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A Study of Some Tense Logics by Gentzen's Sequential Method

By

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§1. Introduction

Gentzen-style formulations of several fundamental modal logics like T, $S4$, $S5$, etc. are well-known now. See, e.g., Ohnishi and Matsumoto [6], Sato [9], Zeman [10], etc. Especially, Sato has established a close relationship between Gentzen-style formulations of modal calculi and Kripke-type semantics in a decisive way.

By the way, there is a strong analogy between classical tense logics and modal logics, which is also well-known. Indeed many techniques originally developed in modal calculi have been applied fruitfully to tense logics. For example, Gabbay [1] has used the so-called Lemmon-Scott or Makinson method to establish the completeness of many tense logics.

The main objective of the present paper is to present Gentzen-style formulations of some fundamental tense logics, say, K_t and K_t4, and then to prove the completeness of these logics with due regard to Gentzen-style formulations after the manner of Sato. Since the completeness of K_t and K_t4 is well-known, our main concern here rests in the relationship between our Gentzen-style systems and the ordinal semantics of tense logics.

Roughly speaking, traditional tense logics may be regarded as modal logics with two necessity-like operators, say, G and H. However, we will see that the relationship of these two operators is much subtler than that of so-called bi-modal logics.

§2. Hilbert-type Systems

Our formal language L consists of the following symbols:

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(1) a countable set P of propositional variables: $p, q, p', \dots$
(2) classical connectives: $\neg, \supset$
(3) tense operators: G, H
(4) parentheses: (,)

The notion of a *well-formed formula* (or simply, a *wff*) is defined inductively as follows:

(1) Any propositional variable p is a wff.
(2) If α and β are wffs, so too are $(\neg\alpha)$, $(\alpha\supset\beta)$, $(G\alpha)$ and $(H\alpha)$.

In the rest of this paper our usage of parentheses is very loose, in so far as there is no danger of possible confusion.

For any wff α, we define $\mathrm{Sub}(\alpha)$, the set of all subformulas of α, inductively as follows:

(1) $\mathrm{Sub}(p)=\{p\}$
(2) $\mathrm{Sub}(\neg\alpha)=\mathrm{Sub}(\alpha)\cup\{\neg\alpha\}$
(3) $\mathrm{Sub}(\alpha\supset\beta)=\mathrm{Sub}(\alpha)\cup\mathrm{Sub}(\beta)\cup\{\alpha\supset\beta\}$
(4) $\mathrm{Sub}(G\alpha)=\mathrm{Sub}(\alpha)\cup\{G\alpha\}$
(5) $\mathrm{Sub}(H\alpha)=\mathrm{Sub}(\alpha)\cup\{H\alpha\}$

Now we review the traditional tense logics K_t and K_t4. We begin with the definition of K_t.

Axioms:

(A1) $\alpha\supset(\beta\supset\alpha)$
(A2) $(\alpha\supset(\beta\supset\gamma))\supset((\alpha\supset\beta)\supset(\alpha\supset\gamma))$
(A3) $(\neg\beta\supset\neg\alpha)\supset(\alpha\supset\beta)$
(G1) $G(\alpha\supset\beta)\supset(G\alpha\supset G\beta)$
(H1) $H(\alpha\supset\beta)\supset(H\alpha\supset H\beta)$
(G2) $\neg H\neg G\alpha\supset\alpha$
(H2) $\neg G\neg H\alpha\supset\alpha$

Rules:

(MP) $\dfrac{\vdash\alpha \quad \vdash\alpha\supset\beta}{\vdash\beta}$

(RG) $\dfrac{\vdash\alpha}{\vdash G\alpha}$

(RH) $\dfrac{\vdash\alpha}{\vdash \mathrm{H}\alpha}$

Now K_t4 is defined to be the system obtained from K_t by adding the following axioms.

(G3) $G\alpha \supset GG\alpha$

(H3) $H\alpha \supset HH\alpha$

For more information on traditional tense logics in Hilbert style, see, e.g., Gabbay [1, 2], Prior [7] and Rescher and Urquhart [8].

