Limiting distribution of geodesics
in a geometrically finite quotients
of regular trees

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Abstract. Let $\mathcal{T}$ be a $(q + 1)$-regular tree and let Γ be a geometrically finite discrete subgroup of the group $\mathrm{Aut}(\mathcal{T})$ of automorphisms of $\mathcal{T}$. In this article, we prove an extreme value theorem on the distribution of geodesics in a non-compact quotient graph $\Gamma \backslash \mathcal{T}$. Main examples of such graphs are quotients of a Bruhat–Tits tree by non-cocompact discrete subgroups Γ of $\mathrm{PGL}(2, \mathbf{K})$ of a local field $\mathbf{K}$ of positive characteristic.

We investigate, for a given time T, the measure of the set of Γ-equivalent classes of geodesics with distance at most $N(T)$ from a sufficiently large fixed compact subset D of $\Gamma \backslash \mathcal{T}$ up to time T. We show that there exists a function $N(T)$ such that for Bowen–Margulis measure μ on the space $\Gamma \backslash \mathcal{GT}$ of geodesics and the critical exponent δ of Γ,

$$\lim_{T \to \infty} \mu(\{[l] \in \Gamma \backslash \mathcal{GT}: \max_{0 \le t \le T} d(D, l(t)) \le N(T) + y\}) = e^{-q^y/e^{2\delta y}}.$$

In fact, we obtain a precise formula for $N(T)$: there exists a constant C depending on Γ and D such that

$$N(T) = \log_{e^{2\delta/q}} \left(\frac{T(e^{2\delta - q})}{2e^{2\delta} - C(e^{2\delta} - q)} \right).$$

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1. Introduction

Extreme value in probability theory has been developed for the last several decades. For a given random process, one can consider probabilistic questions such as law of large numbers, central limit theorem, local limit theorem, etc. One of the probabilistic questions is the distribution of an extreme value of a random

variable: what is the threshold $N(T)$ of extreme events so that the probability that X_t has value at most $N(T)$ for $1 \leq t \leq T$ is not 0 nor 1 as T tends to infinity?

Recently, there has been a series of results on stationary stochastic processes arising from various chaotic dynamical systems such as random walks starting from [5] (see a survey paper [6] and references therein).

One can ask a similar question for geodesics in a non-compact manifold with respect to a measure on the set of geodesics:

> *What is the threshold function $N(T)$ for which the measure of the set of geodesic rays visiting the region of distance at most $N(T)$ from a fixed point in time $[0, T]$ is neither 0 nor 1 as T tends to ∞?*

For the modular surface $\mathbb{H}^2 / \mathrm{SL}_2(\mathbb{Z})$, a related question on continued fraction expansion was answered by Galambos [7] whose result was used by Pollicott [15] for the analogous question on geodesics.

In this article, we address the question for quotients of regular trees, which are the non-Archimedean analog of hyperbolic surfaces. We obtain an extreme value distribution for geometrically finite quotients, which will be defined, of regular trees. These include all the algebraic quotients of Bruhat–Tits tree of the group PGL_2 over positive characteristic local fields.

Let us state our main result. For a $(q + 1)$-regular tree $\mathcal{T}$, the group $\mathrm{Aut}(\mathcal{T})$ of automorphisms of $\mathcal{T}$ is equipped with the compact-open topology. Let Γ be a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$. The quotient graph $X = \Gamma \backslash \mathcal{T}$ has a structure of a graph of groups, which is a graph together with a family of groups G_x, G_e attached to each vertex x and oriented edge e of X together with injective homomorphisms $\alpha_e \colon G_e \to G_{i(e)}$. (These groups G_x are usually given by the stabilizers of vertices and edges in the fundamental domain of Γ in $\mathcal{T}$). We call $[G_{i(e)} \colon \alpha_e(G_e)]$ the *edge-index of e*. See Section 2 for details.

Suppose that the quotient graph $\Gamma \backslash \mathcal{T}_{\min}$ of the minimal Γ-invariant subtree $\mathcal{T}_{\min}$ of $\mathcal{T}$ is a union of a finite graph with finitely many rays each of which is a ray of Nagao type as in Figure 1, with edge-indices alternating between q and 1. We call such Γ *geometrically finite* ([14]). For instance, if G is an F-rank one simple algebraic group over local field F whose Bruhat–Tits tree is $(q + 1)$-regular, then every lattice Γ of G is geometrically finite (see [12]). The Bruhat–Tits tree of G is regular if $[P_1 : B] = [P_2 : B]$ for a minimal parabolic subgroup B of G and two maximal proper parabolic subgroups P_1 and P_2 containing B.

We denote by $\delta = \delta_\Gamma$ the critical exponent of Γ, which is defined by

$$\delta = \delta_\Gamma = \varlimsup_{n \to \infty} \frac{\log \#\{\gamma \in \Gamma \colon d(x, \gamma x) \leq n\}}{n}$$

for any fixed vertex $x \in V\mathcal{T}$. The value does not depend on the choice of $x \in V\mathcal{T}$.
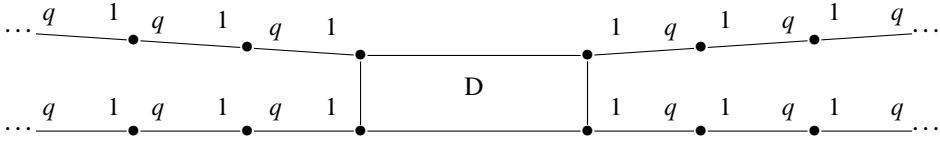
Figure 1. The quotient graph of a geometrically finite subgroup with compact part D

Let $h_T^{(l)}$ be the maximum of the *height* of l among $t \in [0, T]$, which is the distance from a fixed compact part D as in Figure 1:

$$h_T^{(l)} = \max_{0 \le t \le T} d(D, l(t)).$$

Let μ be the Bowen–Margulis measure. (See Definition 3.2.)

Theorem 1.1. *Let Γ be a geometrically finite discrete subgroup of the group* $\mathrm{Aut}(\mathcal{T})$ *of automorphisms of a $(q+1)$-regular tree $\mathcal{T}$. Then there exists a constant* $C = C(\Gamma)$ *such that for*

$$N = \log_{e^{2\delta}/q} \left(\frac{T(e^{2\delta} - q)}{2e^{2\delta} - C(e^{2\delta} - q)} \right),$$

we have

$$\lim_{T \to \infty} \mu(\{[l] \in \Gamma \backslash \mathcal{GT} : h_T^{(l)} \le N + y\}) = e^{-q^y/e^{2\delta y}}.$$

Recall that if Γ is a geometrically finite discrete subgroup for which $\Gamma \backslash \mathcal{T}$ is non-compact, then $\delta_\Gamma > \frac{1}{2} \log q$ (Proposition 4.5 of [11], see also Lemma 5.2). Therefore, as y goes to infinity, the measure of the left-hand side clearly tends to 1.

If Γ is a lattice subgroup of $\mathrm{Aut}(\mathcal{T})$, then $\delta = \log q$ and μ is $\mathrm{Aut}(\mathcal{T})$-invariant. Moreover, if the quotient itself is a ray of Nagao type, then the constant C is equal to 0. (See the proof in Section 4). The main example of such Γ is $\mathrm{PGL}_2(\mathbb{F}_q[t])$ sitting in $\mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))$ (see Section 4).

Corollary 1.2. *Suppose that Γ is a discrete subgroup of* $\mathrm{Aut}(\mathcal{T})$ *such that the edge-indexed graph associated to $\Gamma \backslash\backslash \mathcal{T}$ is equal to the ray $\mathcal{X}$ of Nagao type. Then we have*

$$\lim_{T \to \infty} \mu(\{[l] \in \Gamma \backslash \mathcal{GT} : h_T^{(l)} \le N + y\}) = e^{-1/q^y},$$

with

$$N = \log_q \left(\frac{T(q-1)}{2q} \right).$$

We remark that for quotient spaces of lattices in Lie groups, Kirsebom [10] showed some estimates for the limiting distribution of the maximum height over a specific interval of indices with respect to certain sparse subsequences of the one-parameter action.

