n$ . Observe that  $\tilde{g}_n \in V_n^*$  and  $\tilde{g}_n \tilde{g}_{n-1} \dots \tilde{g}_0 y = \varepsilon_n g_n \dots g_0 y \in X_0 \cap U_{n+1}$ .

It is clear from the construction that setting  $\tilde{g}_n := g_n(U_1, \dots, U_n, V_1, \dots, V_n)$  gives a new winning strategy for Player II which satisfies the required additional conditions. □

We now show that the notion of Becker-embeddability from Definition 2.2 is actually equivalent to the notion of  $r$ -embeddability from Definition 2.1.

**Lemma 2.4.** *Let  $X$  be a Polish  $G$ -space. If  $x, y$  are points of  $X$ , then the following statements are equivalent:*

- (1)  $x \preceq_{\mathcal{B}} y$ ;
- (2)  $x$  is  $r$ -embeddable in  $y$ .

*Proof.* We fix a right-invariant metric  $d$  on  $G$ . For a subset  $A$  of  $G$  we let  $\text{diam}(A)$  be the diameter of  $A$  with respect to  $d$ .

(1)  $\implies$  (2) Suppose that Player II has a winning strategy for the Becker-embedding game  $\text{Emb}(x, y)$ . Let Player I play a sequence  $(U_n)$  which forms a basis of open neighborhoods of  $x$  and a sequence  $(V_n)$  which forms a basis of symmetric open neighborhoods of the identity of  $G$  with  $\text{diam}(V_n) < 2^{-n}$ . Let  $(g_n)_{n \in \omega}$  be the sequence of elements of  $G$  given by a winning strategy for Player II. Then the sequence  $(h_n)_{n \in \omega}$  obtained by setting  $h_n := g_n \dots g_0$  is  $d$ -Cauchy, and  $h_n y \rightarrow x$ .

(2)  $\implies$  (1) Suppose that there exists a  $d$ -Cauchy sequence  $(h_n)_{n \in \omega}$  in  $G$  such that  $h_n y \rightarrow x$ . We describe a winning strategy for Player II. Set  $h_{-1} := 1$ . Suppose that in the first turn Player I plays an open neighborhood  $U_1$  of  $x$  and an open neighborhood  $V_1$  of the identity of  $G$ . Player II replies with  $g_0 := h_{k_0}$ , where  $k_0 \in \omega$  is such so that:

- (1)  $h_k h_{k_0}^{-1} \in V_1$  for all  $k \geq k_0$ , and
- (2)  $h_{k_0} y \in U_1$ .

The first condition is satisfied by a large enough  $k_0 \in \omega$  because  $(h_n)_{n \in \omega}$  is  $d$ -Cauchy. The second condition is satisfied by a large enough  $k_0 \in \omega$  because  $h_n x \rightarrow y$ . Suppose that in the  $n$ -turn Player I plays an open neighborhood  $U_n$  of  $x$  and an open neighborhood  $V_n$  of the identity in  $G$ . Inductively, assume also that  $g_{n-2}$  is of the form  $h_{k_{n-2}} h_{k_{n-3}}^{-1}$  for some  $k_{n-2} \in \omega$  such that  $h_k h_{k_{n-2}}^{-1} \in V_{n-1}$  for all  $k \geq k_{n-2}$ . Player II replies with  $g_{n-1} := h_{k_{n-1}} h_{k_{n-2}}^{-1}$ , where  $k_{n-1} \in \omega$  is such that:

- (1)  $h_k h_{k_{n-1}}^{-1} \in V_n$  for all  $k \geq k_{n-1}$ , and
- (2)  $h_{k_{n-1}} y \in U_n$ .

Again, our assumptions on the sequence  $(h_n)_{n \in \omega}$  guarantee that a large enough  $k_{n-1} \in \omega$  satisfies both these conditions. Then we have that  $g_{n-1} \in V_{n-1}$  by the inductive assumption on  $k_{n-2}$ . Therefore this procedure describes a winning strategy for Player II in the Becker-embedding game  $\text{Emb}(x, y)$ .  $\square$

We let  $X/G$  be the space of  $G$ -orbits of points of  $X$ . The Becker-embeddability preorder defines a directed graph structure on  $X/G$  obtained by declaring that there is an arrow from the orbit  $[x]$  of  $x$  to the orbit  $[y]$  of  $y$  if and only if  $x \preceq_{\mathcal{B}} y$ . We will call this the *Becker digraph*  $\mathcal{B}(X/G)$  of the Polish  $G$ -space  $X$ . Similarly, for a  $G$ -invariant subset  $X_0$  of  $X$  we let  $\mathcal{B}(X_0/G)$  the induced subgraph of  $\mathcal{B}(X/G)$  only containing vertices corresponding to orbits from  $X_0$ . Suppose that  $G, H$  are Polish groups,  $X$  is a Polish  $G$ -space, and  $Y$  is a Polish  $H$ -space. Any  $(E_G^X, E_H^Y)$ -homomorphism  $f: X \rightarrow Y$  induces a function  $[f]: X/G \rightarrow Y/H$ ,  $[x] \mapsto [f(x)]$ . We will show below that, when  $f$  is Baire-measurable, such a function is *generically* a digraph homomorphism with respect to the Becker digraph structures on  $X/G$  and  $Y/H$ .

We now describe the notion of Becker-embedding in case of Polish  $G$ -spaces arising from classes of countable models. Suppose that  $\mathcal{L} = (R_i)_{i \in I}$  is a countable first order relational language, where  $R_i$  is a relation symbol with arity  $n_i$ . Let  $\text{Mod}(\mathcal{L})$  be the space of countable  $\mathcal{L}$ -structures having  $\mathbb{N}$  as support,  $F$  be a countable fragment of  $\mathcal{L}_{\omega_1, \omega}$ , and  $S_\infty$  be the group of permutations of  $\mathbb{N}$ . As usual, one can regard  $\text{Mod}(\mathcal{L})$  as the product  $\prod_{i \in I} 2^{\mathbb{N}^{n_i}}$ . Any  $\mathcal{L}_{\omega_1, \omega}$  formula  $\varphi(x_1, \dots, x_n)$  defines a function  $[\varphi]: \text{Mod}(\mathcal{L}) \rightarrow \{0, 1\}^{\mathbb{N}^n}$  given by its interpretation. A set  $F$  of  $\mathcal{L}_{\omega_1, \omega}$  formulas defines a topology  $t_F$  on  $\text{Mod}(\mathcal{L})$ , which is

the weakest topology that makes the functions  $[\varphi]$  for  $\varphi \in F$  continuous. The canonical action  $S_\infty \curvearrowright \text{Mod}(\mathcal{L})$  turns  $(\text{Mod}(\mathcal{L}), t_F)$  into a Polish  $G$ -space. If  $x, y \in \text{Mod}(\mathcal{L})$ , then we have that  $x \preceq_{\mathcal{B}} y$  if and only if there exists an injective function  $f: \mathbb{N} \rightarrow \mathbb{N}$  that represents an  $F$ -embedding from  $x$  to  $y$ , in the sense that  $f$  preserves the value of formulas  $\varphi$  in  $F$  with parameters. In the particular case when  $F$  is the collection of atomic first-order formulas, the topology  $t_F$  coincides with the product topology, and an  $F$ -embedding is the same as an embedding as  $\mathcal{L}$ -structure. When  $F$  is the collection of all first-order formulas, an  $F$ -embedding is an elementary embedding. It is shown in [3, Proposition 2.D.2], in the case when  $F$  is a *fragment* in the sense define therein, that the same conclusions holds for  $l$ -embeddability.

