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Generic free subgroups and statistical hyperbolicity

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Abstract. This paper studies the generic behavior of k-tuples of elements for $k \geq 2$ in a proper group action with contracting elements, with applications toward relatively hyperbolic groups, CAT(0) groups and mapping class groups. For a class of statistically convex-cocompact action, we show that an exponential generic set of k elements for any fixed $k \geq 2$ generates a quasi-isometrically embedded free subgroup of rank k. For $k = 2$, we study the sprawl property of group actions and establish that statistically convex-cocompact actions are statistically hyperbolic in the sense of M. Duchin, S. Lelièvre, and C. Mooney.

For any proper action with a contracting element, if it satisfies a condition introduced by Dal'bo-Otal-Peigné and has purely exponential growth, we obtain the same results on generic free subgroups and statistical hyperbolicity.

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Contents

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1. Introduction

1.1. Motivation and background. Suppose that a group G admits a proper and isometric action on a proper geodesic metric space (Y, d). The group G is assumed to be *non-elementary.* It is not virtually cyclic. An element $h \in G$ is called *contracting* if for some basepoint $o \in Y$, an orbit $\{h^n \cdot o : n \in \mathbb{Z}\}$ is a contracting subset, and the map $n \in \mathbb{Z} \mapsto h^n o \in Y$ is a quasi-isometric embedding. Here a subset X is called *contracting* if any metric ball disjoint from X has a uniformly bounded projection to X (see [23, 7]). It is clear that this definition does not depend on the choice of the basepoint.

The prototype of a contracting element is a hyperbolic isometry on Gromov-hyperbolic spaces, but more interesting examples are provided by the following:

- hyperbolic elements in relatively hyperbolic groups or groups with nontrivial Floyd boundary (see [17, 18]);
- rank-1 elements in CAT(0) groups (see [3, 7]);
- certain infinite order elements in certain small cancellation groups (see [2]);
- pseudo-Anosov elements in mapping class groups of closed oriented surfaces with genus greater than two acting on Teichmüller space (see [23]).

In [37], the second-named author proved that, for a class of *statistically convex-cocompact* actions defined below, the set X of contracting elements is *exponentially generic* in the ball model:

$$\frac{|X \cap B_n|}{|B_n|} \longrightarrow 1$$

exponentially fast, where $B_n := \{g \in G : d(o, go) \leq n\}$.

Along this line, the goal of this paper is to continue the study of generic properties for k-tuples of elements in G for a fixed $k \geq 2$. To that end, we introduce a few more notations. We fix a basepoint $o \in Y$ and denote $|g| = d(o, go)$ for easy notation. Denote $G^{(k)} = \{(u_1, \dots, u_k) : u_i \in G\}$ and $B_n^{(k)} = \{(u_1, \dots, u_k) \in G^{(k)} : |u_i| \leq n\}$. When k is understood, we write $\vec{u}$ for $(u_1, \dots, u_k)$, and $|\vec{u}|$ for $\max\{|u_i| : 1 \leq i \leq k\}$.

The *asymptotic density* of a subset $X \subseteq G^{(k)}$ in the ball model is defined as

$$\mu(X) = \lim_{n \to \infty} \frac{|X \cap B_n^{(k)}|}{|B_n^{(k)}|}$$

if the limit exists. If the convergence happens exponentially fast, we denote $\mu(X) \overset{\exp}{=} \lambda \in [0, 1]$. We shall be interested in the extreme cases $\mu(X) \overset{\exp}{=} 1$ (resp. $\mu(X) = 1$) which are called *exponentially generic* (resp. *generic*). By definition, the complement of an (exponentially) generic set is called (*exponentially*) *negligible*.

The generic properties of k-tuples of elements have been studied using random walks in various classes of groups with negative curvature. Let μ be a probability measure with finite support on the group G so that the support generates G as a semi-group. A μ-random walk is a product of a sequence of independent identical μ-distributed random variables on G. In our setting, Sisto [31] proved that the n-th step of a simple random rank lands on a contracting element with asymptotic probability one. In mapping class groups, this was obtained by Maher for pseudo-Anosov elements. And the most general result is, as far as we know, due to Maher and Tiozzo [22] which says that random elements are loxodromic for any non-elementary group action on a hyperbolic space. When $k = 2$, Gilman, Miasnikov, and Osin [19] proved that in hyperbolic groups, two simple random walks on the Cayley graph, with asymptotic probability one, stay at a ping-pong position in n-steps so that they generate an undistorted free group of rank 2. The same result holds in non-virtually solvable linear groups [1] and in mapping class groups [28, 34, 21] for two independent μ-random walks. In fact, most of these works are stated in a general class of groups with hyperbolic embedded subgroups as defined by Dahmani, Guirardel and Osin [10], and equivalently, the class of acylindrically hyperbolic groups in the sense of Osin [25]. It is worth pointing out that a non-elementary group admitting a proper action with a contracting element is acylindrically hyperbolic by a result of Sisto [31]. However, our first goal is to address the analogue of generic free subgroups using counting measures defined as above instead of probability measures from random walks.

