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Groups, Geometry, and Dynamics

Relatively extra-large Artin groups

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Abstract. Let $n \geq 2$ be an integer and let N be an $n \times n$ symmetric matrix with 1's on the main diagonal and natural numbers $n_{ij} \neq 1$ as off-diagonal entries. (0 is a natural number). Let $X = \{x_1, \dots, x_n\}$ and let F be the free group on X. For every non-zero off-diagonal entry n_{ij} of N define a word $R_{ij} := UV^{-1}$ in F, where U is the initial subword of $(x_i x_j)^{n_{ij}}$ of length n_{ij} and V is the initial subword of $(x_j x_i)^{n_{ij}}$ of length n_{ij}, $1 \leq i, j \leq n$. Let A be the group given by the presentation $\langle X \mid R_{ij}, n_{ij} \geq 2\rangle$. A is called the *Artin group defined by* N, *with standard generators* X. Let $Y = \{x_1, \dots, x_k\}, k < n$ and let N_Y be the submatrix of N corresponding to Y. Let $H = \langle Y \rangle$. We call A *extra-large relative to* H if N subdivides into submatrices N_Y, B, C and D of sizes $k \times k, k \times l, l \times k, l \times l$, respectively $(l + k = n)$ such that every non zero element of B and C is at least 4 and every off-diagonal non-zero entry of D is at least 3. No condition on N_Y. In this work we solve the word problem for such A, show that A is torsion free and show that A has property $K(\pi, 1)$, provided that H has these properties, correspondingly. We also compute the homology and cohomology of A, relying on that of H. The two main tools used are Howie diagrams corresponding to relative presentations of A with respect to H and small cancellation theory with mixed small cancellation conditions.

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Contents

Introduction

An *Artin group* is a group given by a presentation

$$\mathcal{P} = \langle a_1, \ldots, a_n \mid \mathcal{R} \rangle \tag{0}$$

where $n \geq 2$, $\mathcal{R} = \{R_{i,j}\}, 1 \leq i < j \leq n$, and either $R_{i,j}$ is empty, in which case we define $n_{ij} = \infty$, or $R_{i,j}$ has length $2n_{i,j}$, $n_{i,j} \in \mathbb{N} \setminus \{0, 1\}$, and is given by $R_{i,j} = UV^{-1}$, where U is a head of $(a_i a_j)^{n_{ij}}$ of length n_{ij} and V is a head of $(a_j a_i)^{n_{ij}}$ of length n_{ij}, $n_{ij} < \infty$.

Artin groups are subject to an intensive study at least for fifty years. Yet, the most basic decision problem, the word problem (W.P.) is not known to be solvable, except for some special cases. Recall, that an Artin group A is called of *finite type*, if the corresponding Coxeter group, a presentation of which is obtained from (0) by adding relations a_i^2, $i = 1, \ldots, n$, is finite [7]. Also, recall that A given by (0) is called *large* if $n_{ij} \geq 3$ for all $1 \leq i < j \leq n$ (see [2]) and *extra-large* if $n_{ij} \geq 4$, for all $1 \leq i < j \leq n$ (see [2]). A is called *right angled* if $n_{ij} = 2$, for $n_{ij} < \infty$, $1 \leq i, j \leq n$. Most notably the W.P. is known to be solvable for finite type, see [6] and [18], FC type [1], right angled [21], extra large [2], large [3] and locally aspherical [11]. (For definitions of FC type and locally aspherical Artin groups see the cited references [1] and [11], respectively). Another problem in Artin groups which is known to have an affirmative answer for these classes of Artin groups is the $K(\pi, 1)$ conjecture. (See Section 5 for a very brief introduction of the $K(\pi, 1)$ conjecture, following [32] and for a full discussion in [32]).

In the present work we introduce a framework which allows to deal with Artin groups in which certain subgroups are from one class and certain other subgroups are from other classes in a uniform way, this in contrast with the above mentioned classes, where for each class a specific method is used.

A different type of result which implies solvability of the W.P. and a proof of the $K(\pi, 1)$ conjecture is [20], where it is proved that if a certain type of subgroups have solvable W.P. then the Artin group A has solvable W.P. Also, if these subgroups satisfy the $K(\pi, 1)$ conjecture, then A satisfies it. See [32] for a full discussion of the $K(\pi, 1)$ conjecture and see [15] for this last result. Our main result is of similar type, but has essentially different flavour: it is a kind of relative hyperbolicity result. As is well known, relative hyperbolicity was introduced by M. Gromov [22], and further clarified by B. Farb and others, see [16], [4], [29], [30], and [33]. It is natural to ask when is an Artin group hyperbolic relative to a certain set of subgroups. Indeed, in [25] it is shown that A is hyperbolic relative to a certain set of two-generated subgroups if $n_{ij} \geq 7$, for every $1 \leq i < j \leq n$. Also, a related result is [8], in which relative hyperbolicity is considered with respect to the subgroups of finite type.

In the present work we consider relative hyperbolicity via van Kampen diagrams with various small cancellation conditions. A basic observation, made for the first time in [2], shows that the relations R_{ij} in (0) are not suitable for application of small cancellation theory. The right way is to consider certain products of them in a systematic way. See Section 1.2. This leads to diagrams in which the original regions are replaced by subdiagrams which realise these products. We call them *modified regions* and the diagram with these regions, we call *modified diagram*. Now, if A is extra-large then these modified diagrams have linear isoperimetric functions, hence in this context extra-largeness can be considered as a substitute for hyperbolicity. Similarly, largeness of Artin groups is a form of non-positive curvature. In this work we introduce Artin groups which are extra-large relative to a parabolic subgroup H, thus resembling relative hyperbolicity.

Before describing our main results we need to introduce some notions.

Let A be an Artin group given by presentation (0). The *corresponding* (*Coxeter*) *graph* is the graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E such that if $e \in E$ has endpoints v_i and v_j $(i \neq j)$ then e is labeled by the number n_{ij}; $\lambda(e) := n_{ij}$ and if v_i and v_j are not connected by an edge then $n_{ij} = 0$. Thus, the vertices of Γ are in one-to-one correspondence with the generators of A given by (0), $a_i \mapsto v_i$, $i = 1, \ldots, n$ and the edges of Γ are in one-to-one correspondence with the relators $R_{i,j}$, such that $R_{i,j}$ corresponds to the edge e connecting v_i with v_j and $\lambda(e) = n_{i,j}$.

Suppose now that V_0 is a non-empty subset of V and let $V = V_0 \dot\cup V_1$ be the disjoint union of V_0 and V_1. Denote by E_i, $i = 0, 1$ the set of all edges of Γ which have both their endpoints in V_i and denote by $E_{0,1}$ the set of all the edges of Γ which have one endpoint in V_0 and the other in V_1. Thus, $E = E_0 \dot\cup E_{0,1} \dot\cup E_1$. Further, denote by $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_{0,1}$ the sets of the relators corresponding to E_0, E_1 and $E_{0,1}$, respectively. Clearly, we have $\mathcal{R} = \mathcal{R}_0 \dot\cup \mathcal{R}_{0,1} \dot\cup \mathcal{R}_1$. Denote by $\widehat{\mathcal{R}}_0$, $\widehat{\mathcal{R}}_{0,1}$ and $\widehat{\mathcal{R}}_1$ the symmetric closure of $\mathcal{R}_0$, $\mathcal{R}_{0,1}$ and $\mathcal{R}_1$ respectively. i.e. the set of all the cyclic conjugates of relations and their inverses.

Denote $H = \langle a_i \mid v_i \in V_0 \rangle$. Then H is a standard parabolic subgroup of A and by [31] H is presented by $\langle a_i \mid v_i \in V_0, \mathcal{R}_0 \rangle$.

Definition. Let notation be as above. Call A *extra-large relative to* H if each of the following holds:

(a) $\lambda(e) \geq 3$, for every $e \in E_1$;

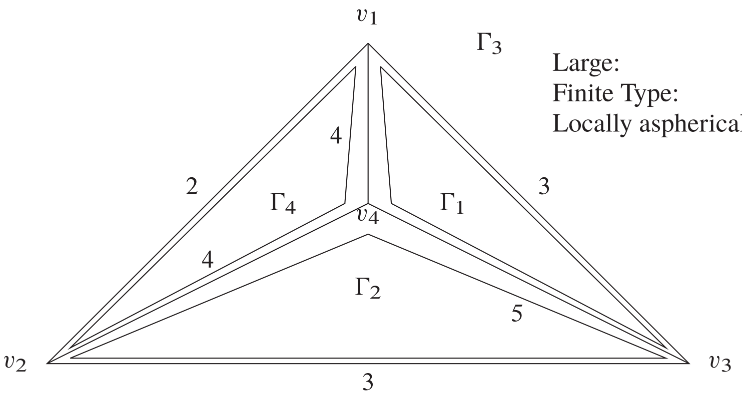
(b) $\lambda(e) \geq 4$, for every $e \in E_{0,1}$.

It is convenient to express this definition in terms of the adjacency matrix $N_A = (n_{ij})$, where, for $i \neq j$, n_{ij} is the natural number which labels the edge of the defining graph of A which connects v_i with v_j if such an edge exists, $n_{ii} = 1$, and is 0 otherwise and $n_{ij} = n_{ji}$. In these terms, A is called *extra-large relative to* H, if N_A splits into 4 submatrices N_H, B, C and D, where the size of N_H is

$k \times k$ where $k = |V_0|$, the sizes of B, C and D are respectively $k \times l$, $l \times k$, $l \times l$, where $l + k = n$ such that all the non-zero entries of B and C are at least 4 and all non-zero off-diagonal entries of D are at least 3.

Example 1. Let A be given by N_A below. Then A is extra-large relative to $H = \langle a_1, a_2, a_3 \rangle$. The Coxeter graph of A is given in Figure 1.

$$N_A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 3 & 1 & 5 \\ 4 & 4 & 5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} \quad C = \begin{pmatrix} 4 & 4 & 5 \end{pmatrix} \quad D = (1)$$



Large: Γ_1 and Γ_2
Finite Type: $\Gamma_3(a_1, a_2, a_3)$
Locally aspherical: Γ_4

Figure 1

With these definitions, our main results are the following theorems.

Theorem A. *Let notation be as above. Assume A is extra-large relative to H. If H has solvable word problem, then A has.*

Theorem B. *Assume A is extra-large relative to H. Then the following hold*:

(a) *if H is torsion free then A is*;

(b) $H^n(A, *) \cong H^n(H, *)$ *and* $H_n(A, *) \cong H_n(H, *)$ *for every* $n \geq 2$.