**2.2. The orbit continuity lemma.** Recall that if  $E, F$  are equivalence relations on Polish spaces  $X, Y$  respectively, then a  $(E, F)$ -homomorphism is a function  $f: X \rightarrow Y$  mapping  $E$ -classes to  $F$ -classes. In this subsection we isolate a lemma to be used in the rest of the paper. It states that a Baire-measurable homomorphism between orbit equivalence relations admits a restriction to a dense  $G_\delta$  set which is continuous at the level of orbits, in a suitable sense. Variations of such a lemma are well known. The starting point is essentially [15, Lemma 3.17] modified as in the beginning of the proof of [15, Theorem 3.18]; see also [11, Lemma 10.1.4 and Theorem 10.4.2].

**Lemma 2.5.** *Suppose that  $G, H$  are Polish groups,  $X$  is a Polish  $G$ -space, and  $Y$  is a Polish  $H$ -space. Let  $f: X \rightarrow Y$  be a Baire-measurable  $(E_G^X, E_H^Y)$ -homomorphism. Then there exists a dense  $G_\delta$  subset  $C$  of  $X$  such that*

- *the restriction of  $f$  to  $C$  is continuous;*
- *for any  $x \in C$ ,  $\{g \in G: gx \in C\}$  is a comeager subset of  $G$ ;*
- *for any  $x_0 \in C$  and for any open neighborhood  $W$  of the identity in  $H$  there exists an open neighborhood  $U$  of  $x_0$  and an open neighborhood  $V$  of the identity of  $G$  such that for any  $x \in U \cap C$  and for a comeager set of  $g \in V$ , one has that  $f(gx) \in Wf(x)$  and  $gx \in C$ .*

*Proof.* Fix a neighborhood  $W_0$  of the identity in  $H$ . We first prove the following claim: for all  $x_0 \in X \forall^* g_0 \in G$ , there is an open neighborhood  $V$  of the identity in  $G$  such that  $\forall^* g_1 \in V$ ,  $f(g_1 g_0 x_0) \in W_0 f(g_0 x_0)$ .

Fix a neighborhood  $W$  of the identity of  $H$  such that  $WW^{-1} \subset W_0$ . Let  $(h_n)$  be a sequence in  $H$  such that  $\{Wh_n: n \in \mathbb{N}\}$  is a cover of  $H$ . Since  $Wh_n f(x_0)$  is analytic, the set of elements  $x$  of the orbit of  $x_0$  such that  $f(x) \in Wh_n f(x_0)$  has the Baire property. Therefore we can find a sequence  $(O_n)$  of open subsets of  $G$  with dense union  $O$  and a comeager subset  $D$  of  $O$  such that for all  $g \in D \cap O_n$ ,  $f(gx_0) \in Wh_n f(x_0)$ . Suppose now that  $g_0 \in D$ . Let  $n \in \mathbb{N}$  be such that  $g_0 \in O_n$ . Then there exists a neighborhood  $V$  of the identity of  $G$  such that  $Vg_0 \subset O_n$ .



Observe that  $(D \cap O_n)g_0^{-1} \cap V$  is a comeager subset of  $V$ . If  $g_1 \in (D \cap O_n)g_0^{-1} \cap V$ , then we have

$$f(g_1g_0x_0) \in Wh_n f(x_0) \quad \text{and} \quad f(g_0x_0) \in Wh_n f(x_0).$$

Therefore

$$f(g_1g_0x_0) \in WW^{-1}f(g_0x_0) \subset W_0f(g_0x_0).$$

This concludes the proof of the claim.

From the claim and the Kuratowski–Ulam theorem, one deduces that there exists a dense  $G_\delta$  subset  $C_0$  of  $X$  such that for every  $x \in C_0$  there exists an open neighborhood  $V$  of the identity of  $G$  such that for all  $\forall^*g \in V$ ,  $f(gx) \in Wf(x)$ . Since  $f$  is Baire-measurable, we can furthermore assume that the restriction of  $f$  to  $C_0$  is continuous.

Fix now a countable basis  $(W_k)$  of open neighborhoods of the identity of  $H$  and a countable basis  $(V_n)$  of open neighborhoods of the identity in  $G$ . Let  $N: X \times \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$  be the function that assigns to  $(x, k)$  the least  $n \in \mathbb{N}$  such that  $\forall^*g \in V_n$ ,  $f(gx) \in W_k f(x)$  if such an  $n$  exists and  $x \in C_0$ , and  $\infty$  otherwise. Then  $N$  is an analytic function, and hence one can find a dense  $G_\delta$  subset  $C_1$  of  $X$  contained in  $C_0$  such that  $N|_{C_1 \times \mathbb{N}}$  is continuous. By [11, Proposition 3.2.5 and Theorem 3.2.7] the set  $C := \{x \in C_1: \forall^*g \in G, gx \in C_1\}$  is a dense  $G_\delta$  subset of  $X$  such that for all  $x \in C$  and  $\forall^*g \in G$ ,  $gx \in C$ . Therefore  $C$  satisfies the desired conclusions. □

**2.3. Generic homomorphisms between Becker graphs.** In this section we use the Becker-embedding game and the orbit continuity lemma to address Problem 1.1 for the class of CLI groups.

**Definition 2.6.** An equivalence relation  $E$  on a Polish space  $X$  is CLI-classifiable if it is Borel reducible to  $E_H^Y$  for some CLI group  $H$  and Polish  $H$ -space  $Y$ .

We will obtain below an obstruction to CLI-classifiability in terms of the Becker digraph. This will be based upon the following properties of the Becker digraph:

- (1) the Becker digraph contains only loops in the case of CLI group actions (Lemma 2.7), and
- (2) a Baire-measurable homomorphism between orbit equivalence relations induces, after restricting to an invariant dense  $G_\delta$  set, a homomorphism at the level of Becker digraphs (Proposition 2.8).

**Lemma 2.7.** *If  $Y$  is a Polish  $H$ -space and  $H$  is a CLI group, then the Becker digraph  $\mathcal{B}(Y/H)$  contains only loops.*

*Proof.* Fix a compatible complete right-invariant metric  $d$  on  $H$ . For a subset  $A$  of  $H$  we let  $\text{diam}(A)$  be the diameter of  $A$  with respect to  $d$ . Let  $x, y$  be elements of  $Y$  with different  $H$ -orbits. We show that Player I has a winning strategy in  $\text{Emb}(x, y)$ . In the  $n$ -th round Player I plays some symmetric open neighborhood  $V_{n+1}$  of the identity of  $H$  with  $\text{diam}(V_{n+1}) < 2^{-n}$  and an open neighborhood  $U_n$  of  $x$  such that the sequence  $(U_n)$  forms a decreasing basis of neighborhoods of  $x$ . Let  $(g_n)$  be the sequence of group elements chosen by Player II, and set  $h_n := g_n \dots g_0$ . We claim that such a sequence does not satisfy the winning condition for Player II in the Becker-embedding game. Suppose by contradiction that this is the case, and hence  $\lim_n h_n y = x$ . For every  $n > m$  we have by right invariance of  $d$  that

$$d(h_n, h_m) = d(g_n \dots g_{m+1}, 1) \leq d(g_n, 1) + d(g_{n-1}, 1) + \dots + d(g_{m+1}, 1) < 2^{-m}.$$

Therefore  $h_n$  is a  $d$ -Cauchy sequence with respect to  $d$ . Since by assumption  $d$  is complete,  $h_n$  converges to some  $h \in H$ . From  $\lim_n h_n y = x$  and continuity of the action, we deduce that  $hy = x$ . This contradicts the assumption that the  $H$ -orbits of  $x$  and  $y$  are different.  $\square$

Using the orbit continuity lemma (Lemma 2.5) one can then show that a Baire-measurable homomorphism preserves Becker embeddability on a comeager set. This is the content of the following proposition.