Remark that Theorem 1.1 implies the logarithm law, which is a well known result of Sullivan for geodesic excursions on non-compact hyperbolic surfaces of finite area [18] and Heronsky–Paulin for geodesic excursions on negatively curved manifold [8] and geometrically finite quotient of trees [9].

Corollary 1.3. *For μ-almost every equivalence class of geodesics $[l] \in \Gamma \backslash \mathcal{GT}$, we have*

$$\limsup_{T \to +\infty} \frac{h_T^{(l)}}{\log T} = \frac{1}{2\delta - \log q}.$$

Proof. For simplicity, let us denote $\alpha = \frac{e^{2\delta}}{q}$. By Theorem 1.1, we have

$$\varlimsup_{T \to \infty} \mu\Big(\Big\{[l] \in \Gamma\backslash\mathcal{GT} : \frac{h_T^{(l)}}{\log_\alpha T} \le 1 + \log_T\Big(\frac{q(\alpha-1)\alpha^y}{(2-C)q\alpha + Cq}\Big)\Big\}\Big) = e^{-\frac{1}{\alpha^y}}.$$

Given any $\epsilon > 0$, choose sufficiently large y and T such that

$$e^{-\frac{1}{\alpha^y}} > 1 - \epsilon \quad \text{and} \quad \log_T\Big(\frac{q(\alpha-1)\alpha^y}{(2-C)q\alpha + Cq}\Big) < \epsilon.$$

This yields

$$\varlimsup_{T \to \infty} \frac{h_T^{(l)}}{\log T} \le \frac{1}{\log \alpha}.$$

Let us prove that

$$\varlimsup_{T \to \infty} \frac{h_T^{(l)}}{\log T} \ge \frac{1}{\log \alpha}.$$

Assume for a contradiction that there exists $\epsilon > 0$ such that

$$\mu\Big(\Big\{[l] \in \Gamma\backslash\mathcal{GT} : \limsup_{T \to +\infty} \frac{h_T^{(l)}}{\log T} \le \frac{1}{\log \alpha} - \epsilon\Big\}\Big) \ge \epsilon.$$

Then, we can choose $T_0 > 0$ such that whenever $T \ge T_0$,

$$\mu\Big(\Big\{[l] \in \Gamma\backslash\mathcal{GT} : h_T^{(l)} \le \Big(\frac{1}{\log \alpha} - \epsilon\Big)\log T\Big\}\Big) \ge \frac{\epsilon}{2}.$$

Equivalently,

$$\mu\Big(\Big\{[l] \in \Gamma\backslash\mathcal{GT} : h_T^{(l)} - \frac{\log T}{\log \alpha} \le -\epsilon \log T\Big\}\Big) \ge \frac{\epsilon}{2}.$$

Meanwhile, by Theorem 1.1, given y there exists $T_1 > 0$ such that if $T \geq T_1$, then

$$\left| \mu\left(\left\{[l] \in \Gamma\backslash\mathcal{G}\mathcal{T}: h_T^{(l)} - \frac{\log T}{\log \alpha} \leq \log_\alpha \frac{q(\alpha-1)\alpha^y}{(2-C)q\alpha + Cq}\right\}\right) - e^{-\frac{1}{\alpha^y}} \right| < \frac{\epsilon}{4}.$$

If y tends to $-\infty$, then we $e^{-\frac{1}{\alpha^y}} < \frac{\epsilon}{4}$ and for such y, there exists large T for which

$$-\epsilon \log T < \log_\alpha \left(\frac{q(\alpha-1)\alpha^y}{(2-C)q\alpha + Cq}\right).$$

This gives a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The article is organized as follows. We recall the covering theory of graph of groups developed by Bass and Serre in Section 2. In Section 3, we recall the Markov chain associated to the discrete time geodesic flow of edge-indexed graphs and the construction of Gibbs measures. In Section 4, we prove the extreme value distribution for the simplest case, the ray of Nagao type. We prove the extreme value distribution of geometrically finite quotients in Section 5 using the theory of a countable Markov chain and the result for the ray of Nagao type. We tried to write Section 4 as self-contained as possible (without a Markov chain argument) for the readers who are mainly interested in the modular ray.

2. Preliminaries: Bass–Serre theory of graphs of groups

We briefly review Bass–Serre theory, the essential features of the covering theory for graphs of groups. We mainly follow [17] and refer to [2] for further details.

Let A be an undirected graph which is allowed to have loops and multiple edges. We denote by VA the set of vertices of A and by EA the set of *oriented* edges of bi-directed graph obtained from A.

For $e \in EA$, let $\bar{e} \in EA$ be the opposite edge of e and let $\partial_0 e$ and $\partial_1 e$ be the initial vertex and the terminal vertex of e, respectively.

Definition 2.1. Let $i: EA \to \mathbb{Z}_{>0}$ be a map assigning a positive integer to each oriented edge. We say (A, i) an *edge-indexed graph*.

Definition 2.2. By a *graph of groups* $\mathbf{A} = (A, \mathcal{A})$, we mean a connected graph A together with attached groups $\mathcal{A}_a$ ($a \in VA$), $\mathcal{A}_e = \mathcal{A}_{\bar{e}}$ ($e \in EA$), and monomorphisms $\alpha_e: \mathcal{A}_e \to \mathcal{A}_{\partial_1 e}$ ($e \in EA$).

An *isomorphism* between two graph of groups $\mathbf{A} = (A, \mathcal{A})$ and $\mathbf{A}' = (A', \mathcal{A}')$ is an isomomorphism $\phi: A \to A'$ between two underlying graphs together with the set of isomomorphisms $\phi_a: \mathcal{A}_a \to \mathcal{A}'_{\phi(a)}$ and $\phi_e: \mathcal{A}_e \to \mathcal{A}'_{\phi(e)}$ satisfying the following property: for each $e \in EA$, there is an element $h_e \in \mathcal{A}'_{\phi(\partial_1 e))}$ such that

$$\phi_{\partial_1 e}(\alpha_e(g)) = h_e \cdot (\alpha'_{\phi(e)}(\phi_e(g))) \cdot h_e^{-1}$$

for all $g \in \mathcal{A}_e$ ([2], Section 2).

Given an edge-indexed graph (A, i), a graph of groups $(A, \mathcal{A})$ is called a *grouping* of (A, i) if $i(e) = [\mathcal{A}_{\partial_1 e} : \alpha_e \mathcal{A}_e]$ and called a *finite grouping* if all $\mathcal{A}_a$ $(a \in VA)$ are finite.

Suppose that we have a graph of groups $\mathbf{A}$. Choosing a base point $a_0 \in VA$, we can define a *fundamental group* $\pi_1(\mathbf{A}, a_0)$ ([17] Section 5.1), a *universal covering tree* $\widetilde{(\mathbf{A}, a_0)}$ and an action without inversion of $\pi_1(\mathbf{A}, a_0)$ on $\widetilde{(\mathbf{A}, a_0)}$ with a morphism $p : \widetilde{(\mathbf{A}, a_0)} \to A$ which can be identified with the quotient projection ([17], Section 5.3).

Definition 2.3. Given a graph of groups $\mathbf{A} = (A, \mathcal{A})$, we denote by $F(\mathcal{A}, E)$ the group generated by the groups $\mathcal{A}_a$, $(a \in VA)$ and the elements $e \in E$, subject to the relations

$$\bar{e} = e^{-1} \quad \text{and} \quad e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g) \quad \text{for } e \in EA \text{ and } a \in \mathcal{A}_e.$$

Let $\pi_1(\mathbf{A}, a_0)$ be the set of elements of $F(\mathcal{A}, E)$ of the form

$$g_0 e_1 g_1 e_2 \cdots e_{n-1} g_{n-1},$$

where $o(e_{i+1}) = t(e_i) \pmod{n}$ and $g_i \in o(e_{i+1})$. It is a subgroup of $F(\mathcal{A}, E)$, called *fundamental group* of $\mathbf{A}$ based at a_0.