In fact, studying the generic properties of k-tuples of elements in a counting measure is not a new idea. In [13], M. Duchin, S. Lelièvre, and C. Mooney initiated a study of the sprawl property of pairs of points in the space. The notion of statistical hyperbolicity is then introduced to capture negative curvature in a statistical sense. Roughly speaking, the intuitive meaning could be explained as follows. Consider the annular set

$$A(n, \Delta) = \{g \in G : ||g| - n| \le \Delta\}$$

for $\Delta > 0$. On average, a random pair of points x, y on an annular set $A(n, \Delta)$ of the group has distance $d(xo, yo)$ nearly equals to $2n$. We formulate this concept using both annuli and balls.

Definition 1.1. Let G admit a proper action on a geodesic metric space (Y, d). Define

$$E_B(G) = \lim_{n\to+\infty} \frac{1}{|B_n|^2} \sum_{x,y\in B_n} \frac{d(x, y)}{n},$$

and for a constant $\Delta > 0$,

$$E_A(G, \Delta) = \lim_{n\to+\infty} \frac{1}{|A(n, \Delta)|^2} \sum_{x,y\in A(n,\Delta)} \frac{d(x, y)}{n},$$

if the limit exists. The action is called *statistically hyperbolic* in annuli (resp. in balls) if $E_A(G, \Delta) = 2$ for any sufficiently large $\Delta > 0$ (resp. $E_B(G) = 2$).

Remark. In [13] this definition was introduced using annular model with $\Delta = 0$ in Cayley graphs of groups. Here we consider also the quantity $E_B(G)$ without involving the extra parameter Δ. In our results, we obtain $E_A(G, \Delta) = E_B(G) = 2$ along the same line of proofs.

The non-examples include elementary groups, $\mathbb{Z}^d$ for $d \geq 2$, and the integer Heisenberg group for any finite generating set among others (cf. [13]). In the opposite, the exact value of $E_B(G) = 2$ indeed happens for many groups with certain negative curvature from the point of view of coarse geometry. For instance, non-elementary relatively hyperbolic groups are statistically hyperbolic for any finite generating set (cf. [13, 24]). Moreover, the statistical hyperbolicity is preserved under certain direct product of a relatively hyperbolic group and a group. And the lamplighter groups $\mathbb{Z}_m \wr \mathbb{Z}$ where $m \geq 2$ are statistically hyperbolic for certain generating sets [13].

The notion of statistical hyperbolicity could be considered for any metric space with a measure as in [13], rather than our definition using a counting measure. In this direction, it was proved in the same paper that for any $m, p \geq 2$, the Diestel-Leader graph $DL(m, p)$ is statistically hyperbolic. The statistical hyperbolicity for Teichmüller space with various measures was proved by Dowdall, Duchin and Masur in [12].

The second goal of this paper is to generalize these results in a very general class of proper actions using counting measures from orbits in Definition 1.1. In what follows, we shall describe our results in detail.

1.2. Main results. In order to state our results, we first give a quick overview of the various classes of actions under consideration in this study. First of all, we consider the class of statistically convex-cocompact actions introduced in [37] which generalizes a convex-cocompact action in a statistical sense. Making this idea precise requires the notion of *growth rate* of a subset X in G:

$$\delta_X = \limsup_{n \to \infty} \frac{\ln |X \cap B_n|}{n}.$$

It is clear that the value δ_X does not depend on the choice of the basepoint. By abuse of language, a geodesic between two sets A and B is a geodesic between $a \in A$ and $b \in B$.

Given constants $0 \leq M_1 \leq M_2$, let $\mathcal{O}_{M_1,M_2}$ be the set of element $g \in G$ such that there exists some geodesic γ between $N_{M_2}(o)$ and $N_{M_2}(go)$ with the property that the interior of γ lies outside $N_{M_1}(Go)$.

Definition 1.2 (SCC action). If there exist positive constants $M_1, M_2 > 0$ such that $\delta_{\mathcal{O}_{M_1,M_2}} < \delta_G < \infty$, then the proper action of G on Y is called *statistically convex-cocompact* (SCC).