**Proposition 2.8.** *Suppose that  $G, H$  are Polish groups,  $X$  is a Polish  $G$ -space, and  $Y$  is a Polish  $H$ -space. Let  $f: X \rightarrow Y$  be a Baire-measurable  $(E_G^X, E_H^Y)$ -homomorphism. Then there exists a  $G$ -invariant dense  $G_\delta$  subset  $X_0$  of  $X$  such that the function  $[f]: X_0/G \rightarrow Y/H$ ,  $[x] \mapsto [f(x)]$  is a digraph homomorphism from the Becker digraph  $\mathcal{B}(X_0/G)$  to the Becker digraph  $\mathcal{B}(Y/G)$ .*

*Proof.* Let  $C$  be a dense  $G_\delta$  subsets of  $X$  obtained from  $f$  as in Lemma 2.5. Set  $X_0 := \{x \in X : \forall^* g \in G, gx \in C\}$ , which is a  $G$ -invariant dense  $G_\delta$  set by [II, Proposition 3.2.5 and Theorem 3.2.7]. We claim that  $[f]: X_0/G \rightarrow Y/H$ ,  $[x] \mapsto [f(x)]$  is a digraph homomorphism from the Becker digraph  $\mathcal{B}(X_0/G)$  to the Becker digraph  $\mathcal{B}(Y/G)$ .

Fix  $x_0, y_0 \in X_0$  such that  $x_0 \preceq_{\mathcal{B}} y_0$ . We want to prove that  $f(x_0) \preceq_{\mathcal{B}} f(y_0)$ . Observe that  $\forall^* g \in G, gx_0 \in C \cap X_0$ . Therefore after replacing  $x_0$  with  $gx_0$  for a suitable  $g \in G$  we can assume that  $x_0 \in C \cap X_0$ . Let us consider the Becker-embedding game  $\text{Emb}(f(x_0), f(y_0))$ . At the same time we consider the Becker-embedding game  $\text{Emb}(x_0, y_0)$  and use the fact that Player II has a winning strategy for such a game.

In the first turn of  $\text{Emb}(f(x_0), f(y_0))$ , Player I plays an open neighborhood  $\hat{U}_1$  of  $f(x_0)$  and an open neighborhood  $\hat{V}_1$  of the identity of  $H$ . Consider an open neighborhood  $U_1$  of  $x_0$  and an open neighborhood  $V_1$  of the identity of  $G$  such that for any  $x \in U_1 \cap C \cap X_0$  and a comeager set of  $g \in V_1$  one has that

$f(gx) \in \widehat{V}_1 f(x)$ . Consider now the round of the game  $\text{Emb}(x_0, y_0)$  where, in the first turn, Player I plays the neighborhood  $U_1$  of  $x_0$  and the neighborhood  $V_1$  of the identity of  $G$ . Since by assumption Player II has a winning strategy for  $\text{Emb}(x_0, y_0)$ , we can consider an element  $g_0$  of  $V_1$  which is obtained from such a winning strategy. By Lemma 2.3, we can also insist that  $g_0$  belongs to the comeager set of  $g \in V_1$  such that  $gy_0 \in U_1 \cap C \cap X_0$  and  $f(gy_0) \in \widehat{V}_1 f(x)$ . We can then let Player II play, in the first turn of the game  $\text{Emb}(f(x_0), f(y_0))$ , an element  $h_0$  of  $\widehat{V}_1$  such that  $f(g_0 y_0) = h_0 f(y_0)$ .

At the  $n$ -th turn of  $\text{Emb}(f(x_0), f(y_0))$ , Player I plays an open neighborhood  $\widehat{U}_n$  of  $f(x_0)$  and an open neighborhood  $\widehat{V}_n$  of the identity of  $H$ . Consider now an open neighborhood  $U_n$  of  $x_0$  and an open neighborhood  $V_n$  of the identity of  $G$  such that for any  $x \in U_n \cap C \cap X_0$  and a comeager set of  $g \in V_n$  one has that  $f(gx) \in \widehat{V}_n f(x)$ . Let Player I play, in the  $n$ -turn of  $\text{Emb}(x_0, y_0)$ , the open neighborhoods  $U_n$  of  $x_0$  and  $V_n$  of the identity of  $G$ . Let  $g_{n-1} \in V_n$  be obtained from a winning strategy for Player II. By Lemma 2.3 we can insist that  $g_{n-1}$  belongs to the comeager set of  $g \in V_n$  such that  $gg_{n-2} \dots g_1 g_0 y \in U_n \cap C \cap X_0$  and  $f(gx) \in \widehat{V}_1 f(x)$ . Therefore we can let Player II play, in the  $n$ -th turn of the game  $\text{Emb}(f(x_0), f(y_0))$ , an element  $h_{n-1} \in \widehat{V}_{n-1}$  such that  $f(g_{n-1} \dots g_0 y) = h_{n-1} f(g_{n-1} \dots g_0 y) = h_{n-1} \dots h_0 y \in \widehat{U}_n$ . Such a construction witness that Player II has a winning strategy for the game  $\text{Emb}(f(x_0), f(y_0))$ .  $\square$

From Lemma 2.7 and Proposition 2.8 one can immediately deduce the following criterion to show that the orbit equivalence relation of a Polish group action is not Borel reducible to the orbit equivalence relation of CLI group action.

**Theorem 2.9.** *Suppose that  $X$  is a Polish  $G$ -space. If for any  $G$ -invariant dense  $G_\delta$  subset  $C$  of  $X$  there exist  $x, y \in C$  with different  $G$ -orbits such that  $x \preceq_{\mathcal{B}}$   $y$ , then for any  $G$ -invariant dense  $G_\delta$  subset  $C$  of  $X$  the relation  $E_C^X$  is not CLI-classifiable.*

*Proof.* Suppose that  $H$  is a CLI group, and  $Y$  is a Polish  $H$ -space. Suppose that  $D$  is a  $G$ -invariant dense  $G_\delta$  subset of  $X$ , and  $f: D \rightarrow Y$  is a Borel  $(E_G^D, E_H^Y)$ -homomorphism. Then by Proposition 2.8 there exists a  $G$ -invariant dense  $G_\delta$  subset  $C$  of  $D$  such that  $[f]: C/G \rightarrow Y/H$  is a digraph homomorphisms for the Becker digraphs  $\mathcal{B}(C/G)$  and  $\mathcal{B}(Y/H)$ . By assumption there exist elements  $x, y$  of  $C$  with different  $G$ -orbits such that  $x \preceq_{\mathcal{B}}$   $y$ . Therefore  $f(x) \preceq_{\mathcal{B}}$   $f(y)$ . Since  $H$  is CLI we have by Lemma 2.7 that  $f(x)$  and  $f(y)$  belong to the same  $H$ -orbit. Therefore  $f$  is not a reduction from  $E_G^D$  to  $E_H^Y$ .  $\square$

**2.4. Applications.** Suppose that  $E$  is an equivalence relation on a Polish space  $X$ . Recall that the Friedman–Stanley jump  $E^+$  of  $E$  [11, Definition 8.3.1]—see also [9]—is the equivalence relation on the standard Borel space  $X^{\mathbb{N}}$  of sequences

of elements of  $X$  defined by  $(x_n)E^+(y_n)$  if and only if  $\{(x_n)_E : n \in \mathbb{N}\} = \{(y_n)_E : n \in \mathbb{N}\}$ .

