

Relations between counting functions on free groups and free monoids

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Abstract. We study counting functions on the free groups F_n and free monoids M_n for $n \geq 2$, which we introduce for combinatorial approach to famous Brooks quasimorphisms on free groups. Two counting functions are considered equivalent if they differ by a bounded function. We find the complete set of linear relations between equivalence classes of counting functions and apply this result to construct an explicit basis for the vector space of such equivalence classes. Moreover, we provide a simple graphical algorithm to determine whether two given counting functions are equivalent. In particular, this yields an algorithm to decide whether two linear combinations of Brooks quasimorphisms on F_n represent the same class in bounded cohomology.

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1. Introduction

1.1. Counting functions on free groups and free monoids. Let $S = \{a_1, \dots, a_n\}$ be a finite set of cardinality $n \geq 2$. We denote by M_n the free monoid on S , i.e. the collection of all words over S (including the empty word e), and by F_n the free group on S , i.e. the collection of all reduced words over the extended alphabet $\bar{S} := \{a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}\}$.

Formally, a non-trivial element $v = s_1 \cdots s_l \in M_n$ is called a *subword* of $w = r_1 \cdots r_m \in M_n$ if there exists $j \in \{1, \dots, m - l\}$ such that

$$s_i = r_{j+i} \quad \text{for all } i = 1, \dots, l. \quad (1.1)$$

Similarly, an element $v \in F_n$ is called a *reduced subword* of an element $w \in F_n$ if the reduced word over \bar{S} representing v is a subword (in the above sense) of the reduced word representing w . These subword relations are among the most basic relations in combinatorial (semi-)group theory.

In this article we are interested in the following quantitative refinement of the subword relation. Given $v = s_1 \cdots s_l \in M_n \setminus \{e\}$ and $w = r_1 \cdots r_m \in M_n$, we denote by $\rho_v(w)$ the number of $j \in \{1, \dots, m - l + 1\}$ such that (1.1) holds. This then defines a function $\rho_v: M_n \rightarrow \mathbb{N}_0$ called the *elementary v -counting function*.¹ For example, $\rho_{a_1 a_2 a_1}(a_1 a_2 a_1 a_2 a_1) = 2$. Restricting to reduced words, we similarly obtain an elementary counting function $\rho_v: F_n \rightarrow \mathbb{N}_0$ for every $v \in F_n \setminus \{e\}$. It is convenient to extend the definitions to the empty word by defining $\rho_e(w)$ to be the word length $|w|_S$ of w with respect to S .

In the sequel we refer to a finite linear combination of elementary counting functions with \mathbb{R} -coefficients simply as a *counting function*. We denote by $\mathcal{C}(M_n)$ respectively $\mathcal{C}(F_n)$ the spaces of counting functions on M_n respectively F_n . Our starting point is the following simple observation which goes back (at least) to [4].

Proposition 1.1. *The elementary counting functions $\{\rho_v \mid v \in M_n \setminus \{e\}\}$ form a basis for $\mathcal{C}(M_n)$, and similarly the counting functions $\{\rho_v \mid v \in F_n \setminus \{e\}\}$ form a basis for $\mathcal{C}(F_n)$.*

Proof. By definition, the elementary counting functions span the space of counting functions, and we can remove ρ_e from this generating set since

$$\rho_e = \sum_{|w|_S=1} \rho_w.$$

Concerning linear independence, let $\alpha = \sum_{w \in W \setminus \{e\}} \alpha_w \rho_w$ be a finite sum and let v be an element of minimal length in W with $\alpha_v \neq 0$. Then $\alpha(v) = \alpha_v$, hence it is possible to compute the coefficients α_v inductively, and the desired linear independence follows. \square

¹ Since we allow the occurrences of v in w to overlap, the function ρ_v is sometimes called the *overlapping v -counting function* or the *big v -counting function*, cf. [6].

From now on we call elements $f, g \in \mathcal{C}(M_n)$ *equivalent* if they differ by a bounded function, and similarly for elements of $\mathcal{C}(F_n)$. We then denote by $\widehat{\mathcal{C}}(M_n)$ respectively $\widehat{\mathcal{C}}(F_n)$ the corresponding spaces of equivalence classes. These quotient spaces appear naturally in a number of applications, e.g. in bounded cohomology. They also admit natural interpretations as function spaces spanned by certain “cyclic counting functions”, see Theorem A.6 in the appendix. In analogy with Proposition 1.1 we are going to study the following problem.

Problem 1.2. *Find explicit bases for the quotient spaces $\widehat{\mathcal{C}}(M_n)$ and $\widehat{\mathcal{C}}(F_n)$.*

We present a complete solution to the problem in Theorem 1.5 below. Initially, our interest in this problem was motivated from a specific problem concerning the second bounded cohomology of free groups, which we describe in the next subsection. However, we believe that the problem is also of independent interest within the theory of combinatorics of words.

1.2. Motivation from bounded cohomology. Historically, the need to understand the quotient space $\widehat{\mathcal{C}}(F_n)$ first arose from the study of the second bounded group cohomology of F_n in the sense of [12, 18]. We briefly recall this motivation here. A function $\varphi: F_n \rightarrow \mathbb{R}$ is called a *quasimorphism*² if

$$\sup_{g,h \in F_n} |\varphi(gh) - \varphi(g) - \varphi(h)| < \infty.$$

Now if $\mathcal{Q}(F_n)$ denotes the space of all quasimorphisms on F_n and $H_b^2(F_n; \mathbb{R})$ denotes the second bounded cohomology of F_n with trivial real coefficients, then there is an isomorphism (see e.g. [6])

$$H_b^2(F_n; \mathbb{R}) \cong \mathcal{Q}(F_n) / (\text{Hom}(F_n, \mathbb{R}) \oplus \ell^\infty(F_n)). \tag{1.2}$$

In his famous paper [4], Brooks pointed out that the symmetrized elementary counting functions on the free group

$$\varphi_w := \rho_w - \rho_{w^{-1}}: F_n \rightarrow \mathbb{Z}$$

are quasimorphisms. (Similar ideas, albeit in a different language, appeared already in earlier work of Rhemtulla [22].) It thus follows from (1.2) that if we denote by $\widehat{\mathcal{B}}(F_n)$ the subspace of $\widehat{\mathcal{C}}(F_n)$ spanned by the equivalence classes of the *Brooks quasimorphisms* φ_w , then the quotient $\widehat{\mathcal{B}}(F_n) / \text{Hom}(F_n, \mathbb{R})$ embeds into $H_b^2(F_n; \mathbb{R})$. In particular,

$$\dim H_b^2(F_n; \mathbb{R}) \geq \dim \widehat{\mathcal{B}}(F_n) - n.$$

² In the Russian literature such functions are sometimes called *quasi-characters* (see [9],[11]), apparently following a suggestion by Shtern [24]. Also, as pointed out in [5], the notion of a quasimorphism is closely related to Ulam’s notion of a δ -homomorphism [25, Chapter 6.1].

Brooks claimed in [4] that (by an argument similar to the proof of Proposition 1.1) the classes $[\varphi_w] \in \widehat{\mathcal{C}}(F_n)$ were linearly independent except for the obvious anti-symmetry relations

$$\varphi_w = -\varphi_{w^{-1}}, \tag{1.3}$$

and deduced that $\dim H_b^2(F_n; \mathbb{R}) = \infty$, thereby providing the first example of a group with infinite-dimensional second bounded cohomology. However, as Grigorchuk pointed out in [11, p.139], the linear combination

$$\varphi_{a_1 a_2} + \varphi_{a_1^{-1} a_2} + \varphi_{a_1 a_2^{-1}} + \varphi_{a_1^{-1} a_2^{-1}}$$

is bounded in absolute value by 1, hence yields a counterexample to the claim of Brooks. Nevertheless it is true that $\dim H_b^2(F_n; \mathbb{R}) = \infty$. Historically, the first complete proof was given by Mitsumatsu [20] who proved linear independence of an infinite collection of equivalence classes of Brooks quasimorphisms (see e.g. [21] for a modern treatment). Mitsumatsu’s result was later extended by Faiziev [9] and Grigorchuk [11] who exhibited larger collection of linearly independent elements. Despite these efforts, the problem of finding a basis of $\widehat{\mathcal{B}}(F_n)$ remained open ever since.

It turns out that a basis for $\widehat{\mathcal{B}}(F_n)$ can be constructed quite easily from a suitable basis of $\widehat{\mathcal{C}}(F_n)$. This motivated us to study the space $\widehat{\mathcal{C}}(F_n)$ and, by analogy, $\widehat{\mathcal{C}}(M_n)$.

1.3. Relations between counting functions. To state our results, we introduce the following notation. Denote by $\mathbb{R}[M_n]$ the space of finitely supported real-valued functions on M_n , and note that every element of $\mathbb{R}[M_n]$ can be written uniquely as a sum

$$\sum_{g \in M_n} \lambda_g \delta_g,$$

where δ_g is the function taking value 1 at g and 0 elsewhere and $\lambda_g = 0$ for all but finitely many $g \in M_n$. We thus have a canonical linear surjection

$$\mathfrak{q}: \mathbb{R}[M_n] \longrightarrow \widehat{\mathcal{C}}(M_n), \quad \mathfrak{q}\left(\sum_{g \in M_n} \lambda_g \delta_g\right) = \left[\sum_{g \in M_n} \lambda_g \rho_g\right],$$

and we can think of the kernel $K(M_n)$ of \mathfrak{q} as the space of relations in the quotient space $\widehat{\mathcal{C}}(M_n)$. By the same formula we also define a map $q: \mathbb{R}[F_n] \rightarrow \widehat{\mathcal{C}}(F_n)$, whose kernel $K(F_n)$ describes the relations satisfied by elementary counting functions in $\widehat{\mathcal{C}}(F_n)$. Finally, there is also a symmetrized version of this map, which parameterizes the Brooks space $\widehat{\mathcal{B}}(M_n)$ and is given by

$$q_{\text{sym}}: \mathbb{R}[F_n] \longrightarrow \widehat{\mathcal{B}}(F_n), \quad q_{\text{sym}}\left(\sum_{g \in G} \lambda_g \delta_g\right) = \left[\sum_{g \in G} \lambda_g \varphi_g\right].$$

Its kernel $K_{\text{sym}}(F_n)$ parametrises the relations between Brooks quasimorphisms. Our first result describes the relation spaces $K(M_n)$, $K(F_n)$ and $K_{\text{sym}}(F_n)$ explicitly.

Theorem 1.3 (Linear relations between elementary counting functions). *Given a word $w \in M_n$, define $\{l_w, r_w\} \subset \mathbb{R}[M_n]$ by*

$$l_w := \delta_w - \sum_{s \in S} \delta_{sw} \quad \text{and} \quad r_w := \delta_w - \sum_{s \in S} \delta_{ws}. \tag{1.4}$$

Given a reduced word $w \in F_n$ with initial letter w_1 and final letter w_{fin} , define

$$l_w := \delta_w - \sum_{s \in \bar{S} \setminus \{w_1^{-1}\}} \delta_{sw}, \quad r_w := \delta_w - \sum_{s \in \bar{S} \setminus \{w_{\text{fin}}^{-1}\}} \delta_{ws}, \quad s_w := \delta_w + \delta_{w^{-1}}. \tag{1.5}$$

Then the relation spaces $K(M_n)$, $K(F_n)$ and $K_{\text{sym}}(F_n)$ defined above admit the following spanning sets:³

- (i) *the space $K(M_n) = \ker(q)$ is spanned by the set $\bigcup_{w \in M_n} \{l_w, r_w\}$;*
- (ii) *the space $K(F_n) = \ker(q)$ is spanned by the set $\bigcup_{w \in F_n} \{l_w, r_w\}$;*
- (iii) *the space $K_{\text{sym}}(F_n) = \ker(q_{\text{sym}})$ is spanned by the set $\bigcup_{w \in F_n} \{l_w, r_w, s_w\}$.*

Theorem 1.3 will be proved in Section 2 below.

Remark 1.4. (1) For $w = e$ the definitions of r_w, s_w and l_w have to be understood as follows. In the monoid case we define

$$l_e := r_e := \delta_e - \sum_{s \in S} \delta_s.$$

In the group case we define

$$l_e := r_e := \delta_e - \sum_{s \in \bar{S}} \delta_s$$

and $s_e := 2\delta_e$.

(2) Relations similar to the relations l_w, r_w and s_w appear under many different names in the literature. We prefer the terms *left-extension relation*, *right-extensions relations* and *symmetry relations* respectively. In different contexts, the left- and right-extension relations are sometimes called (left- and right-) *Kirchhoff laws* or *laws of total probability*.

³ When we reported this result to Danny Calegari, he kindly pointed out to us that Part (iii) of the Theorem can also be deduced from results presented in the preprint version (but not in the published version) of his joint article with Alden Walker [7].

(3) All of these relations are essentially obvious.⁴ The theorem can thus be stated informally by saying that “there are no other relations than those following from the obvious ones.”