The idea to define the set $\mathcal{O}_{M_1,M_2}$ is to look at the action of the fundamental group of a finite volume Hadamard manifold on its universal cover. It is then easy to see that for appropriate constants $M_1, M_2 > 0$, the set $\mathcal{O}_{M_1,M_2}$ coincides with the union of cusp subgroups up to a finite Hausdorff distance. The assumption in SCC actions was called a *parabolic gap condition* by Dal'bo, Otal, and Peigné in [11]. One of the motivations of this study is to push forward the analogy between the concave set $\mathcal{O}_{M_1,M_2}$ and the (union of) parabolic cusp regions. This allows us to draw conclusions for the SCC actions through the analogy with the geometrically finite actions, which have been well studied in last twenty years.

Moreover, our study suggests considering a class of proper actions satisfying a more general condition introduced in the same paper [11]. The condition, reformulated below, is proved to be equivalent to the finiteness of Bowen–Margulis–Sullivan (BMS) measure on the geodesic flow of the unit tangent bundle of a geometrically finite Hadamard manifold in [11], and later for any Hadamard manifold by Pit and Shapira [27, Theorem 2].

Definition 1.3 (DOP condition). The group action of G on Y satisfies the *Dal'bo–Otal–Peigné* (*DOP*) *condition* if there exist two positive constants $M_1, M_2 > 0$ such that

$$\sum_{g \in \mathcal{O}_{M_1,M_2}} |g| \exp(-\delta_G |g|) < \infty$$

Remark. We remark that, in the setting of negatively curved manifolds, the DOP condition is called *positive recurrence* by Pit and Shapira in [27], whereas the notion of SCC actions is called *strongly positive recurrence* by Shapira and Tapie in [30]. We thank Rémi Coulon for bringing these references to our attention.

The concept of the geodesic flow is non-applicable in a general geodesic metric space with coarse negative curvature features such as the contracting property. However, the definition of the DOP condition could be always made, and so could be understood as substitute of finite BMS measures in a general metric space. One of Roblin's results [29, Théoreme 4.1] stated that in the setting of a geometrically finite Hadamard manifold, the finiteness of BMS measures is characterized by the *purely exponential growth* (PEG) of the action

$$|B_n| \asymp \exp(\delta_G n).$$

Hence, proper actions with purely exponential growth should be viewed as equivalent to DOP conditions. We expect that this relation persists in a very general setting, and remark that it is indeed true for the class of geometrically finite actions on a δ-hyperbolic space in [36] (weaker than the setting of Roblin).

Our first main result establishes a *generic free basis property* for the actions with the DOP and PEG condition stated as follows.

Theorem 1.4. *Assume that a non-elementary group G acts properly on a geodesic metric space (Y, d) with a contracting element. If G satisfies the DOP and PEG conditions, then for any $k \geq 2$, the set of all tuples $(u_1, \ldots, u_k) \in G^{(k)}$ generating a free subgroup H of rank k in G is generic in $G^{(k)}$.*

Moreover, we can further require that all non-trivial elements of H are contracting, and that the orbital map $h \in H \mapsto ho \in Y$ is a quasi-isometric embedding with the image Ho being a contracting subset in Y.

When the action is SCC, the above DOP and PEG conditions are satisfied, and moreover, we can obtain an exponential convergence rate for the above conclusion.

Theorem 1.5. *Assume that a non-elementary group G admits a SCC action on a geodesic metric space (Y, d) with a contracting element. Then for any $k \geq 2$, the set of all $(u_1, \ldots, u_k) \in G^{(k)}$ generating a free subgroup H of rank k in G is exponentially generic in $G^{(k)}$.*

The "moreover" statement in Theorem 1.4 holds as well for the free subgroup H.

A group generated by a finite set acts cocompactly on its Cayley graph, so our results apply for this particular case. A finitely generated subgroup H is called *undistorted* if the inclusion $H \subset G$ is a quasi-isometric embedding with respect to word metrics.

Corollary 1.6. *Let G be a non-elementary group with a finite generating set S. If G has a contracting element, then for any $k \geq 2$, the set of all $(u_1, \ldots, u_k) \in G^{(k)}$ which generates an undistorted free subgroup H of rank k in G is exponentially generic in $G^{(k)}$, and we can further require that all non-trivial elements of H are contracting.*

To illustrate consequences of previous results, we now list examples of groups with contracting elements with respect to some Cayley graph:

(1) any relatively hyperbolic group G acting on a Cayley graph $\mathcal{G}(G, S)$ with respect to a finite generating set S, see [17];

(2) any group G with non-trivial Floyd boundary acting on a Cayley graph $\mathcal{G}(G, S)$ with respect to a finite generating set S, see [17];

(3) any right-angled Artin (Coxeter) group with respect to the standard generating set, if it is not virtually direct product, see [4, 6, 9];

(4) any $\mathrm{Gr}'\left(\frac{1}{6}\right)$-labeled graphical small cancellation group G with finite components labeled by a finite set S acting on its Cayley graph $\mathcal{G}(G, S)$ (cf. [2]). In particular, this includes any classical $\mathrm{C}'\left(\frac{1}{6}\right)$ small cancellation group G with a finite generating set S acting on its Cayley graph $\mathcal{G}(G, S)$. See [32] for a definition of $\mathrm{C}'\left(\frac{1}{6}\right)$ groups.