In particular one can start with the relation  $=$  of equality on a perfect Polish space  $X$ . The corresponding Friedman–Stanley jump is the relation  $=^+$  on  $X^\omega$  defined by  $(x_n) =^+ (y_n)$  if and only if the sequences  $(x_n)$  and  $(y_n)$  have the same range. With respect to Borel reducibility,  $=^+$  is the most complicated (essentially)  $\Pi_3^0$  equivalence relation [11, Theorem 12.5.5]; see also [16].

Hjorth has proven in [14, Theorem 5.19] that  $=^+$  is not Borel reducible to the orbit equivalence relation of a continuous action of an *abelian* Polish group. As remarked in [14, page 663], Hjorth’s proof uses a metamathematical argument involving forcing and Stern’s absoluteness principle. Similar methods are used in [19, Theorem 17.1.3] to prove that  $=^+$  is not Borel reducible to the orbit equivalence relation of a Borel action of a CLI group. This is obtained as a consequence of a general result concerning pinned equivalence relations; see [19, Definition 17.1.2]. To our knowledge, the argument below provides the first entirely classical proof of this result.

Let  $\sigma: X^\mathbb{N} \rightarrow X^\mathbb{N}$  be the *unilateral shift*  $(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$ . We consider the restriction of  $=^+$  to the dense  $G_\delta$  subset  $Y$  of  $X^\mathbb{N}$  that consists of injective sequences. Observe that this is the orbit equivalence relation of the canonical action of  $S_\infty$  on  $X^\mathbb{N}$  obtained by permuting the indices.

**Theorem 2.10.** *Let  $Z \subset Y$  be a nonempty  $S_\infty$ -invariant  $G_\delta$  set such that  $\sigma[Z] = Z$ . The restriction of  $=^+$  to any  $S_\infty$ -invariant dense  $G_\delta$  subset of  $Z$  is not Borel reducible to a Borel action of a CLI group on a standard Borel space.*

*Proof.* Let  $E$  be the restriction of  $=^+$  to  $Z$ . As observed before,  $E$  is the orbit equivalence relation of the canonical action  $S_\infty \curvearrowright Z \subset Y \subset X^\mathbb{N}$  given by permuting the coordinates. We apply Proposition 2.9. Let  $C$  be an  $S_\infty$ -invariant dense  $G_\delta$  subset of  $Z$ . We need to prove that there exist  $x, y \in C$  with different orbits such that  $x \preceq_{\mathcal{B}} y$ . For  $x = (x_n) \in Y$  we let  $\text{Ran}(x)$  be the set  $\{x_n : n \in \mathbb{N}\}$ . It is not difficult to see that, for  $x, y \in Y$ ,  $x \preceq_{\mathcal{B}} y$  if and only if  $\text{Ran}(x) \subset \text{Ran}(y)$ . Observe that  $\sigma: Z \rightarrow Z$  is continuous, open, and surjective. Therefore, since  $C$  is a dense  $G_\delta$  subset of  $Z$ , we have that there exists a comeager subset  $C_0$  of  $C$  such that, for every  $x \in C_0$ ,  $\sigma^{-1}(x) \cap C$  is a comeager subset of  $\sigma^{-1}(x)$ ; see [29, Theorem A.1]. Pick now  $x \in C_0$  and  $y \in \sigma^{-1}(x) \cap C$ . It is clear that  $x \preceq_{\mathcal{B}} y$  and  $x, y$  lie in different  $S_\infty$ -orbits. This concludes the proof.  $\square$

We now apply Theorem 2.10 to obtain information about the orbit equivalence relation of some canonical actions of the unitary group  $\mathcal{U}(\mathcal{H})$ . Let  $\mathcal{H}$  be the separable infinite-dimensional Hilbert space, and let  $\mathcal{U}(\mathcal{H})$  be the group of unitary operators on  $\mathcal{H}$ . This is a Polish group when endowed with the weak operator topology; see [5, Proposition I.3.2.9]. The group  $\mathcal{U}(\mathcal{H})$  admits an action by conjugation on itself and on the space  $\mathcal{B}(\mathcal{H})_{sa}$  of selfadjoint operators.

**Theorem 2.11.** *The following relations are not Borel reducible to a Borel action of a CLI group on a standard Borel space:*

- (1) *unitary equivalence of unitary operators;*
- (2) *unitary equivalence of selfadjoint operators.*

*Proof.* As in Theorem 2.10 we consider the equivalence relation  $=^+$  on the set  $X^{\mathbb{N}}$  of sequences of elements of a perfect Polish space  $X$ . Fix an orthonormal basis  $(e_n)$  of  $\mathcal{H}$ . Let  $X$  be the circle group  $\mathbb{T}$ , and  $Y \subset \mathbb{T}^{\mathbb{N}}$  be the set of injective sequences. The map  $f: Y \rightarrow \mathcal{U}(\mathcal{H})$  which sends an element  $(\lambda_n) \in Y$  to the unitary operator

$$(e_n) \mapsto (\lambda_n e_n)$$

is a Borel reduction from  $=^+|_Y$  to unitary equivalence of unitary operators. The proof of selfadjoint operators is the same, where one replaces  $\mathbb{T}$  with  $[0, 1]$ .  $\square$

### 3. A game-theoretic approach to turbulence

**3.1. Hjorth’s turbulence theory.** Suppose that  $\mathcal{L} = (R_i)_{i \in I}$  is a countable first order relational language, where  $R_i$  is a relation symbol with arity  $n_i$ . We denote as above by  $\text{Mod}(\mathcal{L})$  the Polish  $S_\infty$ -space of  $\mathcal{L}$ -structures with support  $\mathbb{N}$ . Recall that a Polish group  $G$  is called *non-Archimedean* if it admits a neighborhood basis of the identity of open subgroups or, equivalently, it is isomorphic to a closed subgroup of  $S_\infty$ ; see [4, Theorem 1.5.1]. A relation  $E$  is *classifiable by countable structures* if it is Borel reducible to the isomorphism relation in  $\text{Mod}(\mathcal{L})$  for some countable first order relational language  $\mathcal{L}$ . This is equivalent to the assertion that  $E$  is Borel reducible to the orbit equivalence relation of a Borel action of a non-Archimedean Polish group  $G$  on a standard Borel space by [4, Theorem 5.1.11] and [11, Theorem 3.5.2, Theorem 11.3.8].