§3. Gentzen-type Systems I

We now define Gentzen-type systems GK_t and GK_t4, which are equivalent to K_t and K_t4 respectively. We denote the set of all wffs by WFF. Following Sato [9], we define a *sequent* as an element in the set $2^{\mathrm{WFF}} \times 2^{\mathrm{WFF}}$. Namely, it is a pair of (possibly infinite) sets of wffs. In order to match with Gentzen's original notation, we will denote a sequent $\Gamma \to \Delta$ rather than (Γ, Δ). Some other notational conventions of Sato, which are almost self-explanatory, are adopted here. For example, $\Gamma \to \Delta, \Pi$ stands for $\Gamma \to \Delta \cup \Pi$.

We will also use the following notation:

(1) $\Gamma_0 \to \Delta_0 \subseteq \Gamma \to \Delta$ iff $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$.

(2) $\Gamma_0 \Subset \Gamma$ iff $\Gamma_0 \subseteq \Gamma$ and Γ_0 is finite.

(3) $\Gamma_0 \to \Delta_0 \Subset \Gamma \to \Delta$ iff $\Gamma_0 \Subset \Gamma$ and $\Delta_0 \Subset \Delta$.

We now give the definition of the system GK_t.

Axioms: $\alpha \to \alpha$

Rules:

$$\frac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Sigma} \ \text{(extension)}$$

$$\frac{\Gamma \to \Delta, \alpha \quad \alpha, \Pi \to \Sigma}{\Gamma, \Pi \to \Delta, \Sigma} \ \text{(cut)}$$

$$\frac{\Gamma \to \Delta, \alpha}{\neg\alpha, \Gamma \to \Delta} \ (\neg \to)$$

$$\frac{\alpha, \Gamma \to \Delta}{\Gamma \to \Delta, \neg\alpha} \ (\to \neg)$$

$$\frac{\Gamma \to \Delta, \alpha \quad \beta, \Pi \to \Sigma}{\alpha \supset \beta, \Gamma, \Pi \to \Delta, \Sigma} \ (\supset \to)$$

$$\frac{\alpha, \Gamma \to \Delta, \beta}{\Gamma \to \Delta, \alpha \supset \beta} \ (\to \supset)$$

$$\frac{\Gamma \to \alpha, H\Delta}{G\Gamma \to G\alpha, \Delta} \ (\to G)$$

$$\frac{\Gamma \to \alpha, G\Delta}{H\Gamma \to H\alpha, \Delta} \ (\to H)$$

In the above rules $G\Gamma=\{G\alpha \mid \alpha\in\Gamma\}$ and $H\Gamma=\{H\alpha \mid \alpha\in\Gamma\}$ for any $\Gamma\subseteq\mathrm{WFF}$.

Now GK_t4 is obtained from GK_t by replacing the rules $(\rightarrow G)$ and $(\rightarrow H)$ by the following rules respectively.

$$\frac{G\Gamma,\ \Gamma\rightarrow\alpha,\ H\Delta,\ H\Sigma}{G\Gamma\rightarrow G\alpha,\ \Delta, H\Sigma}\ (\rightarrow G)_4$$

$$\frac{H\Gamma,\ \Gamma\rightarrow\alpha,\ G\Delta,\ G\Sigma}{H\Gamma\rightarrow H\alpha,\ \Delta,\ G\Sigma}\ (\rightarrow H)_4$$

It is important to notice that $(\rightarrow G)$ and $(\rightarrow H)$ are admissible rules in GK_t4.

We call a sequent $\Gamma\rightarrow\Delta$ *finite* if both Γ and Δ are finite. Then it is easy to prove the following lemma.

Lemma 3.1. *If a finite sequent $\Gamma\rightarrow\Delta$ is provable in GK_t (in GK_t4, resp.), then each sequent occurring in any proof of $\Gamma\rightarrow\Delta$ is finite.*

Theorem 3.2. *If $\vdash\Gamma\rightarrow\Delta$ in GK_t (in GK_t4, resp.), then there exist some $\Gamma_0\rightarrow\Delta_0\Subset\Gamma\rightarrow\Delta$ such that $\vdash\Gamma_0\rightarrow\Delta_0$ in GK_t (in GK_t4, resp.).*

Proof. By induction on the number n of sequents occurring in the proof of $\Gamma\rightarrow\Delta$.