(4) The statement of (iii) contains some redundancy. Namely, since the right-extension relations follow from the left-extension relations and the symmetry relations, the space $K_{\text{sym}}(F_n)$ is already spanned by the set $\bigcup_{w \in W} \{l_w, s_w\}$ (or, equivalently, $\bigcup_{w \in W} \{r_w, s_w\}$). We stated (iii) in the above redundant form to stress the analogy with (i) and (ii).

1.4. Explicit bases. Using the description of the space of relations provided in the last subsection we are able to provide an explicit basis for each of the spaces $\widehat{\mathcal{C}}(M_n)$, $\widehat{\mathcal{C}}(F_n)$ and $\widehat{\mathcal{B}}(F_n)$. The final result is as follows:

Theorem 1.5 (basis theorem). (i) *Denote by W the set of all words in M_n which do not start or end with a_1 (including the empty word). Then the classes represented by the elementary counting functions $\{\rho_w \mid w \in W\}$ form a basis for the space $\widehat{\mathcal{C}}(M_n)$.*

(ii) *Denote by W' the set of all reduced words in F_n which do not start with a_1 or $a_2a_1^{-1}$ and do not end with a_1^{-1} or $a_1a_2^{-1}$ (including the empty word), and let $W := W' \cup \{a_1^{-1}\}$. Then the classes represented by the elementary counting functions $\{\rho_w \mid w \in W\}$ form a basis for the space $\widehat{\mathcal{C}}(F_n)$.*

(iii) *Let W as in (ii) and let $W_0 := W \cup \{a_1\} \setminus \{e\}$. Let W_+ be a subset of W_0 which intersects each pair $\{w, w^{-1}\} \subset W$ in precisely one element. Then the classes represented by the Brooks quasimorphisms $\{\phi_w \mid w \in W\}$ form a basis for the space $\widehat{\mathcal{B}}(F_n)$.*

We will establish Theorem 1.5 in Section 3 below. Parts (i) and (ii) solve Problem 1.2, and Part (iii) solves the long-standing problem of finding an explicit basis for the Brooks space $\widehat{\mathcal{B}}(F_n)$. There are of course many possible choices for W^+ . Concretely, one can choose an order on \bar{S} and order F_n lexicographically. For any such choice, the classes represented by the counting functions associated with the words

$$W^+ = \{w \in W_0 \mid w < w^{-1}\}$$

form a basis.

1.5. Algorithms for comparing counting functions. For efficient computations in $\widehat{\mathcal{C}}(M_n)$ and $\widehat{\mathcal{C}}(F_n)$ (and its subspace $\widehat{\mathcal{B}}(F_n)$) it is crucial to be able to decide efficiently whether two given counting functions are equivalent. Since subtraction

⁴ For the convenience of the reader we establish them in Subsection 2.1 below.

of counting functions can be done efficiently, this problem amounts to deciding efficiently whether a counting function of the form

$$f = \sum \alpha_w \rho_w \tag{1.6}$$

represents the zero class in $\widehat{\mathcal{C}}(M_n)$ or $\widehat{\mathcal{C}}(F_n)$.

In Section 5 we provide a visualization of counting functions of the form (1.6) by vertex-labelled finite weighted trees and then describe a simple graphical procedure to decide whether the counting function represented by a given tree represents the zero class. It turns out that for many trees one can see immediately from the picture that they do not represent the zero class (see Theorem 4.2 below). Moreover, we describe several basic moves to replace a weighted tree by an equivalent one and show that every tree can be transformed by finitely many such moves into a tree which either obviously represents a non-zero class or obviously represents the zero class. This leads to a “pen-and-paper algorithm” to decide triviality of a given class of counting functions.

It is possible to implement our pen-and-paper algorithm on a computer. To formally describe such a computer algorithm one has to discuss how to store counting functions of the form (1.6) in a computer, and how precisely to implement the basic moves on the given data structures. Once such a formalization of our algorithm is given, one can analyze its runtime. This requires some technical arguments in complexity theory and is beyond the scope of the present article.

In the companion article [15] we describe a modification of the present algorithm, which is more intricate (and hence less comprehensible), but it is “very efficient”, when one runs it on a computer. To give a sample result, we show that if the coefficients of f are integers (stored as binary expansions), then triviality of the class represented by f can be decided in linear time in the size of the input (except the monoid case with $n = 2$, where we can give only a quadratic estimate). For details and statements involving more general coefficients, see [15].

1.6. Outlook and open problems. The present article is a first major step towards efficient computation with counting functions, and in particular, towards efficient computations in the Brooks space $\widehat{\mathcal{B}}(F_n)$. We would like to mention that while there are many good reasons why one would want to carry out computations in $\widehat{\mathcal{B}}(F_n)$, the present work is motivated by some specific problems arising from work of the first author with P. Schweitzer [14] concerning the $\text{Out}(F_n)$ -action on bounded cohomology of free groups. Namely, the automorphism group of F_n acts naturally on the space $\mathcal{Q}(F_n)/\ell^\infty(F_n)$, and this action factors through $\text{Out}(F_n)$. There is a natural $\text{Out}(F_n)$ -invariant locally-convex (non-complete) topology on $\mathcal{Q}(F_n)/\ell^\infty(F_n)$ given by pointwise convergence of homogeneous representatives. The following equivariant version of a classical result of Grigorchuk [11] was established in [14, Section 2].

Theorem 1.6 (Grigorchuk, Hartnick–Schweitzer). *The Brooks space $\widehat{\mathcal{B}}(F_n)$ is a dense subspace of $\mathcal{Q}(F_n)/\ell^\infty(F_n)$ and invariant under the action of $\text{Out}(F_n)$. In particular, $\widehat{\mathcal{B}}(F_n)$ is independent of the free generating set used to define it, and the action of $\text{Out}(F_n)$ on $\mathcal{Q}(F_n)/\ell^\infty(F_n)$ (and thus also the $\text{Out}(F_n)$ -action on $H_b^2(F_n; \mathbb{R})$) is uniquely determined by its restriction to $\widehat{\mathcal{B}}(F_n)$.*

This motivates a closer study of the action of $\text{Out}(F_n)$ on $\widehat{\mathcal{B}}(F_n)$. For example, one would like to know whether a given class $[f] \in \widehat{\mathcal{B}}(F_n)$ is stabilized by some $g \in \text{Out}(F_n)$, i.e. whether $g.f - f$ is bounded. By means of the algorithm developed in this article, it is now possible to decide this efficiently, and this has been applied recently by Hase to study the dynamics of $\text{Out}(F_n)$ on $\widehat{\mathcal{B}}(F_n)$, see [16].

All this is just the tip of a much larger iceberg. Analogues of Brooks quasimorphisms have been defined for Gromov-hyperbolic groups [8], various classes of groups acting on hyperbolic spaces [10, 13], mapping class groups [2] and most recently for general acylindrically hyperbolic groups [17, 1], comprising all previous constructions. In all these situations it is known that there is an infinite-dimensional subspace of the second bounded cohomology which is analogous to the Brooks space. The combinatorial fine-structure of these generalized Brooks spaces is not at all understood at this point. Even for relatively simple examples such as surface groups, we have currently no idea how a basis for the generalized Brooks space should look like.

1.7. Organization of the article. This article is organized as follows. In Section 2 we establish Theorem 1.3, and in Section 3 we establish Theorem 1.5. In both cases, we first consider the monoid case, and then deal with the additional complications in the group case. In Section 4 we explain how sums of counting functions can be represented graphically as finite weighted trees. Here the main result is Theorem 4.2 which singles out a large class of such trees which represent non-trivial elements in $\widehat{\mathcal{C}}(M_n)$ and $\widehat{\mathcal{C}}(F_n)$. Based on this result, we present in Section 5 an algorithm to decide whether a given counting function is bounded. The appendix collects some basic facts about homogenizations of counting functions used throughout the body of the text.

2. Relations between elementary counting functions

2.1. Basic relations between elementary counting functions. The goal of this section is to establish Theorem 1.3, i.e. to determine all relations between elementary counting functions and counting quasimorphisms. Parts (i), (ii), and (iii) of the theorem will be established in Subsections 2.3, 2.4, and 2.5 respectively. Informally, the theorem states that every relation between elementary counting

functions is a consequence of certain basic relations. In this subsection we briefly explain these basic relations. We start with the case of monoids.

Lemma 2.1. *For every $w \in M_n$ the left-extension relation l_w and the right-extension relation r_w as defined in (1.4) are contained in the relation space $K(M_n)$.*

Proof. For $w = e \in M_n$ we have $q(l_w) = q(r_w) = 0$, since adding the counts of letters of a word yields the word length. If $w \in M_n$ with $|w|_S \geq 1$ then for every $v \in M_n$ the difference

$$\rho_w(v) - \sum_{s \in S} \rho_{sw}(v)$$

takes value 1 or 0 depending on whether v starts with w or not. Similarly,

$$\rho_w(v) - \sum_{s \in S} \rho_{ws}(v)$$

takes value 1 or 0 depending on whether v ends with w or not. Since these are bounded functions we deduce that $q(l_w) = q(r_w) = 0$. □

In the group case we have the following similar result.

Lemma 2.2. *For every $w \in F_n$ the left-extension relation l_w and the right-extension relation r_w as defined in (1.5) are contained in the relation space $K(F_n)$. Moreover, the relations l_w , r_w and the relation s_w defined in (1.5) are contained in $K_{\text{sym}}(F_n)$.*

Proof. The first statement is proved exactly as in the monoid case. We have $q(l_e) = q(r_e) = 0$ since the word length can be obtained by adding up the counts for all possible letter, and if $|w|_S \geq 1$, then $q(l_w)$ and $q(r_w)$ can be represented by a function taking only values 0 and 1.

Concerning $K_{\text{sym}}(F_n)$ we can argue as follows. Since $\varphi_w = \rho_w - \rho_{w^{-1}}$ we have

$$q_{\text{sym}}(l_w) = q(l_w) - q(r_{w^{-1}}) = 0$$

by the result about $K(F_n)$. Symmetrically we obtain $q_{\text{sym}}(r_w) = 0$. Finally, $q_{\text{sym}}(s_w)$ is represented by the function

$$\varphi_w + \varphi_{w^{-1}} = \rho_w - \rho_{w^{-1}} + \rho_{w^{-1}} - \rho_w = 0. \quad \square$$

From now on we denote by $B(M_n) \subset K(M_n)$ and $B(F_n) \subset K(F_n)$ the respective subspaces spanned by corresponding left- and right-extension relations $\{l_w, r_w\}$. We also denote by $B_{\text{sym}}(M_n) \subset K_{\text{sym}}(F_n)$ the subspace generated by the relations $\{l_w, r_w, s_w\}$. In this notation our goal is to establish the equalities $B(M_n) = K(M_n)$, $B(F_n) = K(F_n)$ and $B_{\text{sym}}(F_n) = K_{\text{sym}}(F_n)$.

2.2. Pure elements. From now on we fix an integer $n \geq 2$ and a set $S = \{a_1, \dots, a_n\}$ of cardinality n . We then denote by M_n , respectively F_n , the free monoid, respectively free group, on S . Given an integer $L \geq 0$, an element $f \in \mathbb{R}[M_n]$ will be called *pure of length L* if $|w| = L$ for all $w \in \text{supp}(f)$. Thus for example $\delta_{a_1 a_2} - 3\delta_{a_2 a_1}$ is pure of length 2. Similarly, $\mathbb{R}[F_n]_L$ denotes the space of all finitely supported real-valued functions on F_n which are pure of length L . We also introduce the notations

$$K_L(M_n) := K(M_n) \cap \mathbb{R}[M_n]_L, \quad B_L(M_n) := B(M_n) \cap K_L(M_n),$$

$$K_L(F_n) := K(F_n) \cap \mathbb{R}[F_n]_L, \quad B_L(F_n) := B(F_n) \cap K_L(F_n),$$

where the basic relation spaces $B(M_n)$ and $B(F_n)$ are defined as in the previous subsection. When the monoid or group in question is clear from the context we simply write K, B, K_L, B_L . Note that the spaces K_L and B_L are finite-dimensional for each $L \geq 0$.

Lemma 2.3. *If $\dim B_L \geq \dim K_L$ for all $L \geq 0$, then $B = K$.*

Proof. Since $B_L \subset K_L$, the assumption implies $B_L = K_L$ for all L . Now if $r \in K$ is any function, then by adding elements of B we can always achieve that r is pure. Thus if $B_L = K_L$ for all $L \geq 0$, then

$$K/B \subseteq \left(\sum_L K_L \right) / B = \sum_L K_L / (B \cap K_L) = \sum_L K_L / B_L = 0. \quad \square$$

This reduces the proof of the first two parts of Theorem 1.3 to an estimate of the dimensions of the finite-dimensional vector spaces B_L and K_L . We now carry out the necessary estimates, first in Subsection 2.3 for M_n and then in Subsection 2.4 for F_n . The argument is basically the same in both cases, but in the case of F_n some additional care has to be taken because of potential cancellations. Once the relations between counting functions are determined, it is easy to also determine the relations between counting quasimorphisms. This will be carried out in Subsection 2.5.