*Turbulence* is a dynamical condition on a Polish  $G$ -space  $X$  which is an obstruction of classifiability of  $E_G^X$  by countable structures. We now recall here the fundamental notions of the theory of turbulence, developed by Hjorth in [15]. Suppose that  $X$  is a Polish  $G$ -space,  $x \in X$ ,  $U$  is a neighborhood of  $x$ , and  $V$  is a neighborhood of the identity in  $G$ . The local orbit  $\mathcal{O}(x, U, V)$  is the smallest subset of  $U$  with the property that  $x \in \mathcal{O}(x, U, V)$ , and if  $g \in V$ ,  $x \in \mathcal{O}(x, U, V)$ , and  $gx \in U$ , then  $gx \in \mathcal{O}(x, U, V)$ . A point  $x \in X$  is called *turbulent* if it has dense orbit and, for any neighborhood  $U$  of  $x$  and neighborhood  $V$  of the identity in  $G$ , the closure of  $\mathcal{O}(x, U, V)$  is a neighborhood of  $x$ . A Polish  $G$ -space  $X$  is *preturbulent* if every point  $x \in X$  is turbulent, and *turbulent* if every point  $x \in X$  is turbulent and has meager orbit.

An equivalence relation  $E$  on a Polish space  $X$  is generically  $S_\infty$ -ergodic if, for any Polish  $S_\infty$ -space  $Y$  and Baire-measurable  $(E, E_{S_\infty}^Y)$ -homomorphism, there exists a comeager subset of  $X$  that is mapped by  $f$  to a single  $S_\infty$ -orbit.

By [11, Theorem 3.5.2, Theorem 11.3.8], this is equivalent to the assertion that, for any non-Archimedean Polish group  $H$ , Polish  $H$ -space  $Y$ , and Baire measurable  $(E, E_H^Y)$ -homomorphisms, there exists a comeager subset of  $X$  that is mapped by  $f$  to a single  $H$ -orbit. The following is the main result in Hjorth's turbulence theory, providing a dichotomy for preturbulent Polish  $G$ -spaces.

**Theorem 3.1** (Hjorth). *Suppose that  $X$  is a preturbulent Polish  $G$ -space. Then the associated orbit equivalence relation  $E_G^X$  is generically  $S_\infty$ -ergodic. In particular, either  $X$  has a dense  $G_\delta$  orbit, or the restriction of  $E_G^X$  to any comeager subset of  $X$  is not classifiable by countable structure.*

In this section, for each Polish  $G$ -space  $X$ , we define a graph structure  $\mathcal{H}(X/G)$  with domain the quotient  $X/G = \{[x]: x \in X\}$  of  $X$  via the action of  $G$ . We call this the Hjorth graph associated with the  $G$ -space  $X$ . An (induced) subgraph of  $\mathcal{H}(X/G)$  is of the form  $\mathcal{H}(C/G)$ , where  $C$  is an invariant subset of  $X$ . We view Hjorth's turbulence theorem as a corollary of the following facts:

- (1)  $\mathcal{H}(X/G)$  contains only loops if  $G$  is non-Archimedean;
- (2)  $\mathcal{H}(X/G)$  is a clique if the action of  $G$  on  $X$  is preturbulent;
- (3) given a Polish  $G$ -space  $X$  and a Polish  $H$ -space  $Y$ , a Baire measurable  $(E_G^X, E_H^Y)$ -homomorphism  $f$  induces, after restricting to an invariant dense  $G_\delta$  set, a graph homomorphism between the corresponding Hjorth graphs.

**3.2. The Hjorth-isomorphism game.** We start by defining a game associated with points of a given Polish  $G$ -space, which captures isomorphism in the case of Polish  $S_\infty$ -spaces.

**Definition 3.2.** Suppose that  $X$  is a Polish  $G$ -space, and  $x, y \in X$ . We consider the *Hjorth-isomorphism game*  $\text{Iso}(x, y)$  played between two players as follows. Set  $x_0 := x$ ,  $y_0 := y$ ,  $U_0^y := X$ , and  $V_0^y = G$ .

- (1) In the first turn, Player I plays an open neighborhood  $U_0^x$  of  $x_0$  and an open neighborhood  $V_0^x$  of the identity in  $G$ . Player II replies with an element  $g_0^y$  in  $G$ .
- (2) In the second turn, Player I then plays an open neighborhood  $U_1^y$  of  $y_1 := g_0^y y_0$  and an open neighborhood  $V_1^y$  of the identity of  $G$ , and Player II replies with an element  $g_0^x$  in  $G$ .
- ( $2n + 1$ ) At the ( $2n + 1$ )-st turn, Player I plays an open neighborhood  $U_n^x$  of  $x_n := g_{n-1}^x x_{n-1}$  and an open neighborhood  $V_n^x$  of the identity of  $G$ , and Player II responds with an element  $g_n^y$  of  $G$ .
- ( $2n + 2$ ) At the ( $2n + 2$ )-nd turn, Player I plays an open neighborhood  $U_{n+1}^y$  of  $y_{n+1} := g_n^y y_n$  and an open neighborhood  $V_{n+1}^y$  of the identity of  $G$ , and Player II responds with an element  $g_n^x$  of  $G$ .

The game proceed in this way, producing sequences  $(x_n)$  and  $(y_n)$  of elements of  $X$ , sequences  $(g_n^x)$  and  $(g_n^y)$  of elements of  $G$ , sequences  $(U_n^x)$  and  $(U_n^y)$  of open subsets of  $X$ , and sequences  $(V_n^x)$  and  $(V_n^y)$  of open neighborhoods of the identity in  $G$ . Player II wins the game if, for every  $n \geq 0$ ,

- $y_{n+1} \in U_n^x$  and  $x_n \in U_n^y$ ,
- $g_n^y = h_k \dots h_0$  for some  $k \geq 0$  and  $h_0, \dots, h_k \in V_n^y$  such that  $h_i \dots h_0 y_n \in U_n^y$  for  $i \leq k$ ,
- $g_n^x = h_k \dots h_0$  for some  $k \geq 0$  and  $h_0, \dots, h_k \in V_n^x$  such that  $h_i \dots h_0 x_n \in U_n^x$  for  $i \leq k$ .

We write  $x \sim_{\mathcal{H}} y$  and we say that  $x, y$  are Hjorth-isomorphic if Player II has a winning strategy for the Hjorth game  $\mathcal{H}(x, y)$ .

**Remark 3.3.** If Player II has a winning strategy for the Hjorth game as described above, then it also has a winning strategy for the same game with the additional winning conditions that  $g_n^x = h_k \dots h_0$  for some  $h_0, \dots, h_k$  from a given comeager subset of  $V_n^x$  such that  $h_i \dots h_0 x_n$  belongs to a given comeager subset  $X_0$  of  $X$  for  $i = 0, \dots, k$ , provided that the set of  $h \in G$  such that  $hx \in X_0$  is comeager. Similarly one can add the winning conditions that  $g_n^y = h_k \dots h_0$  for some  $h_0, \dots, h_k$  from a given comeager subset of  $V_n^y$  such that  $h_i \dots h_0 y_n$  belongs to a given comeager subset  $X_0$  of  $X$ , provided that the set of  $h \in G$  such that  $hy \in X_0$  is comeager. This can be proved similarly to Lemma 2.3 using the following version of the Kuratowski–Ulam theorem: suppose that  $X, Y$  are Polish spaces and  $f: X \rightarrow Y$  is a continuous open map. Then a Baire-measurable subset  $A$  of  $X$  is comeager if and only if the set  $\{y \in Y: A \cap f^{-1}\{y\}$  is comeager in  $f^{-1}\{y\}$  is comeager; see [29, Theorem A.1]. One can then apply this fact to the continuous and open map  $G \times X \rightarrow X, (g, x) \mapsto gx$ .