Given a graph of groups $\mathbf{A}$, the graph $\widetilde{X} = \widetilde{(\mathbf{A}, a_0)}$ is defined as

$$V\widetilde{X} = \bigcup_{a \in VA} \pi_1(\mathbf{A}, a_0)/\mathrm{Stab}_{\pi_1(\mathbf{A}, a_0)}(a)$$

and

$$E\widetilde{X} = \bigcup_{e \in EA} \pi_1(\mathbf{A}, a_0)/\mathrm{Stab}_{\pi_1(\mathbf{A}, a_0)}(e).$$

Theorem 2.4 ([17], Theorem 12). *The graph $\widetilde{X}$ defined as above is a tree.*

The tree $\widetilde{X} = \widetilde{(\mathbf{A}, a_0)}$ is called a *universal covering tree* of the graph of groups $\mathbf{A}$.

Definition 2.5. For each $v \in V(\Gamma \backslash \mathcal{T})$ and $e \in E(\Gamma \backslash \mathcal{T})$, we choose any corresponding vertex $\tilde{v} \in V\mathcal{T}$ and edge $\tilde{e} \in E\mathcal{T}$ satisfying the condition $\bar{\tilde{e}} = \widetilde{\bar{e}}$. Fix an element $\gamma_e \in \Gamma$ which satisfies $\gamma_e \widetilde{\partial_1(e)} = \partial_1(\tilde{e})$. Define $\mathcal{A}_v$ and $\mathcal{A}_e$ be the stabilizer of $\tilde{v}$ and $\tilde{e}$ in Γ, respectively, and let $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_1 e}$ be the monomorphism $h \mapsto \gamma_e^{-1} h \gamma_e$. Then the graph of groups $(\Gamma \backslash \mathcal{T}, \mathcal{A})$ is called the *quotient graph of groups*. It does not depend on the choice of $\tilde{v}, \tilde{e}$ and γ_e up to isomorphism of graph of groups ([2], Section 3).

A graph of group $(A, \mathcal{A})$ is called a *ray of Nagao type* if it isomorphic to the following graph of groups with

$$\Gamma_0^+ > \Gamma_0 < \Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots \quad \text{and} \quad [\Gamma_n : \Gamma_{n-1}] < \infty \text{ for } n \geq 1.$$
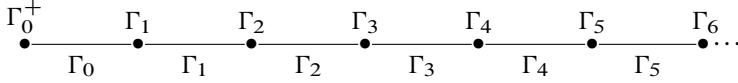

Figure 2. A ray of Nagao type.

As we mentioned in the introduction, a discrete group Γ of $\mathrm{Aut}(\mathcal{T})$ is called *geometrically finite* if for a minimal Γ-invariant subtree $\mathcal{T}_{\min}$ of $\mathcal{T}$, the quotient graph of groups $\Gamma \backslash\backslash \mathcal{T}_{\min}$ is a union of a finite graph with finitely many rays each of which is a ray of Nagao type as in Figure 2. If $\mathcal{T}$ is $(q+1)$-regular, then we have $\mathcal{T} = \mathcal{T}_{\min}$ if and only if $\delta = \log q$. The edge-indices of the rays of $\Gamma \backslash\backslash \mathcal{T}_{\min}$ are alternating between q and 1 if Γ is geometrically finite and $\mathcal{T}$ is q-regular.

3. A Markov chain and Gibbs measures

In this section, $\mathcal{T}$ is a locally finite tree and Γ a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$. In particular, we do not need any assumption on the indices of the quotient graph $\Gamma \backslash \mathcal{T}_{\min}$. Let $\delta = \delta_\Gamma$ be the critical exponent of Γ defined in the introduction.

3.1. Bowen–Margulis measure.
In this subsection, we review the construction of a conformal family $\{\mu_x\}_{x \in V\mathcal{T}}$ of measures on the boundary $\partial_\infty \mathcal{T}$ at infinity and a measure μ on the space $\mathcal{G}\mathcal{X}$ of bi-infinite geodesics invariant under the geodesic flow. Such a measure μ is finite when Γ is geometrically finite ([16]). For lattices of Nagao type, it coincides with Haar measure induced from $Aut(\mathcal{T})$.

The construction below is a special case of the construction of Bowen–Margulis measure from Patterson–Sullivan density, more generally that of Gibbs measures from conformal densities. (See [18] for hyperbolic manifolds, [16] for CAT(-1) spaces, and [3] for trees.)

Let us fix a vertex $x \in V\mathcal{T}$. Let $\mathcal{G}\mathcal{T} = \{l : \mathbb{Z} \to V\mathcal{T}, n \mapsto l_n \text{ isometry}\}$ be the space of bi-infinite geodesics and ϕ the discrete time geodesic flow on $\mathcal{G}\mathcal{T}$ given by $\phi(l)(n) = l(n+1)$. Let $\mathcal{G}\mathcal{T}^+ = \{l : \mathbb{Z}_{\geq 0} \to V\mathcal{T}, n \mapsto l_n \text{ isometry}\}$ be the space of geodesic rays and $\mathcal{G}\mathcal{T}_x^+ = \{l \in \mathcal{G}\mathcal{T}^+ : l_0 = x\}$ be the space of geodesic rays starting at x.

Let $\partial_\infty \mathcal{T}$ be the Gromov boundary at infinity of $\mathcal{T}$. For a fixed a vertex $x \in V\mathcal{T}$, the Gromov boundary $\partial_\infty \mathcal{T}$ can be identified with $\mathcal{G}\mathcal{T}_x^+$.

Let $\pi : \mathcal{T} \to \Gamma \backslash \mathcal{T}$ be the natural projection. It induces the natural projection map $\mathcal{G}\mathcal{T} \to \Gamma \backslash \mathcal{G}\mathcal{T}$ which we will also denote by π.

Definition 3.1 (Patterson–Sullivan density). Given $\omega \in \partial_\infty \mathcal{T}$ and $x, y \in V\mathcal{T}$, the Busemann cocycle $\beta_\omega(x, y)$ is defined as $d(x, z) - d(y, z)$ where $[x, \omega] \cap [y, \omega] = [z, \omega]$.

(1) A *Patterson density* of dimension δ for a discrete group $\Gamma < \mathrm{Aut}(\mathcal{T})$ is a family of finite nonzero positive Borel measures $\{\mu_x\}_{x \in V\mathcal{T}}$ on $\partial_\infty \mathcal{T}$ such that for every $\gamma \in \Gamma$, for all $x, y \in \mathcal{T}$ and $\omega \in \partial_\infty \mathcal{T}$,

$$\gamma_* \mu_x = \mu_{\gamma \cdot x} \quad \text{and} \quad \frac{d\mu_x}{d\mu_y}(\omega) = e^{-\delta \beta_\omega(x, y)}.$$

(2) For Γ geometrically finite, the Patterson density $\{\mu_x\}_{x \in V\mathcal{T}}$ of dimension $\delta = \delta_\Gamma$ is the (unique) weak-limit of $\mu_{x,s}$ as $s \to \delta^+$ where

$$\mu_{x,s} = \frac{1}{\sum_{\gamma \in \Gamma} e^{-sd(s,\gamma x)}} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x)} \delta_{\gamma x}$$

and $\delta_{\gamma x}$ is the Dirac mass at γx ([9]).

Now consider the set $\mathcal{GT}_x$ of bi-infinite geodesics which reaches x at time zero. On the set $\mathcal{GT}_x$, we define μ locally by $\mu_x \times \mu_x$: for $D^-, D^+ \subset \partial_\infty \mathcal{T}$ such that every geodesic line connecting a point in D^- and D^+ passes through x, we define

$$(\mu_x \times \mu_x)(\{l \in \mathcal{GT}_x : l^- \in D^- \text{ and } l^+ \in D^+\}) = C_x \mu_x(D^-) \mu_x(D^+)$$

on $\mathcal{GT}_x$. Here, C_x is the normalizing constant such that $(\mu_x \times \mu_x)(\mathcal{GT}_x) = 1$.