Thus, by Corollary 1.6, the list of these examples all have the generic free basis property. We remark that this result is even new in the class of relatively hyperbolic groups.

We next explain an application of Theorem 1.5 about surface group extensions. Let $\mathbb{M}(\Sigma_g)$ be the mapping class group of a closed oriented surface Σ_g of genus $g \geq 2$. Combining the results of Minsky [23] and Eskin-Mirzakhani-Rafi [14] we know that the action of $\mathbb{M}(\Sigma_g)$ on Teichmüller space $\mathcal{T}(\Sigma_g)$ is a SCC action with a contracting element. By Theorem 1.5, we obtain the exponential genericity of k-tuples of elements $(u_1, u_2, \ldots, u_k)$ that are free bases in the counting measure from the Teichmüller metric. Denote $\Gamma := \langle u_1, u_2, \ldots, u_k \rangle$. Marking a point $p \in \Sigma_g$, the Bireman exact sequence in [8] gives an extension E_Γ in $\mathbb{M}(\Sigma_g, p)$ of the surface group $\pi_1(\Sigma_g, p)$ by Γ as follows

$$1 \longrightarrow \pi_1 \Sigma_g \longrightarrow E_\Gamma \longrightarrow \Gamma \longrightarrow 1.$$

We refer the reader to the reference [15] for related facts about $\mathbb{M}(\Sigma_g)$ and $\mathcal{T}(\Sigma_g)$.

In [16], Farb and Mosher studied when the extension is a hyperbolic group and showed that, when Γ is a Schottky group, this is equivalent to the quasi-convexity of Γ-orbits in $\mathcal{T}(\Sigma_g)$.

In Theorem 1.5, the quasi-isometrically embedded image of the free group Γ are contracting and thus quasi-convex in the sense of Farb and Mosher. Thus, by [16, Theorem 1.1], the free group Γ is convex-cocompact in their sense, so the following result holds.

Theorem 1.7. *The set of k-tuples of elements $(u_1, u_2, \ldots, u_k)$ in $\mathbb{M}(\Sigma_g)$ with hyperbolic extension in $\mathbb{M}(\Sigma_g, p)$ is exponentially generic.*

Remark. It is worth pointing out that the same result has been obtained by Taylor and Tiozzo [34] where the k-tuples are chosen via independent random walks.

Our second main result obtains the statistical hyperbolicity for the same class of actions as in Theorem 1.4, and in particular for statistically convex-cocompact actions.

Theorem 1.8. *Let a non-elementary group G act properly on (Y, d) with a contracting element satisfying the DOP and PEG conditions. Then G is statistically hyperbolic in balls and annuli. In particular, if the action is SCC, then G is statistically hyperbolic in balls and annuli.*

Remark. Motivated by the exponential convergence rate for SCC actions in Theorem 1.5, one may wonder whether there is a significant convergence rate of $E_A(G, \Delta)$ or $E_B(G)$ under SCC actions. This is, however, not true even in free groups: a simple computation as Example 4.5 shows that the convergence rate is of order $\frac{1}{n}$. Hence, we have no assertion on the convergence speed.

Except the class of SCC actions, the actions of discrete groups on CAT(-1) spaces provide a source of examples with DOP condition and purely exponential growth. For example, combining [29] and [27] we obtain that the finiteness of the Bowen–Margulis–Sullivan measure on the geodesic flow is equivalent to either having purely exponential growth or satisfying the DOP condition. Hence, we obtain the following corollary.

Theorem 1.9. *Suppose that the Bowen–Margulis–Sullivan measure on the unit tangent bundle of a Hadamard manifold is finite. Then the fundamental group action on the universal cover is statistically hyperbolic in balls and annuli. Moreover, the generic pair of elements generate a free group of rank* 2 *which is a uniformly quasi-isometric embedding with contracting image.*

If a hyperbolic n-manifold for $n \geq 2$ is geometrically finite, then the BMS measure is always finite [33]. We thus have the following corollary in Kleinian groups, which seems to be not recorded in the literature. Note that examples of non-geometrically finite Kleinian groups with finite BMS measures are constructed for $n \geq 4$ by Peigné in [26].

Corollary 1.10. *Geometrically finite Kleinian groups are statistically hyperbolic and have the generic free basis property.*

For the action of mapping class groups on Teichmüller space, we then have the following corollary, which could be thought of as a discrete analogue of the result in [12].