The relation  $\sim_{\mathcal{H}}$  is an equivalence relation on  $X$  which we call *Hjorth isomorphism*. It is clear that Hjorth isomorphism is a coarsening of the orbit equivalence relation  $E_G$  on  $G$ . Furthermore if  $x \sim_{\mathcal{H}} y, x'$  belongs to the  $G$ -orbit of  $x$ , and  $y'$  belongs to the  $G$ -orbit of  $y$ , then  $x' \sim_{\mathcal{H}} y'$ . Let as before  $X/G$  be the space of  $G$ -orbits of elements of  $X$ . The Hjorth-graph  $\mathcal{H}(X/G)$  associated with the Polish  $G$ -space  $X$  is symmetric, reflexive graph on  $X/G$  given by declaring that there exists an edge between the orbit  $[x]$  of  $x$  and the orbit  $[y]$  of  $y$  if and only if  $x \sim_{\mathcal{H}} y$ . We call  $\mathcal{H}(X/G)$  the Hjorth graph associated with the Polish  $G$ -space  $X$ . One can similarly define the Hjorth graph  $\mathcal{H}(C/G)$  for any invariant subset  $C$  of  $X$ . A comeager subgraph  $\mathcal{G}$  of  $\mathcal{H}(X/G)$  is a graph of the form  $\mathcal{H}(C/G)$ , for some invariant comeager subset  $C$  of  $X$ .

**3.3. Generic homomorphisms between Hjorth graphs.** We now proceed to the proof of the properties of Hjorth graphs stated at the end of Subsection 3.1.

In the following, for a subset  $V$  of  $G$  and  $k \in \mathbb{N}$  let  $V^k$  be the set of elements of  $G$  that can be written as the product of  $k$  elements from  $V$ .

**Lemma 3.4.** *Suppose that  $H$  is a non-Archimedean Polish group, and  $Y$  is a Polish  $H$ -space. Then the Hjorth graph  $\mathcal{H}(Y/H)$  contains only loops.*

*Proof.* Suppose that  $G$  is a non-Archimedean Polish group. Fix a compatible complete metric  $d$  on  $X$ , and a compatible complete metric  $d_G$  on  $G$ . We denote by  $\text{diam}(A)$  the diameter of a subset  $A$  of  $X$  with respect to the metric  $d$ , and by  $\text{cl}(A)$  the closure of  $A$ . Suppose that Player II has a winning strategy for the Hjorth-isomorphism game  $\text{Iso}(x, y)$ . We want to show that  $x$  and  $y$  belong to the same orbit. This can be seen by letting Player I play open subsets  $U_n^x$  and  $U_n^y$  of  $X$  such that  $\text{cl}(U_{n+1}^y) \subset U_n^x$ ,  $\text{cl}(U_n^x) \subset U_n^y$ ,  $\text{diam}(U_n^x) \leq 2^{-n}$ ,  $\text{diam}(U_{n+1}^y) \leq 2^{-n}$ , and open subgroups  $V_n^x$  and  $V_n^y$  of  $G$  such that

$$V_n^x \subset \{g \in G : d_G(gg_{n-1}^x \dots g_0^x, g_{n-1}^x \dots g_0^x) < 2^{-n}\},$$

$$V_n^y \subset \{g \in G : d_G(gg_{n-1}^y \dots g_0^y, g_{n-1}^y \dots g_0^y) < 2^{-n}\}.$$

Let then  $(x_n)$  and  $(y_n)$  be the sequences of elements of  $X$  and  $(g_n^x)$  and  $(g_n^y)$  be the sequences of elements of  $G$  obtained from the corresponding round of the Hjorth game. Then the assumptions on  $U_n^x$  and  $U_n^y$  guarantee that the sequences  $(x_n)$  and  $(y_n)$  converge to the same point  $z$  of  $X$ . The assumptions on  $V_n^x$  and  $V_n^y$  guarantee that the sequences  $(g_n^x g_{n-1}^x \dots g_0^x)_{n \in \omega}$  and  $(g_n^y g_{n-1}^y \dots g_0^y)_{n \in \omega}$  converge in  $H$  to elements  $g_\infty^x$  and  $g_\infty^y$  such that  $g_\infty^x x = z$  and  $g_\infty^y y = z$ . This shows that  $x$  and  $y$  belong to the same orbit. □

**Lemma 3.5.** *Suppose that  $X$  is a preturbulent Polish  $G$ -space. Then the Hjorth graph  $\mathcal{H}(X/G)$  is a clique.*

*Proof.* Suppose that  $X$  is a preturbulent Polish  $G$ -space. Fix  $x, y \in X$ . We want to prove that Player II has a winning strategy for the Hjorth game  $\mathcal{H}(x, y)$ . We begin with a preliminary observation. Suppose that  $z \in X$ ,  $U$  is an open neighborhood of  $z$ , and  $V$  is an open neighborhood of the identity in  $G$ . Let  $\mathcal{J}(z, U, V)$  be the interior of the closure of the local orbit  $\mathcal{O}(z, U, V)$ . Since  $z$  is turbulent,  $\mathcal{J}(z, U, V)$  contains  $z$ . It is not difficult to see that, for any  $w \in \mathcal{J}(z, U, V)$ , the local orbit  $\mathcal{O}(w, \mathcal{J}(z, U, V), V)$  is dense in  $\mathcal{J}(z, U, V)$ . We use this observation to conclude that Player II has a winning strategy, which we proceed to define. As in the definition of the Hjorth game, we let  $x_0 = x, y_0 = y, U_0^y = X$ , and  $V_0^y = G$ . At the  $(2n + 1)$ -st turn Player II plays an element  $g_n^y = h_k \dots h_0 \in (V_n^y)^k$  for some  $k \geq 1$  such that  $y_{n+1} = g_n^y y_n \in \mathcal{J}(x_n, U_n^x, V_n^x)$  and  $h_i \dots h_0 y_n \in U_n^y$  for  $i \leq k$ , while at the  $(2n + 2)$ -nd turn Player II plays an element  $g_n^x = h_k \dots h_0 \in (V_n^x)^k$  for some  $k \geq 1$  such that  $x_{n+1} = g_n^x x_n \in \mathcal{J}(y_{n+1}, U_{n+1}^y, V_{n+1}^y)$  and  $h_i \dots h_0 x_n \in U_n^x$  for  $i \leq k$ . Such a choice is possible at the 1-st turn since  $y$  has dense orbit. It is possible



at the  $(2n + 2)$ -nd turn ( $n \geq 0$ ) since  $y_{n+1} \in \mathcal{J}(x_n, U_n^x, V_n^x)$  and for every  $w \in \mathcal{J}(x_n, U_n^x, V_n^x)$  the local orbit  $\mathcal{O}(w, \mathcal{J}(x_n, U_n^x, V_n^x), V_n^x)$  is dense in  $\mathcal{J}(x_n, U_n^x, V_n^x)$ . It is possible at the  $(2n + 1)$ -st turn ( $n \geq 1$ ) since  $x_n \in \mathcal{J}(y_n, U_n^y, V_n^y)$  and for any  $w \in \mathcal{J}(y_n, U_n^y, V_n^y)$  the local orbit  $\mathcal{O}(w, \mathcal{J}(y_n, U_n^y, V_n^y), V_n^y)$  is dense in  $\mathcal{J}(y_n, U_n^y, V_n^y)$ . This concludes the proof that Player II has a winning strategy for the Hjorth game  $\mathcal{H}(x, y)$ .  $\square$