It is easy to see the equivalence of K_t and GK_t (K_t4 and GK_t4, resp.).

Theorem 3.3. *For any wff α, $\vdash\alpha$ in K_t (in K_t4, resp.) iff $\vdash\rightarrow\alpha$ in GK_t (in GK_t4, resp.).*

Corollary 3.4. *Let $\Gamma\subseteq\mathrm{WFF}$ and $\alpha\in\mathrm{WFF}$. Then $\Gamma\vdash\alpha$ in K_t (in K_t4, resp.) iff $\vdash\Gamma\rightarrow\alpha$ in GK_t (in GK_t4, resp.).*

The following example shows that our sequential systems GK_t and GK_t4 are not cut-free.

$$\dfrac{(\rightarrow\neg)\,\dfrac{(\neg\rightarrow)\,\dfrac{p\rightarrow p}{\neg p,\ p\rightarrow}}{p\rightarrow\neg\neg p} \qquad \dfrac{\dfrac{\dfrac{H\neg p\rightarrow H\neg p}{\rightarrow\neg H\neg p,\ H\neg p}\,(\rightarrow\neg)}{\rightarrow G\neg H\neg p,\ \neg p}\,(\rightarrow G)}{\neg\neg p\rightarrow G\neg H\neg p}\,(\neg\rightarrow)}{p\rightarrow G\neg H\neg p}\ (\mathrm{cut})$$

Thus we conclude this section by the following theorem.

Theorem 3.5. *The cut-elimination theorem does fail for GK_t and GK_t4.*

§4. Completeness

First of all, we review the semantics for tense logic. By a T-*structure*, we mean a triple (S, R, D), where

(1) S is a set (called the "time").
(2) R is a binary relation on S (the earlier-later relation).
(3) D is a function from $P \times S$ to $\{0, 1\}$. That is, D assigns a truth-value to each propositional variable at each moment (an element of S is called a moment).

Given a T-structure (S, R, D), the truth-value $V(\alpha: t)$ of a wff α at a moment t is defined inductively as follows:

(V1) $V(p: t) = D(p, t)$ for any propositional variable p.
(V2) $V(\neg\alpha: t) = 1$ iff $V(\alpha: t) = 0$.
(V3) $V(\alpha \supset \beta: t) = 1$ iff $V(\alpha: t) = 0$ or $V(\beta: t) = 1$.
(V4) $V(G\alpha: t) = 1$ iff for any $s \in S$ such that tRs, $V(\alpha: s) = 1$.
(V5) $V(H\alpha: t) = 1$ iff for any $s \in S$ such that sRt, $V(\alpha: s) = 1$.

We also define $V(\Gamma \to \Delta: t)$, where $\Gamma \to \Delta$ is a sequent, as follows:

(V6) $V(\Gamma \to \Delta: t) = 1$ iff $V(\alpha: t) = 1$ for any $\alpha \in \Gamma$ and $V(\beta: t) = 0$ for any $\beta \in \Delta$.

By a T-*model*, we mean a 4-tuple (S, R, D, o), where

(1) (S, R, D) is a T-structure.
(2) o is an element of S (called the "present moment").

A T-structure (S, R, D) (a T-model (S, R, D, o), resp.) is called a T4-*structure* (T4-*model*, resp.) if R is a transitive relation.

We say that:

(1) A T-model (S, R, D, o) *realizes* a sequent $\Gamma \to \Delta$ if $V(\Gamma \to \Delta: o) = 1$.
(2) A sequent $\Gamma \to \Delta$ is T-*realizable* (T4-*realizable*, resp.) if $\Gamma \to \Delta$ can be realized by some T-model (T4-model, resp.).
(3) A sequent $\Gamma \to \Delta$ is T-*valid* (T4-*valid*, resp.) if it is not T-realizable (T4-realizable, resp.).

We say that:

(1) A sequent $\Gamma \to \Delta$ is G-*provable* (G4-*provable*, resp.) if it is provable in GK_t (in GK_t4, resp.).
(2) A sequent $\Gamma \to \Delta$ is G-*consistent* (G4-*consistent*, resp.) if it is not G-provable (G4-provable, resp.).