Corollary 1.11. *The action of mapping class groups on Teichmüller space is statistically hyperbolic with respect to the Teichmüller metric.*

Of course, the action of a group on the Cayley graph is SCC, so if there exists a contracting element, then it is statistically hyperbolic. This allows us to give new examples of groups with the statistically hyperbolic property in the original sense [13].

Corollary 1.12. *The following classes of groups are statistically hyperbolic with respect to word metrics of special generating set.*

(1) *Any* $\mathrm{Gr}'\left(\frac{1}{6}\right)$*-labeled graphical small cancellation group* (G, S) *with finite components labeled by a finite set* S*. In particular, any classical* $\mathrm{C}'\left(\frac{1}{6}\right)$ *small cancellation group* (G, S) *with a finite generating set* S*.*

(2) *Right-angled Artin* $($*Coxeter*$)$ *groups are statistically hyperbolic with respect to the standard generating set, if they are not virtually direct product.*

We point out that it is not clear to us whether the above two classes of groups are statistically hyperbolic for every generating set. Note that they include non-relatively hyperbolic examples of groups (cf. [5, 20]). Hence, it would be interesting to know to which extent the statistical hyperbolicity for every generating set characterizes the class of relatively hyperbolic groups.

1.3. Sketch of proofs. To conclude the introduction, we give a sketch of proofs of main theorems. We refer the reader to Section 4 for the full detail.

Sketch of proof of Theorem 1.4. We consider $k = 2$ and explain the main ingredients of the proof that a *generic subgroup* H generated by two elements u_1, u_2 of G is free.

The strategy to prove that $\langle u_1, u_2 \rangle$ is free is standard. We construct a quasi-geodesic β out of every freely reduced word $W = x_1 x_2 \dots x_n$ $(n \geq 1)$ over the alphabet $\{u_1, u_2, u_1^{-1}, u_2^{-1}\}$. This quasi-geodesic has the following structure: it is a piecewise geodesic $\beta = q_0 p_1 q_1 \dots p_n q_n$ which travels alternatively into a sequence of contracting subsets X_i, i.e., $(p_i)_+, (p_i)_- \in X_i$. These $\{X_i\}$ are translated axis of a predefined contracting element h so they have uniformly bounded projection.

The path with such a structure is called *admissible* (Definition 2.10). A result in [35], Proposition 2.11 here, shows that if p_i is sufficiently long and each q_i has bounded projection to X_{i-1} and X_i, then β is a quasi-geodesic. The difficulty is thus the *genericity* of pairs (u_1, u_2) which allows to obtain such an admissible path for every word W.

We now explain in more details the construction of this admissible path and the generic choice of (u_1, u_2). We first start with a piecewise geodesic $\gamma = \gamma_1 \dots \gamma_n$ labeled by letters x_i in the word W. To get the admissible path β, we require each γ_i to travel into a contracting subset $X_i \in \mathbb{X}$, where $\mathbb{X} = \{g\mathbf{Ax}(h) \colon g \in G\}$. We then truncate the "angle" at $x_1 \dots x_i o$ between two consecutive X_i and X_{i+1} and replace with a geodesic q_i. See the truncation path as indicated by the red dotted path in Figure 1.

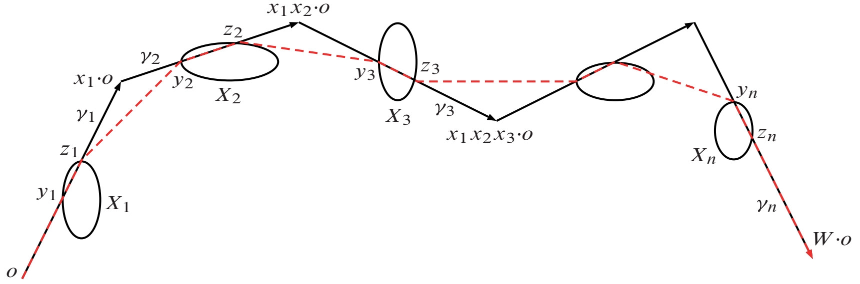

Figure 1. Truncated admissible path.

Keeping in mind the desired bounded projection in Definition 2.10, we are thus lead to consider the following sets.

(1) The set of elements $u \in E$ contains an (ν, h)-barrier (Definition 2.12) in the $(\varepsilon, 1-\varepsilon)$-proportion of $[o, uo]$. See its precise definition in (9). Thus, the barrier in $x_i \in E$ produces a contracting set $X_i \in \mathbb{X}$.