**Proposition 3.6.** *Suppose that  $G, H$  are Polish groups,  $X$  is a Polish  $G$ -space, and  $Y$  is a Polish  $H$ -space. If  $f$  is a Baire-measurable  $(E_G^X, E_H^Y)$ -homomorphism, then there exists a  $G$ -invariant dense  $G_\delta$  subset  $X_0$  of  $X$  such that the function  $X_0/G \rightarrow Y/H, [x] \mapsto [f(x)]$  is a homomorphism from the Hjorth graph  $\mathcal{H}(X_0/G)$  to the Hjorth graph  $\mathcal{H}(Y/H)$ .*

*Proof.* We proceed as in the proof of Proposition 2.8. Let  $C$  be dense  $G_\delta$  subsets of  $X$  obtained from  $f$  as in Lemma 2.5. Set  $X_0 := \{x \in X : \forall^* g \in G, gx \in C\}$ , which is a  $G$ -invariant dense  $G_\delta$  set by [11, Proposition 3.2.5 and Theorem 3.2.7]. We claim that  $X_0/G \rightarrow Y/H, [x] \mapsto [f(x)]$  is a graph homomorphism from the Hjorth graph  $\mathcal{H}(X_0/G)$  to the Hjorth graph  $\mathcal{H}(Y/H)$ .

Fix  $x_0, y_0 \in X_0$  such that  $x_0 \sim_{\mathcal{H}} y_0$ . We want to prove that  $f(x_0) \sim_{\mathcal{H}} f(y_0)$ . Observe that  $\forall^* g \in G, gx_0 \in C \cap X_0$ . Therefore after replacing  $x_0$  with  $gx_0$  for a suitable  $g \in G$  we can assume that  $x_0 \in C \cap X_0$ . In this case one can define, similarly as in the proof of Proposition 2.8, a winning strategy for Player II for  $\text{Iso}(f(x_0), f(y_0))$  from a winning strategy for Player II for  $\text{Iso}(x_0, y_0)$  using Remark 3.3 and the choice of  $C$ .  $\square$

It is now easy to see that Theorem 3.1 is an immediate consequence of Lemma 3.4 and Lemma 3.5 together with Proposition 3.6.

### 4. Groupoids

**4.1. Polish groupoids.** The goal of this section is to observe that the proofs above apply equally well in the setting of Polish groupoids as introduced in [32, 31, 26]. A *groupoid*  $G$  is a small category where every morphism (also called arrow) is invertible. By identifying any object with the corresponding identity arrow, one can regard the set  $G^0$  of objects of  $G$  as a subset of  $G$ . The source and range maps  $s, r: G \rightarrow G^0$  assign to every arrow in  $G$  its domain (or source) and codomain (or range). The set  $G^2$  of composable arrows is the set of pairs  $(\gamma, \rho)$  of arrows from  $G$  such that  $s(\gamma) = r(\rho)$ . Composition of arrows is a function  $G^2 \rightarrow G, (\gamma, \rho) \mapsto \gamma\rho$ . If  $A, B \subset G$ , then we denote by  $AB$  the set  $\{\gamma\rho: (\gamma, \rho) \in G^2 \cap (A \times B)\}$ . If  $x \in G^0$  and  $A \subset G$ , then we let  $Ax := A\{x\} = \{\gamma \in G: s(\gamma) = x\}$  and  $xA := \{x\}A = \{\gamma \in G: r(\gamma) = x\}$ .

A *Polish groupoid* is a groupoid  $G$  endowed with a topology such that

- (1) there exists a countable basis  $\mathcal{B}$  of Polish open sets,
- (2) composition and inversion of arrows are continuous and open,
- (3) the sets  $Gx$  and  $xG$  are Polish subspaces for every  $x \in G^0$ , and
- (4) the set of objects  $G^0$  is a Polish subspace.

A Polish groupoid is not required to be globally Hausdorff. Many Polish groupoids arising in the applications, such as the locally compact groupoids associated with foliations of manifolds, are not Hausdorff; see [30, Chapter 2].

Suppose that  $H$  is a Polish group. One can associate with any Polish  $H$ -space  $X$  a Polish groupoid  $H \times X$ —the *action groupoid*—that completely encodes the action. Such a groupoid has the Cartesian product  $H \times X$  as set of arrows (endowed with the product topology), and  $\{(1_H, x) : x \in X\}$  as set of objects. Source and range maps are defined by  $s(h, x) = (1_H, x)$  and  $r(h, x) = (1_H, hx)$ . Composition is given by  $(h, x)(h', y) = (hh', y)$  whenever  $x = h'y$ . In this way one can regard continuous actions of Polish groups on Polish spaces as a particular instance of Polish groupoids. One can also consider continuous actions of Polish groupoids on Polish spaces, but these can be in turn regarded as Polish groupoids via a similar construction as the one described above. The class of Polish groupoids is also closed under taking restrictions. If  $X$  is a  $G_\delta$  subset of the set of objects of a Polish groupoid  $G$ , then the *restriction*  $G|_X$  is the collection of arrows of  $G$  with source and range in  $X$ , endowed with the induced Polish groupoid structure. More information about Polish groupoids can be found in [26].

Given a Polish groupoid  $G$ , the orbit equivalence relation  $E_G$  is the equivalence relation on  $G^0$  defined by setting  $x E_G y$  if and only if  $x, y$  are source and range of an arrow from  $G$ . The orbit of an object in  $G$  is the  $E_G$ -class of  $x$ .

**4.2. Turbulence for Polish groupoids.** The notion of (pre)turbulence for Polish groupoid has been considered in [12, Section 4]. Suppose that  $G$  is a Polish groupoid,  $x$  is an object of  $G$ , and  $U$  is a neighborhood of  $x$  in  $G$ . The local orbit  $\mathcal{O}(x, U)$  is the smallest subset of  $U \cap G^0$  with the property that  $x \in \mathcal{O}(x, U)$ , and if  $\gamma \in U$  is such that  $s(\gamma) \in \mathcal{O}(x, U)$ , then  $r(\gamma) \in \mathcal{O}(x, U)$ . An object  $x$  is called *turbulent* if it has orbit dense in  $G^0$  and, for any neighborhood  $U$  of  $x$ , the closure of  $\mathcal{O}(x, U)$  is a neighborhood of  $x$  in  $G^0$ . A Polish groupoid is *preturbulent* if every object is turbulent, and *turbulent* if every object is turbulent and has orbit meager in  $G^0$ . It is not difficult to see that these definitions are consistent with the ones for Polish group actions, when a Polish group action is identified with its associated action groupoid.