For $x \in V\mathcal{T}$, let Γ_x be the stabilizer of x in Γ. Choosing a set of representatives $[x] \in \Gamma \backslash V\mathcal{T}$, we have a one-to-one correspondence between $\Gamma \backslash \mathcal{GT}$ and $\bigcup_{[x] \in \Gamma \backslash V\mathcal{T}} \Gamma_x \backslash \mathcal{GT}_x$. (Note that Γ_x acts on $\mathcal{GT}_x$ and $\mathcal{GT} = \bigcup_{[x] \in \Gamma \backslash V\mathcal{T}} \mathcal{GT}_x$.) We take the sum of $\mu_x \times \mu_x$ and normalize to get a ϕ-invariant probability measure as in the following definition.

Definition 3.2 (Bowen–Margulis measure). For a measurable subset $E \subset \Gamma \backslash \mathcal{GT}$, define

$$\mu(E) := C_0 \sum_{[x] \in \Gamma \backslash V\mathcal{T}} \frac{1}{|\Gamma_x|} (\mu_x \times \mu_x)(\pi^{-1} E \cap \mathcal{GT}_x),$$

where

$$C_0 = \left(\sum_{[x]} \frac{1}{|\Gamma_x|} \right)^{-1}.$$

Note that C_0 is chosen so that $\mu(\Gamma \backslash \mathcal{GT}) = 1$ and the quantity above is well-defined i.e., it depends only on the class of x. Furthermore, the measure μ is ϕ-invariant. Indeed, any measurable subset of $\Gamma \backslash \mathcal{GT}$ can be decomposed into projection of cylinders of the form

$$E = \pi(C_E)$$

with

$$C_E = \{l \in \mathcal{GT}_x : l^- \in D^-, l^+ \in D^+\}$$

small enough so that π is one-to-one on C_E. Note that the measure of such cylinders are ϕ-invariant:

$$\begin{aligned}
\mu(\phi^{-1}E) &= \frac{1}{|\Gamma_{x'}|}(\mu_{x'} \times \mu_{x'})(\pi^{-1}\phi^{-1}E \cap \mathcal{GT}_{x'}) \\
&= \frac{1}{|\Gamma_{x'}|}|\Gamma_{x'}|(\mu_{x'} \times \mu_{x'})(C_{\phi^{-1}E} \cap \mathcal{GT}_{x'}) \\
&= \frac{1}{|\Gamma_x|}|\Gamma_x|(\mu_x \times \mu_x)(C_E \cap \mathcal{GT}_x) \\
&= \frac{1}{|\Gamma_x|}(\mu_x \times \mu_x)(\pi^{-1}E \cap \mathcal{GT}_x) \\
&= \mu(E),
\end{aligned}$$

where x' is the base point of the elements of $\phi^{-1}E$. The third equality follows from the definition of $\mu_x \times \mu_x$. Another way of seeing ϕ-invariance is to observe that μ is a Gibbs measure for dicrete time geodesic flow ϕ, comparing with Proposition 4.13 of [3].

3.2. The Markov chain of $\Gamma_f \backslash \mathcal{GT}$. In this subsection, we construct a Markov chain associated to the geodesic flow on the compact part. First enlarge the given geometrically finite group Γ to the full group Γ_f associated to Γ, which is defined as the group maximal with the property that the quotient graph $\Gamma \backslash \mathcal{T}$ coincides with $\Gamma_f \backslash \mathcal{T}$ [4], namely

$$\Gamma_f = \{g \in \mathrm{Aut}(\mathcal{T}) : \pi \circ g = \pi\}.$$

Note that $\Gamma \backslash \mathcal{GT}$ and $\Gamma_f \backslash \mathcal{GT}$ can be very different. We will define a Markov chain of $\Gamma_f \backslash \mathcal{GT}$ coding the geodesic flow. We remark that Γ_f is not necessarily virtually discrete, thus, the Markov chain of $\Gamma_f \backslash \mathcal{GT}$ does not necessarily give a Markov chain of $\Gamma \backslash \mathcal{GT}$ coding the geodesic flow. However, the quotient graphs are identical, thus the extreme value condition for Γ_f holds if and only if the same condition holds for Γ if we consider the measure on $\Gamma \backslash \mathcal{GT}$ induced from $\Gamma_f \backslash \mathcal{GT}$.

More precisely, let μ be the Bowen–Margulis measure defined in Definition 3.2. Denote by p the natural projection $\Gamma \backslash \mathcal{GT} \to \Gamma_f \backslash \mathcal{GT}$ such that $\phi \circ p = p \circ \phi$. An important fact is that the set $\{[l] \in \Gamma \backslash \mathcal{GT} : h_T^{(l)} \leq N + y\}$ is invariant under the associated full group Γ_f. Thus, if we denote by $\bar{\mu}$ the measure on $\Gamma_f \backslash \mathcal{GT}$ given by $\bar{\mu}(E) = \mu(p^{-1}(E))$, then it suffices to consider the limiting distribution of

$$\bar{\mu}(\{[l] \in \Gamma_f \backslash \mathcal{GT} : h_T^{(l)} \leq N + y\}).$$

Now we construct the Markov chain associated to the discrete time geodesic flow. Recall that for an undirected graph A, EA is the set of all oriented edges, thus the cardinality of EA is twice the number of edges of A.

For a given edge-indexed graph (A, i), consider the following subset

$$X_{(A,i)} = \{x = (e_j)_{j \in \mathbb{Z}} : \partial_0 e_{j+1} = \partial_1 e_j \text{ and if } e_{j+1} = \overline{e_j}, \text{ then } i^A(e_j) > 1\}$$

of admissible paths in $(EA)^{\mathbb{Z}}$. The family of *cylinders*

$$[e_0, \ldots, e_{n-1}] := \{x \in X_{(A,i)} : x_i = e_i, i = 0, \ldots, n-1\}$$

is a basis of open sets for the subspace topology on $X_{(A,i)}$ induced from a product topology of $(EA)^{\mathbb{Z}}$. Let $\sigma : X_{(A,i)} \to X_{(A,i)}$ be the shift given by $\sigma(x)_i := x_{i+1}$. If (A, i) is the edge-indexed graph associated with the quotient graph of groups $\Gamma_f \backslash\backslash \mathcal{T}$, then we have a bijection $\Phi : (\Gamma_f \backslash \mathcal{GT}, \phi) \to (X_{(A,i)}, \sigma)$ given by $\Phi([l]) = (e_j)_{j \in \mathbb{Z}}$, $\partial_i e_j = l_{i+j}$ for all $j \in \mathbb{Z}$ and $i = 0, 1$, so that the following diagram commute (cf. [4]).

$$
\begin{array}{ccccc}
\Gamma \backslash \mathcal{GT} & \xrightarrow{p} & \Gamma_f \backslash \mathcal{GT} & \xrightarrow{\Phi} & X_{(A,i)} \\
\downarrow{\phi} & & \downarrow{\phi} & & \downarrow{\sigma} \\
\Gamma \backslash \mathcal{GT} & \xrightarrow{p} & \Gamma_f \backslash \mathcal{GT} & \xrightarrow{\Phi} & X_{(A,i)}
\end{array}
$$

For $f \in E\mathcal{T}$, we denote the *shadow* of an edge f by

$$\mathcal{O}(f) = \{\omega \in \partial_\infty \mathcal{T} : \text{there exists } \xi \in \mathcal{GT} \text{ such that}$$
$$\xi_0 = \partial_0 f, \xi_1 = \partial_1 f \text{ and } \xi^+ = \omega\}.$$

Let $[e_0, \ldots, e_{n-1}]$ be an admissible cylinder of $X_{(A,i)}$. Following [4], we define λ by

$$\lambda([e_0, \ldots, e_{n-1}]) = \frac{\mu_{\partial_0 f_0}(\mathcal{O}(\overline{f_0}))\mu_{\partial_1 f_{n-1}}(\mathcal{O}(f_{n-1}))}{|\Gamma_{f_0,\ldots,f_{n-1}}|} e^{-n\delta}$$

where f_j is an oriented edge of $\mathcal{T}$ for which $\pi(f_j) = e_j$ and $\partial_1 f_j = \partial_0 f_{j+1}$ and $\Gamma_{f_0,\ldots,f_{n-1}}$ is the stabilizer group of $f_0, \ldots, f_{n-1}$ of Γ. This quantity does not depend on the choice of f_j.