2.3. Relations between counting functions on free monoids. The goal of this subsection is to establish Part (i) of Theorem 1.3 concerning the space of relations between counting functions on monoids. We fix $n \geq 2$ and consider the free monoid M_n with generating set $S = \{a_1, \dots, a_n\}$. We also use the notation introduced in Subsection 2.2 and write K, B, K_L and B_L for $K(M_n), B(M_n), K_L(M_n)$ and $B_L(M_n)$. By Lemma 2.3 it suffices to establish $\dim B_L \geq \dim K_L$. For $L = 0$ we have $\dim K_L = 0$, so there is nothing to show. For $L \geq 1$ we are going to show that

$$\dim B_L \geq n^{L-1} - 1 \geq \dim K_L \tag{2.1}$$

by first establishing a lower bound for $\dim B_L$, and then establishing an upper bound for $\dim K_L$.

STEP 1: LOWER BOUND FOR $\dim B_L$. We now establish the first inequality in (2.1) for all $L \geq 1$. For $L = 1$ there is nothing to show, thus we will assume $L \geq 2$. Given any word w of length $L - 1$ we have

$$b_w := r_w - l_w \in B_L.$$

This defines n^{L-1} elements in B_L , and we claim that their span B_L^0 has dimension precisely $n^{L-1} - 1$. If we write each of the elements b_w as

$$b_w = \sum_{|v|=L} \lambda_{w,v} \delta_v,$$

then this amounts to showing that the $n^{L-1} \times n^L$ -matrix

$$A_L(M_n) = (\lambda_{w,v}) \tag{2.2}$$

has rank $n^{L-1} - 1$.

Example 2.4. The matrix $A_3(M_2)$ has the form

	$a_1 a_1 a_1$	$a_1 a_1 a_2$	$a_1 a_2 a_1$	$a_1 a_2 a_2$	$a_2 a_1 a_1$	$a_2 a_1 a_2$	$a_2 a_2 a_1$	$a_2 a_2 a_2$
$a_1 a_1$	0	-1	0	0	1	0	0	0
$a_1 a_2$	0	1	-1	-1	0	1	0	0
$a_2 a_1$	0	0	1	0	-1	-1	1	0
$a_2 a_2$	0	0	0	1	0	0	-1	0

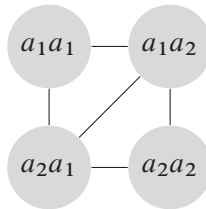
As is apparent in the example, the structure of the matrix A_L is very special: observe that $\lambda_{w,v} \neq 0$ only if w is a maximal proper subword of v . Thus each column contains either one $+1$ and one -1 (if deleting the first and the last letter lead to different words), or no non-zero entry at all (if deleting the first and last letter lead to the same word). The latter actually happens only if $v = w$ is a power of some a_j . As a first consequence of this special structure, we see that the sum of the rows is 0, whence $\text{rank}(A_L(M_n)) \leq n^{L-1} - 1$. The converse inequality can be reformulated in graph theoretic terms.

Lemma 2.5. *Let $\Gamma_L(M_n)$ be the (non-oriented) graph whose vertices are words w of length $L - 1$, and in which two different words w and w' are joined by an edge iff there exists a column of the matrix $A_L(M_n)$ with non-zero entries in the rows corresponding to both w and w' . Then $\text{rank}(A_L(M_n)) = n^{L-1} - 1$ if and only if the graph $\Gamma_L(M_n)$ is connected.*

Proof. Assume that the graph $\Gamma_L(M_n)$ is connected and that some linear combination of rows involving the w -th row is 0. Then every non-zero entry of the w -th row of the matrix $A_L(M_n)$ has to be cancelled. However, since for each of these entries there is only one other row containing it, this row has to be involved in

the linear combination. The upshot is that if v and w are connected by an edge in $\Gamma_L(M_n)$, then every linear combination of rows involving w which adds up to 0-row must also involve v , and the coefficients for v and w in this sum have to be the same. We deduce that if $\Gamma_L(M_n)$ is connected then no proper subset of rows is linearly dependent, whence $\text{rank}(A_L(M_n)) = n^{L-1} - 1$. Conversely, if Γ' is a connected component of $\Gamma_L(M_n)$, then adding up the rows corresponding to the vertices of Γ' yields 0. In particular $\text{rank}(A_L(M_n)) = n^{L-1} - 1$ implies that there is only one connected component. \square

The following picture shows the graph $\Gamma_3(M_2)$ corresponding to the matrix $A_3(M_2)$ above:



The graphs $\Gamma_L(M_n)$ are closely related to a family of classical examples in finite graph theory called *De Bruijn graphs* [3].⁵ Recall that the L -th *De Bruijn graph* over S is the graph $\Gamma_L(S)$ whose vertices are words of length $L - 1$ and whose edges are words of length L connecting the subwords obtained by deleting the first, respectively last letter. We claim that the graph $\Gamma_L(M_n)$ can be obtained from $\Gamma_L(S)$ by erasing all loops. Indeed, vertices w and w' of $\Gamma_L(M_n)$ are connected by an edge if and only if there exist $s_1, s_2 \in S$ such that either $ws_1 = s_2w'$ or $s_1w = w's_2$, but not both. Then the claim follows from the fact that the latter case can only happen if $w = w'$ is a power of some letter a_j . We thus refer to $\Gamma_L(M_n)$ as a *loop-erased De Bruijn graph*.

Since erasing loops does not change connectivity of a graph, it remains only to show connectedness of the De Bruijn graphs. This is a folklore fact from finite graph theory. Explicitly, two words $w = s_1 \cdots s_{L-1}$ and $w' = r_1 \cdots r_{L-1}$ can be connected through the path

$$s_1 \cdots s_{L-1} \sim r_{L-1}s_1 \cdots s_{L-2} \sim r_{L-2}r_{L-1}s_1 \cdots s_{L-3} \sim \cdots \sim r_1 \cdots r_{L-1}.$$

We thus deduce from Lemma 2.5 that

$$\dim B_L \geq \dim B_L^0 = \text{rank}(A_L(M_n)) = n^{L-1} - 1.$$

This finishes Step 1.

⁵ All graphs in this paper are *non-oriented* graphs, and what we call De Bruijn graphs are sometimes call non-oriented De Bruijn graphs to distinguish them from a similarly defined oriented version.

STEP 2: UPPER BOUND FOR $\dim K_L$. We are now going to establish the second inequality in (2.1) for all $L \geq 1$. Rather than showing directly that $\dim K_L \leq n^{L-1} - 1$ we will show that the codimension of K_L in $\mathbb{R}[M_n]_L$ is bounded below by

$$\text{codim } K_L \geq \dim \mathbb{R}[M_n]_L - (n^{L-1} - 1) = (n - 1) \cdot n^{L-1} + 1. \tag{2.3}$$

In order to establish (2.3) we will construct $(n - 1) \cdot n^{L-1} + 1$ linearly independent linear functionals on $\mathbb{R}[M_n]_L$, which vanish on K_L . Such a linear functional will be called a *certificate*.

We now describe a way to construct certificates using homogenization. Recall from the appendix that a function $f: M_n \rightarrow \mathbb{R}$ is called *homogenizable* if the limit

$$\hat{f}(x) := \lim_{n \rightarrow \infty} \frac{f(x^n)}{n} \tag{2.4}$$

exists for every $x \in M_n$, and in this case the function $\hat{f}: M_n \rightarrow \mathbb{R}$ defined by (2.4) is called the *homogenization* of f . By Corollary A.5 every function $f \in \mathcal{C}(M_n)$ is homogenizable. Given $c \in M_n$ we may thus define a linear functional

$$\langle c \rangle_L: \mathbb{R}[M_n]_L \longrightarrow \mathbb{R}, \quad \sum \lambda_g \delta_g \longmapsto \sum \lambda_g \hat{\rho}_g(c) \tag{2.5}$$

by evaluation of the homogenization at c .

Lemma 2.6. *For every $c \in M_n$ and $L \geq 1$ the functional $\langle c \rangle_L \in (\mathbb{R}[M_n]_L)^*$ is a certificate, i.e. it vanishes on K_L .*

Proof. Let $f_0 = \sum \lambda_g \delta_g \in K_L$. By definition, this means that the function

$$f := \sum \lambda_g \rho_g \in \mathcal{C}(M_n)$$

is bounded. Consequently, the homogenization \hat{f} satisfies $\hat{f} \equiv 0$. We deduce that, for every $c \in M_n$,

$$0 = \hat{f}(c) = \sum \lambda_g \hat{\rho}_g(c) = \langle c \rangle_L \left(\sum \lambda_g \delta_g \right) = \langle c \rangle_L(f_0),$$

which shows that $f_0 \in \ker(\langle c \rangle_L)$ and finishes the proof. □

In view of the lemma we refer to $\langle c \rangle_L$ as the *L-certificate of c*. It remains to show that there exists $(n - 1) \cdot n^{L-1} + 1$ elements of M_n whose corresponding L-certificates are linearly independent. For this we start from the set

$$W_L := \{a_i w \mid i \neq 1, |w| = L - 1\} \cup \{a_1^L\}$$

of *special words* and define the associated set of certificates to be

$$C_L := \{\langle ca_1^L \rangle_L \mid c \in W_L\}.$$

Note that $|C_L| = |W_L| = (n - 1) \cdot n^{L-1} + 1$. We will show that the certificates in C_L are linearly independent, thereby finishing the proof. For our computations in the dual space $\mathbb{R}[M_n]_L^*$ we will denote by $\{[w] \mid w \in S^L\}$ the dual basis to the basis $\{\delta_w \mid w \in S^L\}$, i.e.

$$[w](\delta_v) = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

As an immediate consequence of Lemma A.2, we can write our certificates in terms of this dual basis as follows.

Lemma 2.7. *Let $c \in M_n$. Then*

$$\langle c \rangle_L = \sum [w],$$

where w runs through all cyclic subwords of c of length L with multiplicity.

For example,

$$\langle a_1 a_2 a_1^2 a_2^2 \rangle_3 = [a_1 a_2 a_1] + [a_2 a_1 a_1] + [a_1 a_1 a_2] + [a_1 a_2 a_2] + [a_2 a_2 a_1] + [a_2 a_1 a_2].$$

Now we can finish the proof by the following lemma.

Lemma 2.8. *The set $C_L \subset (\mathbb{R}[M_n]_L)^*$ is linearly independent.*

Proof. Using again our set W_L of special words we introduce a test space $T_L \subset \mathbb{R}[M_n]_L$ as

$$T_L := \text{span}\{\delta_w \mid w \in W_L\}.$$

We will show that already the restrictions of the certificates in C_L to T_L are linearly independent. For this we observe that if $w = a_i s_1 \cdots s_{L-1} \in W_L \setminus \{a_1^L\}$, then $i \neq 1$ and by Lemma 2.7

$$\begin{aligned} \langle wa_1^L \rangle_L &= [a_i s_1 \cdots s_{L-1}] + [s_1 \cdots s_{L-1} a_1] + \cdots + [s_{L-1} a_1^{L-1}] + [a_1^L] \\ &\quad + [a_1^{L-1} a_i] + [a_1^{L-2} a_i s_1] + \cdots + [a_1 a_i s_1 \cdots s_{L-2}], \end{aligned}$$

Now, by definition, the words $a_1^{L-1} a_i, a_1^{L-2} a_i s_1, \dots, a_1 a_i s_1 \cdots s_{L-2}$ appearing in the second row are not contained in W_L . It follows that

$$\langle wa_1^L \rangle_{T_L} = ([a_i s_1 \cdots s_{L-1}] + [s_1 \cdots s_{L-1} a_1] + \cdots + [s_{L-1} a_1^{L-1}] + [a_1^L])|_{T_L}.$$

Concerning the final certificate we have

$$\langle a_1^{2L} \rangle_L = \langle a_1^{2L} \rangle_L |_{T_L} = 2L \cdot [a_1^L].$$

Now we introduce a total order on M_2 as follows. We first order S by declaring that

$$a_1 < a_2 < \dots < a_n$$

and then extend to M_n in a *right*-lexicographic way. Given $w, w' \in M_2$ and $s, s' \in S$ we set $ws < w's'$ if and only if $s < s'$ or $s = s'$ and $w < w'$. Note in particular that $w_1 > w_2$ if and only if $w_1 a_1^L > w_2 a_1^L$. With this order understood the largest element in the support of $\langle w a_1^L \rangle_L |_{T_L}$ is precisely w ; indeed this follows from

$$a_i s_1 \dots s_{L-1} \geq s_1 \dots s_{L-1} a_1 \geq \dots \geq s_{L-1} a_1^{L-1} \geq a_1^L$$

and the above computation of supports. We deduce that the matrix obtained by evaluating the certificates from C_L on the basis $\{\delta_w \mid w \in W_L\}$ of T_L is of lower triangular form for the given order with non-trivial diagonal entries. It therefore has full rank, and the lemma follows. \square

This finishes Step 2 and thereby the proof of Theorem 1.3.(i).