Theorem C. *Let notation be as above. If H satisfies the $K(\pi, 1)$ conjecture then A satisfies it.*

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1. Preliminary results

1.0. Results on Artin groups. The main tool we use is van Kampen diagrams. We collect here the known results on diagrams of Artin groups, which we shall need. For diagrams in general we follow [28]. We denote by Φ the labelling function of a diagram M over a free group F (thus $\Phi\colon M \to F$). In addition, denote by $\mathrm{Reg}(M)$ the set of all the regions of M. We denote by $\mathrm{Int}(M)$ the interior of M and by $|M|$ the number of regions of M (thus $|M| = |\mathrm{Reg}(M)|$).

Remark 1.0. We consider ∂D, the boundary of a region D as a path, rather than just a set in $\mathbb{E}^2$ and similarly for ∂M, the boundary of a diagram M.

We shall use the following notation and convention, as well.

(a) Let W be a non-empty reduced word in $F(a_1, \ldots, a_n)$ (the free group, freely generated by $\{a_1, \ldots, a_n\}$). Denote by $\mathrm{Supp}(W)$ the set of all the letters a_i such that either a_i or a_i^{-1} occurs in W.

(b) Let W be a reduced word in $F(a_1, \ldots, a_n)$. Denote, as usual, by $\|W\|$ its *syllable length* (or free-product length, see [28, p. 176]) and denote by $l(W)$ its *word length*. Thus, if $W = a_1^2 a_2^3 a_1^{-1} a_4^4$ then $\|W\| = 4$, $l(W) = 2 + 3 + |-1| + 4 = 10$ and $\mathrm{Supp}(W) = \{a_1, a_2, a_4\}$. If μ is a path in M we write $l(\mu)$ for $l(\Phi(\mu))$ and write $\mathrm{Supp}(\mu)$ for $\mathrm{Supp}(\Phi(\mu))$. Also, for a region D in M write $\mathrm{Supp}(D)$ for $\mathrm{Supp}(\Phi(\partial D))$. We denote $\|\mu\| = \|\Phi(\mu)\|$. If $\Phi(\partial D) = R \in \mathcal{R}$ we write $n(D)$ for $\frac{1}{2} l(R)$.

(c) Let $\nu\colon F \to A$ be the natural map from F onto the Artin group A. If no ambiguity is caused, then we shall not distinguish between a subgroup H of F and its image by ν in A.

Basic to what follows is the theorem of L. Paris [31].

Theorem 1.1 (H. van der Lek [26]; L. Paris [31]). *Let A be an Artin group given by* (0) *and let H be a standard parabolic subgroup of A. Then the natural map of H into A is an embedding.*

Fix a set of Artin relators $\mathcal{R}$ as in (0) which defines the Artin group A and let $H = \langle V_0 \rangle$, as in the introduction. From now on we shall write "parabolic" for "standard parabolic."

This theorem has several consequences which we shall need.

Corollary 1.1. *Let M_0 be a van Kampen $\mathcal{R}$-diagram with* $\mathrm{Supp}(\partial M_0) \subseteq H$. *Then there exists a van Kampen $\mathcal{R}$-diagram M with* $\mathrm{Supp}(D) \subseteq H$, *for every* $D \in \mathrm{Reg}(M)$, *such that M and M_0 have the same boundary labels.*

Proof. Let W be a boundary label of M_0. Then $W \in H$ and W represents 1 in A. Since the natural map of H into A is an embedding, by Theorem 1.1, W also represents 1 in H. Hence by [28, Chapter V] there is a van Kampen $\mathcal{R}$-diagram M *over* H with boundary label W. □

Definition. Let H be a parabolic subgroup of A and let M be a van Kampen $\mathcal{R}$-diagram. Call M *H-normal* if every subdiagram L with connected interior in which $\mathrm{Supp}(D) \subseteq H$ for every $D \in \mathrm{Reg}(L)$ and which is maximal with respect to this property, has simply connected interior.

Corollary 1.2. *Let M_0 be a reduced van Kampen $\mathcal{R}$-diagram and let H be a parabolic subgroup of A. Then there exists an H-normal reduced van Kampen $\mathcal{R}$-diagram M with $\Phi(\partial M) = \Phi(\partial M_0)$.*

Proof. If M_0 is already H-normal then take $M = M_0$. Assume M_0 is not H-normal and L is a subdiagram of M_0 which satisfies the conditions of the definition but its interior is not simply connected. Then $\mathbb{E}^2 \setminus L$ has a finite number of bounded simply connected components $J_1, \ldots, J_k$, $k \geq 1$. Now, each J_i is a van Kampen $\mathcal{R}$-diagram with $\mathrm{Supp}(\partial J_i) \subseteq H$. Hence, by Corollary 1.1, J_i can be replaced by an $\mathcal{R}_0$ van Kampen diagram J_i' with the same boundary label and with $\mathrm{Supp}(\mathcal{R}_0) \subseteq H, \mathcal{R}_0 \subseteq \mathcal{R}$. Cutting out J_i and sewing back $J_i', i = 1, \ldots, k$ we get a diagram L' which has connected and simply connected interior with $\mathrm{Supp}(L') \subseteq H$ and with the same boundary label as L. If L' is reduced we are done. If L' is not reduced then we can reduce it by diamond moves (See [12]), which, as well-known do not change the boundary label, yet they may identify boundary vertices. See Figure 2.

In Figure 2, L'' is the diagram obtained from L' by diamond moves. In case(a) of Figure 2, $\mathrm{Int}(L'')$ is the disjoint union of subdiagrams Δ_i each homeomorphic to an open disc: $\mathrm{Int}(L'') = \Delta_1 \dot\cup \ldots \dot\cup \Delta_s$, $s \geq 1$. (In Figure 2, $s = 2$) Now, by construction, Δ_i satisfies the conditions of the definition and has simply connected interior, as required.

In case (b) of Figure 2, although L'' is not simply connected, its interior is, and the conditions of the definition are satisfied. There is also the possibility that conjunctions of case (a) and case (b) occur. In these cases the result follows by the above arguments of cases (a) and (b). □

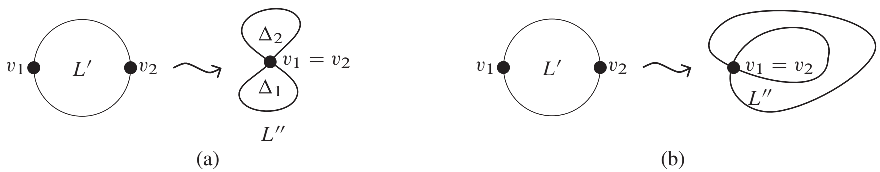


Figure 2

Remark 1.1. Notice that the procedure of Corollary 1.2 is applied only to regions with boundary labels in H and leaves all other regions unaltered.

Corollary 1.3. *Every standard generator a_i of A has infinite order.*

Proof. Apply Theorem 1.1 to $H = \langle a_i \rangle$. □

We close this section with the following result due to R. Charney and L. Paris, see [10].

Theorem 1.2 ([10]). *Let W be a reduced word in $H \subseteq F$ and let Z be a shortest representative of W in A. Then* $\mathrm{Supp}(Z) \subseteq \mathrm{Supp}(H)$

1.1. Two-generator Artin groups and related results. The following are well known results from [3].

Lemma 1.1 ([3]). *Let $A = \langle x, y \mid R\rangle$ be a two generated Artin group and let M be a van Kampen diagram for A with connected interior. Then the following holds.*

(a) $\|\partial M\| \geq \|R\|$. (*See* [3, Lemma 6].)

(b) *If $\mu\nu$ is a cyclically reduced boundary cycle of M and $\|\mu\| \leq \frac{1}{2}\|R\|$ then $l(\mu) \leq l(\nu)$. Moreover, if $\|\mu\| < \frac{1}{2}\|R\|$ then $l(\mu) < l(\nu)$.* (*See* [3, Lemma 7].)

(c) *Every $\mathcal{R}$-diagram satisfies conditions* C(4) *and* T(4). (*See* [3].)

(d) $|M| \leq (l(\partial M))^2$. (*See* [28, Chapter V].)

Part (b) of Lemma 1.1 together with Theorem 1.2 have the following consequence.

Lemma 1.2. *Let A be an Artin group defined by (0) and let $a_i^{\alpha_i} a_j^{\alpha_j} K^{-1} = 1$ in A, $\alpha_i, \alpha_j \in \mathbb{Z}$, where K is an arbitrary reduced word in $F(a_1, \ldots, a_n)$. Then $l(K) \geq |\alpha_i| + |\alpha_j|$.*

Proof. Without loss of generality we may assume that K is a shortest representative of $a_i^{\alpha_i} a_j^{\alpha_j}$ in A. Let $U = a_i^{\alpha_i} a_j^{\alpha_j} K^{-1}$. Then U represents 1 in $\langle a_i, a_j\rangle$ due to Theorem 1.2 and $l(K) \leq l(a_i^{\alpha_i} a_j^{\alpha_j})$. If U is cyclically reduced then $l(a_i^{\alpha_i} a_j^{\alpha_j}) \leq l(K)$, By Lemma 1.1(b), hence $l(a_i^{\alpha_i} a_j^{\alpha_j}) = l(K)$, i.e. $a_i^{\alpha_i} a_j^{\alpha_j}$ is a shortest representative of itself in A, as required. If U is not cyclically reduced then $K = K_1 K_2 K_3$ and $a_i^{\alpha_i} a_j^{\alpha_j} = K_1 L K_3$ reduced as written such that $LK_2^{-1} = 1$ in $\langle a_i^{\alpha_i} a_j^{\alpha_j}\rangle$ and LK_2^{-1} is cyclically reduced. If $L = 1$ in F then $K_2 = 1$ in F, since K is a shortest representative of $a_i^{\alpha_i} a_j^{\alpha_j}$. But then $l(a_i^{\alpha_i} a_j^{\alpha_j}) = l(K)$, again. If $L \neq 1$ in F then L has form $a_i^{\beta_i} a_j^{\beta_j}$, not both β_i and β_j are zero, hence the result follows from the cyclically reduced case. □

Corollary 1.4. *Let $W \neq 1$ be a word in F which represents 1 in A. Then $\|W\| \geq 4$.*

Proof. Without loss of generality we may assume that W is cyclically reduced. By Corollary 1.3, $\|W\| \geq 2$. If $\|W\| = 2$ then clearly $|\operatorname{Supp}(W)| = 2$, hence by Lemma 1.1(a), $\|W\| \geq 4$, a contradiction. Finally, suppose that $\|W\| = 3$ and that W has no cyclic conjugate W' with $\|W'\| = 2$. Then $|\operatorname{Supp}(W)| = 3$, hence $a_i^{\alpha_i} a_j^{\alpha_j} a_k^{\alpha_k} = 1$ in A, $\alpha_i, \alpha_j, \alpha_k$ all non-zero. Therefore, by Lemma 1.2 $|\alpha_i| \geq |\alpha_j| + |\alpha_k|$, $|\alpha_j| \geq |\alpha_i| + |\alpha_k|$ and $|\alpha_k| \geq |\alpha_j| + |\alpha_i|$. Summing up we get $\alpha \geq 2\alpha$, where $\alpha = |\alpha_i| + |\alpha_j| + |\alpha_k|$, a contradiction, which completes the proof of the Corollary. □

1.2. Modified diagrams. Diagrams of Artin groups with more than two generators are not suitable for studying them directly via small cancellation theory. Let M be a van Kampen diagram over a set of Artin relators. It was observed in [2] that if we replace the regions of M by certain subdiagrams which are diagrams for two generator subgroups, then we have control on the pieces in the new diagram obtained. These subdiagrams can be defined as follows.