It has the Markov property, namely

$$\sum_{e_j : \partial_0 e_k = \partial_1 e_j} \lambda([e_j, e_k]) = \lambda([e_k]) \quad \text{and} \quad \sum_{e_k : \partial_0 e_k = \partial_1 e_j} \lambda([e_j, e_k]) = \lambda([e_j]). \quad (3.1)$$

We also have

$$\int_{\Gamma_f \backslash \mathcal{GT}} f \, d\bar{\mu} = \int_{X_{(A,i)}} \Phi^*(f) \, d\lambda \quad (3.2)$$

where $\Phi^*(f)[(x_i)_{i \in \mathbb{Z}}] = f[\Phi^{-1}((x_i)_{i \in \mathbb{Z}})]$. In other words, the measure λ is a Markov measure and two dynamical systems $(\Gamma_f \backslash \mathcal{GT}, \phi, \bar{\mu})$ and $(X_{(A,i)}, \sigma, \lambda)$ are isomorphic ([4]).

Let us briefly recall a positive recurrent Markov chain following [13]. Let Z_n be a Markov chain with phase space $\mathcal{S} = \{s_1, s_2, \dots\}$ and transition probabilities

$$p_{ij} = p_{s_i s_j} = P\{Z_{n+1} = s_j : Z_n = s_i\}, \quad \sum_j p_{ij} = 1.$$

For a subset $B \subset \mathcal{S}$ of alphabets, let

$$f_{ij}^{(n)} = P\{Z_1 \neq s_j, \dots, Z_{n-1} \neq s_j, Z_n = s_j : Z_0 = s_i\}, \qquad (3.3a)$$

$$p_{ij}^{(n)} = P\{Z_n = s_j : Z_0 = s_i\}. \qquad (3.3b)$$

and set $f_{ij}^{(0)} = 0$ and $p_{ij}^{(0)} = \delta_{ij}$. Observe the following convolution relation

$$p_{s_i s_j}^{(n)} = \sum_{r=1}^{n} f_{s_i s_i}^{(r)} p_{s_i s_j}^{(n-r)}. \qquad (3.4)$$

Suppose that the Markov chain Z_n is irreducible, i.e., for any $s_i, s_j \in \mathcal{S}$, there exists $n > 0$ such that $p_{ij}^{(n)} > 0$. We say π_j is a *stationary* distribution if it satisfies $\pi_j = \sum_{i \in \mathcal{S}} \pi_i p_{ij}$. A Markov chain $(\mathcal{S}, p_{ij})$ is *recurrent* if a stationary distribution exists and furthermore it is called *positive recurrent* if $\sum_{n=1}^{\infty} n f_{jj}^{(n)} < \infty$. When $(\mathcal{S}, p_{ij}, \pi_j)$ is positive recurrent, π_j is unique and we have

$$\pi_j = \frac{1}{\sum_{n=1}^{\infty} n f_{jj}^{(n)}}.$$

An irreducible Markov chain is called *aperiodic* if for some (and hence every) state $s_i \in \mathcal{S}$, its period $\gcd\{n : p_{ii}^{(n)} > 0\}$ is 1.

The Markov chain we consider in this article is (see [11]) $(\mathcal{S}, p_{ij})$ for

$$\mathcal{S} = E(\Gamma \backslash\backslash \mathcal{T}), \quad p_{ij} = p_{e_i e_j} = \frac{\lambda([e_i, e_j])}{\lambda([e_i])}. \qquad (3.5)$$

Note that the stationary distribution is $\pi_j = \lambda([e_j])$ which is positive recurrent and aperiodic. If a positive recurrent Markov chain Z_n is aperiodic, then $\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$ and π_j does not depend on the choice of $i \in \mathcal{S}$ (Chapter 8-10, [13]). We will use this fact in Section 5.

4. Extreme value distribution for rays of Nagao type

Let $\mathcal{T}$ be a $(q + 1)$-regular tree and let $G = \mathrm{Aut}(\mathcal{T})$. Let $\mathcal{X}$ be the edge-indexed graph described in Figure 3 and Γ be the fundamental group of a finite grouping

of $\mathfrak{X}$. In other words, Γ is a discrete subgroup of G for which the edge-indexed graph associated to the quotient graph of groups $\Gamma \backslash\backslash \mathfrak{T}$ is $\mathfrak{X}$. Let us denote the vertices of $\mathfrak{X}$ by $v_0, v_1, v_2, \ldots$ as in Figure 3.

$$\begin{array}{c} \underset{v_0}{\bullet} \overset{q+1}{} \overset{q}{} \underset{v_1}{\bullet} \overset{1}{} \overset{q}{} \underset{v_2}{\bullet} \overset{1}{} \overset{q}{} \underset{v_3}{\bullet} \overset{1}{} \overset{q}{} \underset{v_4}{\bullet} \overset{1}{} \overset{q}{} \underset{v_5}{\bullet} \overset{1}{} \overset{q}{} \underset{v_6}{\bullet} \cdots \end{array}$$
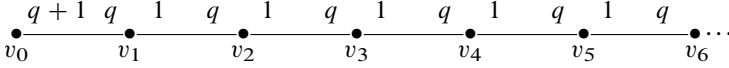
Figure 3. An edge-indexed ray of Nagao type.

The main motivating example is the modular ray: let $\mathbf{K} = \mathbb{F}_q((t^{-1}))$ and $\mathbf{Z} = \mathbb{F}_q[t]$. Let $G = \mathrm{PGL}(2, \mathbf{K})$ and $\Gamma = \mathrm{PGL}(2, \mathbf{Z})$. The group G acts transitively on the $(q + 1)$-regular tree $\mathfrak{T}$ which is called the Bruhat–Tits tree associated to G [17]. Let us fix a vertex $x \in V\mathfrak{T}$.

Recall that $\mathcal{GT}, \mathcal{GT}^+, \mathcal{GT}^+_x$ are the space of biinfinite geodesics, geodesic rays and geodesic rays starting at x, respectively.

Let $\pi \colon \mathfrak{T} \to \Gamma \backslash \mathfrak{T}$ be the natural projection. It induces the natural projection map $\mathcal{GT} \to \Gamma \backslash \mathcal{GT}$ which we will also denote by π.

Definition 4.1. Let us denote $\pi^{-1} v_0 \cap \{l_n \colon n \in \mathbb{Z}_{\geq 0}\} = \{l_{t_1}, l_{t_2}, \ldots\}$. For a fixed geodesic l, such t_i will be denoted by $t_i(l)$.

(1) The sequence of vertices $l_{t_n} l_{t_n+1} \cdots l_{t_{n+1}}$ is called the *n-th excursion of l*. For given l, such t_i will be denoted by $t_i(l)$.

(2) Since the quotient graph of groups has indices alternating between q and 1 as in Figure 2, all the neighbors of $\pi^{-1}(v_0)$ are mapped to v_1 under π. As for $i \geq 1$, all but one neighbors of $\pi^{-1}(v_i)$ are mapped to v_{i-1} under π and the remaining one is mapped to v_{i+1}. As any geodesic has no back-tracking, any n-th excursion of geodesic $l_{t_{n-1}} \cdots l_{t_n}$ projects to $v_0 \cdots v_m v_{m-1} \cdots v_1 v_0$. Call such m the *height* of n-th excursion of l and denote it by $a_n(l)$. The *n-th excursion time* is
$$t_{i+1}(l) - t_i(l) = 2a_i(l).$$

Definition 4.2. Let μ_x be the probability measure on $\mathcal{GT}^+_x$ defined as follows. The subsets $E_y = \{l \in \mathcal{GT}^+_x \colon l \text{ passes through } y\}$ for $y \in \mathfrak{T}$ form a basis for a topology on $\mathcal{GT}^+_x$. Let $\mathcal{B}$ be the associated Borel σ-algebra.

The probability measure μ_x is given by
$$\mu_x(E_y) = \frac{1}{(q + 1)q^{d(x,y)-1}}.$$

The measure μ_x is invariant under every element of $\mathrm{Aut}(\mathfrak{T})$ which fixes x.