With these definitional preparations, we can present the following theorem.

Theorem 4.1 (Soundness Theorem). *Any* G-*provable sequent* (G4-*provable sequent, resp.*) *is* T-*valid* (T4-*valid, resp.*).

Corollary 4.2. *If* $\vdash \alpha$ *in* K_t (*in* K_t4, *resp.*), *then* $V(\alpha : o) = 1$ *for any* T-*model* (T4-*model, resp.*) (S, R, D, o).

Proof. This is immediate from Theorem 3.3 and Theorem 4.1.

Corollary 4.3 (Consistency of GK_t and GK_t4). *The empty sequent* $\rightarrow$ *is not provable in* GK_t (*in* GK_t4, *resp.*).

We now deal with completeness theorems. It is easy to see that the following lemma holds.

Lemma 4.4 (Lindenbaum's Lemma). *Let it be that* $\not\vdash \Gamma \rightarrow \Delta$ *in* GK_t (*in* GK_t4, *resp.*) *and* Ω *is a set of wffs such that* $\Gamma \cup \Delta \subseteq \Omega$. *Then there exist* $\tilde{\Gamma}$, $\tilde{\Delta}$ *such that*:

(1) $\not\vdash \tilde{\Gamma} \rightarrow \tilde{\Delta}$ *in* GK_t (*in* GK_t4, *resp.*).

(2) $\tilde{\Gamma} \rightarrow \tilde{\Delta} \supseteq \Gamma \rightarrow \Delta$.

(3) $\tilde{\Gamma} \cup \tilde{\Delta} = \Omega$.

A set Ω of wffs is said to be *closed under subformulas* if $\mathrm{Sub}(\alpha) \subseteq \Omega$ for any $\alpha \in \Omega$. Now take any such Ω and fix it. A sequent $\Gamma \rightarrow \Delta$ is said to be Ω, G-*complete* (Ω, G4-*complete*, resp.) if $\Gamma \rightarrow \Delta$ is G-consistent (G4-consistent, resp.) and $\Gamma \cup \Delta = \Omega$. We define $C(\Omega)$ and $C_4(\Omega)$ as follows:

(1) $C(\Omega) = \{\Gamma \rightarrow \Delta \mid \Gamma \rightarrow \Delta$ is Ω, G-complete$\}$.

(2) $C_4(\Omega) = \{\Gamma \rightarrow \Delta \mid \Gamma \rightarrow \Delta$ is Ω, G4-complete$\}$.

It is easy to see that for any $\Gamma \rightarrow \Delta \in C(\Omega)$, $\Gamma \cap \Delta = \emptyset$ because $\Gamma \rightarrow \Delta$ is G-consistent. Similarly for any $\Gamma \rightarrow \Delta \in C_4(\Omega)$, $\Gamma \cap \Delta = \emptyset$. For any $\Gamma \subseteq \mathrm{WFF}$, we denote by Γ_G and Γ_H the sets $\{\alpha \mid G\alpha \in \Gamma\}$ and $\{\alpha \mid H\alpha \in \Gamma\}$ respectively. We now define the *universal* T-*structure* $U(\Omega) = (S, R, D)$ as follows:

(1) $S = C(\Omega)$.

(2) $(\Gamma \rightarrow \Delta) R (\Gamma' \rightarrow \Delta')$ iff $\Gamma_G \subseteq \Gamma'$ and $\Gamma'_H \subseteq \Gamma$.

(3) $D(p, \Gamma \rightarrow \Delta) = 1$ iff $p \in \Gamma$.

Similarly we define the *universal* T4-*structure* $U_4(\Omega) = (S', R', D')$ as follows:

(1) $S' = C_4(\Omega)$.

(2) $(\Gamma \rightarrow \Delta) R' (\Gamma' \rightarrow \Delta')$ iff $\Gamma_G \subseteq \Gamma'$, $\Gamma_G \subseteq \Gamma'_G$, $\Gamma'_H \subseteq \Gamma$ and $\Gamma'_H \subseteq \Gamma_H$.