Suppose that  $G$  is a Polish groupoid, and  $x, y \in G^0$  are two objects of  $G$ . The Hjorth-isomorphism game  $\text{Iso}(x, y)$  can be defined similarly as in Definition 3.2. Set  $x_0 := x$ ,  $y_0 := y$ ,  $U_0^y = G$ , and  $V_0^y = G$ . In this case, in the first turn Player I plays an open neighborhood  $U_0^x$  of  $x_0$  in  $G$  and Player II replies with an

element  $\gamma_0^y$  of  $G$  with  $s(\gamma_0^y) = y_0$ . In the second turn, Player I plays an open neighborhood  $U_1^y$  of  $y_1 := r(\gamma_0^y)$  in  $G$  and an element  $\gamma_0^x$  of  $G$  with  $s(\gamma_0^x) = x_0$ . At the  $(2n + 1)$ -st turn, Player I plays an open neighborhood  $U_n^x$  of  $x_n := r(\gamma_{n-1}^x)$  in  $G$ , and Player II responds with an element  $\gamma_n^y$  of  $G$  with  $s(\gamma_n^y) = y_n$ . At the  $(2n + 2)$ -nd turn, Player I plays an open neighborhood  $U_{n+1}^y$  of  $y_{n+1} := r(\gamma_n^y)$  in  $G$ , and Player II responds with an element  $\gamma_n^x$  of  $G$ .

The game then produces sequences  $(x_n), (y_n)$  of objects of  $G$ , sequences  $(\gamma_n^x), (\gamma_n^y)$  of arrows in  $G$ , and sequences  $(U_n^x), (U_n^y)$  of open subsets of  $G$ . Player II wins the game if, for every  $n \geq 0$ ,

- $y_{n+1} \in U_n^x$  and  $x_n \in U_n^y$ ,
- $\gamma_n^y = \rho_1^y \rho_2^y \dots \rho_k^y$  for some  $k \geq 1$  and  $\rho_i^y \in V_n^x$  for  $i = 1, 2, \dots, k$ , and  $\gamma_n^x = \rho_1^x \dots \rho_k^x$  for some  $k \geq 1$  and  $\rho_i^x \in V_n^y$  for  $i = 1, 2, \dots, k$ .

As in the case of Polish group actions, this defines an equivalence relation  $\sim_{\mathcal{H}}$  (Hjorth-isomorphism) on the set of objects of  $G$ , by letting  $x \sim_{\mathcal{H}} y$  whenever Player II has a winning strategy for the Hjorth-isomorphism game  $\text{Iso}(x, y)$ . Adding to the winning conditions in the Hjorth-isomorphism game the requirement that  $r(\gamma_n^x)$  belongs to a given comeager subset  $X$  of  $G^0$  and that  $\gamma_n^x$  belongs to a given comeager subset of  $Gx_n$  yields an equivalent game, provided that the set of  $\gamma \in Gx$  such that  $r(\gamma) \in X$  is comeager. The same applies to  $y$ . The Hjorth-isomorphism relation on  $G^0$  defines a graph structure  $\mathcal{H}(G)$  on the space of  $G$ -orbits, which we call the Hjorth graph of  $G$ . The same proof as Lemma 3.5 shows that if  $G$  is a preturbulent Polish groupoid, then the Hjorth graph  $\mathcal{H}(G)$  is a clique. The analogue of Lemma 2.5 for Polish groupoids has been proved in [12, Lemma 4.5]. Using this one can then prove the analog of Proposition 3.6 and deduce the following result.

**Theorem 4.1.** *Suppose that  $G$  is a preturbulent Polish groupoid. Then the associated orbit equivalence relation  $E_G$  is generically  $S_\infty$ -ergodic.*

Theorem 4.1 recovers [12, Theorem 4.3], and can be seen as the groupoid version of Theorem 3.1 for Polish groupoids.

Since the operations in the groupoid  $G$  are continuous and open, one can reformulate the Hjorth-isomorphism game  $\text{Iso}(x, y)$  as presented above by letting Player II play open sets rather than groupoid elements. Fix a countable basis  $\mathcal{B}$  of Polish open subsets of  $G$ . In this formulation of the game, Player I plays elements  $U_n^x, U_{n+1}^y$  of  $\mathcal{B}$  for  $n \geq 0$  and player II plays elements  $W_n^x, W_n^y$  of  $\mathcal{B}$  for  $n \geq 0$ . The winning conditions are then, setting  $U_0^y = G$ ,

- $r[W_{n+1}^y] \subset U_n^x$  and  $r[W_n^x] \subset U_n^y$ ,
- $W_n^y \subset (U_n^y)^k$  for some  $k \geq 1$  and  $W_n^x \subset (U_n^x)^k$  for some  $k \geq 1$ ,
- $y \in s[W_n^y \dots W_0^y]$  and  $x \in s[W_n^x \dots W_0^x]$ .

Such a version of the Hjorth-isomorphism game fits in the framework of Borel games as described in [21, Section 2.A]. In fact, this is an *open* game for Player I and closed for Player II, which allows one to define an  $\omega_1$ -valued *rank* for strategies of Player I [21, Exercise 20.2]. Insisting that Player I only has winning strategies of rank at least  $\alpha \in \omega_1$  (or no winning strategy at all) gives a hierarchy of equivalence relations  $\sim_\alpha$  indexed by countable ordinals, whose intersection is the Hjorth isomorphism relation.

**4.3. Becker-embeddings for Polish groupoids.** Similarly as for the Hjorth-isomorphism game, the Becker-embedding game  $\text{Emb}(x, y)$  can be defined whenever  $x, y$  are objects in a Polish groupoid  $G$ . This gives a notion of Becker embedding for objects  $G$ , by letting  $x \preceq_{\mathcal{B}} y$  if and only if Player II has a winning strategy for  $\text{Emb}(x, y)$ . In turn this induces a digraph structure  $\mathcal{B}(G)$  on the space of  $G$ -orbits.

One can prove the groupoid analog of Proposition 2.8 in a similar fashion, by replacing Lemma 2.5 with [12, Lemma 4.5]. One can then deduce the following generalization of Theorem 2.9 to Polish groupoids.

**Theorem 4.2.** *Suppose that  $G$  is a Polish groupoid. If for any invariant dense  $G_\delta$  subset  $C$  of  $G^0$  there exist  $x, y \in C$  with different orbits such that  $x \preceq_{\mathcal{B}} y$ , then the orbit equivalence relation  $E_G$  is not CLI-classifiable.*

As for the case of the Hjorth-isomorphism game, one can also describe the Becker-embedding game  $\text{Emb}(x, y)$  for objects  $x, y$  in a Polish groupoid  $G$  as an open game for Player I and closed for Player II. This allows one to define an  $\omega_1$ -valued rank for strategies for Player I. Again, insisting that Player I only has winning strategies of rank at least  $\alpha \in \omega_1$  gives a hierarchy or preorder relations  $\preceq_\alpha$  indexed by countable ordinals, whose intersection is the Becker-embeddability preorder.

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Martino Lupini, Mathematics Department, Caltech, 1200 E. California Blvd,  
MC 253-37 Pasadena, CA 91125, USA

home page: <http://www.lupini.org/>

e-mail: [lupini@caltech.edu](mailto:lupini@caltech.edu)

Aristotelis Panagiotopoulos, Department of Mathematics, 1409 W. Green St.,  
University of Illinois, Urbana, IL 61801, USA

*Current address:* Mathematics Department, Caltech, 1200 E. California Blvd,  
MC 253-37 Pasadena, CA 91125, USA

home page: <http://www.its.caltech.edu/~panagio/>

e-mail: [panagio@caltech.edu](mailto:panagio@caltech.edu)