Proposition 4.3 (independence of excursions). *Let x be a vertex of $\mathfrak{T}$ which is a lift of v_0. For any $k \geq 1$ and for any $1 \leq i_1 < \cdots < i_k$,*
$$\mu_x(\{l \in \mathcal{GT}^+_x \colon \max_{1 \leq j \leq k} a_{i_j}(l) \leq N\}) = \left(1 - \frac{1}{q^N}\right)^k.$$

Proof. We prove by induction. Let us denote

$$A_{i,N} = \{l \in \mathcal{GT}_x^+ : \max_{t_i \le t \le t_{i+1}} d(v_0, \pi(l_t)) \le N\}$$

and $A_{i,N}^c$ its complement in $\mathcal{GT}_x^+$ so that

$$\{l \in \mathcal{GT}_x^+ : \max_{1 \le j \le k} a_{i_j}(l) \le N\} = \bigcap_{j=1}^{k} A_{i_j,N}.$$

We first consider the case $k = 1$ by computing $\mu_x(A_{i_1,N}^c)$. Let

$$V_{i_1} = \{y \in \pi^{-1}(v_0) : |[xy] \cap v_0\Gamma| = i_1\}$$

be the set of starting vertices of i_1-th excursions of geodesics. The geodesic rays with $a_{i_1}^{(l)} > N$ have $l_{t_{i_1}} \in V_{i_1}$ and the i_1-th excursion projects to a ray on $\mathcal{X}$ starting with $v_0 v_1 \cdots v_N v_{N+1}$.

The following observation is the keypoint: for each $y \in V_{i_1}$, there exist $q + 1$ lifts of v_1 which are neighbors of y. However, one of them is visited by the geodesic just before it arrives at y. Thus, there are exactly q lifts of $v_0 v_1$ starting from y not backtracking the geodesic $l_{t_{i_1}-1} l_{t_{i_1}}$. For each of these lifts, there is a unique lift of $v_0 \cdots v_{N+1}$ starting with the lift. Call the endpoints of these lifts $z_j, j = 1, \ldots, q$. It follows that

$$\frac{\mu_x(\{l : l_{t_{i_1}} = y, \pi(l_{t_{i_1}} \cdots l_{t_{i_1}+N+1}) = v_0 \cdots v_{N+1}\})}{\mu_x(\{l : l_{t_{i_1}} = y\})}$$

$$= \frac{\sum_{i=1}^{q} \mu_x(E_{z_i})}{\mu_x(E_y)} = \frac{q \cdot \frac{1}{(q+1)q^{N+1+d(x,y)-1}}}{\frac{1}{(q+1)q^{d(x,y)-1}}} = \frac{1}{q^N}. \tag{4.1}$$

By definition, $l_{t_{i_1}} \in V_{i_1}$ for any l. Thus, summing over $y \in V_{i_1}$, we have

$$\mu_x(A_{i_1,N}^c) = \sum_{y \in V_{i_1}} \mu_x(\{l : l_{t_{i_1}} = y\} \cap A_{i_1,N}^c) = \sum \mu_x\{l : l_{t_{i_1}} = y\}\frac{1}{q^N} = \frac{1}{q^N}.$$

Now suppose the proposition holds up to $k - 1$. Replacing $y \in V_{i_1}$ by $z \in V_{i_k}$ in the equations (4.1), the equation

$$\frac{\mu_x(\{l : l_{t_{i_k}} = z\} \cap \bigcap_{j=1}^{k-1} A_{i_j,N} \cap A_{i_k,N})}{\mu_x(\{l : l_{t_{i_k}} = z\} \cap \bigcap_{j=1}^{k-1} A_{i_j,N})} = 1 - \frac{1}{q^N} \tag{4.2}$$

holds if and only if both the numerator and the denominator of the left hand side are not zero. Equivalently, $[xz]$ is the beginning of a geodesic in $\bigcap_{j=1}^{k-1} A_{i_j,N}$, i.e. $[xz]$ does not project to a ray starting with $v_0 \cdots v_{N+1}$ on the $\cdots i_{k-1}$-th excursions.

By induction hypothesis, it follows that

$$\mu_x(\{l \in \mathcal{GT}_x^+ : \max_{1 \leq j \leq k} a_{i_j}^{(l)} \leq N\})$$

$$= \mu_x\left(\bigcap_{j=1}^{k-1} A_{i_j,N} \cap A_{i_k,N}\right)$$

$$= \sum_{z \in V_{i_k}} \mu_x\left(\{l : l_{t_{i_k}} = z\} \cap \bigcap_{j=1}^{k-1} A_{i_j,N} \cap A_{i_k,N}\right)$$

$$= \left(1 - \frac{1}{q^N}\right) \sum_{z \in V_{i_k}} \mu_x\left(\{l : l_{t_{i_k}} = z\} \cap \bigcap_{j=1}^{k-1} A_{i_j,N}\right)$$

$$= \left(1 - \frac{1}{q^N}\right) \mu_x\left(\bigcap_{j=1}^{k-1} A_{i_j,N}\right)$$

$$= \left(1 - \frac{1}{q^N}\right)^k.$$

This completes the proof of the proposition. $\qquad \square$

Although the proof is lengthy, the main idea of the proof above is that each excursion is independent. We will use this fact again in Section 5 for more general discrete subgroups.

Proposition 4.4. *For any x,*

$$\lim_{N \to \infty} \mu_x(\{l \in \mathcal{GT}_x^+ : \max_{1 \leq j \leq q^N} a_j(l) \leq N + y\}) = e^{-q^y}.$$

Proof. By Lemma 4.3, we have

$$\mu_x(\{l \in \mathcal{GT}_x^+ : \max_{1 \leq j \leq n} a_j(l) \leq N + y\}) = \left(1 - \frac{1}{q^{N+y}}\right)^n.$$

Letting $n = q^N$, and $N \to \infty$, we obtain the proposition. $\qquad \square$

We now prove a similar result for bi-infinite geodesics. Note that

$$\lim_{N \to \infty} (\mu_x \times \mu_x)(\{l \in \mathcal{GT}_x : \max_{1 \leq j \leq q^N} a_j(l) \leq N + y\})$$

$$= \lim_{N \to \infty} \mu_x(\{l \in \mathcal{GT}_x^+ : \max_{1 \leq j \leq q^N} a_j(l) \leq N + y\})$$

$$= e^{-q^y}.$$

Proposition 4.5. $\lim_{N\to\infty} \mu(\{l \in \mathcal{GT}: \max_{1\le j \le q^N} a_j(l) \le N + y\}) = e^{-q^y}.$

Proof. Choose a lift x_i in $V\mathcal{T}$ of v_i. Recall from Definition 4.1 that t_1 is the smallest non-negative integer satisfying $\pi(l_{t_1}) = v_0$. For $i \ne 0$, we have

$$C_0(\mu_{x_i} \times \mu_{x_i})(\{l \in \mathcal{GT}_{x_i}: \max_{1\le j \le q^N} a_j(l) \le N + y\})$$

$$= C_0 \sum_{k=0}^{\infty} (\mu_{x_i} \times \mu_{x_i})(\{l \in \mathcal{GT}_{x_i}: \max_{1\le j \le q^N} a_j(l) \le N + y, t_1 = i + 2k\})$$

$$= \sum_{k=0}^{\infty} \mu(\{[l] \in \Gamma\backslash\mathcal{GT}: \max_{1\le j \le q^N} a_j(l) \le N + y, t_1 = i + 2k\} \cap \pi(\mathcal{GT}_{x_i}))$$

$$= \sum_{k=0}^{\infty} \mu(\phi^{i+2k}[\{[l] \in \Gamma\backslash\mathcal{GT}: \max_{1\le j \le q^N} a_j(l) \le N + y, t_1 = i + 2k\} \cap \pi(\mathcal{GT}_{x_i})])$$

$$= \mu(\{[l] \in \Gamma\backslash\mathcal{GT}: \max_{1\le j \le q^N} a_j(l) \le N + y\} \cap \pi(\mathcal{GT}_x)$$

$$\cup \bigcup_{k=0}^{\infty} \{l \in \Gamma\backslash\mathcal{GT}: t_1 = i + 2k\})$$

$$= C_0(\mu_x \times \mu_x)(\{l \in \mathcal{GT}_x: \max_{1\le j \le q^N} a_j(l) \le N + y\}).$$

The ϕ-invariance of μ gives the third equality. $\square$

Using Proposition 4.5, we prove the main theorem for the rays of Nagao type. Recall that $h_T^{(l)} = \max_{0\le t \le T} d(\pi(l_t), v_0)$ and that t_n is the starting time of the n-th excursion of l.