Remark 2.9. Note that as a by-product of the proof we also see that $B_L = B_L^0$.

2.4. Relations between counting functions on free groups. We are now going to extend the results of the previous subsection to the group case, thereby establishing Part (ii) of Theorem 1.3. The proof is in close analogy with the monoid case and we will only highlight the necessary modifications. Throughout we fix $n \geq 2$ and write K, B, K_L and B_L as short hands for the spaces $K(F_n), B(F_n), K_L(F_n)$ and $B_L(F_n)$ as introduced in Section 2.2. By Lemma 2.3 it suffices again to establish $\dim B_L \geq \dim K_L$.

Assume first that $L \leq 1$. We claim that in this case $\dim K_L = 0$, whence the desired inequality hold automatically. For $L = 0$ the claim follows from the fact that the function ρ_e is unbounded. Now let $L = 1$. We have to show that the functions $\{\rho_{a_1}, \rho_{a_1^{-1}}, \dots, \rho_{a_n}, \rho_{a_n^{-1}}\}$ are linearly independent modulo bounded functions. For this it suffices to observe that

$$\left(\sum_{i=1}^n \lambda_i \rho_{a_i} + \sum_{i=1}^n \mu_i \rho_{a_i^{-1}} \right) (a_j^n) = \begin{cases} n \cdot \lambda_j & \text{if } n > 0, \\ -n \cdot \mu_j, & \text{if } n < 0. \end{cases}$$

We have thus established the desired inequality for $L \leq 1$. Next we are going to show that

$$\dim B_L \geq 2n(2n - 1)^{L-2} - 1 \geq \dim K_L. \tag{2.6}$$

for all $L \geq 2$.

STEP 1. Concerning the lower bound on $\dim B_L$ we observe that, as in the case of monoids, every reduced word w of length $L - 1$ gives rise to a basic relation $b_w = r_w - l_w$ of length L , and there are $2n(2n - 1)^{L-2}$ such words. Again we write each of the elements b_w as

$$b_w = \sum_{|v|=L} \lambda_{w,v} \delta_v,$$

and obtain a matrix $A_L(F_n) = (\lambda_{w,v})$ of size $2n(2n - 1)^{L-2} \times 2n(2n - 1)^{L-1}$. We then have to show that the matrix $A_L(F_n)$ has rank $2n(2n - 1)^{L-2} - 1$. This amounts again to showing connectedness of a certain graph.

More precisely, let $\bar{S} := \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\}$ be the symmetrization of the free generating set of F_n . Then the L -th De Bruijn–Martin graph⁶ over \bar{S} is the graph $\Gamma_L(\bar{S})$ with vertices given by reduced words of length $L - 1$ over \bar{S} and edges given by reduced words of length L over \bar{S} , where the edge labelled by some word w connects the two vertices labelled by the words which are obtained by cancelling the first, respectively last, letter of w . Then Lemma 2.5 and its proof carry over to the present setting in the following form:

Lemma 2.10. *Let $\Gamma_L(F_n)$ be the loop-erased De Bruijn–Martin graph $\Gamma_L(\bar{S})$. Then $\text{rank}(A_L(F_n)) = 2n(2n - 1)^{L-2} - 1$ if and only if the graph $\Gamma_L(F_n)$ (or, equivalently, $\Gamma_L(\bar{S})$) is connected.*

Connectedness of the De Bruijn–Martin graph $\Gamma_L(\bar{S})$ is again well-known and easy to see as follows. If $w = s_1 \cdots s_{L-1}$ and $w' = r_1 \cdots r_{L-1}$ with $r_{L-1} \neq s_1^{-1}$, then as in the monoid case,

$$s_1 \cdots s_{L-1} \sim r_{L-1} s_1 \cdots s_{L-2} \sim r_{L-2} r_{L-1} s_1 \cdots s_{L-3} \sim \cdots \sim r_1 \cdots r_{L-1}.$$

Thus in this case, w and w' are in the same connected component of the graph. If, however, $r_{L-1} = s_1^{-1}$, then we can choose a_j with $a_j^{\pm 1} \neq r_{L-1}$, and by the previous case, both w and w' are in the same connected component as a_j^{L-1} .

This shows connectedness of the De Bruijn–Martin graph and thereby finishes Step 1.

STEP 2. In analogy with the monoid case we have to construct $\dim \mathbb{R}[F_n]_L - 2n(2n - 1)^{L-2} + 1$ linearly independent certificates for K_L . By Lemma A.3 the counting functions $\rho_w \in \mathcal{C}(F_n)$ are homogenizable. We can thus define, as in the monoid case, for every $c \in F_n$ a certificate $\langle c \rangle_L$ for K_L by the same formula as in (2.5). If we assume in addition that c is cyclically reduced, then we have the following analogue of Lemma 2.7, which is again a direct consequence of Lemma A.3.

⁶ These graphs were popularized through the PhD of Martin [19], who pointed out that they are Eulerian and that this can be used to show that integral measured currents on F_n can be written as sums of counting currents.

Lemma 2.11. *Let $c \in F_n$. If c is cyclically reduced, then*

$$\langle c \rangle_L = \sum [w],$$

where w runs through all cyclic subwords of c of length L with multiplicity.

In order to establish the second inequality in (2.6) along the same lines as in the monoid case we will thus have to find $\dim \mathbb{R}[F_n]_L - 2n(2n - 1)^{L-2} + 1$ cyclically reduced words with linearly independent L -certificates. For this we choose our set of special words as follows. Denote by $\bar{S}^{(L)} \subset \bar{S}^L$ the set of all reduced words of length L , by $A_L^{(1)}$ the set of all such words starting with a_1 and by $A_L^{(2)}$ the set of all such words starting with $a_2a_1^{-1}$. Then we define

$$W_L := (\bar{S}^{(L)} \setminus (A_L^{(1)} \cup A_L^{(2)})) \cup \{a_1^L\}.$$

Note that

$$\begin{aligned} |W_L| &= |\bar{S}^{(L)}| - |A_L^{(1)}| - |A_L^{(2)}| + 1 \\ &= \dim \mathbb{R}[F_n]_L - (2n - 1)^{L-1} - (2n - 1)^{L-2} + 1 \\ &= \dim \mathbb{R}[F_n]_L - 2n(2n - 1)^{L-2} + 1. \end{aligned}$$

We are now going to define a certificate for every $w \in W_L$. Since we want to avoid cyclic cancellation, the definition of the set of certificates C_L is more complicated than in the monoid case. Given $w \in W_L$ we define a reduced word $s(w) \in \bar{S}^{(L)}$ as follows:

- (i) if w does not start with a_1^{-1} and does not end with a_1^{-1} , then $s(w) := a_1^L$;
- (ii) if w does not start with a_1^{-1} but ends with a_1^{-1} , then $s(w) := a_2a_1^L$;
- (iii) if w starts with a_1^{-1} but does not end with a_1^{-1} , then $s(w) := a_1^L a_2$;
- (iv) if w starts and ends with a_1^{-1} , then $s(w) := a_2a_1^L a_2$.

This definition is made in such a way that for every $w \in W_L$ the word $ws(w)$ is cyclically reduced, and we define

$$C_L := \{\langle ws(w) \rangle_L \mid w \in W_L\}.$$

It remains to show only that the set C_L is linearly independent. We will in fact show that the certificates in C_L are linearly independent when restricted to the test space $T_L := \text{span}\{\delta_w \mid w \in W_L\}$. We order the words in W_L as follows. First we order the letters according to

$$a_1 < a_2 < \dots < a_n < a_1^{-1} < a_2^{-1} < \dots < a_n^{-1}$$

and extend this to a total ordering on W_L in the right-lexicographic way. We now claim that the matrix N_L formed by the evaluations of the certificates in C_L on the

canonical basis of the test space T_L is a lower triangular matrix with respect to the given order on W_L with non-zero diagonal entries. Indeed, this follows from the following two basic observations.

- (i) Consider the cyclic L -subwords of $ws(w)$ with initial letter in w . These form a strictly decreasing sequence, so w is the biggest of them.
- (ii) It remains to deal with those cyclic L -subwords v of $ws(w)$ whose initial letter is contained in $s(w)$. Here there are several cases: If w is among the words of Type (i) or (iii), then v is not in W_L , so we can ignore it. In cases (ii) and (iv), v is either not in W_L or $v = a_2 a_1^{L-1}$. Since w is not of type (i) we have $w \neq a_1^L$. But v is the second smallest element of W_L after a_1^L , so $w \geq v$ also in this case.

This shows that the matrix is indeed lower triangular with non-zero coefficients on the diagonal, which finishes Step 2 and thereby the proof of Theorem 1.3.(ii).

2.5. Relations between counting quasimorphisms. In this subsection we finish the proof of Theorem 1.3 by deducing Part (iii) of the theorem from Part (ii). Throughout we fix $n \geq 2$ and, using the notation introduced in Subsection 2.2, write $B, K, \hat{C}, K_{\text{sym}}$ and \hat{B} for $B(F_n), K(F_n), \hat{C}(F_n), K_{\text{sym}}(F_n)$ and $\hat{B}(F_n)$ respectively. We also denote by $B_{\text{sym}} \subset K_{\text{sym}}$ the space spanned by the symmetry relations $\{s_w \mid w \in F_n\}$. We have already seen in Subsection 2.1 that $B + B_{\text{sym}} \subset K_{\text{sym}}$, and we would like to show the opposite inclusion $K_{\text{sym}} \subset B + B_{\text{sym}}$.

For this we first define a linear involution

$$\sigma: \mathbb{R}[F_n] \longrightarrow \mathbb{R}[F_n], \quad f \longmapsto f^*$$

by demanding that $\delta_w^* := -\delta_{w^{-1}}$. Then the natural inclusion map $\iota: \hat{B} \rightarrow \hat{C}$, which sends the class $[\rho_w]$ to the class $[\rho_w - \rho_{w^{-1}}]$, lifts to a map

$$i: \mathbb{R}[F_n] \longrightarrow \mathbb{R}[F_n], \quad f \longmapsto f + f^*,$$

i.e. we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{\text{sym}} & \longrightarrow & \mathbb{R}[F_n] & \xrightarrow{q_0} & \hat{B} \longrightarrow 0 \\ & & & & \downarrow i & & \downarrow \iota \\ 0 & \longrightarrow & B & \longrightarrow & \mathbb{R}[F_n] & \xrightarrow{q} & \hat{C} \longrightarrow 0. \end{array}$$

Note that the top row is exact by definition, whereas the bottom row is exact by Theorem 1.3.(ii). We deduce that i restricts to a linear map $j: K_{\text{sym}} \rightarrow B$.

Now in order to show the desired inclusion $K_{\text{sym}} \subset B + B_{\text{sym}}$ it suffices to show that $j(K_{\text{sym}}) \subset j(B) + j(B_{\text{sym}})$ and that $\ker(j) \subset \ker(j|_{B+B_{\text{sym}}})$. We will actually establish the stronger inclusions $j(K_{\text{sym}}) \subset j(B)$ and $\ker(j) \subset B_{\text{sym}}$. The latter inclusion is actually immediate from the description of $\ker(j)$ as

$$\ker(j) = \{f \in K_{\text{sym}} \mid j(f) = f + f^* = 0\}.$$

It thus remains to show only that $j(K_{\text{sym}}) \subset j(B)$. We claim that

$$j(B) = \{b \in B \mid b^* = b\}. \tag{2.7}$$

The inclusion \subseteq follows from $(f + f^*)^* = f + f^*$. Conversely, if $b^* = b \in B \subset K_{\text{sym}}$, then $j(b/2) = b/2 + b^*/2 = b$, which shows the opposite inclusion and proves (2.7).

Now if $f \in K_{\text{sym}}$, then $j(f) \in B$ and $j(f)^* = (f + f^*)^* = f + f^* = j(f)$. We then deduce from (2.7) that $j(f) \in j(B)$. This proves the remaining inclusion $j(K_{\text{sym}}) \subset j(B)$.

We have thus shown that Part (ii) of Theorem 1.3 implies Part (iii). This concludes the proof of Theorem 1.3.

3. Bases for $\widehat{\mathcal{C}}(M_n)$, $\widehat{\mathcal{C}}(F_n)$ and $\widehat{\mathcal{B}}(F_n)$

3.1. Pure bases and compatible bases. The purpose of this section is to construct bases for each for the spaces $\widehat{\mathcal{C}}(M_n)$, $\widehat{\mathcal{C}}(F_n)$ and $\widehat{\mathcal{B}}(F_n)$ and thereby to establish Theorem 1.5. Given $L \geq 0$ we will denote by $\widehat{\mathcal{C}}(M_n)_L$ the image of $\mathbb{R}[M_n]_L$ in $\widehat{\mathcal{C}}(M_n)$. The spaces $\widehat{\mathcal{C}}(F_n)_L$ and $\widehat{\mathcal{B}}(F_n)_L$ will be defined similarly. We are going to relate bases of the spaces $\widehat{\mathcal{C}}(M_n)_L$ to bases of $\widehat{\mathcal{C}}(M_n)$, and similarly for $\widehat{\mathcal{C}}(F_n)$.