Let M be a reduced $\mathcal{R}$-diagram, $\mathcal{R}$ a set of Artin relators. Say that two regions are *neighbours* or are *adjacent* if they have a common edge. Say that two adjacent regions D_1 and D_2 are *friends*, if $\operatorname{Supp}(D_1) = \operatorname{Supp}(D_2)$. Let "$\sim$" be the transitive closure of friendness. Then "$\sim$" is an equivalence relation on $\operatorname{Reg}(M)$. For a region $D \in \operatorname{Reg}(M)$ denote by $[D]_M$ the equivalence class of D in M and let

$$\Delta(D) = \bigcup_{E \in [D]_M} \bar{E}, \quad \text{where } \bar{E} \text{ denotes the closure of } E \text{ in } \mathbb{E}^2.$$

Let $\operatorname{Supp}(D) = \{a_i, a_j\}$ and let $H_{i,j}$ be the subgroup of F generated by a_i and a_j, $H_{i,j} = \langle a_i, a_j \rangle \not\subseteq H$. Using Corollary 1.2 and Remark 1.1, it follows that we may get by the procedure of Corollary 1.2 a diagram with the same boundary label as M which is H-normal and $H_{i,j}$-normal for $H_{i,j} \not\subseteq H$. Therefore we may consider $\Delta(D)$ as a region. (If the label of $\partial\Delta(D)$ is not cyclically reduced, then we can apply a sequence of diamond moves. They do not alter $\operatorname{Reg}(\Delta(D))$, considering $\Delta(D)$ as a subdiagram of M). We denote by M' the diagram obtained by considering $\Delta(D)$, for $D \in \operatorname{Reg}(M)$, as regions and call M' the *modified diagram* of M . We call $\{\Delta(D) \mid D \in \operatorname{Reg}(M)\}$ the *modified regions of M'*. We shall denote a typical modified region by Δ' and denote by Δ the corresponding subdiagram of M. Thus, as sets Δ and Δ' are the same. We shall denote paths in M' by $\mu', \nu', \dots$ and the corresponding paths in M by $\mu, \nu, \dots$. Again, as sets μ and μ' are the same. Denote by $\|\mu'\|$ the syllable length of μ' in M' and by $\|\mu\|$ the syllable length of μ in M and similarly for $l(\mu)$ and $l(\mu')$. These distinctions are needed because we are going to work sometimes in M' and then pass to M and back. We shall introduce in Section 2 a further diagram constructed from M and extend these notations.

Observe that $l(\partial M') = l(\partial M)$ and $\Phi(\partial M') = \Phi(\partial M)$, since by definition $\partial M = \partial M'$.

The main property of these diagrams is given by the lemma below.

Lemma 1.3 ([3, Lemma 3]). *Let notation be as above and let Δ_1 and Δ_2 be modified regions of M'. Then $\|\partial\Delta_1 \cap \partial\Delta_2\| \leq 1$.*

Combining Lemma 1.1(a) with Lemma 1.3 implies that modified diagrams of Artin groups of large type satisfy the small cancellation condition C(6). See [3]. However, we shall need a more flexible small cancellation condition which we recall in the next subsection.

1.3. Diagrams with small cancellation condition V(6). Recall that the (underlying) map corresponding to a diagram M is obtained by removing labels of M.

Definition. Let M be a map. Say that M satisfies the *small cancellation condition* V(6) if each of the following holds:

(i) each inner region has at least 4 neighbours;

(ii) if an inner region has less than 6 neighbours, then every vertex on its boundary has valency at least 4.

Remark 1.2. Condition V(6) is a special case of the condition W(6). See [24], where a structure theorem for simply connected maps with the condition W(6) was developed. Hence we can use the results of [24] for V(6)-maps and we do so below.

The first basic result concerning V(6) maps is given by Theorem * on p. 61 of [24], which states that in a V(6) map M, for every region D, ∂D is a simple closed curve and if D_1 and D_2 are regions of M such that $\partial D_1 \cap \partial D_2 \neq \emptyset$ then $\partial D_1 \cap \partial D_2$ is a connected simple curve. Consequently, following Definition 1.1.4 on p. 58 of [24] we can reformulate Definition 2.1 in [24, p. 59] as follows.

Definition. Let E be a boundary region of M. Call E a *k-corner region of M*, $k \in \{1, 2, 3\}$ if the following holds:

(1) $\partial E \cap \partial M$ is connected;

(2) E has k neighbours in M;

(3) if $k = 2$ then at least one endpoint of $\partial E \cap \partial M$ has valency 3 in M;

(4) if $k = 3$ then at most one inner vertex of ∂E has valency 4 or more.

Definition (one layer maps). (a) Let M be a connected, simply connected regular map (i.e. every edge is on the boundary of a region) with connected interior. M is called a *one layer map* if its dual is a line segment.

(b) Let A be an annular regular map with connected interior. A is called a *one layer map* if its dual is a circle.

Following Definition 2.7 on p. 63 of [24], for every submap K of a map M we define $S_M(K)$ to be the set of all the regions in $\mathrm{Reg}(M) \setminus \mathrm{Reg}(K)$ which have a common vertex with ∂K. Further, we define $\mathrm{St}_M(K) = \{K\} \cup S_M(K)$ and inductively by $\mathrm{St}^0_M(K) = K$ and for $t \geq 1$, $\mathrm{St}^t_M(K) = \mathrm{St}_M(\mathrm{St}^{t-1}_M(K))$. Finally, define $L^t_M(K)$ to be the closure of the submap obtained by deleting $\mathrm{St}^{t-1}_M(K)$ from St^t_M. We shall need this construction for the special case when K is a region or a vertex. Following [24, p. 64, Definition 2], we say that M has *convex layer structure* if for every $D \in \mathrm{Reg}(M)$ and for $i \geq 1$, every submap S of $\mathrm{St}^t_M(D)$ which contains $\mathrm{St}^{t-1}_M(D)$ is simply connected.

The main result of [24] is Theorem 2.7 on p. 64 saying that every simply connected V(6) map with connected interior has convex layer structure. We call $L^t_M(D)$ the *t-th layer* (relative to D) and denote $\Lambda_M(D) = (L^0(D), \ldots, L^p(D))$.

We shall need the following results on V(6)-maps. If M is fixed, we shall write $L_t(D)$ or just L_t, if D is clear, for $L^t_M(D)$.

Theorem 1.3. *Let M be a simply connected* V(6)*-map with connected interior containing at least two regions and let $\Lambda = (L^0(D), \ldots, L^p(D))$ be a convex layer structure for M.*

(a) *Either* $\mathrm{Int}(L_t)$ *is annular and is a one-layer map, or it is the disjoint union of simply connected one-layer maps.*

(b) *Let $\omega(L_t) = \partial L_t \cap \partial L_{t+1}$. Then every vertex in $\omega(L_t)$ has valency at most* 3 *in L_t.*

(c) *M has at least two corner regions. If M has exactly two corner regions then M is a one-layer map.*

(d) $|M| \leq 2[l(\partial M)]^2$.

(e) *If M is reduced then M cannot be mapped onto a* 2*-sphere.*

Proof. (a) follows from the definition of L_t, Theorem 2.7 and Lemma 2.7(b) in [24], p. 64.

(b) is Lemma 2.7(a) in [24], p. 64.

(c) follows from Theorem 2.2 in [24], p. 59 and Corollary 1 on p. 60.

(d) follows from the area Theorem in [24, p. 76].

(e) is Corollary 2 in [24, p. 60]. □

For an inner region D of M denote by $d_M(D)$, as usual, the number of neighbours (neighbouring regions). For a boundary region D of M denote by $i_M(D)$ the number of neighbouring regions of D in M. For a vertex v in M denote, as usual, by $d_M(v)$ the valency of v in M.

We denote for a boundary path μ the number of segments it contains by $|\mu|$, where a segment is a path with endpoints having valency at least 3 and every inner vertex has valency 2.

2. The relative diagram

The basic idea of the proofs of Theorems A, B, and C is to consider a relative presentation of A, rather than its free presentation (0).

Recall from [5] that a *relative presentation* $\mathbb{P}$ consists of a group U, a set Y, with $Y \cap U = \emptyset$, and a set $\mathcal{R}$ of cyclically reduced words in $U * F(Y)$ which are not in U. We denote

$$\mathbb{P} = \langle U, Y \mid \mathcal{R} \rangle$$

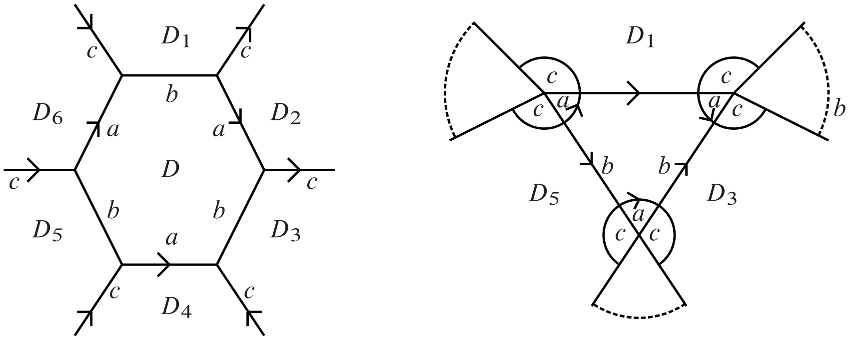
The group defined by $\mathbb{P}$ is $U * F(Y)/ \ll \mathcal{R} \gg$, where $\ll \mathcal{R} \gg$ is the normal closure of $\mathcal{R}$ in $U * F(Y)$.