(2) For each pair $(u_1, u_2) \in E^{(2)}$, any segment $[o, so]$ labeled by an element $s \in S := \{u_1, u_2, u_1^{-1}, u_2^{-1}\}$ stays far from the (ν, h)-barrier of the other $t \neq s^{-1} \in S$. The contracting property then gives the bounded projection of the "angle" and thus of the geodesic q_i, to the barrier X_i of x_i.

The main technical contribution of this paper is showing the negligibility of several sets of independent interests under actions with the DOP and PEG conditions. Among those, the genericity of the set E is proved in Lemma 3.3, and the genericity of the set $E^{(2)}$ is proved under two cases treated separately in Lemmas 3.4 and 3.5. The proof of these results uses often Lemma 3.1 which says that the set of elements with an ε-proportion subsegment without (ν, h)-barrier is negligible. In the proof of this lemma, the DOP and PEG conditions are used crucially to get the negligibility, and if the action is SCC, then the set is exponentially negligible. □

Sketch of proof of Theorem 1.8. We give a sketch of the proof for the annuli case; the ball case is similar. The proof of statistical hyperbolicity boils down to the fact that for every $\varepsilon > 0$, the generic pair $(x, y) \in A(n, \Delta)^{(2)}$ stays at an *opposite position*:

$$d(xo, yo) \geq d(o, xo) + d(o, yo) - 10\varepsilon n \geq 2n(1 - 5\varepsilon).$$

By the genericity of pairs (x, y), this inequality implies $E_A(G) = 2$.

Consider the geodesics $\alpha = [o, xo], \beta = [o, yo]$. Fix a contracting element h with the C-contracting axis $\mathbf{Ax}(h)$ for some $C > 0$. To achieve the above inequality, we make use of a generic choice of pairs (x, y) with the properties very similar to the ones as above in the set E.

(1) The geodesic α contains an (ν, h^m)-barrier in the $(2\varepsilon, 3\varepsilon)$-proportion of α. The barrier produces a contracting set $g\mathbf{Ax}(h) := gX$. This is proved in Lemma 3.3.

(2) The geodesic β stays far from gX: $\beta \cap N_C(gX) = \emptyset$, by Lemma 3.5.

(3) Let v, w be the entry and exit points of α in $N_C(gX)$. Then the distance $d(u, v)$ is bigger than $d(o, h^m o) - 2\nu$ by the first property, but is not larger than $\varepsilon|x|$, which is guaranteed by Lemma 3.2. Thus, v, w lie in $(2\varepsilon, 3\varepsilon)$-proportion of α.

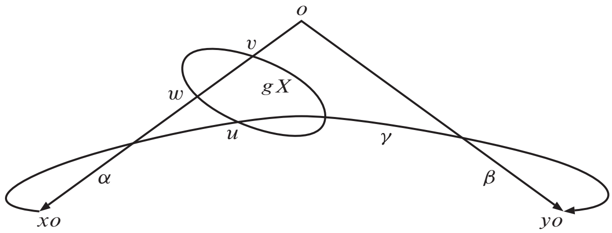

Figure 2. statistical hyperbolicity.

A consequence of these properties is that $\gamma \cap N_C(gX) \neq \emptyset$ for $\gamma := [xo, yo]$. Indeed, if $\gamma \cap N_C(gX) = \emptyset$, then by the first and second properties, we get

$$\begin{aligned} d(v,w) &\le \|\Pi_X(\{v,o\})\| + \|\Pi_X(\{o,yo\})\| + \|\Pi_X(\{yo,xo\})\| + \|\Pi_X(\{xo,w\})\| \\ &\le 8C. \end{aligned}$$

If m is chosen sufficiently large, then this contradicts with the third property that $d(u,v) \ge d(o,h^m o) - 2\nu$. Thus, we proved that γ must intersect $N_C(gX)$ at the entry point u. In Lemma 4.4, a further estimate shows $d(u,w) \le 4C$, and consequently,

$$d(xo, yo) \ge 2(n - 4\varepsilon n - 4\varepsilon\Delta - \Delta - 4C).$$

Since ε can be made arbitrary small, we obtain $E_A(G) = 2$. □

The structure of this paper. The preliminary Section 2 discusses the notions such as contracting elements, SCC actions, the DOP condition etc. and list a few useful results needed later on. The main technical contribution of this paper is contained in Section 3 which provides several negligible sets with property required in the sketches of proofs in Introduction. Section 4 explains the choice of a generic set of pairs whose properties are used to complete the proofs of Theorems 1.4, 1.5, and 1.8.

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2. Preliminaries

In this section, we will introduce some preliminaries. First we fix some notations and conventions.