(3) $D'(p, \Gamma \rightarrow \Delta) = 1$ iff $p \in \Gamma$.

It is easy to see that $U(\Omega)$ ($U_4(\Omega)$, resp.) is indeed a T-structure (T4-structure, resp.).

Theorem 4.5 (Fundamental Theorem of Universal Structure). *For any* $\alpha\in\Omega$ *and* $\Gamma\to\Delta\in U(\Omega)$ ($U_4(\Omega)$, *resp.*), $V(\alpha:\Gamma\to\Delta)=1$ *if* $\alpha\in\Gamma$ *and* $V(\alpha:\Gamma\to\Delta)=0$ *if* $\alpha\in\Delta$.

Proof. By induction on the construction of wffs.

(a) α is a propositional variable: Immediate from the definition of D (D', resp.).

(b) $\alpha=\neg\beta$: Suppose $\alpha\in\Gamma$. It is sufficient to show that $\nvdash\Gamma\to\Delta,\ \beta$, which implies $\beta\in\Delta$ because of the maximality of $\Gamma\to\Delta$. In this case we can conclude that $V(\alpha:\Gamma\to\Delta)=1$ by induction hypothesis. Suppose, for the sake of contradiction, that $\vdash\Gamma\to\Delta,\ \beta$. Then we can show that $\vdash\Gamma\to\Delta$ as follows:

$$\frac{\Gamma\to\Delta,\ \beta}{\neg\beta,\ \Gamma\to\Delta}\ (\neg\to)$$

This is a contradiction. The case $\alpha\in\Delta$ can be treated in a similar manner.

(c) $\alpha=\beta\supset\gamma$: Suppose $\alpha\in\Gamma$. It is sufficient to show that $\nvdash\Gamma\to\Delta,\ \beta$ or $\nvdash\gamma,\ \Gamma\to\Delta$, which implies that $\beta\in\Delta$ or $\gamma\in\Gamma$. In any case $V(\alpha:\Gamma\to\Delta)=1$ by induction hypothesis. Suppose, for the sake of contradiction, that $\vdash\Gamma\to\Delta,\ \beta$ and $\vdash\gamma,\ \Gamma\to\Delta$. Then we can show that $\vdash\Gamma\to\Delta$ as follows:

$$\frac{\Gamma\to\Delta,\ \beta\qquad\gamma,\ \Gamma\to\Delta}{\beta\supset\gamma,\ \Gamma\to\Delta}\ (\supset\to)$$

This is a contradiction. Suppose $\alpha\in\Delta$. It is sufficient to show that $\nvdash\beta,\ \Gamma\to\Delta,\ \gamma$, which implies $\beta\in\Gamma$ and $\gamma\in\Delta$, because of the maximality of $\Gamma\to\Delta$. So we can conclude $V(\alpha:\Gamma\to\Delta)=0$ by induction hypothesis. Suppose, for the sake of contradiction, that $\vdash\beta,\ \Gamma\to\Delta,\ \gamma$. Then we can show that $\vdash\Gamma\to\Delta$ as follows:

$$\frac{\beta,\ \Gamma\to\Delta,\ \gamma}{\Gamma\to\Delta,\ \beta\supset\gamma}\ (\to\supset)$$

This is a contradiction.

(d) $\alpha=G\beta$: Suppose $\alpha\in\Gamma$. That $V(\alpha:\Gamma\to\Delta)=1$ follows directly from the definition of R or R'.

Now suppose that $\alpha\in\Delta$.