Now we consider the relative presentation of an Artin group A, which is extra-large relative to a parabolic subgroup H, generated by V_0. Let $F(V_j)$ be the free group, freely generated by $\{a_i \mid v_i \in V_j\}$, $j = 0, 1$. Recall that $V = V_0 \cup V_1$. Then $F(a_1, \dots, a_n) = F(V_0) * F(V_1)$ and $U = H = F(V_0)/ \ll \mathcal{R}_0 \gg$. The relative presentation of A (relative to H) is given by

$$\mathbb{P} = \langle H, V_1 \mid \mathcal{R}_1 \cup \mathcal{R}_{0,1} \rangle \tag{1}$$

Let M be an $\mathcal{R}$-diagram over F, where $\mathcal{R}$ is a set of Artin relators which define the Artin group A. We consider the corresponding Howie diagrams, $\widetilde{M}$ and their modified diagrams. (See [23] for definition and [13] for definition in a more general context). They are constructed from M by shrinking each edge which is labelled by a letter from V_0. (In particular, such boundary edges are shrunk.) The regions of $\widetilde{M}$ are $\mathrm{Reg}_1(\widetilde{M}) \cup \mathrm{Reg}_{0,1}(\widetilde{M})$, where $\mathrm{Reg}_1(\widetilde{M})$ is the set of regions of $\widetilde{M}$ with boundary label from $\mathcal{R}_1$ and $\mathrm{Reg}_{0,1}(\widetilde{M})$ is the set of regions of $\widetilde{M}$ with boundary label from $\mathcal{R}_{0,1}$, after shrinking the edges with labels in V_0 to vertices and introducing them as *corner labels*. (This notion has nothing to do with corner regions.) See Figure 3. We call such vertices, with corner labels from $F(V_0)$, *coloured vertices* and from $F(V_1)$ *uncoloured vertices*.

Our aim in the present subsection is to show that $\widetilde{M'}$ satisfies condition V(6).



$c, a \in V_0, \quad b \in V_{0,1}$

Figure 3

Remark 2.1. Consider Figure 4. We have the following diagram, where δ is the operation of forming the modified diagram and σ is the operation of forming the Howie diagram. These two operations are independent, hence $\sigma \circ \delta = \delta \circ \sigma$, i.e. $\widetilde{(M')} = (\widetilde{M})'$. From now on we shall denote the common value by $\widetilde{(M')}$

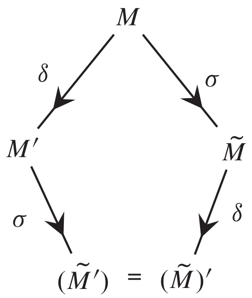


Figure 4

Notation. For a region D in M we denote the corresponding region(or vertex) in $\widetilde{M}$ by $\widetilde{D}$. For a modified region Δ' of M' we denote the corresponding region in $\widetilde{(M')}$ by $\widetilde{(\Delta')}$. Thus, $\widetilde{\Delta}'$ is obtained from Δ' by shrinking each edge of Δ' labelled by a word from H to a point. If $\Delta' \in \mathrm{Reg}_0(M')$ then $\widetilde{\Delta}'$ shrinks to a vertex v such that $\Delta' \subseteq N_v' \subseteq M'$, where N_v is the diagram of the set of all the regions of M which are shrunk to $\tilde{v}'$. If $\Delta' \in \mathrm{Reg}_{0,1}(M')$ then in $\widetilde{\Delta}'$ all the edges with labels from H shrink to a point ,hence $\|\partial\Delta'\|_{M'} = 2\|\partial\widetilde{\Delta}'\|_{\widetilde{M'}}$, where the subscripts M' and $\widetilde{M'}$ indicate that we consider the syllable length in M' and $\widetilde{M'}$, respectively. For simplicity we shall avoid the subscripts, understanding that $\|\partial\widetilde{\Delta}'\|$ is naturally measured in $\widetilde{M'}$. If $\Delta' \in \mathrm{Reg}_1(M')$ then $\widetilde{\Delta}' = \Delta'$ because the boundary label of

Δ does not contain letters from H, yet we consider $\widetilde{\Delta}'$ as a region of $\widetilde{M'}$, while we consider Δ' as a region of M'. We use similar notation for paths $\widetilde{\mu}'$ in $\widetilde{M}'$ and corresponding paths μ' and μ in M' and M respectively.

Lemma 2.1. *Let M be a van Kampen diagram corresponding to* (0) *such that A is extra-large relative to H. Let Δ' be an inner region of M'.*

(a) *If $\widetilde{\Delta}' \in \mathrm{Reg}_1(\widetilde{M}')$ then each boundary vertex is uncoloured, the corner label of $\widetilde{\Delta}'$ at each vertex is* 1*, and $d_{\widetilde{M'}}(\widetilde{\Delta}') \geq 6$, for every inner region $\widetilde{\Delta}'$ of $\widetilde{M}'$.*

(b) *If $\widetilde{\Delta}' \in \mathrm{Reg}_{0,1}(\widetilde{M'})$ with $\mathrm{Supp}(\widetilde{\Delta}') = \{a_i, a_j\}$, $a_i \in V_0$, $a_j \in V_1$ then $\|\partial\widetilde{\Delta}'\| \geq 4$ and every boundary vertex of $\widetilde{\Delta}'$ is coloured and the corner label is $a_i^{\alpha_i}$, $\alpha_i \in \mathbb{Z} \setminus \{0\}$.*

Proof. (a) This is clear from the definition of $\widetilde{M'}$ and the fact that $\mathrm{Supp}(\Delta) \subseteq \{a_i \mid a_i \in V_1\}$, since $\mathrm{Supp}(\Delta) \cap V_0 = \emptyset$.

(b) Again, this is clear from the definition of $\widetilde{M'}$ and the fact that $\mathrm{Supp}(\Delta) \cap V_0 = \{a_i\}$, and $n(\Delta') \geq 4$, due to the relative extra-large condition. Hence $\|\partial\Delta'\| \geq 4$. □

Remark 2.2. Observe that if μ is a path in M and $\|\Phi(\widetilde{\mu}')\| \geq 2$ then μ contains a subpath $\theta = \theta_1\theta_2\theta_3$ with $\Phi(\theta_1), \Phi(\theta_3) \in F(V_1)$ and $\Phi(\theta_2) \in F(V_0)$, since then $\widetilde{\mu}'$ contains a subpath $\widetilde{\mu_1}'\widetilde{v}'\widetilde{\mu_2}'$, $\widetilde{v}'$ a vertex, $\theta_1 = \mu_1, \theta_3 = \mu_2$ and $\theta_2 \subseteq \partial N_v \cap \mu$. Consequently, $\Phi(\mu)$ contains a subword $a^\alpha b^\beta c^\gamma$, with $a, c \in F(V_1)$ and $b \in F(V_0)$, α, β, γ non-zero integers.

Lemma 2.2 (corner regions). *Let M be an H-normal van Kampen $\mathcal{R}$-diagram* (*with connected interior*)*. Assume that*

(1) *$\widetilde{M}'$ has connected interior and*

(2) *every inner coloured vertex of $\widetilde{M}'$ has valency at least* 4.

Let $\widetilde{\Delta}'$ be a k-corner region of $\widetilde{M}'$, $k = 1, 2, 3$ and let $\widetilde{\mu}' = \partial\widetilde{\Delta}' \cap \partial\widetilde{M}'$. Then $\|\widetilde{\mu}'\| \geq 2$.

Proof. Assume first that $\widetilde{\Delta}' \in \mathrm{Reg}_{0,1}(\widetilde{M}')$. By Lemma 2.1(b) $\|\partial\widetilde{\Delta}'\| \geq 4$, hence $\|\widetilde{\mu}'\| \geq 4 - k$. Therefore, if $k = 1, 2$, then $\|\widetilde{\mu}'\| \geq 2$, as required. If $k = 3$ then by assumption 2, all the inner vertices of $\partial\widetilde{\Delta}'$ have valency at least 4, violating the definition of 3-corner regions. Next, if $\widetilde{\Delta}' \in \mathrm{Reg}_1(\widetilde{M}')$, then $\|\partial\widetilde{\Delta}'\| \geq 6$, by the relative extra-large condition, hence $\|\widetilde{\mu}'\| \geq 6 - k \geq 3 > 2$. □

Lemma 2.3 ($\partial\Delta' \cap \partial N_v'$ is connected). *Let M be an H-normal van Kampen $\mathcal{R}$-diagram and assume that in every proper subdiagram $\widetilde{M}_1'$ of $\widetilde{M}'$, every coloured inner vertex has valency at least 4. Let Δ' be a modified region in* $\mathrm{Reg}_{0,1}(\widetilde{M}')$ *with $\widetilde{v}_0'$ a coloured vertex on $\partial\widetilde{\Delta}'$. If $\partial\Delta' \cap \partial N_{v_0}'$ is not connected then no bounded connected component $\widetilde{C}'$ of $\mathbb{E}^2 \setminus (\widetilde{N}_{v_0}' \cup \widetilde{\Delta}')$ satisfies condition* V(6).