Concerning bases of $\widehat{\mathcal{C}}(M_n)_L$ we adapt the following language. A basis of $\widehat{\mathcal{C}}(M_n)_L$ is called *pure* if its elements are of the form $[\rho_w]$ for some $w \in S^L$. Note that there are only finitely many pure bases for a given L . If B_L is a basis for $\widehat{\mathcal{C}}(M_n)_L$, then the sequence of bases $(B_L)_{L \geq 0}$ is called *compatible* if $B_L \cap \widehat{\mathcal{C}}(M_n)_{L-1} = B_{L-1}$. Pure bases of $\widehat{\mathcal{C}}(F_n)$ and compatible sequences of bases of $\widehat{\mathcal{C}}(F_n)$ are defined similarly.

It follows from the left- and right-extension relations that every $[\rho_w]$ with $|w| = L$ can be written as a linear combination of $[\rho_v]$ for some v of length $L + 1$. This implies that

$$\widehat{\mathcal{C}}(M_n)_0 \subset \widehat{\mathcal{C}}(M_n)_1 \subset \dots \subset \widehat{\mathcal{C}}(M_n)_L \subset \dots,$$

whence $\widehat{\mathcal{C}}(M_n)$ is the ascending union

$$\widehat{\mathcal{C}}(M_n) = \bigcup_{L \geq 0} \widehat{\mathcal{C}}(M_n)_L.$$

Consequently, if $(B_L)_{L \geq 0}$ is a compatible sequence of bases of $\widehat{\mathcal{C}}(M_n)_L$, then $B := \bigcup B_L$ defines a basis of $\widehat{\mathcal{C}}(M_n)$. Similarly, compatible sequences of bases for $\widehat{\mathcal{C}}(F_n)_L$ give rise to bases of $\widehat{\mathcal{C}}(F_n)$.

It turns out that pure bases of $\widehat{\mathcal{C}}(M_n)_L$ can be classified in graph theoretic terms. Given pure bases for each $\widehat{\mathcal{C}}(M_n)_L$ one can then easily modify them to obtain a compatible sequence of bases, and thereby a basis for $\widehat{\mathcal{C}}(M_n)$. This will be carried out in Subsection 3.2, leading to a proof of Part (i) of Theorem 1.5. The analogous constructions in the group case and in particular the proof of Part (ii) of Theorem 1.5 will be given in Subsection 3.3. Deducing Part (iii) of Theorem 1.5 from Part (ii) is essentially a triviality, since our basis for $\widehat{\mathcal{C}}(F_n)$ can easily be modified as to possess the necessary symmetries. We will give the details in Subsection 3.4.

3.2. A basis for $\widehat{\mathcal{C}}(M_n)$. Our first task is to classify pure bases for $\widehat{\mathcal{C}}(M_n)_L$. The only pure basis for $\widehat{\mathcal{C}}(M_n)_0$ is given by $B_0 = \{\rho_e\}$. We will now parametrize pure bases for $\widehat{\mathcal{C}}(M_n)_L$ for $L \geq 1$. Recall from Subsection 2.3 that the L -th De Bruijn graph $\Gamma_L(S)$ over $S = \{a_1, \dots, a_n\}$ has vertices labelled by S^{L-1} and edges labelled by S^L where the edge labelled w connects the two vertices by the subwords obtained from w by deleting the first respectively last letter. The graph $\Gamma_L(M_n)$ is obtained from this graph by erasing all loops. We observe:

Proposition 3.1. *Let $L \geq 1$ and let W be a set of words of length L in M_n of cardinality $|W| = (n-1)n^{L-1} + 1$. Then the following are equivalent:*

- (i) *the set $B(W) := \{\rho_w \mid w \in W\}$ is a pure basis of $\widehat{\mathcal{C}}(M_n)_L$;*
- (ii) *the subgraph of $\Gamma_L(S)$ (or equivalently of $\Gamma_L(M_n)$) with vertices S^{L-1} and edges labelled by $S^L \setminus W$ is connected;*
- (iii) *the subgraph of $\Gamma_L(S)$ with vertices S^{L-1} and edges labelled by $S^L \setminus W$ is a spanning tree of $\Gamma_L(S)$.*

In particular, $\dim \widehat{\mathcal{C}}(M_n)_L = (n-1)n^{L-1} + 1$ and pure bases of $\widehat{\mathcal{C}}(M_n)_L$ are in bijection with spanning trees of the De Bruijn graph $\Gamma_L(S)$.

Proof. For $L = 1$ the only pure basis of $\widehat{\mathcal{C}}(M_n)_L$ is $B(W)$ with $W = S$. This is in accordance with (ii) and (iii), since $\Gamma_1(S)$ has a single vertex. We may thus assume from now on that $L \geq 2$. By Remark 2.9 any relations between the ρ_w are consequences of the basic relations b_w . This can be expressed in terms of the matrix $A_L(M_n)$ given by (2.2) as follows. Let us enumerate the words of length L

by $\{w_1, \dots, w_{n^L}\}$ and the words of length $L - 1$ by $\{v_1, \dots, v_{n^{L-1}}\}$. Then

$$\sum_{i=1}^{n^L} \alpha_i [\rho_{w_i}] = 0 \iff \text{for all } j = 1, \dots, n^{L-1} \text{ there exist } \lambda_j \text{ such that}$$

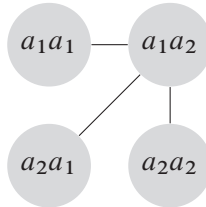
$$\alpha_i = \sum_{j=1}^{n^{L-1}} \lambda_j a_{ji} \text{ for all } i, \dots, n^L,$$

where a_{ji} is the entry of $A_L(M_n)$ corresponding to the row v_i and the column w_j . It follows that the set $\{\rho_w \mid w \in W\}$ is linearly independent if and only if the $n^{L-1} \times (n^{L-1} - 1)$ submatrix A_W of $A_L(M_n)$ formed by the columns corresponding to words in $S^L \setminus W$ has full rank $n^{L-1} - 1$. The matrix A_W has the same structure as A , i.e. every column contains at most two non-zero entries which are contained in ± 1 and sum up to 0. Thus we can argue as in Step 1 of Subsection 2.3 to conclude that A_W has full rank if and only if the subgraph of $\Gamma_L(M_n) \subset \Gamma_L(S)$ with edges in $S^L \setminus W$ is connected. This shows the equivalence (i) \iff (ii) and also implies that $\dim \widehat{C}(M_n)_L = (n - 1)n^{L-1} + 1$. Then the equivalence (ii) \iff (iii) is an immediate consequence of the fact that a graph with k vertices and $k - 1$ edges is connected if and only if it is a tree. \square

For example let $S = \{a_1, a_2\}$ and consider the subset

$$W := \{a_1^3, a_2a_1^2, a_2a_1a_2, a_2^2a_1, a_2^3\} \subset S^3.$$

Then $S^3 \setminus W = \{a_1^2a_2, a_1a_2a_1, a_1a_2^2\}$ corresponds to the following spanning tree of $\Gamma_3(S)$:



It follows that $B(W)$ is a pure basis of $\widehat{C}(M_2)_3$. This example generalizes as follows.

Corollary 3.2. *Given $L \geq 0$ define $W(a_1; L)$ as follows. If $L = 0$, then $W(a_1; L) := \{e\}$. If $L > 0$, then*

$$W(a_1; L) := (S^L \setminus \{w \in S^L \mid w \in a_1 S^{L-1}\}) \cup \{a_1^L\}.$$

Then $B(W(a_1; L)) := \{\rho_w \mid w \in W(a_1; L)\}$ is a pure basis of $\widehat{C}(M_n)_L$.

Proof. Let $w \in S^{L-1}$. We show that w can be connected in $\Gamma_L(S)$ to a_1^{L-1} by a path using only edges from $S^L \setminus W(a_1; L)$. For this we write $w = a_1^k s_1 \dots s_{L-k-1}$

with $s_1 \neq a_1$. Then

$$w = a_1^k s_1 \dots s_{L-k-1} \sim a_1^{k+1} s_1 \dots s_{L-k-2} \sim \dots \sim a_1^{L-2} s_1 \sim a_1^{L-1}$$

is an admissible path. □

We can now modify these pure bases to obtain a family of compatible bases as follows.

Corollary 3.3. *Let $j \in \{1, \dots, n\}$ and let W_L denote the set of (possibly empty) words of length $\leq L$ not starting or ending in a_j . Then $B(W_L) := \{\rho_w \mid w \in W_L\}$ is a basis of $\widehat{\mathcal{C}}(W_n)_L$, and these bases are compatible.*

Proof. By symmetry we may assume without loss of generality that $j = 1$. Observe first that $|W_L| = \dim \widehat{\mathcal{C}}(M_n)_L$. For $L = 0$ this is obvious, and for $L \geq 1$ it follows from the formula $\dim \widehat{\mathcal{C}}(M_n)_L = (n - 1)n^{L-1} + 1$. In view of this observation it suffices to show that each element of the basis $B(W(a_1; L))$ can be expressed as a linear combination of elements in $B := B(W_L)$. The case $L = 0$ is again obvious, so we may assume $L \geq 1$. In this case the set $W(a_1; L)$ can be written as the disjoint union

$$W(a_1; L) = W_1 \cup W_2 \cup \{a_1^L\},$$

where W_1 is the set of words of length L neither starting nor ending with a_1 , and W_2 is the set of words of length L ending but not starting in a_1 . If $w \in W_1$, then $\rho_w \in B$. If $w \in W_2$ then $w = va_1^k$, where v does not end in a_1 . If $k = 1$, then

$$\rho_w = \rho_v - \sum_{s \notin \{w_{\text{fin}}^{-1}, a_1\}} \rho_{vs}$$

is contained in the span of B . For $k \geq 1$ we obtain $w \in B$ by induction on k , applying again the right-extension relations. Finally, using again the right-extension relation and induction on k one shows that $\rho_{a_1^k}$ is contained in the span of B for all $k \leq L$. This shows that every element of $B(W(a_1; L))$ is contained in the span of B and finishes the proof. □

We deduce that the ascending union $\bigcup B(W_L)$ is a basis for $\widehat{\mathcal{C}}(M_n)$. This finishes the proof of Theorem 1.5.(i).

3.3. A basis for $\widehat{\mathcal{C}}(F_n)$. We now modify the argument of the last subsection so that it works also for free groups instead of free monoids. Let F_n be the free group with basis $S = \{a_1, \dots, a_n\}$ and let $\bar{S} = \{a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}\}$. We write $\bar{S}^{(L)} \subset \bar{S}^L$ for the subset of *reduced* words of length L . With this notation, the De Bruijn–Martin graph $\Gamma_L(\bar{S})$ has vertex set $\bar{S}^{(L-1)}$ and edge set $\bar{S}^{(L)}$, where each edges connects the two vertices obtained by deleting the first respectively last letter. In complete analogy with Proposition 3.1 one proves:

Proposition 3.4. *Let W be a set of reduced words of length $L \geq 2$ in F_n of cardinality $|W| = 2n(2n - 1)^{L-2}(2n - 2) + 1$. Then the following are equivalent:*

- (i) *the set $B(W) := \{\rho_w \mid w \in W\}$ is a pure basis of $\widehat{\mathcal{C}}(F_n)_L$;*
- (ii) *the subgraph of $\Gamma_L(\bar{S})$ with vertices $\bar{S}^{(L-1)}$ and edges labelled by $\bar{S}^{(L)} \setminus W$ is connected;*
- (iii) *the subgraph of $\Gamma_L(\bar{S})$ with vertices $\bar{S}^{(L-1)}$ and edges labelled by $\bar{S}^{(L)} \setminus W$ is a spanning tree of $\Gamma_L(\bar{S})$.*

In particular, $\dim \widehat{\mathcal{C}}(F_n)_L = 2n(2n - 1)^{L-2}(2n - 2) + 1$ and pure bases of $\widehat{\mathcal{C}}(F_n)_L$ are in bijection with spanning trees of the De Bruijn–Martin graph $\Gamma_L(S)$.

There are two exceptional cases $L = 0$ and $L = 1$. The unique pure basis of $\widehat{\mathcal{C}}(F_n)_0$ is given by $\{\delta_e\}$, and the unique pure basis of $\widehat{\mathcal{C}}(F_n)_1$ is given by $B(\bar{S})$. Both statements are immediate from our earlier computation $\dim K(F_n)_0 = \dim K(F_n)_1 = 0$. In general we can choose the following pure basis.