For $U(\Omega)$: We show that the sequent $\Gamma_G\to\beta,\ \{H\gamma\mid H\gamma\in\Omega \text{ and } \gamma\in\Delta\}$ is G-consistent. We assume, for the sake of contradiction, that $\vdash\Gamma_G\to\beta,\ \{H\gamma\mid H\gamma\in\Omega \text{ and } \gamma\in\Delta\}$. Then we can show that $\vdash\Gamma\to\Delta$ as follows:

$$\dfrac{\dfrac{\Gamma_G \rightarrow \beta,\ \{H\gamma \mid H\gamma \in \Omega \text{ and } \gamma \in \Delta\}}{G\Gamma_G \rightarrow G\beta,\ \{\gamma \mid H\gamma \in \Omega \text{ and } \gamma \in \Delta\}}\ (\rightarrow G)}{\Gamma \rightarrow \Delta}$$

This is a contradiction. So we can conclude that the sequent $\Gamma_G \rightarrow \beta$, $\{H\gamma \mid H\gamma \in \Omega$ and $\gamma \in \Delta\}$ is G-consistent. By Lemma 6.4 this sequent can be extended to some Ω, G-complete sequent $\Gamma' \rightarrow \Delta'$. It is easy to see that $(\Gamma \rightarrow \Delta)R(\Gamma' \rightarrow \Delta')$ and $V(\beta: \Gamma' \rightarrow \Delta')=0$. Therefore $V(\alpha: \Gamma \rightarrow \Delta)=0$.

For $U_4(\Omega)$: We show that the sequent Γ_G, $G\Gamma_G \rightarrow \beta$, $\{H\gamma \mid H\gamma \in \Omega$ and $\gamma \in \Delta\}$, $H\Delta_H$ is G4-consistent. We assume, for the sake of contradiction, that this sequent is G4-provable. Then we see that $\Gamma \rightarrow \Delta$ is also G4-provable as the following proof-figure shows:

$$\dfrac{\dfrac{G\Gamma_G,\ \Gamma_G \rightarrow \beta,\ \{H\gamma \mid H\gamma \in \Omega \text{ and } \gamma \in \Delta\},\ H\Delta_H}{G\Gamma_G \rightarrow G\beta,\ \{\gamma \mid H\gamma \in \Omega \text{ and } \gamma \in \Delta\},\ H\Delta_H}\ (\rightarrow G)_4}{\Gamma \rightarrow \Delta}$$

This is a contradiction. So we can conclude that the sequent $G\Gamma_G$, $\Gamma_G \rightarrow \beta$, $\{H\gamma \mid H\gamma \in \Omega$ and $\gamma \in \Delta\}$, $H\Delta_H$ is G4-consistent. By Lemma 6.4 this sequent can be extended to some Ω, G4-complete sequent $\Gamma' \rightarrow \Delta'$. It is easy to see that $(\Gamma \rightarrow \Delta)R'(\Gamma' \rightarrow \Delta')$ and $V(\beta: \Gamma' \rightarrow \Delta')=0$. Therefore $V(\alpha: \Gamma \rightarrow \Delta)=0$.

(e) $\alpha = H\beta$: Similar to the case (d).

Several results follow directly from this theorem.

Theorem 4.6 (Generalized Completeness Theorem). *Any* G-*consistent* (G4-*consistent, resp.*) *sequent is* T-*realizable* (T4-*realizable, resp.*).

Proof. Immediate from Lemma 4.4 and Theorem 4.5.

Theorem 4.7 (Compactness Theorem). *For any sequent* $\Gamma \rightarrow \Delta$, $\Gamma \rightarrow \Delta$ *is* T-*realizable* (T4-*realizable, resp.*) *iff for any* $\Gamma_0 \rightarrow \Delta_0 \Subset \Gamma \rightarrow \Delta$ *is* T-*realizable* (T4-*realizable, resp.*).

Theorem 4.8 (Completeness and Decidability Theorem). *For any finite sequent* $\Gamma \rightarrow \Delta$, $\Gamma \rightarrow \Delta$ *is* G-*provable* (G4-*provable, resp.*) *iff* $\Gamma \rightarrow \Delta$ *holds in all* T-*models* (T4-*models, resp.*) *whose cardinality* $\leq 2^n$, *where n is the cardinality of* $\bigcup_{\alpha \in \Gamma \cup \Delta} \mathrm{Sub}(\alpha)$.