Proof of Lemma 2.3. Consider Figure 5. We have in M', $\partial C' = u\mu' v\nu'$, where u, v are vertices $\mu' = \partial N'_v \cap \partial C'$ and $\nu' = \partial \Delta'_v \cap \partial C'$. Suppose by way of contradiction that $\widetilde{C}'$ satisfy condition V(6). If $\widetilde{C}'$ consists of a single region $\widetilde{\Delta}'_0$ then $\widetilde{\Delta}'_0 \in \mathrm{Reg}_{0,1}(\widetilde{M}')$ and $\|\mu'\| = 1$, since Δ' is a modified region and $\nu'\mu'$ is a boundary cycle of C'. Hence $\|\nu'\| \geq 2n(\Delta') - 1 \geq 2 \cdot 4 - 1 = 7$, i.e $\|\nu'\| \geq 7$. See Figure 5(a). This, however, implies that all the regions of Δ_0 in M are equivalent, in the sense of Section 1.2, to all the regions of Δ in M, a contradiction to the definition of Δ_0 (and Δ). Assume therefore that $\widetilde{C}'$ contains at least two regions. See Figure 5(b). Then by Theorem 1.3, $\widetilde{C}'$ contains at least two corner regions, say $\widetilde{\Delta}'_1$ and $\widetilde{\Delta}'_2$ and, by Lemma 2.2, $\|\partial\widetilde{\Delta}'_1 \cap \partial\widetilde{C}'\| \geq 2$ and $\|\partial\widetilde{\Delta}'_2 \cap \partial\widetilde{C}'\| \geq 2$. Therefore by Remark 2.2, $\partial\widetilde{\Delta}'_i \cap \partial\widetilde{C}'$, $i = 1, 2$, contains a subpath with label $a_i^{\alpha_i} b_i^{\beta_i} c_i^{\gamma_i}$, where $|\alpha_i|, |\beta_i|, |\gamma_i| > 0$, $a_i, c_i \in V_1$, $b_i \in V_0$. Since $\|\partial\Delta'_i \cap \mu'\| \leq 1$, this implies that both Δ'_1 and Δ'_2 contain μ' on their boundary. But this violates the fact that M' is planar. Hence, $\widetilde{M}'$ does not satisfy condition V(6). The Lemma is proved. □

Figure 5

Lemma 2.4 ($d_{\widetilde{M}'}(\tilde{v}') \geq 4$). *Let $\tilde{v}'$ be a coloured inner vertex of $\widetilde{M}'$ and suppose that for every region $\widetilde{\Delta}'$ of $\widetilde{M}'$ which contains $\tilde{v}'$ on its boundary, $\partial N'_v \cap \partial\Delta'$ is connected. Then $d_{\widetilde{M}'}(\tilde{v}') \geq 4$*

Proof. By Corollary 1.4, $\|\partial N'\| \geq 4$ and due to Lemma 1.3, if θ is a connected component of $\partial N'_v \cap \partial\Delta'$ then $\|\Phi(\theta)\| \leq 1$. Since by assumption, $\partial N'_v \cap \partial\Delta'$ is connected, $d_{\widetilde{M}'}(\tilde{v}') \geq \|\partial N'\| \geq 4$ (each neighbour of N'_v from $\mathrm{Reg}_{0,1}(\widetilde{M}')$ contributes at least 1 to $d_{\widetilde{M}'}(\tilde{v}')$), as required. □

Proposition 2.5 ($\widetilde{M}'$ satisfies V(6)). *Let M be an H-normal van Kampen $\mathcal{R}$-diagram with connected interior. Then $\widetilde{M}'$ satisfies condition* V(6) *and every coloured inner vertex has valency at least* 4.

Proof. By induction on $|\widetilde{M}'|$. The case $|\widetilde{M}'| = 1$ being clear. Suppose the Lemma holds for every proper van Kampen subdiagram with connected interior $\widetilde{L}'$ of $\widetilde{M}'$. We show that if $\tilde{v}'$ is a coloured inner vertex of $\widetilde{M}'$ then $d_{\widetilde{M}'}(\tilde{v}') \geq 4$. By Lemma 2.4 it is enough to show that $\partial N'_v \cap \partial\Delta'$ is connected for every region $\widetilde{\Delta}'$ of $\widetilde{M}'$ which contains $\tilde{v}'$ on its boundary. Suppose not. Then by Lemma 2.3, $\widetilde{M}'$ contains a proper van Kampen subdiagram $\widetilde{C}'$ which either doesn't satisfy condition V(6), or it contains a coloured inner vertex with valency ≤ 3, both in contrast with the induction hypothesis. Hence $\partial N'_v \cap \partial\Delta'$ is connected and $d_{\widetilde{M}'}(v) \geq 4$, for every coloured inner vertex of $\widetilde{M}'$, by Lemma 2.4. Consequently, if $\widetilde{\Delta}'$ is an inner region in $\mathrm{Reg}_{0,1}(\widetilde{M}')$ then $d_{\widetilde{M}'}(\widetilde{\Delta}') \geq 4$ due to Lemma 2.1, $d_{\widetilde{M}'}(\tilde{v}') \geq 4$ for every boundary vertex of $\widetilde{\Delta}'$. On the other hand, if $\widetilde{\Delta}'$ is a region in $\mathrm{Reg}_1(\widetilde{M}')$, then, by Lemma 2.1, $d_{M'}(\Delta') = d_{\widetilde{M}'}(\widetilde{\Delta}') \geq 6$. Hence condition V(6) is satisfied by $\widetilde{M}'$. The Proposition is proved. □

Corollary 2.1. *Let notation and assumptions be as in Lemma 2.4. Then $\partial N'_v \cap \partial\Delta'$ is connected and $d_{\widetilde{M}'}(\tilde{v}') \geq 4$.*

Proof. By Proposition 2.5, $\widetilde{M}'$ satisfies condition V(6), and every coloured inner vertex has valency at least 4. Hence by Lemma 2.3, $\partial N'_v \cap \partial\Delta'$ is connected. □

3. Proof of Theorem A

Let $\widetilde{\Delta}'$ be a corner region of $\widetilde{M}'$, let $\tilde{\mu}' = \partial\widetilde{\Delta}' \cap \partial\widetilde{M}'$ and let $\tilde{u}'\tilde{\mu}'\tilde{v}'\tilde{v}'$ be a boundary cycle of $\widetilde{\Delta}'$; see Figure 6(a). We consider Δ' in M'; see Figure 6(b), where K_1, K_2 and K_3 are the connected components of N'_u and L_1 and L_2 are the connected components of N'_v. Thus μ' is the longest subpath of $\partial\Delta' \cap \partial M'$ with $\mu' \cap \partial N'_v = \emptyset$ and $\mu' \cap \partial N'_u = \emptyset$. ($\mu'$ is an open path.) Hence, if ν' is the complement of μ' on $\partial\Delta'$ then $\nu' = \nu'_v\nu'_1\nu'_u$ where $\nu'_v = \partial\Delta' \cap \partial N'_v$ and $\nu'_u = \partial\Delta' \cap \partial N'_u$.

Lemma 3.1. *Let M be an H-normal van-Kampen $\mathcal{R}$-diagram with $|\widetilde{M}'| \geq 2$ and let $\widetilde{\Delta}'$ be a k-corner region of $\widetilde{M}'$, $k = 1, 2, 3$. Let $\mu = \partial\Delta \cap \partial M$ and let ν be the complement of μ on $\partial\Delta$. Let $\widetilde{u'}$ and $\widetilde{v'}$ be the endpoints of $\tilde{\mu}'$ in $\widetilde{M}'$, such that $\tilde{u}'\tilde{\mu}'\tilde{v}'\tilde{v}'_1$ is a boundary cycle of $\widetilde{\Delta}'$. (Thus $\nu' = \nu'_v\nu'_1\nu'_u$, where $\nu'_u = \partial N'_u \cap \partial\Delta'$ and $\nu'_v = \partial N'_v \cap \partial\Delta'$ and ν'_1 is an edge.) Suppose that $\widetilde{\Delta}' \in \mathrm{Reg}_{0,1}(\widetilde{M}')$. Then one of the following holds.*

(a) *$\tilde{\mu}' = \tilde{\mu}'_1\tilde{z}'\tilde{\mu}'_2$, where $\tilde{z}'$ a vertex with $\mathrm{Reg}(N'_z) \neq \emptyset$. Let $\alpha' = \partial N'_z \cap \partial M'$ and let β' be the complement of α' on $\partial N'_z$. In this case $l(\beta') < l(\alpha')$.*

(b) $\tilde{v}'$ (*or* $\tilde{u}'$) *has valency* 3 *in* $\widetilde{M}'$ *and* $\mathrm{Reg}(N'_v) \neq \emptyset$ $(\mathrm{Reg}(N'_u) \neq \emptyset)$. *In this case* $l(\partial N'_v \cap \partial M') \geq l(\zeta)$, *where* ζ *is the complement of* $\partial N'_v \cap \partial M'$ *on* N'_v $(l(\partial N'_u \cap \partial M') \geq l(\zeta))$.

(c) $l(\nu) \leq l(\mu)$.

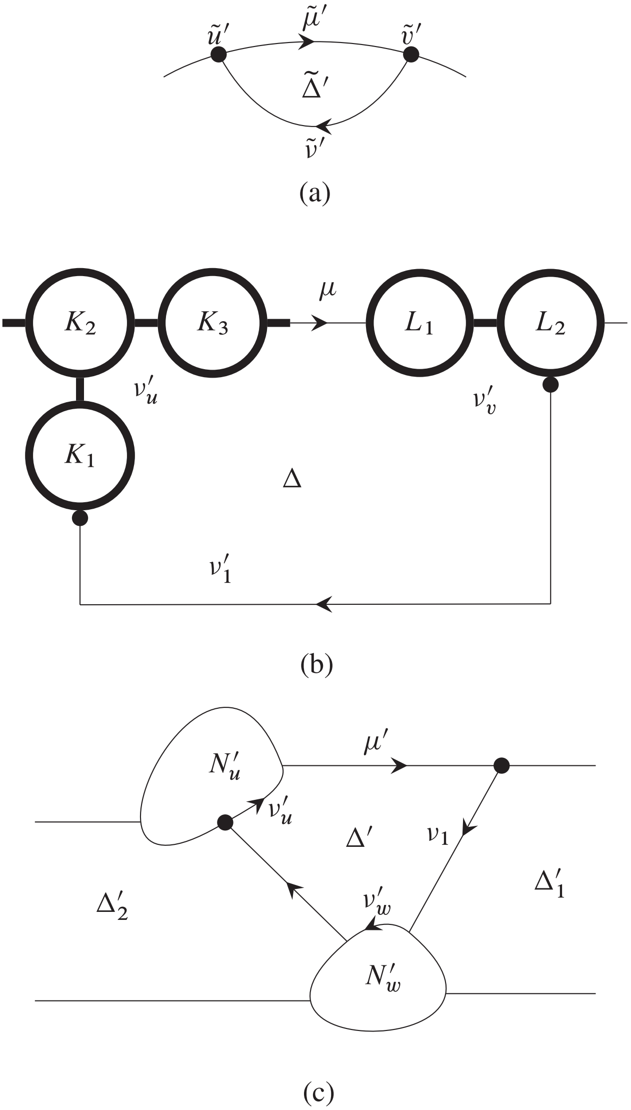


Figure 6

Proof. It is clear from Figure 6(b) (where $\mathrm{Int}(N'_v)$ is not connected), that in order to prove Lemma 3.1 we may assume without loss of generality that $\mathrm{Int}(N'_v)$ is connected and we shall do so.

Suppose (a) and (b) do not hold. We show that necessarily (c) holds. Assume first that $\Delta' \in \mathrm{Reg}_1(M')$. Then $n(\Delta') \geq 3$. Since $k \leq 3$ and $\|\tilde{v}\| \leq k$, we get $\|\tilde{v}\| \leq n(\Delta')$. Hence by Lemma 1.1(b) $l(\nu) \leq l(\mu)$.
Assume now that $\Delta' \in \mathrm{Reg}_{0,1}(M')$. Then $n(\Delta') \geq 4$ by the relative extra-large assumption. Consider three cases according to the values of k.