For each geodesic l, let $S_n^{(l)} = 2(a_1^{(l)} + \cdots + a_n^{(l)}) = t_{n+1} - t_1$ be the total time of the first n excursions and T_n its expectation with respect to μ. Note that

$$T_n = \mathbb{E}_\mu\Big(\sum_{i=1}^{n} 2a_i\Big) = 2n\mathbb{E}_\mu(a_1) = 2n\sum_{k=1}^{\infty} k\mu(a_1 = k) = \sum_{k=1}^{\infty} \frac{2nk(q-1)}{q^k}$$

$$= \frac{2qn}{q-1}.$$

Theorem 4.6. $\lim_{T\to\infty} \mu(\{[l] \in \Gamma\backslash\mathcal{GT}: h_T^{(l)} \le \log_q\left(\frac{T(q-1)}{2q}\right) + y\}) = e^{-1/q^y}.$

Proof. By the law of large numbers, we have $\frac{S_n^{(l)} - T_n}{n} \to 0$ for μ-almost every $l \in \mathcal{GT}$. Moreover, if we denote by $B_{n,C}$ the set $\{l \in \mathcal{GT}: |S_n^{(l)} - T_n| \le C\sqrt{n}\}$, then by the central limit theorem of the shift map, for any $\epsilon > 0$, there exists $C > 0$ such that

$$\mu_x(B_{n,C}) > 1 - \epsilon$$

holds for all $n \ge 1$.

Let $A_{T,N+y} = \{l \in \mathcal{GT} : \max_{1 \le t \le T} d(v_0, \pi(l_t)) \le N + y\}$. Note that

$$A_{T_{q^N} + Cq^{N/2}, N+y} \subseteq A_{T_{q^N}, N+y} \subseteq A_{T_{q^N} - Cq^{N/2}, N+y}.$$

Therefore,

$$\mu(A_{T_{q^N} + Cq^{N/2}, N+y} \cap B_{n,C}) \le \mu(A_{S^{(l)}_{q^N}, N+y} \cap B_{n,C})$$

$$\le \mu(A_{T_{q^N} - Cq^{N/2}, N+y} \cap B_{n,C}).$$

Meanwhile,

$$\mu(A_{T_{q^N} - Cq^{N/2}, N+y}) - \mu(A_{T_{q^N} + Cq^{N/2}, N+y})$$

$$= \mu(\{l \in \mathcal{GT} : \max_{T_{q^N} - Cq^{N/2} \le t \le T_{q^N} + Cq^{N/2}} d(v_0, \pi(l_t)) > N + y\})$$

$$= \mu(\{l \in \mathcal{GT} : \max_{0 \le t \le 2Cq^{N/2}} d(v_0, \pi(l_t)) > N + y\}) \quad (\mu \text{ is } \phi\text{-invariant})$$

$$\le \mu(\{l \in \mathcal{GT} : \max_{0 \le t \le \frac{2q^{(2N/3)+1}}{q-1} - Cq^{N/3}} d(v_0, \pi(l_t)) > N + y\})$$

$$\left(\frac{2q^{(2N/3)+1}}{q - 1} - Cq^{N/3} \ge 2Cq^{N/2} \text{ for sufficiently large } N\right)$$

$$\le \mu(\{l \in \mathcal{GT} \cap B_{n,C} : \max_{0 \le t \le S^{(l)}_{q^{2N/3}}} d(v_0, \pi(l_t)) > N + y\}) + \epsilon.$$

Hence, for any given $\epsilon > 0$, there exists $M > 0$ such that

$$\mu(A_{S^{(l)}_{q^N}}, N + y) - 2\epsilon \le \mu(A_{T_{q^N}}, N + y) \le \mu(A_{S^{(l)}_{q^N}}, N + y) + 2\epsilon$$

holds for all $N \ge M$. By Proposition 4.5, we have

$$\lim_{N \to \infty} \mu(\{l \in \mathcal{GT} : \max_{0 \le t \le \frac{2q^N+1}{q-1}} d(v_0, \pi(l_t)) \le N + y\}) = e^{-q^y}$$

which completes the proof. □

5. Extreme value distribution for geometrically finite quotient

In this section, we prove extreme value distribution for geometrically finite quotient graphs of regular trees using the Markov chain on the compact part and the extreme value distribution result for each ray proved in the previous section.

We remark that an alternative approach might be to use general extreme value theorem [6] using the ϕ-mixing property, i.e., the error term of mixing $|\mu(A \cap T^{-n}B) - \mu(A)\mu(B)|$ is bounded by the measures of the sets A and B) of the measure-preserving transformation T, which is not available here. Note that exponential mixing is known [3] based on a result of Young [19] (see also the paper by the first author [11]).

Let us first fix some notations on the quotient graph. Given a $(q + 1)$-regular tree $\mathcal{T}$, let $\mathrm{Aut}(\mathcal{T})$ be the group of automorphisms of $\mathcal{T}$ and Γ be a geometrically finite discrete subgroup of $\mathrm{Aut}(\mathcal{T})$ (see Section 1). There are finite edge-indexed rays $C_1, \ldots, C_k$ and a finite edge-indexed graph D such that

(1) $V(\Gamma \backslash \mathcal{T}_{\min}) = VD \cup VC_1 \cup \cdots \cup VC_k$;

(2) $|VD \cap VC_j| = 1$ and $VC_i \cap VC_j = \phi$ if $i \neq j$;

(3) each C_i is a Nagao ray of index $(1, q, 1, q, \ldots,)$, i.e., $i(\overline{e_n}) = q, i(e_n) = 1$ for all $n \geq 0$.

Fix such C_j and D. Let us denote by $v_{i,0}$ the unique element of $VD \cap VC_i$ $(1 \leq i \leq k)$. Then, we obtain the Figure 4.
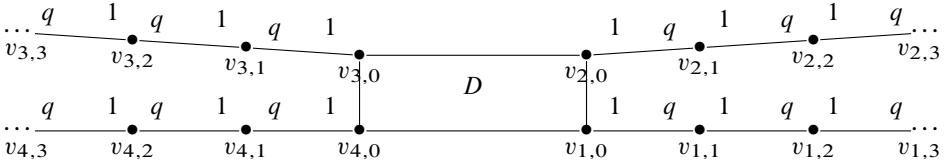

Figure 4. The quotient graph of a geometrically finite subgroup.

Definition 5.1. Let $l \in \mathcal{GT}$. We can write

$$\pi^{-1}(\{v_{1,0}, \ldots, v_{k,0}\}) = \{\ldots, l_{t_{-1}}, l_{t_0}, l_{t_1}, l_{t_2}, \ldots,\}$$

with t_1 be the smallest positive time when l leaves the compact part. Note that $\pi(l_{t_{2n-1}}) = \pi(l_{t_{2n}})$. The sequence of vertices $l_{t_{2n-1}} l_{t_{2n-1}+1} \cdots l_{t_{2n}}$ is called the *n-th excursion of l*.

Comparing with Definition 4.1, note that the starting time of the n-th excursion is now t_{2n-1}. As explained in Definition 4.1 (2), any n-th excursion of geodesic projects to $v_{i,0} \cdots v_{i,m} v_{i,m-1} \cdots v_{i,1} v_{i,0}$ for some i and m. We call such m the *height* of n-th excursion of l and denote it by $a_n(l)$.