§5. Gentzen-type Systems II

In Section 3 we have introduced Gentzen-type systems GK_t and GK_t4, which was shown to be deductively equivalent to traditional tense logics K_t and K_t4 in Hilbert style respectively in Section 4. However, strictly speaking, GK_t and GK_t4 are somewhat crude since in the rules $(\to G)$, $(\to H)$, $(\to G)_4$ and $(\to H)_4$ some subformulas of the upper sequent may disappear in the lower sequent. That is, GK_t and GK_t4 can not necessarily enjoy the usual property of ordinal Gentzen-type systems that the totality of subformulas of a sequent increase as we proceed downward in a proof-figure without a cut. But this defect of GK_t and GK_t4 is rather superficial than crucial, and the main purpose of this section is to introduce more elaborated Gentzen-type systems GHK_t and GHK_t4, which are to be shown to be deductively equivalent to GK_t and GK_t4 (and so to K_t and K_t4) respectively.

We now define GHK_t. In GHK_t, a *sequent* is defined to be an element of the set $2^{\mathrm{WFF}} \times 2^{\mathrm{WFF}} \times 2^{\mathrm{WFF}} \times 2^{\mathrm{WFF}} \times 2^{\mathrm{WFF}} \times 2^{\mathrm{WFF}}$. Thus a sequent is of the form $(\Pi_1, \Gamma, \Pi_2, \Sigma_1, \Delta, \Sigma_2)$. However, we denote this as $\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; \Delta; \Sigma_2$. Moreover we denote $;\Gamma; \to\ ;\Delta;$ $(=(\emptyset, \Gamma, \emptyset, \emptyset, \Delta, \emptyset))$ simply as $\Gamma \to \Delta$. A sequent of this form will be called *proper*. Other sequents will be called *improper*.

We define GHK_t as follows:

Axioms: $\alpha \to \alpha$

$\alpha;\ ;\ \to \alpha;\ ;$

$;\ ;\alpha \to\ ;\ ;\alpha$

Rules:

$$\frac{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; \Delta; \Sigma_2}{\Pi_1', \Pi_1; \Gamma', \Gamma; \Pi_2', \Pi_2 \to \Sigma_1', \Sigma_1; \Delta', \Delta; \Sigma_2', \Sigma_2} \text{ (extension)}$$

$$\frac{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; \Delta, \alpha; \Sigma_2 \quad \Pi_1'; \alpha, \Gamma'; \Pi_2' \to \Sigma_1'; \Delta'; \Sigma_2'}{\Pi_1, \Pi_1'; \Gamma, \Gamma'; \Pi_2, \Pi_2' \to \Sigma_1, \Sigma_1'; \Delta, \Delta'; \Sigma_2, \Sigma_2'} \text{ (cut)}$$

$$\frac{\Pi_1; \Gamma; \alpha, \Pi_2 \to \Sigma_1; \Delta; \Sigma_2}{\Pi_1; \Gamma, G\alpha; \Pi_2 \to \Sigma_1; \Delta; \Sigma_2} \ (G\to : \text{out})$$

$$\frac{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; \Delta; \alpha, \Sigma_2}{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; \Delta, G\alpha; \Sigma_2} \ (\to G : \text{out})$$

$$\frac{\Pi_1, \alpha; \Gamma; \Pi_2 \to \Sigma_1; \Delta; \Sigma_2}{\Pi_1; H\alpha, \Gamma; \Pi_2 \to \Sigma_1; \Delta; \Sigma_2} \ (H\to : \text{out})$$

$$\frac{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1, \alpha; \Delta; \Sigma_2}{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; H\alpha, \Delta; \Sigma_2} \ (\to H : \text{out})$$

$$\frac{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; \Delta, \alpha; \Sigma_2}{\Pi_1; \neg\alpha, \Gamma; \Pi_2 \to \Sigma_1; \Delta; \Sigma_2} \ (\neg\to)$$

$$\frac{\Pi_1; \alpha, \Gamma; \Pi_2 \to \Sigma_1; \Delta; \Sigma_2}{\Pi_1; \Gamma; \Pi_2 \to \Sigma_1; \Delta, \neg\alpha; \Sigma_2} \ (\to\neg)$$