Case 1: $k = 1$. Let $\nu'_u = \partial N'_u \cap \partial\Delta'$ and let $\nu'_v = \partial N'_v \cap \partial\Delta'$. Then $\mu'\nu'_v\nu'_1\nu'_u$ is a boundary cycle of Δ', as above. Due to Lemma 1.2 and Lemma 1.3 $\|\nu'_v\|$, $\|\nu'_1\|$ and $\|\nu'_u\|$ are all at most 1, hence $\|\nu'\| = \|\nu'_v\nu'_1\nu'_u\| \leq 3 < n(\Delta')$, hence due to Lemma 1.1(b) $l(\nu) < l(\mu)$.

Case 2: $k = 2$. Consider Figure 6(c). Since (a) and (b) do not hold, we may assume $\mathrm{Reg}(N'_v) = \emptyset$, and for every vertex $\tilde{z}'$ on $\tilde{\mu}'$ we have $\mathrm{Reg}(N'_z) = \emptyset$. Hence $\nu'_v = \emptyset$ and $\nu' = \nu'_1\nu'_w\nu'_2\nu'_u$, where ν'_1 and ν'_2 are common edges of Δ' with neighbours and $\nu'_w = \partial\Delta' \cap \partial N'_w$ and $\nu'_v = \partial\Delta' \cap \partial N'_v$. Consequently, due to Lemma 1.2 $\|\nu'\| \leq 4 \leq n(\Delta')$. Hence the result follows from Lemma 1.1(b).

Case 3: $k = 1$. Since $\widetilde{\Delta}'$ is coloured, its boundary vertices which are inner vertices of $\widetilde{M}'$ have valency at least 4, by Corollary 2.1. But this contradicts the definition of 3-corner regions, which may have at most one such vertex with valency 4 or more, see Section 1.2. Hence this case cannot occur.

Finally, we prove the "in this case" of parts (a) and (b). In part (a) $\|\beta\| = 1$, hence it follows from Lemma 1.1 that $l(\beta) \leq l(\alpha)$. In part (b) let $\widetilde{\Delta}'$ and $\widetilde{\Delta}'_1$ be the regions of $\widetilde{M}'$ which contain $\tilde{v}'$ on their boundaries.

Then $\zeta' = \nu'_0\nu'_1$, where $\nu'_0 = \partial\Delta' \cap \partial N'_v$ and $\nu'_1 = \partial\Delta'_1 \cap \partial N'_v$, hence $\|\zeta'\| \leq 2$. Therefore Lemma 1.1 applies and implies $l(\zeta) \leq l(\mu_v)$, where $\mu_v = \partial N_v \cap \partial M$. □

Lemma 3.2. *If* $\Delta' \in \mathrm{Reg}_{0,1}(M') \cup \mathrm{Reg}_1(M')$ *then* $l(\partial\Delta') \leq 2l(\partial M')$.

Proof. By induction on $|M|$. If $|\mathrm{Reg}_{0,1}(M')| = 0$ then either the result vacuously holds or $\mathrm{Reg}(M') = \mathrm{Reg}_1(M)$. If $|\mathrm{Reg}(M')| = 1$ then clearly the result follows. Assume therefore that $|\mathrm{Reg}(M')| \geq 2$. Then $\widetilde{M}' = M'$ and M' is C(6) diagram which has at least 2 corner regions. Removing one of them easily gives the result. (See later on in the proof details of the argument) Assume therefore that $|\mathrm{Reg}_{0,1}(M')| \geq 1$. If $|\mathrm{Reg}_{0,1}(M')| = 1$ and $\widetilde{M}'$ contains no boundary vertices v with $|N'_v| \geq 1$ and $|\mathrm{Reg}_1(M'_0)| = 0$ and also $|\mathrm{Reg}_{0,1}(M')| = 1$ then $\partial M = \partial\Delta$, hence the result is clear. If $|\mathrm{Reg}_1(M')| \neq 0$ then M' has corner region $\Delta'_1 \in \mathrm{Reg}_1(M')$, in which case the result follows by an easy induction argument, by removing Δ'_1. If $|N'_v| \neq 0$ (or $|N'_u| \neq 0$) then since $\|\partial N'_v \cap \partial\Delta'\| = 1$ due to Lemma 1.1, it follows from Lemma 1.2 that $l(\mu_v) > l(\nu_v)$, where $\mu_v = \partial N_v \cap \partial M$ and $\nu_v = \partial N_v \cap \partial\Delta$; see Figure 7. Hence the result follows.

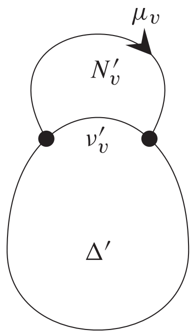


Figure 7

Assume now that $|\mathrm{Reg}_{0,1}(M')| \geq 2$ and the result holds for every diagram with fewer regions. Then $\mathrm{Int}(\widetilde{M}')$ has a connected component $\widetilde{Z}'$, the boundary of which contains at most one double point $\tilde{w}'$. See Figure 8(a).

If $|\mathrm{Reg}(\widetilde{Z}')| = 1$ then remove Z' from M' and by the above arguments, the result follows. Assume therefore that $|\mathrm{Reg}(\widetilde{Z}')| \geq 2$. Then due to Proposition 2.5 and Theorem 1.3, $\widetilde{Z}'$ contains at least 2 corner regions, $\widetilde{\Delta}'_1$ and $\widetilde{\Delta}'_2$ such that if $\widetilde{\Delta}'_i \in \mathrm{Reg}_{0,1}(\widetilde{Z}')$ then $k(\widetilde{\Delta}'_i) \leq 2$ (Observe that $k = 3$ would contradict Proposition 2.5 by the definition of 3-corner regions. See Section 1.3). Assume that one of them, denote it by $\widetilde{\Delta}'$, does not contain $\tilde{w}'$ on its boundary. See Figure 8(b). If $\widetilde{\Delta}' \in \mathrm{Reg}_1(\widetilde{Z}')$ then remove Δ' from M'. It follows easily, due to lemmas 2.1(a) and 1.1(b) that $l(\partial\Delta) \leq 2l(\partial M)$ and for the rest of the regions of M' in $\mathcal{R}_{0,1}(M') \cup \mathcal{R}_1(M')$ the result follows by induction. Assume therefore that $\widetilde{\Delta}' \in \mathrm{Reg}_{0,1}(\widetilde{Z}')$. Then Lemma 3.1 applies. If case (a) of Lemma 3.1 holds then remove N'_z from M' and if case (b) of Lemma 3.1 holds then remove N'_v from M'. (We follow the notation of Lemma 3.1). Then by Lemma 3.1(a, b) the length of the boundary of the diagram obtained, M'_1, does not exceed $l(\partial M)$ and M'_1 contains all the regions of $\mathrm{Reg}_{0,1}(M')$, hence the result follows by the induction hypothesis. If case(c) of Lemma 3.1 occurs, then remove Δ'. It follows from Lemma 3.1(c) that the length of the boundary of the diagram obtained, M'_1, does not exceed $l(\partial M)$ and M'_1 contains all the regions of $\mathrm{Reg}_{0,1}(M')$, except Δ'. Hence the result holds for all regions in $\mathrm{Reg}_{0,1}(M')$, except perhaps Δ'_1, by the induction hypothesis. But $l(\nu') \leq l(\mu') \leq l(\partial M')$, hence $l(\partial\Delta') = l(\mu') + l(\nu') \leq 2l(\partial M')$. (We followed the notation of Lemma 3.1.)

Finally, assume that every corner region of $\widetilde{Z}'$ contains $\tilde{w}'$ on its boundary. Then $\widetilde{Z}'$ has 1-corner regions $\widetilde{\Delta}'_1$ and $\widetilde{\Delta}'_2$, see Figure 8(c). It is easy to see that if we remove one of them then the diagram obtained has shorter boundary than M has. (See proof of Lemma 3.1(a)). Hence the above argument for the case when Lemma 3.1 was applied applies here and the result follows. □

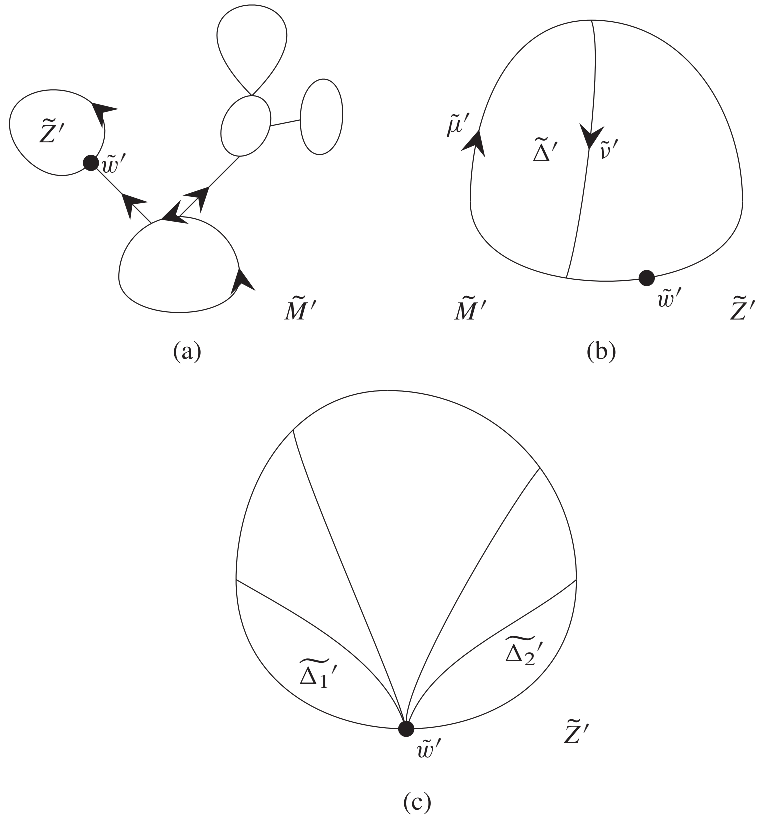


Figure 8

Lemma 3.3. *Let A be an Artin group, extra-large relative to H as in the introduction, and assume that $g : \mathbb{N} \to \mathbb{N}$ is the Dehn function of H. Let $\tilde{v}'$ be a coloured vertex in $\widetilde{M}'$ and let $q = l(\partial M)$. Then,*

(a) $l(\partial N_v') \leq 6q^4$;

(b) $|N_v| \leq g(6q^4)$;

(c) $|\operatorname{Reg}_0(M)| \leq 2q^3 g(6q^4)$.