2.1. Notations and conventions. Let (Y, d) be a proper geodesic metric space. The r closed neighborhood of a subset $X \subseteq Y$ is denoted by $N_r(X)$. We denote by $\|X\|$ the diameter of a subset $X \subseteq Y$ and by $d_{\text{Haus}}(X_1, X_2)$ the Hausdorff distance of two subsets $X_1, X_2 \subseteq Y$. Given a point $y \in Y$, and a subset $X \subseteq Y$, let $\Pi_X(y)$ be the set of point x in X such that $d(y,x) = d(y,X)$. The projection $\Pi_X(y)$ is non-empty if X is closed, which is the case in our applications. The *projection* of a subset $A \subseteq Y$ to X is then $\Pi_X(A) := \bigcup_{a \in A} \Pi_X(a)$.

The path γ in Y under consideration is always assumed to be rectifiable with arc-length parametrization $[0, |\gamma|] \to \gamma$, where $|\gamma|$ denotes the length of γ. Denote by γ_-, γ_+ the initial and terminal points of γ respectively. For any two parameters $a < b \in [0, |\gamma|]$, we denote by $[\gamma(a), \gamma(b)]_\gamma := \gamma([a,b])$ and $(\gamma(a), \gamma(b))_\gamma := \gamma((a,b))$ the closed (resp. open) subpath of γ between a and b. For any $x, y \in Y$, we denote by $[x, y]$ a choice of geodesic in Y from x to y.

Given a property (P), a point z on γ is called the *entry point* satisfying (P) if $|[\gamma_-, z]_\gamma|$ is minimal among the points z on γ with the property (P). A point w on γ is called the *exit point* satisfying (P) if $|[w, \gamma_+]_\gamma|$ is minimal among the points w on γ with the property (P).

A path γ is called a (λ, c)-quasi-geodesic for $\lambda \geq 1, c \geq 0$ if the following holds

$$|\beta| \leq \lambda \cdot d(\beta_-, \beta_+) + c$$

for any rectifiable subpath β of α.

Let β, γ be two paths in Y. Denote by $\beta \cdot \gamma$ (or simply $\beta\gamma$) the concatenated path provided that $\beta_- = \gamma_+$.

Let f, g be real-valued functions with domain understood in the context. Then $f \prec_{c_i} g$ means that there is a constant $a > 0$ depending on parameters c_i such that $f < ag$. The symbols $\succ_{c_i}$ and $\asymp_{c_i}$ are defined analogously. For simplicity, we shall omit c_i if they are universal constants.

We say a sequence $\{a_n\} \subseteq \mathbb{R}$ of numbers converges to a number $\lambda \in \mathbb{R}$ *exponentially fast*, denoted by $a_n \xrightarrow{\exp} \lambda$, if

$$|\lambda - a_n| \leq c\theta^n$$

for some constant $\theta \in (0,1)$ and a positive constant $c > 0$.

Remark. (1) It is clear that the (exponential) genericity is preserved by taking any finite intersection and finite union. This fact shall be often used implicitly.

(2) If X is exponentially negligible, then $\delta_X < \delta_G$, in which case we call X *growth tight* in [35]. Note that if G has purely exponential growth, then a growth tight set is exponentially negligible. In this paper, the group actions under consideration always have purely exponential growth, so we do not distinguish these two notions.

2.2. Contracting property. We fix a preferred class of quasi-geodesics $\mathcal{L}$, which contains at least all geodesics in Y.

Definition 2.1 (contracting subset). A subset $X \subseteq Y$ is called *κ-contracting* with respect to $\mathcal{L}$ if for any quasi-geodesic $\gamma \in \mathcal{L}$ with $d(\gamma, X) \geq \kappa$, we have $\|\Pi_X(\gamma)\| \leq \kappa$. A collection of κ-contracting subsets is referred to as a *κ-contracting system* (with respect to $\mathcal{L}$).

We first note the following examples in various contexts.

Examples 2.2. Let $\lambda \geq 1, c \geq 0$ be any fixed numbers.

(1) Quasi-geodesics and quasi-convex subsets are contracting with respect to the set of all (λ, c)-quasi-geodesics in hyperbolic spaces.

(2) Fully quasi-convex subgroups (and in particular, maximal parabolic subgroups) are contracting with respect to the set of all (λ, c)-quasi-geodesics in relatively hyperbolic groups (see [18, Proposition 8.2.4]).

(3) The subgroup generated by a hyperbolic element is contracting with respect to the set of all (λ, c)-quasi-geodesics in groups with non-trivial Floyd boundary (see [35, Section 7]).

(4) Contracting segments in CAT(0)-spaces in the sense of Bestvina and Fujiwara are contracting with respect to the set of all geodesics (see [7, Corollary 3.4]).

(5) The axis of any pseudo-Anosov element is contracting with respect to all geodesics in Teichmüller spaces [23].