$$\frac{\Pi_1;\Gamma;\Pi_2\to\Sigma_1;\Delta,\alpha;\Sigma_2 \quad \Pi_1';\beta,\Gamma';\Pi_2'\to\Sigma_1';\Delta';\Sigma_2'}{\Pi_1,\Pi_1';\alpha\supset\beta,\Gamma,\Gamma';\Pi_2,\Pi_2'\to\Sigma_1,\Sigma_1';\Delta,\Delta';\Sigma_2,\Sigma_2'}\ (\supset\to)$$

$$\frac{\Pi_1;\alpha,\Gamma;\Pi_2\to\Sigma_1;\Delta,\beta;\Sigma_2}{\Pi_1;\Gamma;\Pi_2\to\Sigma_1;\Delta,\alpha\supset\beta;\Sigma_2}\ (\to\supset)$$

$$\frac{;\Gamma;\to\Delta;\alpha;}{;\,;\Gamma\to;\Delta;\alpha}\ \text{(r-trans)}$$

$$\frac{;\Gamma;\to;\alpha;\Delta}{\Gamma;\,;\to\alpha;\Delta;}\ \text{(l-trans)}$$

Now, GHK_t4 is obtained from GHK_t by replacing the rules (r-trans) and (l-trans) by the following rules respectively.

$$\frac{;\Gamma;\Gamma\to\Delta,\Sigma;\alpha;}{;\,;\Gamma\to\Sigma;\Delta;\alpha}\ \text{(r-trans)}_4$$

$$\frac{\Gamma;\Gamma;\to;\alpha;\Delta,\Sigma}{\Gamma;\,;\to\alpha;\Delta;\Sigma}\ \text{(l-trans)}_4$$

Now we should prove the equivalence of GK_t and GHK_t (GK_t4 and GHK_t4, resp.).

Theorem 5.1. *Let $\Gamma\to\Delta$ be a proper sequent. Then $\vdash\Gamma\to\Delta$ in GK_t (in GK_t4, resp.) iff $\vdash\Gamma\to\Delta$ in GHK_t (in GHK_t4, resp.).*

Proof. We deal only with the equivalence of GK_t4 and GHK_t4.
If part: By induction on the construction of the proof, we prove that if $\vdash\Pi_1;\Gamma;\Pi_2\to\Sigma_1;\Delta;\Sigma_2$ in GHK_t4, then $\vdash H\Pi_1,\Gamma,G\Pi_2\to H\Sigma_1,\Delta,G\Sigma_2$ in GK_t4. The proof is almost straight, and so is left to the reader. But we comment that (r-trans)$_4$ and (l-trans)$_4$ correspond to $(\to G)_4$ and $(\to H)_4$ respectively.
Only if part: For any $\Gamma\subseteq$ WFF, we denote the sets $\{\alpha\mid G\alpha\in\Gamma\}$ and $\{\alpha\mid H\alpha\in\Gamma\}$ by Γ_G and Γ_H respectively. We denote $\Gamma-\Gamma_G\cup\Gamma_H$ by Γ_J. By induction on the construction of the proof, we prove that $\vdash\Gamma\to\Delta$ in GK_t4, then $\vdash\Gamma_H;\Gamma_J;\Gamma_G\to\Delta_H;\Delta_J;\Delta_G$ in GHK_t4. Since the proof is almost direct, it is left to the reader.

As corollaries of this theorem,

Corollary 5.2. *$\vdash\Pi_1;\Gamma;\Pi_2\to\Sigma_1;\Delta;\Sigma_2$ in GHK_t (in GHK_t4, resp.) iff $\vdash H\Pi_1,\Gamma,G\Pi_2\to H\Sigma_1,\Delta,G\Sigma_2$ in GK_t (in GK_t4, resp.).*

Corollary 5.3. *A sequent $\Pi_1;\Gamma;\Pi_2\to\Sigma_1;\Delta;\Sigma_2$ is provable without a cut in GHK_t (in GHK_t4, resp.) iff the sequent $H\Pi_1,\Gamma,G\Pi_2\to H\Sigma_1,\Delta,G\Sigma_2$ is provable without a cut in GK_t (in GK_t4, resp.).*

Corollary 5.4. *The cut-elimination theorem does fail for GHK_t and GHK_t4.*

Proof. Immediate from Corollary 5.3 and Theorem 3.5.

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