Proof. Let r be the length of the longest relator in $\mathcal{R}$, and let p be the maximal valency of a coloured vertex $\tilde{v}'$ in $\widetilde{M}'$. Then N_v' has at most p neighbours in $\widetilde{M}'$, because each neighbour is coloured and contributes 1 to the valency of the vertex $\tilde{v}'$. Since $l(\partial N_v' \cap \partial\Delta') \leq l(\partial\Delta')$, where Δ' is a coloured region, each neighbour contributes at most $2q (= 2l(M))$ to $l(\partial N_v')$, by Lemma 3.2. Thus, if $\tilde{v}'$ is a coloured inner vertex of $\widetilde{M}'$ then $l(\partial N_v') \leq 2pq$ and if v is a boundary vertex then $l(\partial N') \leq 2pq + q \leq 3pq$. Hence

$$l(\partial N') \leq 3pq. \tag{1}$$

Consequently, if H has Dehn function $g : \mathbb{N} \to \mathbb{N}$, then

$$|N_v| \leq g(3pq). \tag{2}$$

Since all the neighbours of N'_v in M' are from $\mathrm{Reg}_{0,1}(M')$, the number of subdiagrams N'_v of M' is at most as the number of coloured vertices in $\widetilde{M}'$. Now, by Theorem 1.3, the number of regions in $\widetilde{M}'$ is bounded by q^2 and by Lemma 3.2, $l(\partial\widetilde{\Delta}') \leq 2q$. Hence the number of vertices in $\widetilde{M}'$ is at most $2q^3$. Hence by (2)

$$|\,\mathrm{Reg}_0(M)| \leq 2q^3 g(3pq). \tag{3}$$

Finally, we estimate p. Let $\tilde{v}'$ be a vertex of $\widetilde{M}'$ with valency p. Then p edges are incident at $\tilde{v}'$ in $\widetilde{M}'$. Hence it follows that p is at most as the sum of the lengths of all the edges in $\widetilde{M}'$. By Theorem 1.3 the number of regions in $\widetilde{M}'$ is at most q^2 and by Lemma 3.2 each region $\widetilde{\Delta}'$ contributes at most $2q$ to the total length of edges. Hence $p \leq 2q^3$. Substituting this to (1), (2), and (3) we get:

(a) $l(\partial N_v) \leq 6q^4$,

(b) $|N_v| \leq g(6q^4)$,

(c) $|\,\mathrm{Reg}_0(M)| \leq 2q^3 g(6q^4)$,

as required. □

Remark 3.1. We do not claim that our estimates are best possible.

We are now ready to prove Theorem A.

Proof of Theorem A. We have to show that the group A presented by (0) has a recursive isoperimetric function $f : \mathbb{N} \to \mathbb{N}$, see [19]. We estimate $|\,\mathrm{Reg}(M)|$ as a function of $q(= l(\partial M))$. To simplify notation we shall write in the proof of Theorem A, $\mathcal{R}(M)$ for $\mathrm{Reg}(M)$ and similarly for $\mathrm{Reg}_0(M), \mathrm{Reg}_{0,1}(M)$ and $\mathrm{Reg}_1(M)$. We have

$$\mathcal{R}(M) = \mathcal{R}_0(M)\dot{\cup}(\mathcal{R}_{0,1}(M)\dot{\cup}\mathcal{R}_1(M)). \tag{1}$$

By Lemma 3.3(c),

$$|\mathcal{R}_0(M)| \leq 2q^3 g(6q^4), \quad \text{where } g \text{ is the Dehn function of } H. \tag{2}$$

Now due to Theorem 1.3(d) and Proposition 2.5,

$$|\mathcal{R}_{0,1}(M')\dot{\cup}(\mathcal{R}_1(M')| \leq 2q^2. \tag{3}$$

and by Lemma 3.2 for each $\Delta' \in \mathcal{R}_{0,1}(M')\dot{\cup}(\mathcal{R}_1(M')$ we have

$$l(\partial\Delta') \leq 2q. \tag{4}$$

Since each Δ is a subdiagram of M with the conditions C(4) and T(4) condition by Lemma 1.1, it follows by (4) and Theorem 1.3(d) that

$$|\Delta| \leq 4q^2. \tag{5}$$

It follows from (3) and (5) that

$$|\mathcal{R}_{0,1}(M) \dot{\cup} \mathcal{R}_1(M)| \leq 8q^4. \tag{6}$$

Hence, by (1), (2), and (6),

$$|\mathcal{R}(M)| \leq 2q^3 g(6q^4) + 8q^4. \tag{7}$$

Let $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(x) = 2x^3 g(6x^4) + 8x^4$. Then f is a recursive function, since by assumption g is, and by (7) g is an isoperimetric function for $\mathcal{P}$ in (0).

The theorem is proved. □

4. Proof of Theorem B

In Section 2 we observed that the Artin group A given by the free presentation (0) can be also presented by the relative presentation (1): $\mathbb{P} = \langle H, V_1 \mid \hat{\mathcal{R}}_3 \rangle$, where $\hat{\mathcal{R}}_3 = \hat{\mathcal{R}}_1 \cup \hat{\mathcal{R}}_{0,1}$, $V_1 = \{a_{k+1}, \dots, a_n\}$ and $H = \langle a_1, \dots, a_k \rangle$.

Without loss of generality we may assume that each $R \in \mathcal{R}_3$ starts with a V_1 letter, and we shall do so. The main argument in the proof of Theorem B is that $\mathbb{P}$ is aspherical in the sense of [5]. We start by recalling the necessary notions.

Definitions 4.1. (a) Let Σ be a Howie diagram over $\mathbb{P}$. For a region D in Σ and a vertex v on ∂D denote the boundary label of ∂D which starts at v by $w_D(v)$.

(b) Let D_1 and D_2 be regions in Σ. Call (D_1, D_2) a *cancelling-pair* in Σ if $\partial D_1 \cap \partial D_2 \neq \emptyset$ and for every vertex $v \in \partial D_1 \cap \partial D_2$, with $w_{D_1}(v) = w_{D_2}(v)$, holds.

(c) Σ is *reduced* if there are no regions E_1 and E_2 in Σ such that, by performing a sequence of diamond moves if necessary, along a path the label of which reduces in F to 1_F, (E_1, E_2) becomes a cancelling-pair. See Figure 9.

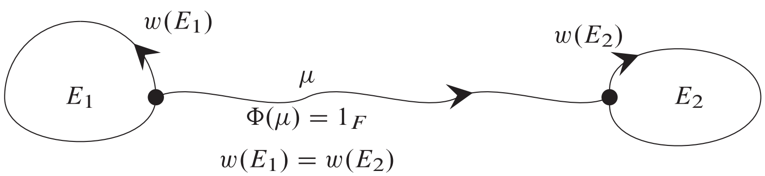


Figure 9

Definition 4.2′ ([5, p. 7]). A relative presentation $\mathbb{P}$ is aspherical if there are no reduced spherical pictures over $\mathbb{P}$.

We are not going to introduce pictures over relative presentations, (For pictures see [5]) since we work with their duals, the corresponding Howie diagrams. In terms of Howie diagrams, following this definition, we have

Definition 4.2. a relative presentation $\mathbb{P}$ is *aspherical* if there are no reduced spherical Howie diagrams over $\mathbb{P}$.

Notice that the operation on pictures corresponding to diamond moves are the bridge moves. (see [12]) and a picture is reduced in the sense of [5] if and only if the corresponding Howie diagram is reduced, as defined above.

We need one more definition from [5].

Definition 4.3 ([5, p.5]). $\mathbb{P}$ is *orientable* if no element of $\mathcal{R}$ is a cyclic conjugate of its inverse and for every $R \in \mathcal{R}$, with $\{R\}^* \cap \mathcal{R} = \{R\}$, where $\{R\}^*$ is the set of all the cyclic conjugates of R and R^{-1}.

Lemma 4.1. *Let A be an Artin group given by $\mathbb{P}$ in (I). Then $\mathbb{P}$ is orientable.*

Proof. Let $R_1, R_2 \in \mathcal{R}_3$, with $R_1 \neq R_2$. Then $\mathrm{Supp}(R_1) \neq \mathrm{Supp}(R_2)$ by definition of Artin relations, hence R_1 is neither a cyclic conjugate of R_2, nor of R_2^{-1}. If R_1 is cyclic conjugate of its inverse, then $R_1 = KL$ and $R_1^{-1} = LK$, with $K, L \in H * F(V_1)$, where not both K and L are empty. Thus $K = K^{-1}$ and $L = L^{-1}$, hence if $K \neq 1$ then $K = K_0 q K_0^{-1}$, reduced as written, $q \in H$ or $q \in F(V_1)$, $q^2 = 1, q \neq 1$ and q is cyclically reduced. Since R_1 and R_2 are Artin relators $q \in V_0$ or $q \in V_1$. But this contradicts Corollary 1.3. Hence $\mathbb{P}$ is orientable as required. □

Proposition 4.1. *Let A be a relatively extra-large Artin group given by the relative presentation in* (1). *Then $\mathbb{P}$ is aspherical in the sense of* [5].

Proof. Suppose not. Then by definition, there is a reduced spherical Howie-diagram $\tilde{M}$ over $\mathbb{P}$ which contains at least one region. Consider the modified diagram $(\tilde{M})'$ of $\tilde{M}$. By Remark 2.1, $(\tilde{M})' = \widetilde{(M')}$, hence $(\tilde{M})'$ satisfies condition V(6) by Proposition 2.5. Since $\tilde{M}$ is spherical, clearly $\tilde{M}'$ is spherical. (It defines a tessellation on S^2 which is coarser than $\tilde{M}$.) By Theorem 1.3(e) $\tilde{M}'$ is not reduced.

Consider the collection of all such reduced diagrams with al least one region which have non-reduced spherical modified diagrams and assume that $\tilde{M}$ contains minimal number of regions among them. We show that then $\tilde{M}$ is not reduced, contrary to assumtion.

Since $\widetilde{M}'$ is not reduced there are modified regions $\widetilde{\Delta}'_1$ and $\widetilde{\Delta}'_2$ in $\widetilde{M}'$ which constitutes a cancelling-pair. Therefore they have exactly the same boundary labels, supported by 2 letters, say a_i and a_j, $i \neq j$. Hence we may cut out $\widetilde{\Delta}'_1$ and $\widetilde{\Delta}'_2$ from $\widetilde{M}'$ and sew the diagram obtained along $\partial\widetilde{\Delta'_1} \cup \partial\widetilde{\Delta'_2}$, to give a diagram $\widetilde{M}'$. By minimality $\widetilde{M'}$ is reduced. This however contradicts Theorem 1.3(e). Hence $\mathrm{Reg}(\widetilde{M'_1}) = \emptyset$, i.e. $\mathrm{Reg}(\widetilde{M'}) = \{\widetilde{\Delta'_1}, \widetilde{\Delta'_2}\}$ and the $\mathcal{R}$-diagram M satisfies C(4) and T(4), since $\mathrm{Supp}(\widetilde{\Delta}_1) = \mathrm{Supp}(\widetilde{\Delta}_2) = \{a_i, a_j\}$.