Corollary 3.5. *Given $L \geq 0$ define $W(a_1; L)$ as follows. If $L = 0$, then $W(a_1; 0) := \{e\}$, if $L = 1$, then $W(a_1; 1) := \bar{S}$ and if $L \geq 2$, then*

$$W(a_1; L) := (\bar{S}^{(L)} \setminus \{w \in \bar{S}^{(L)} \mid w \text{ starts with } a_1 \text{ or } a_2a_1^{-1}\}) \cup \{a_1^L\}.$$

Then $B(W(a_1; L)) := \{\rho_w \mid w \in W(a_1; L)\}$ is a pure basis of $\widehat{\mathcal{C}}(F_n)_L$.

Proof. For $L \leq 1$ there is nothing to show. For $L \geq 2$ we have to show that every vertex $w \in \bar{S}^{(L-1)}$ can be connected to a_1^{L-1} using only edges starting with a_1 (but not equal to a_1^L) and $a_2a_1^{-1}$. If w does not start with a_1^{-1} we can argue as in the monoid case, using only edges of the first kind. If $w = a_1^{-1}s_2 \dots s_{L-1}$ then an admissible path to a_1^{L-1} is given by

$$\begin{aligned} a_1^{-1}s_2 \dots s_{L-1} &\sim a_2a_1^{-1}s_2 \dots s_{L-2} \sim a_1a_2a_1^{-1}s_2 \dots s_{L-3} \sim \dots \\ &\sim a_1^{L-3}a_2a_1^{-1} \sim a_1^{L-2}a_2^{-1} \sim a_1^{L-1}. \end{aligned} \quad \square$$

Again it is easy to pass from a family of pure bases to a family of compatible bases:

Corollary 3.6. *Given $L \geq 1$ let W'_L denote the set of words of length $\leq L$ (including the empty word) not starting in a_1 or $a_2a_1^{-1}$ and not ending in a_1^{-1} or $a_1a_2^{-1}$. Let $W_L := W'_L \cup \{a_1^{-1}\}$. Then $B(W_L) := \{\rho_w \mid w \in W_L\}$ is a basis of $\widehat{\mathcal{C}}(W_n)_L$, and these bases are compatible.*

Proof. Again we have to produce every element ρ_w in $B(a_1; L)$ as a linear combination of elements in $B := B(W_L)$. If w does not end in a_1^{-1} or $a_1a_2^{-1}$ then $\rho_w \in B$. Otherwise we can apply right-extension relations and argue inductively just as in the proof of Proposition 3.3. □

Now Theorem 1.5.(ii) follows.

3.4. A basis for $\mathcal{B}(F_n)$. We conclude this section by pointing out that Part (iii) of Theorem 1.5 follows from Part (ii). The proof is based on the following simple observation:

Lemma 3.7. *Let $W \subset F_n$ be any set of reduced words such that $B(W) := \{\rho_w \mid w \in W\}$ is a basis of $\widehat{\mathcal{C}}(F_n)$ and such that $w^{-1} \in W$ for all $w \in W$. Let W^+ be a subset of W which intersects each of the sets $\{w, w^{-1}\}$ in a single element. Then $B(W^+) := \{\varphi_w \mid w \in W^+\}$ is a basis of $\widehat{\mathcal{B}}(F_n)$.*

Proof. This is immediate from the fact that $\widehat{\mathcal{B}}(F_n) \subset \widehat{\mathcal{C}}(F_n)$ is the fixed point set of the linear involution mapping ρ_w to $-\rho_{w^{-1}}$. □

The basis of $\widehat{\mathcal{C}}(F_n)$ constructed in Corollary 3.6 does not satisfy the assumptions of the lemma. However, we can modify it as follows. Let $W_0 := \bigcup W_L$, where W_L is defined as in Corollary 3.5 and let $W := (W_0 \setminus \{e\}) \cup \{a_1\}$. Then $B(W)$ is another basis for $\widehat{\mathcal{C}}(F_n)$ and W satisfies the assumptions of Lemma 3.7. This shows that Part (iii) of Theorem 1.5 follows indeed from Part (ii), and thereby finishes the proof of the theorem.

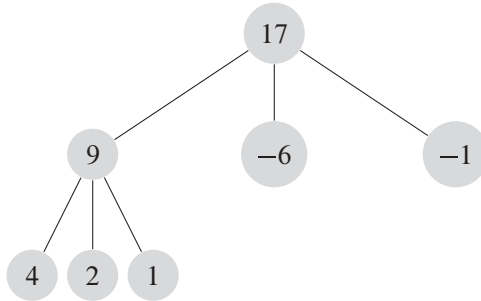
4. Counting functions and weighted trees

4.1. Representing counting functions by weighted trees. The goal of this subsection is to provide a graphical representation counting functions on free groups and monoids. This will help us to visualize certain operations on counting functions and allow us to decide whether a given counting function is bounded, i.e. represents the trivial element in $\widehat{\mathcal{C}}(M_n)$ or $\widehat{\mathcal{C}}(F_n)$. We start by discussing the case of monoids.

Denote by $T(M_n)$, or T_n for short, the right-Cayley tree of M_n with respect to S , i.e. the vertex set $V(T_n)$ of T_n is given by $V(T_n) = M_n$ and $w \in M_n$ is connected by an edge to wa_j for each $j = 1, \dots, n$. We think of T_n as a coloured rooted tree with root e , where edges are coloured by the generating set S . We define the *depth* of a vertex w as the word length of w or, equivalently, the distance of the vertex from the root. By a *weight* on T_n we mean a finitely supported real-valued function $\alpha: V(T_n) \rightarrow \mathbb{R}$. We can visualize the pair (T_n, α) by drawing the finite subtree of T_n spanned by the union of the support of α together with the root and labelling every vertex w by $\alpha(w)$. The *depth* of the weighted tree (T_n, α) is defined as $-\infty$ if $\alpha = 0$, and as

$$L(T_n, \alpha) := \max\{|w| \mid \alpha(w) \neq 0\} \tag{4.1}$$

otherwise. The following picture shows an example of a weighted tree of depth 2 for $n = 3$:



Given such a weighted tree (T_n, α) we define the associated counting function by

$$c(T_n, \alpha) := \sum_{w \in M_n} \alpha(w) \cdot \rho_w \in \mathcal{C}(M_n).$$

We then say that the weighted trees (T_n, α) and (T_n, α') are *equivalent*, denoted $(T_n, \alpha) \sim (T_n, \alpha')$, if $[c(T_n, \alpha)] = [c(T_n, \alpha')] \in \hat{\mathcal{C}}(M_n)$. Our goal is to understand geometrically what it means for two weighted trees to be equivalent. Since there is an obvious geometric way to subtract weighted trees, it suffices to understand geometrically whether $[c(T_n, \alpha)] = [0]$ for a given weighted tree (T_n, α) . To this end it is convenient to define the *depth* of an equivalence class $[T]$ of weighted trees by

$$L([(T_n, \alpha)]) := \min\{L(T_n, \alpha') \mid (T_n, \alpha') \sim (T_n, \alpha)\}.$$

We then say that a weighted tree (T_n, α) is *minimal* if

$$L((T_n, \alpha)) = L([(T_n, \alpha)]),$$

i.e. if it is not equivalent to a shorter tree. Note that $[c(T_n, \alpha)] = [0]$ if and only if $L([(T_n, \alpha)]) = -\infty$. In particular, if (T_n, α) is equivalent to a minimal tree (T_n, α') with $\alpha' \neq 0$, then it is not equivalent to the trivial weighted tree.

Everything we said so far carries over verbatim to the case of a free group F_n , if we replace the tree $T(M_n)$ by the right-Cayley tree $T_n := T(F_n)$ of F_n with respect to the generating set S . As in the monoid case, we also would like to understand in the group case the geometric meaning of the condition $[c(T_n, \alpha)] = [0]$.

4.2. Operations on weighted trees. Throughout this subsection let T_n be either $T(M_n)$ or $T(F_n)$. We describe some operations which transform a weighted tree (T_n, α) into an equivalent weighted tree (T_n, α') .

Let $V(T_n)$ be the vertex set of T_n . Given $w \in V(T_n)$ of depth ≥ 1 , we refer to the vertices on the geodesic between e in w (including e , but excluding w) as the *ancestors* of w . The unique ancestor $v = \text{Fa}(w)$ of w adjacent to w is called its *father* and the vertices with the same father as w are called its *brothers*. Their collection is called the *brotherhood* of w and denoted by $\text{Br}(w)$. We say that a brotherhood B is a *constant brotherhood* with respect to α if $\alpha|_B$ is constant, and

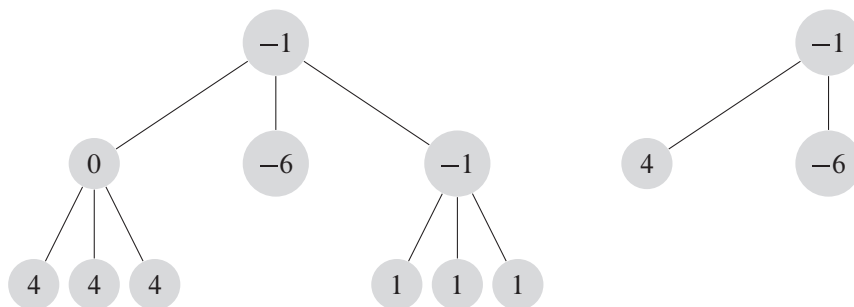
a *non-constant brotherhood* otherwise. We also define the *depth of a brotherhood* B as the depth of any of its members and denote by $\text{Fa}(B)$ the common father of the brotherhood. Note that by definition the depth of a brotherhood is ≥ 1 .

If two vertices $u, v \in M_n$ have the same depth $|u| = |v| = L \geq 1$ and differ only by the first letter, then we say that they are *related* and write $u \smile v$. In this case we also say that the brotherhood $B_1 := \text{Br}(u)$ and $B_2 := \text{Br}(v)$ are related and write $B_1 \smile B_2$. If $B_1 \smile B_2$, then there is a unique bijection $\iota_{B_1, B_2}: B_1 \rightarrow B_2$ with the property that $\iota_{B_1, B_2}(w) \smile w$.

In the monoid case, each brotherhood has exactly n elements, and every brotherhood of depth ≥ 2 has exactly n related brotherhoods including itself. In the group case, every brotherhood of depth ≥ 2 has $2n - 1$ elements, and there is a unique exceptional brotherhood of depth 1 containing $2n$ elements. In this case, every brotherhood of depth ≥ 2 has $2n - 1$ related brotherhoods including itself. We now introduce the following two types of operations.

Firstly, let B be a brotherhood of depth ≥ 1 with father w . In the monoid case, let $s \in S$, and in the group case let $s \in \bar{S} \setminus \{w_{\text{fin}}^{-1}\}$, where w_{fin} denotes the last letter of w . Then the *partial reduction* of B along s is the operation $(T_n, \alpha) \mapsto \text{Red}_{B,s}(T_n, \alpha) := (T_n, \alpha')$, where $\alpha' \in \mathbb{R}[M_n]$ is given as follows. Let $v_0 \in B$ be the unique element with final letter s . Then $\alpha'(w) := \alpha(w) + \alpha(v_0)$, $\alpha'(v) := \alpha(v) - \alpha(v_0)$ for all $v \in B$ and $\alpha'(v) := \alpha(v)$ in all other cases. Then, by the right extension relations r_w , the operation $\text{Red}_{B,s}$, transforms every weighted tree into an equivalent weighted tree. Note that α' differs from α only along the brotherhood B and its father.

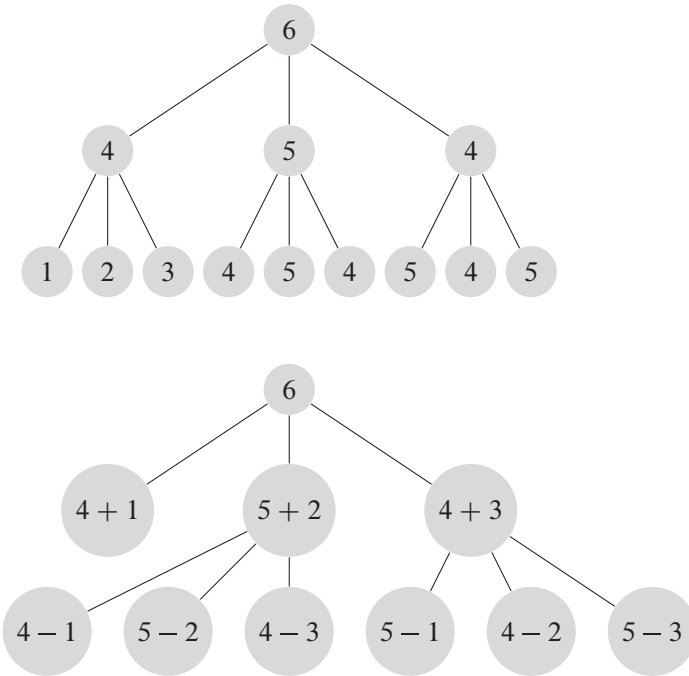
A special case appears if B is a constant family with respect to α . In this case all the partial reductions Red_{B,a_j} have the same effect on (T_n, α) , and we have $\text{Red}_{B,a_j} \alpha|_B \equiv 0$. In this case we refer to $\text{Red}_{B,a_j}(T_n, \alpha)$ as the *pruning* of (T_n, α) along B . The following pictures show an effect of two subsequent prunings:



We now define a second operation called *transfer* which corresponds to the left-extension relation. Since we chose to work with *right-Cayley graphs*, the geometric meaning of this operation is less natural.