Recall that $h_T^{(l)} = \max_{0 \leq t \leq T} d(D, l(t))$ and the Markov chain $(\mathcal{S}, p_{ij})$ is given by

$$\mathcal{S} = E(\Gamma \backslash\backslash \mathcal{T}), \quad p_{ij} = p_{e_i e_j} = \frac{\lambda([e_i, e_j])}{\lambda([e_i])}. \tag{5.1}$$

The stationary distribution is given by $\pi_j = \lambda([e_j])$. Recall also that μ_x is the Patterson density for Γ based at x (Definition 3.1) and μ is the Gibbs measure constructed in Definition 3.2.

Lemma 5.2. *If Γ is non-elementary and $\Gamma \backslash \mathcal{T}$ has at least one Nagao ray, then $\delta > \frac{1}{2} \log q$ and*

$$\bar{\mu}(\{[l] \in \Gamma_f \backslash \mathcal{GT} : a_n(l) \leq N\}) = 1 - \frac{q^N}{e^{2\delta N}}.$$

Proof. The proof is verbatim to the proof of Proposition 4.3, except that we need to obtain the general version of (4.1). For $j = 0, \ldots, m$, let $x_j \in V\mathcal{T}$ be the vertices satisfying $\pi(x_j) = v_{i,j}$ and x_j, x_{j+1} are adjacent. For $j = 1, \ldots, m$, let $f_j \in E\mathcal{T}$ such that $\partial_0 f_j = x_{j-1}$ and $\partial_1 f_j = x_j$. We need to show that for any integer $N > 0$,

$$\frac{\mu_{x_i}(\mathcal{O}(f_{i+N}))}{\mu_{x_i}(\mathcal{O}(f_i))} = \left(\frac{q}{e^{2\delta}}\right)^N.$$

Let $v_{i,0} \cdots v_{i,m} v_{i,m-1} \cdots v_{i,1} v_{i,0}$ be the projection of n-th excursion of some geodesic $l \in \mathcal{GT}$ under π.

Let $\alpha_j = \mu_{x_j}(\mathcal{O}(\overline{f_j}))$. (Note that this does not depend on the choice of x_j). Since Γ is non-elementary, it follows that μ has no atoms and hence $\alpha_j \neq 0$. The conformal property of μ implies that $\mu_{x_j}(\mathcal{O}(\overline{f_j})) = \mu_{x_{j+1}}(\mathcal{O}(\overline{f_j}))e^\delta$. Since there are q neighbors of x_j which projects to $v_{i,j}$, and Γ_{x_j} acts transitively on these neighbors, we have

$$\alpha_{j+1} = q\mu_{x_j+1}(\mathcal{O}(\overline{f_j})) = q\mu_{x_j}(\mathcal{O}(\overline{f_j}))e^{-\delta} = q\alpha_j e^{-\delta}. \qquad (5.2)$$

Let e_i be the edge given by $\partial_0 e_i = v_{i-1}$ and $\partial_1 e_i = v_i$. Let us decompose the shadow $\mathcal{O}(f_{j+N})$ into countable disjoint union of sets: $\mathcal{O}(f_{j+N})$ is the union of $(q-1)$ shadows $O(g_0)$ of lifts g_0 of $\overline{e_{j+N}}$ adjacent to f_{j+N} and the shadow $\mathcal{O}(f_{j+N+1})$.

The shadow $\mathcal{O}(f_{j+N+1})$ is in turn the union of $q-1$ shadows $\mathcal{O}(g_1)$ of lifts g_1 of $\overline{e_{j+N+1}}$ adjacent to f_{j+N+1} and the shadow $\mathcal{O}(f_{j+N+2})$. We repeat this decomposition. For any $l \geq 0$, we obtain $q - 1$ shadows $\mathcal{O}(g_l)$'s such that

$$\mu_{x_j}(\mathcal{O}(g_l)) = \mu_{x_j}(\mathcal{O}(\overline{f_{j+N+l}}))e^{-2(N+l)\delta} = \alpha_j q^{N+l} e^{-2(N+l)\delta}$$

by the conformal property and (5.2)

Therefore, for any $j, N \geq 0$, we have

$$\mu_{x_j}(\mathcal{O}(f_{j+N})) = (q-1)\alpha_j \sum_{n=N}^{\infty} q^n e^{-2n\delta} = \frac{(q-1)\alpha_j \left(\frac{q}{e^{2\delta}}\right)^N}{1 - \frac{q}{e^{2\delta}}}.$$

Since $\mu_{x_j}(\mathcal{O}(f_j)) < \infty$, the above series must converge thus $\delta_\Gamma > \frac{1}{2} \log q$ and

$$\frac{\mu_{x_j}(\mathcal{O}_{f_{j+N}})}{\mu_{x_j}(\mathcal{O}_{f_j})} = \left(\frac{q}{e^{2\delta}}\right)^N. \qquad \square$$

Therefore, by independence of excursions (similar to the proof of Proposition 4.3), we obtain a limiting *Galambos type formula*

$$\bar{\mu}(\{[l] \in \Gamma_f \backslash \mathcal{GT} : \max_{1 \le j \le k} a_j(l) \le N + y\}) = \left(1 - \frac{q^{N+y}}{e^{2\delta(N+y)}}\right)^k.$$

Given a geodesic l in $\mathcal{GT}$, let us denote by $C(l)$ the expectation

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (l_{t_{2n+1}} - l_{t_{2n}})$$

of the difference between the starting time of the $(n+1)$-th excursion time and the ending time of the n-th excursion of l (the time living in the compact part) over $n \in \mathbb{Z}_{>0}$. Note that this does not depend on the choice of representative in $\Gamma \backslash \mathcal{GT}$ and the limit exists for μ-almost every $[l] \in \Gamma \backslash \mathcal{GT}$.

Lemma 5.3. *Let $C_\Gamma = \int_{\Gamma \backslash \mathcal{GT}} C([l]) d\mu$. The expectation with respect to μ of the time of n excursions $t_{2n} - t_1$ over $\Gamma \backslash \mathcal{GT}$ is*

$$\left(\frac{2e^{2\delta}}{e^{2\delta} - q} + C_\Gamma\right)n.$$

Proof. Since the Markov chain associated to the compact part is finite, it is positive recurrent (Chapter 10, [13]). Hence, the constant C_Γ is finite and depends only on the structure of quotient graph $\Gamma \backslash \mathcal{T}$ and the choice of the compact part D.

The expectation with respect to μ of $t_{2n} - t_1$ of l is

$$\mathbb{E}_\mu\left(\sum_{i=1}^{n} (2a_i^{(l)} + C^{(l)})\right) = n(2\mathbb{E}_\mu(a_1) + \mathbb{E}_\mu C^{(l)})$$

$$= C_\Gamma n + 2n \sum_{k=1}^{\infty} k\mu(a_1 = k)$$

$$= C_\Gamma n + \sum_{k=1}^{\infty} \frac{2nk(\frac{e^{2\delta}}{q} - 1)}{(\frac{e^{2\delta}}{q})^k}$$

$$= C_\Gamma n + \sum_{k=1}^{\infty} \frac{2nk(e^{2\delta} - q)q^{k-1}}{e^{2\delta k}}$$

$$= \left(\frac{2e^{2\delta}}{e^{2\delta} - q} + C_\Gamma\right)n.$$

This completes the proof of the lemma. $\qquad \square$

By the similar argument deriving Theorem 4.6 from Proposition 4.5, we finally have that

$$\lim_{N\to\infty} \mu(\{[l] \in \Gamma\backslash\mathcal{GT} : \max_{0\le t\le T} h_T^{(l)} \le N + y\}) = e^{-q^y/e^{2\delta y}}$$

with

$$T = \left(\frac{2e^{2\delta}}{e^{2\delta} - q} + C_\Gamma\right)\frac{e^{2\delta N}}{q^N}.$$

Therefore,

$$\lim_{T\to\infty} \mu\left(\left\{[l] \in \Gamma\backslash\mathcal{GT} : h_T^{(l)} \le \log_{e^{2\delta}/q}\left(\frac{T(e^{2\delta} - q)}{2e^{2\delta} - C_\Gamma(e^{2\delta} - q)}\right) + y\right\}\right) = e^{-q^y/e^{2\delta y}}.$$

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