Consequently, M is not reduced, due to Lyndon's Lemma [27]. In particular, $\widetilde{M}$ contains a cancelling-pair (after performing a finite sequence of diamond moves, if needed) $(\widetilde{D}_1, \widetilde{D}_2)$, where $\widetilde{D}_1$ and $\widetilde{D}_2$ are regions of $\widetilde{M}$. By the minimality of $|\widetilde{M}|$ it follows that $\mathrm{Reg}(\widetilde{M}) = \{\widetilde{D_1}, \widetilde{D_2}\}$. But then $\widetilde{M}$ is not reduced, contrary to assumption. Therefore $\mathbb{P}$ is aspherical. □

We are now ready to prove Theorem B.

Proof of Theorem B. By Lemma 4.1, $\mathbb{P}$ is orientable and, by Proposition 4.1, $\mathbb{P}$ is aspherical. Hence, part (a) follows directly from [5, 0.4] and part (b) from [5, 0.3], noticing that no Artin relator is a proper power. □

5. Proof of Theorem C

5.0. The $K(\pi, 1)$ conjecture. We start with a very brief introduction of the $K(\pi, 1)$ conjecture. We follow [32], where a comprehensive introduction is given. Let W be a Coxeter group. Then W acts faithfully on an open non-empty convex cone I such that the union of the regular orbits is the complement in I of a possibly infinite family $\mathcal{A}$ of linear hyperplanes.

It was proved by van der Lek that the Artin group A_W corresponding to W (A_W has the same defining graph as W has) is the fundamental group of the space

$$\mathcal{N}(W) = (I \times I) \setminus \bigcup_{K \in \mathcal{A}} (K \times K)/W$$

The $K(\pi, 1)$ *conjecture* says that $\mathcal{N}(W)$ is an Eilenberg–Mac Lane space for A_W. i.e. $\mathcal{N}(W)$ is aspherical, see [6, p. 15].

5.1. Presentation complex for $\mathbb{P}$. Let $\mathcal{P} = \langle X \mid \mathcal{R}\rangle$ be a free group-presentation. Recall from [17, p. 5] that the *presentation complex* $\kappa(\mathcal{P})$ consists of a single vertex v, for every generator $x \in X$, a simple closed curve starting and ending at $v (\approx S^1)$ which is labelled by x, forming a bouquet. For every relator $R \in \mathcal{R}$ we attach to the bouquet of circles obtained a 2-cell, the boundary of which is labelled by R. It is known that the group defined by $\mathcal{P}$ is isomorphic to the fundamental group of $\kappa(\mathcal{P})$. [28] More formally, $\kappa(\mathcal{P})$ is defined by the following push-out (attaching) diagram in Figure 10.

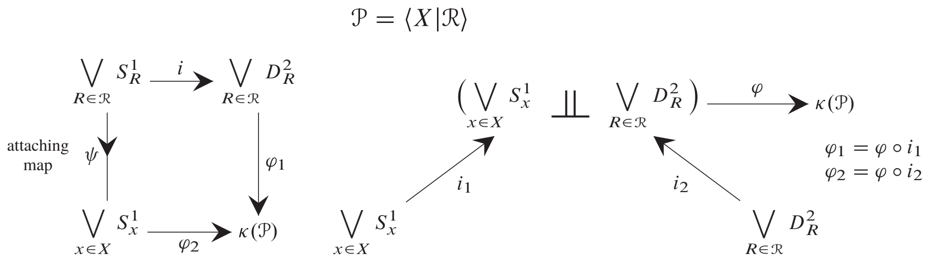


Figure 10

Definition 5.1. $\mathcal{P}$ is called *aspherical* if $\kappa(\mathcal{P})$ is aspherical as a CW-complex([12]).

In [5, p. 36] a similar construction is given for relative presentations $\mathbb{P}$ given by (1), which we give below.

Let K be an Eilenberg–Mac Lane space for H and define

$$K_1 = K \bigvee_{y \in V_1} S^1_y.$$

Let $\kappa(\mathbb{P})$ be defined by the pushout diagram in Figure 11, where $\mathcal{R}_3 = \mathcal{R}_1 \cup \mathcal{R}_{0,1}$ ($\kappa(\mathbb{P})$ is denoted in [5, p. 36] by M).

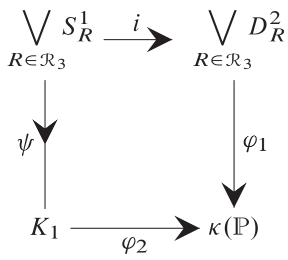


Figure 11

5.2. The asphericity of $\kappa(\mathbb{P})$

Theorem 5.1 ([5, p. 36]). *Let* $\mathbb{P}$ *be an orientable aspherical relative presentation* (*see definitions* 4.2 *and* 4.3). *Then* $\kappa(\mathbb{P})$ *is aspherical.* (*as a CW-complex*)

Corollary 5.1. *Let* $\mathbb{P}$ *be the relative presentation* (1) *of a relative extra-large Artin group* A. *Then* $\kappa(\mathbb{P})$ *is aspherical.*

Proof of Corollary 5.1. By Lemma 4.1, $\mathbb{P}$ is orientable and, by Proposition 4.1, $\mathbb{P}$ is aspherical. Hence by Theorem 5.1, $\kappa(\mathbb{P})$ is aspherical. □

5.3. $\kappa(\mathbb{P})$ is homotopy-equivalent to Z_W. The idea of the proof of Theorem C is to show that $\kappa(\mathbb{P})$ is homotopy equivalent to $\mathcal{N}(W)$, where W is the Coxeter group obtained from A by adding relators $a_i^2 = 1, i = 1, \dots, n$. (It is easy to see that $A = A_W$) We do this in 3 steps. We follow the notation of [32].

Step 1. *$\mathcal{N}(W)$ is homotopy equivalent to a complex $\overline{\mathrm{Salv}}(A)$.* See [32, Corollary 3.4, p. 28]; the notation is the same as in p. 7 and p. 31 of [32].

Step 2. *$\overline{\mathrm{Salv}}(A)$ is homotopy equivalent to a CW complex Z_W, obtained from the presentation complex $\kappa(\mathcal{P})$, with $\mathcal{P}$ given by* (0) *of A, by attaching one i-cell, $i \geq 3$, for every finite type standard parabolic subgroup generated by i elements from $\{a_1, \dots, a_n\}$.* See R. Charney and M. Davis, [9, Corollary 2.23]; the author is grateful to R. Charney for this reference.

Step 3

Proposition 5.1. *Let A be a relatively extra-large Artin group given by* (0) *and let $\mathbb{P}$ be the relative presentation* (1) *of A. Assume that the $K(\pi, 1)$ conjecture holds for H. Then $\kappa(\mathbb{P})$ is homotopy equivalent to Z_W*

We need the following lemma to prove Proposition 5.1.

Lemma 5.1. *Let $\mathcal{F}(A)$ and $\mathcal{F}(H)$ be the sets of finite-type parabolic subgroups of A on at least 3 generators and H, respectively. Then $\mathcal{F}(A) = \mathcal{F}(H)$.*

Proof of Lemma 5.1. It is enough to show that every 3 generated finite type parabolic subgroup of A is already in H. Let $G = \langle a, b, c\rangle$ be a parabolic subgroup of A, $\{a, b, c\} \subseteq \{a_1, \dots, a_n\}$. G is of finite type if and only if $\frac{1}{n_{a,b}} + \frac{1}{n_{b,c}} + \frac{1}{n_{c,a}} > 1$. In particular, if $\{a, b, c\} \subseteq V_1$ then G is large hence G is not of finite type. Hence if G is of finite type then at least one of the generators of G is in H. Suppose $a \in H$ and $b \notin H$. Then by the relative extra-large condition, $n_{a,b} \geq 4$ and if $c \notin H$ then $n_{a,c} \geq 4$. If G is of finite type then $\frac{1}{n_{b,c}} > \frac{1}{2}$, a contradiction since $n_{i,j} \geq 2$ by definition. If $c \in H$ then $n_{b,c} \geq 4$ and $n_{a,b} \geq 4$, which leads to the same contradiction. Hence $\{a, b\} \subseteq H$. if $c \notin H$ then the above calculation shows that G is not of finite type. Thus $G \in \mathcal{F}(A)$ implies that $G \in \mathcal{F}(H)$.

The Lemma is proved. □

Proof of Proposition 5.1. Let

$$Q = \bigvee_{y \in V} S^1_y,$$

$$Y_1 = \bigvee_{R \in \mathcal{R}_0} D_R^2,$$

$$Y_2 = \bigvee_{R \in \mathcal{R}_3} D_R^2,$$

$$Y_3 = \bigvee_{E \in \mathcal{F}(A)} C_E,$$

where C_E is the i-cell corresponding to the i-generated finite type parabolic subgroup E of A, mentioned in the definition of Z_W above. Then,

$$Z_W \text{ is obtained from } Q \text{ by attaching } Y_1 \cup Y_2 \cup Y_3 \text{ to it.} \tag{1}$$

Now $\kappa(\mathbb{P})$ is obtained from K_1 by attaching Y_2 to it (See attaching diagram Figure 11). Since by assumption H satisfies the $K(\pi, 1)$ conjecture, it follows from the last Lemma that K_1 is obtained from Q by attaching $Y_1 \cup Y_3$ to it. Thus:

$$\kappa(\mathbb{P}) \text{ is obtained from } Q \text{ by attaching } Y_1 \cup Y_3 \text{ to it and then attaching } Y_2. \tag{2}$$

Noticing that we use the same attaching maps in both cases (up to the difference in the domains), it follows from (1) and (2) that $\kappa(\mathbb{P})$ and Z_W are homotopy equivalent. □

Now we are ready to prove Theorem C.

Proof of Theorem C. The conjunction of steps 1, 2, and 3 implies that $\mathcal{N}(W)$ is homotopy equivalent to $\kappa(\mathbb{P})$. By Corollary 5.1, $\kappa(\mathbb{P})$ is aspherical. Consequently $\mathcal{N}(W)$ is aspherical, as required. □

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