To define transfer, let B be a brotherhood of depth $L \geq 2$. Then the *transfer of B* is the operation $(T_n, \alpha) \mapsto \text{Tr}_B(T_n, \alpha) := (T_n, \alpha')$, where α' is given as follows: If $v \in B$ and v' is obtained from v by deleting the first letter, then $\alpha'(v) = 0$ and $\alpha'(v') = \alpha(v') + \alpha(v)$. Moreover, if w is contained in a brotherhood B' related to B , then $\alpha'(w) := \alpha(w) - \alpha(\iota_{B',B}(w))$. Finally, $\alpha'(w) := \alpha(w)$ for all other vertices w .

The following picture shows the effect of transfer applied to the brotherhood labelled 1, 2, 3.



By definition, transfer has the following properties: Tr_B maps every weighted tree to an equivalent one. It changes the values of α only on (certain) elements of depth $L - 1$ and on those elements of depth L which are related to a member of B . Moreover, $\text{Tr}_B \alpha|_B \equiv 0$. We emphasize that we will only apply transfer to brotherhoods of depth at least 2.

4.3. Unbalanced weighted trees. We now define a special class of weighted trees; as in the last subsection T_n denotes either $T(M_n)$ or $T(F_n)$.

Definition 4.1. A weighted tree (T_n, α) of depth $L \geq 2$ is called *unbalanced* if there exist two related brotherhoods $B_1 \sim B_2$ of depth L such that $\alpha|_{B_1} \equiv 0$ and B_2 is non-constant with respect to α .

It is easy to see from the picture, whether a given weighted tree is unbalanced. For example, the weighted tree given in Subsection 4.1 is unbalanced. Note that in order to decide whether a given weighted tree of depth $L \geq 2$ is unbalanced, we only need to inspect its bottom level (i.e. the vertices of depth L).

Theorem 4.2. *Every unbalanced weighted tree is minimal, hence represents a non-trivial element in $\widehat{\mathcal{C}}(M_n)$ or $\widehat{\mathcal{C}}(F_n)$.*

To prove Theorem 4.2, it actually suffices to show the (a priori weaker) statement, that every unbalanced weighted tree represents a non-trivial element in $\widehat{\mathcal{C}}(M_n)$ or $\widehat{\mathcal{C}}(F_n)$. Indeed, assume this statement and let (T_n, α) be unbalanced. If (T_n, α) was equivalent to a weighted tree (T_n, α') of smaller depth, then $(T_n, \alpha - \alpha')$ would represent 0 in $\widehat{\mathcal{C}}(M_n)$ or $\widehat{\mathcal{C}}(F_n)$. But $(T_n, \alpha - \alpha')$ has the same bottom level as (T_n, α) , hence is also unbalanced by the previous remark. We thus obtain a contradiction, and it remains to prove only the weaker statement that unbalanced weighted trees represent non-trivial elements. We discuss the proof of this Theorem separately in the monoid case and in the group case.

Proof of Theorem 4.2 in the monoid case. Using the operations defined in Subsection 4.2 we will transform (T_n, α) into an equivalent weighted tree, which is non-trivial for obvious reasons. Let $B_1 \smile B_2$ be brotherhoods of depth $L := \text{depth}(T_n, \alpha)$ such that $\alpha|_{B_1} \equiv 0$ and B_2 is non-constant with respect to α , and let a_i , respectively a_j be the first letters of $\text{Fa}(B_1)$ and $\text{Fa}(B_2)$.

Firstly, we transfer all brotherhoods of depth L in the subtree $a_i M_n$ except for the brotherhood B_1 . The weights of B_2 remain the same, because only the transfer of B_1 could affect B_2 , but B_1 was not transferred. After these transfers, all coefficients in the level L in $a_i M_n$ are equal to 0. Secondly, we perform a partial reduction of all brotherhoods in the level L with respect to the ending a_i . Since B_2 was non-constant, it remains non-constant under these partial reductions. Since all the brotherhoods in $a_i M_n$ had coefficients 0, they also remain 0.

Now we repeat the same procedure in the levels $l = L - 1, L - 2, \dots, 2$. Namely, first we transfer all brotherhoods having depth l from the subtree $a_i M_n$. This affects values in levels $\leq l$ and makes all coefficients in the subtree $a_i M_n$ in the levels l, \dots, L equal to 0. Secondly, we apply partial reduction with respect to the letter a_i in all brotherhoods (except those of the subtree $a_i M_n$) of level l . This affects level $l - 1$ and makes all coefficients in level l of words ending with a_i equal to 0. The brotherhood B_2 remains non-constant throughout.

Finally we reach a weighted tree (T_n, α') equivalent to (T_n, α) with the following properties. If w is any word of depth ≥ 2 which starts or ends in a_i , then $\alpha'(w) = 0$. Moreover, B_2 is non-constant with respect to α' . We now do one final reduction of the brotherhood $\text{Br}(a_1)$ with respect to a_i to obtain yet another equivalent weighted tree (T_n, α'') . Now α'' vanishes on all words w starting or ending in a_i , but is not equal to 0 (since B_2 is non-constant). It then follows from Corollary 3.3 that $[\alpha] = [\alpha''] \neq 0$. \square

The strategy of the proof can be described as clearing out all coefficients of vertices starting or ending in a_i . This strategy works because of Corollary 3.3. In the group case we have to replace Corollary 3.3 by Corollary 3.6. We therefore have to clear out all coefficients of vertices starting in a_1 or $a_2a_1^{-1}$ or ending in a_1^{-1} or $a_1a_2^{-1}$, except for a_1^{-1} . This is slightly more complicated than in the monoid case, but ultimately works the same way.

Proof of Theorem 4.2 in the group case. By assumption we have two brotherhoods $B_1 \smile B_2$ of depth $L := \text{depth}(T_n, \alpha)$ such that $\alpha|_{B_1} \equiv 0$ and B_2 is non-constant with respect to α . Without loss of generality we may assume that the first letters of $\text{Fa}(B_1)$ and $\text{Fa}(B_2)$ respectively are a_1 and a_2 . Note that the second letter of $\text{Fa}(B_1)$, and hence also of $\text{Fa}(B_2)$ cannot be a_1^{-1} .

Let α' be obtained from α by applying the following operations. Firstly, apply transfer to all brotherhoods of depth L in the subtree of reduced words starting from a_1 . Secondly, apply transfer to all brotherhoods of depth L in the subtree of reduced words starting from $a_2a_1^{-1}$. Finally, apply a partial reduction $\text{Red}_{B,s}$ for every non-zero brotherhood of depth L , where s depends on the last letter of $\text{Fa}(B)$. If this last letter happens to be a_1 , then we choose $s := a_2^{-1}$, otherwise we choose $s := a_1^{-1}$.

We now consider the values of α' on words w of depth L . Assume first that w starts with a_1 . Then $\alpha'(w) = 0$ after the first transfer step. The second transfer step only transfers into words whose second letter is a_1^{-1} , hence does not change $\alpha'(w)$. Since also the partial reductions do not influence the family of w , we get $\alpha'(w) = 0$. Similarly, if w starts with $a_2a_1^{-1}$, then $\alpha'(w) = 0$. Finally, if w ends with a_1^{-1} or with $a_1a_2^{-1}$, then $\alpha'(w) = 0$, since w gets cleared in the partial reduction steps. On the other hand, we claim that the brotherhood B_2 does not get cleared completely and in fact remains non-constant for α' . The unique brotherhood with initial letter a_1 related to B_2 is B_1 , and this one does not transfer anything over in the first step, since $\alpha|_{B_1} \equiv 0$. Since the second letter of $\text{Fa}(B_2)$ is not a_1^{-1} , the values of α' on B_2 also do not change in the second transfer step. In the partial reduction step, the value of α' on B_2 is changed, however, a partial reduction cannot turn a non-constant family into a non-constant family.

We can now repeat the same procedure on levels $l = L - 1, L - 2, \dots, 2$. Ultimately we end up with a weighted tree (T_n, α') equivalent to (T_n, α) with the following properties. If w is any word of depth ≥ 2 which starts with a_1 or $a_2a_1^{-1}$ or end with a_1^{-1} or $a_1a_2^{-1}$, then $\alpha'(w) = 0$. Moreover, there exists $w_0 \in B_2$ with $\alpha(w_0) \neq 0$.

Now α' is equivalent to α'' as given by $\alpha''(s) := \alpha'(s) - \alpha'(a_1)$ for all $s \in \bar{S}$, $\alpha''(e) := \alpha'(e) + \alpha'(a_1)$ and $\alpha''(w) = \alpha'(w)$ for all words w of length ≥ 2 . Moreover, if W_L is defined as in Corollary 3.6, then $\alpha''(w) = 0$ for all $w \notin \bigcup W_L$ and $\alpha''(w_0) \neq 0$. It then follows from Corollary 3.6 that α'' and hence α does not represent the 0 function. □

5. Deciding boundedness for counting functions

We now present a pen-and-paper algorithm⁷ to decide whether a given counting function is bounded (hence represents the trivial class), which is based on Theorem 4.2. The basic strategy is as follows. Represent the given function by a weighted tree, and try to transform this weighted tree either into the empty weighted tree or an unbalanced weighted tree using the operations discussed in Subsection 4.2. If you reach the empty tree, then the initial function was bounded, and if you reach an unbalanced weighted tree, then the function was unbounded. To obtain an actual algorithm, we have to ensure that we either reach an unbalanced tree or the empty tree within a finite number of operations.

In the monoid case, the algorithm looks as follows:

ALGORITHM DECIDE TRIVIALITY IN $\widehat{\mathcal{C}}(M_n)$

INPUT: Weighted tree (T_n, α) .

OUTPUT: *Trivial* or *Non-trivial* according to whether $[c(T_n, \alpha)] = [0]$ or not.

STEP 1. Let $\alpha' := \alpha$, and let l be the depth of α' .

STEP 2. While $l \geq 2$ repeat the following steps to the tree α' :

- (a) Transfer all brotherhoods which start with a_1 and have depth l .
- (b) Reduce all constant brotherhoods of depth l .
- (c) If the depth of α' is still l , return *non-trivial* and stop the algorithm. Otherwise, replace l by the new length of α' .

STEP 3. If $\alpha'(a_i) = -\alpha'(e)$ for each $i = 1, \dots, n$, then return *trivial*, otherwise return *non-trivial*.

Since l decreases by at least one in each iteration of Step 2, the algorithm terminates. Let us verify correctness of the algorithm. Since α' is equivalent to α at all stages, if the algorithm returns *trivial*, then indeed $[c(T_n, \alpha)] = [0]$. Conversely, if the algorithm returns *non-trivial* in Step 2(c), then we have reached an unbalanced tree, so indeed $[c(T_n, \alpha)] \neq [0]$ by Theorem 4.2. (The theorem applies, since every brotherhood is related to a brotherhood with initial letter a_1 .) Also, if the algorithm returns *non-trivial* in Step 3, then it follows from Corollary 3.2 applied to $L = 1$ that $[\alpha] \neq 0$. Thus the algorithm works correctly.

Almost the same algorithm works in the group case. The main difference appears in Step 2(a), where we have also to clear elements starting with $a_2 a_1^{-1}$. This is because not every brotherhood is related to a brotherhood starting from a_1 ,

⁷By this we mean that we do not discuss in detail how to implement the algorithm on a computer. For a detailed discussion of a possible computer-implementation of our algorithm we refer the reader to the companion paper [15].

but every brotherhood is related to a brotherhood starting from either a_1 (if its second letter is not a_1^{-1}) or $a_2a_1^{-1}$ (otherwise). Also, in Step 3 we have to take the inverses of the generators into account.

This leads to the following algorithm:

ALGORITHM DECIDE TRIVIALITY IN $\widehat{\mathcal{C}}(F_n)$

INPUT: Weighted tree (T_n, α) .

OUTPUT: *Trivial* or *Non-trivial* according to whether $[c(T_n, \alpha)] = [0]$ or not.

STEP 1. Let $\alpha' := \alpha$, and let l be the depth of α' .

STEP 2. While $l \geq 2$ repeat the following steps to the tree α' :

- (a) Transfer all brotherhoods which start with a_1 or $a_2a_1^{-1}$ and have depth l .
- (b) Reduce all constant brotherhoods of depth l .
- (c) If the depth of α' is still l , return *non-trivial* and stop the algorithm. Otherwise, replace l by the new length of α' .

STEP 3. If $\alpha'(a_i) = \alpha'(a_i^{-1}) = -\alpha'(e)$ for each $i = 1, \dots, n$, then return *trivial*, otherwise return *non-trivial*.

The proof for termination and correctness is as in the monoid case.

As mentioned earlier, the algorithms above are merely “pen-and-paper” algorithms, and not formal algorithms in the sense of computer science, since we do not discuss the implementation details (such as data storage and implementation of arithmetic operations). However, it is possible to formalize these algorithms, and once a formalization is given, one can discuss their runtime. Since this requires some considerations in complexity theory which are very different in nature from the arguments in the present article, we defer these discussions to a separate article [15].

Also it turns out that the algorithms arising from formalizing the above pen-and-paper algorithms are not quite optimal as far as their runtime is concerned. The main problem is that in the transfer step Step 2(a), big coefficients in the transferred brotherhood may generate big coefficients in related brotherhoods. For this reason, the algorithms discussed in [15] are based on a modification of the ideas discussed in this subsection. Instead of always applying transfer to the brotherhoods starting with a_1 (or $a_2a_1^{-1}$ in the group case), we will apply transfer to some family of brotherhoods which is carefully chosen to minimize the amount of newly created data. The precise way to choose these brotherhoods is more complicated (especially for non-integer coefficients) and based on a mixed optimization strategy; we refer the reader to [15] for details.

Appendix A. Homogenizations of counting functions

In this appendix we discuss certain classes of homogeneous functions related to counting functions.

Definition A.1. Let M be a monoid. A function $f: M \rightarrow \mathbb{R}$ is called *homogeneous* if $f(g^n) = n \cdot f(g)$ for all $g \in M$ and $n \geq 0$. It is called *homogenizable* if for every $g \in M$ the limit

$$\hat{f}(g) = \lim_{n \rightarrow \infty} \frac{f(g^n)}{n}$$

exists. In this case, $\hat{f}: M \rightarrow \mathbb{R}$ is called the *homogenization* of f .

By basis properties of limits, the homogenizable functions form a real vector space, and homogenization defines a linear endomorphism of this vector space, whose image is given by the subspace of homogeneous functions. Moreover, if two homogenizable functions are at bounded distance, then their homogenizations coincide.

It is well-known that quasimorphisms are homogenizable (see e.g. [6]). Moreover, if f is a quasimorphism, then its homogenization \hat{f} can be characterized as the unique homogeneous function at bounded distance from f . Moreover, two quasimorphisms are at bounded distance if and only if their homogenizations coincide.

In this appendix we will show (following closely an argument from [11] for counting quasimorphisms) that counting functions on free monoids and groups are also homogenizable. However, in the group case it is *not* true that a counting function is at bounded distance from its homogenization. Consequently, some standard arguments from the theory of homogeneous quasimorphisms do not carry over to the setting of counting functions. This caveat is the reason why we work out a couple of otherwise standard arguments in detail.

From now on let $S = \{a_1, \dots, a_n\}$ be a set of cardinality n . We denote by M_n and F_n respectively the free monoid and free group with basis S . In analogy with the subword relation discussed in the introduction we can also introduce a cyclic subword relation as follows. Informally, if $v, w \in M_n$ we say that v is a *cyclic subword* of w if v can be read off by running along (possibly several times) the cyclic word obtained by closing up w (i.e. writing w along a circle). Thus e.g. a_1a_2 is a cyclic subword of $a_2a_1a_3a_1$, but also a_1^3 is a cyclic subword of a_1^2 . To define this more formally, we introduce the following notation.

Given positive integers l, m we denote by $[l]_m$ the unique number in $\{1, \dots, m\}$ which is congruent to l modulo m . Then $v = s_1 \cdots s_l \in M_n$ is a cyclic subword of $w = r_1 \cdots r_m \in M_n$ if there exists $j \in \{1, \dots, m\}$ such that

$$s_i = r_{[j+i]_m} \quad \text{for all } i = 1, \dots, l. \tag{A.1}$$

Given $v \in M_n$ we define the *elementary cyclic counting function* $\widehat{\rho}_v: M_n \rightarrow \mathbb{Z}$ as follows: $\widehat{\rho}_v(w)$ counts the cyclic occurrences of v in w , i.e. if $v = s_1 \cdots s_l$ and $w = r_1 \cdots r_m \in M_n$, then $\widehat{\rho}_v(w)$ is the number of $j \in \{1, \dots, m\}$ such that (A.1) holds.

Lemma A.2. *The elementary counting function $\rho_v: M_n \rightarrow \mathbb{Z}$ is homogenizable, and its homogenization is given by the elementary cyclic counting function $\widehat{\rho}_v: M_n \rightarrow \mathbb{Z}$.*

Proof. The elementary cyclic counting function is obviously homogeneous. Moreover, if (A.1) holds for some $j \in \{1, \dots, m\}$ and $j \leq m - l$, then (1.1) holds for the same j . It follows that $\|\widehat{\rho}_v - \rho_v\|_\infty \leq |v|_S$. Thus for all $w \in M_n$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\rho_v(w^n)}{n} &= \lim_{n \rightarrow \infty} \left(\frac{\widehat{\rho}_v(w^n)}{n} + \frac{\rho_v(w^n) - \widehat{\rho}_v(w^n)}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{\widehat{\rho}_v(w^n)}{n} = \lim_{n \rightarrow \infty} \frac{n \cdot \widehat{\rho}_v(w)}{n} \\ &= \widehat{\rho}_v(w). \end{aligned} \quad \square$$

Note that in the monoid case we have $\|\widehat{\rho}_v - \rho_v\|_\infty < \infty$. We will see in Example A.4 that the corresponding statement fails in the group case. The reason for this failure is given by cyclic cancellations, as we explain next.

Recall that a reduced word $w \in F_n$ is called *cyclically reduced* if its initial letter is not the inverse of its final letter. In this case we can close up w and obtain a reduced cyclic word. Every reduced word $w \in F_n$ is conjugate to a cyclically reduced (and reduced) word w_0 (sometimes called the *cyclic reduction* of w), which is unique up to cyclic permutation. In particular, the cyclic word obtained by closing w_0 depends only on w . Given a reduced word $v \in F_n$ we define an *elementary cyclic counting function* $\widehat{\rho}_v: F_n \rightarrow \mathbb{Z}$ as follows. Given a reduced word $w \in F_n$, let w_0 be its cyclic reduction. Then $\widehat{\rho}_v(w)$ counts the cyclic occurrences of v in the reduced cyclic word obtained by closing w_0 . With this definition understood we have:

Lemma A.3. *The elementary counting function $\rho_v: F_n \rightarrow \mathbb{Z}$ is homogenizable, and its homogenization is given by the elementary cyclic counting function $\widehat{\rho}_v: F_n \rightarrow \mathbb{Z}$.*

Proof. We observe first that if $w = xw_0x^{-1}$ as above, then

$$\lim_{n \rightarrow \infty} \frac{\rho_v(w^n)}{n} = \lim_{n \rightarrow \infty} \frac{\rho_v(xw_0^n x^{-1})}{n} = \lim_{n \rightarrow \infty} \frac{\rho_v(w_0^n)}{n},$$

because $|\rho_v(xw_0^n x^{-1}) - \rho_v(w_0^n)| < 2|x|_S$. It thus suffices to show that the homogenization of ρ_v coincides with $\widehat{\rho}_v$ on cyclically reduced words $w = w_0$. However, on such words we can argue literally as in the monoid case. \square

Example A.4. We have $\rho_{a_1}(a_1^n a_2 a_1^{-n}) = n$, whereas $\widehat{\rho}_{a_1}(a_1^n a_2 a_1^{-n}) = 0$. Thus $\|\rho_{a_1} - \widehat{\rho}_{a_1}\|_\infty = \infty$.

In the body of the text we will apply homogenization in the following form.

Corollary A.5. Let $f = \sum a_w \rho_w$ be a counting function either on the free monoid M_n or on the free group F_n . Then the following hold:

- (i) f is homogenizable;
- (ii) the homogenization of f is given in terms of elementary cyclic counting functions as

$$\hat{f} = \sum \alpha_w \widehat{\rho}_w;$$

- (iii) if f is a bounded function, then $\hat{f} = 0$.

Proof. (i) follows from Lemma A.2 and Lemma A.3 together with the fact that homogenizable functions form a vector space. (ii) follows from these lemmas together with the fact that homogenization is linear. (iii) is immediate from the definition of homogenization. □

Finally, let us relate the spaces $\widehat{\mathcal{C}}(F_n)$ and $\widehat{\mathcal{C}}(M_n)$ to cyclic counting functions. Here by a *cyclic counting function* we mean a linear combination of elementary cyclic counting functions with real coefficients. By Corollary A.5 we have well-defined linear maps

$$\iota_{M_n}: \widehat{\mathcal{C}}(M_n) \longrightarrow \text{span}\{\widehat{\rho}_w \mid w \in M_n\}, \quad \iota_{F_n}: \widehat{\mathcal{C}}(F_n) \longrightarrow \text{span}\{\widehat{\rho}_w \mid w \in F_n\},$$

which send a class $[f]$ to the homogenization \hat{f} .

Theorem A.6. The maps ι_{M_n} and ι_{F_n} are isomorphisms. In particular $\widehat{\mathcal{C}}(F_n)$ and $\widehat{\mathcal{C}}(M_n)$ can be identified with the respective vector spaces of cyclic counting function.

Some parts of Theorem A.6 are obvious. Firstly, surjectivity is immediate from Corollary A.5. In the monoid case, injectivity is also easy. Namely, given $f \in \mathcal{C}(M_n)$ we have $\|f - \hat{f}\|_\infty < \infty$. Thus if $\hat{f} = 0$, then f is bounded and thus $[f] = 0$. This shows that $\ker(\iota_{M_n})$ is trivial, and thus ι_{M_n} is indeed an isomorphism. However, in view of Example A.4 this simple argument does not work in the group case. Instead we have to use the full strength of the proof of Theorem 1.3.

Proof of Theorem A.6. Since $\widehat{\mathcal{C}}(F_n)$ is the union of the pure subspaces $\widehat{\mathcal{C}}(F_n)_L$, it suffices to show that for each fixed $L \geq 2$, the map

$$\iota_L: \widehat{\mathcal{C}}(F_n)_L \hookrightarrow \widehat{\mathcal{C}}(F_n) \xrightarrow{\iota_{F_n}} \text{span}\{\widehat{\rho}_w \mid w \in F_n\}, \quad [f] \mapsto \widehat{f}$$

is injective. Denote by $q_L: \mathbb{R}[F_n]_L \rightarrow \widehat{\mathcal{C}}(F_n)_L$ the natural surjection and define $K'_L(F_n) := \ker(\iota_L \circ q_L)$. Then we have a commuting diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_L(F_n) & \longrightarrow & \mathbb{R}[F_n]_L & \xrightarrow{q_L} & \widehat{\mathcal{C}}(F_n)_L \longrightarrow 0 \\ & & \downarrow (\iota_L)_* & & \downarrow \text{Id} & & \downarrow \iota_L \\ 0 & \longrightarrow & K'_L(F_n) & \longrightarrow & \mathbb{R}[F_n]_L & \xrightarrow{\iota_L \circ q_L} & \text{Im}(\iota_L) \longrightarrow 0. \end{array}$$

Since ι_L is onto, the induced map $(\iota_L)_*$ embeds $K_L(F_n)$ into $K'_L(F_n)$, and we have to show that this embedding is onto. We have seen in Step 2 of Subsection 2.4 that the kernel $K_L(F_n)$ can be characterized as the subset of $\mathbb{R}[F_n]$ on which certain certificates $\langle c \rangle_L$ vanish. However, by definition of these certificates, these also vanish on $K'_L(F_n)$. This yields the desired surjectivity and finishes the proof. \square

Combining Theorem A.6 with Theorem 1.5 we deduce:

Corollary A.7 (basis theorem for elementary cyclic counting functions). (i) *Denote by W the set of all words in M_n which do not start or end with a_1 (including the empty word). Then the elementary cyclic counting functions $\{\widehat{\rho}_w \mid w \in W\}$ form a basis for the space $\text{span}\{\widehat{\rho}_w \mid w \in M_n\}$ of all cyclic counting functions.*

(ii) *Denote by W' the set of all reduced words in F_n which do not start with a_1 or $a_2 a_1^{-1}$ and do not end with a_1^{-1} or $a_1 a_2^{-1}$ (including the empty word), and let $W := W' \cup \{a_1^{-1}\}$. Then the elementary cyclic counting functions $\{\widehat{\rho}_w \mid w \in W\}$ form a basis for the space $\text{span}\{\widehat{\rho}_w \mid w \in F_n\}$ of all cyclic counting functions.*

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