Convention 2.3. In view of the above examples, the preferred collection $\mathcal{L}$ in the sequel will always be the set of all geodesics in Y.

The notion of a contracting subset is equivalent to the following one considered by Minsky [23]. The proof given in [7, Corollary 3.4] for CAT(0) spaces is valid in the general case. In this paper, we will always work with the above definition of the contracting property.

Lemma 2.4. *A subset X is contracting in Y if and only if any open ball B missing X has a uniformly bounded projection to X.*

We collect some properties of contracting sets that will be used later on. The proof is straightforward and is left to the interested reader.

Lemma 2.5. *Let X be a contracting set.*

(1) ***Quasi-convexity.*** *X is σ-quasi-convex for a function $\sigma: \mathbb{R}_{\geq 0} \to \mathbb{R}_+$: given $r \geq 0$, any geodesic with endpoints in $N_r(X)$ lies in the neighborhood $N_{\sigma(r)}(X)$.*

(2) ***Finite neighborhood.*** *Let Z be a set with finite Hausdorff distance to X. Then Z is contracting.*

(3) *There exists a constant $C > 0$ such that for any geodesic segment γ,*

$$\left| \|\Pi_X(\{\gamma_-, \gamma_+\})\| - \|\Pi_X(\gamma)\| \right| \leq C. \tag{1}$$

In most situations, we are interested in a contracting system $\mathbb{X}$ with *ν-bounded intersection* for a function $\nu\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, i.e. the following holds

$$\|N_r(X) \cap N_r(X')\| \leq \nu(r) \quad \text{for all } X \neq X' \in \mathbb{X}$$

for any $r \geq 0$. This property is, in fact, equivalent to a bounded projection property of $\mathbb{X}$: there exists a constant $B > 0$ such that

$$\|\Pi_X(X')\| \leq B$$

for any $X \neq X' \in \mathbb{X}$. See [35] for further discussions.

An infinite subgroup $H < G$ is called *contracting* if for some (hence any by [37, Proposition 2.4.2]) $o \in Y$, the subset Ho is contracting in Y. In fact, we usually deal with a contracting subgroup H with *bounded intersection*: the collection of subsets

$$\{gH \cdot o\colon g \in G\}$$

is a contracting system with bounded intersection in Y. (In [35], a contracting subgroup H with bounded intersection was called strongly contracting.)

An element $h \in G$ is called *contracting* if the subset $\langle h \rangle o$ is contracting, and the orbital map $n \in \mathbb{Z} \mapsto h^n o \in Y$ is a quasi-isometric embedding. The set of contracting elements is preserved under conjugacy.

Let H be a contracting subgroup. We define a group $E(H)$ as follows:

$$E(H) := \{g \in G\colon \text{there exists } r > 0 \text{ such that } gHo \subseteq N_r(Ho),\ Ho \subseteq N_r(gHo)\}.$$

For a contracting element h, we have the following result about $E(h) := E(\langle h \rangle)$ (see [37, Lemma 2.11]).

Lemma 2.6. *Assume that G acts properly on (Y, d). For a contracting element h, the following statements hold*:

(1) $E(h) = \{g \in G\colon \text{there exists } n > 0$ *such that* $gh^n g^{-1} = h^n$ or $gh^n g^{-1} = h^{-n}\}$;

(2) $[E(h)\colon \langle h \rangle] < \infty$, *and $E(h)$ is a contracting subgroup with bounded intersection.*

The contracting subset $\mathbf{Ax}(h) := \{f \cdot o\colon f \in E(h)\}$ shall be referred to as the *axis* of h. In the following discussion, we always fix a contracting element h, so we denote $A = \mathbf{Ax}(h)$ for simplicity.

The following lemma is elementary and well-known. We provide a proof for completeness.

Lemma 2.7. *For any $C > 0$, let γ be a geodesic whose interior does not meet $N_C(A)$. Then*

$$d_{\text{Haus}}(\Pi_{N_C(A)}(\gamma), \Pi_A(\gamma)) \leq C.$$

In particular, if C is a contracting constant of A, then $\|\Pi_{N_C(A)}(\gamma)\| \leq 3C$.

Proof. For any $x \notin N_C(A)$, it is sufficient to prove

$$d_{\text{Haus}}(\Pi_{N_C(A)}(x), \Pi_A(x)) \leq C.$$

For any $y \in \Pi_A(x)$, let z be the point of a geodesic $[x, y]$ such that $d(y, z) = C$. We claim that $z \in \Pi_{N_C(A)}(x)$. Indeed, for each $z' \in N_C(A)$, there exists $y' \in A$ such that $d(y', z') \leq C$. By the definition of $y \in \Pi_A(x)$, we have $d(x, y) \leq d(x, y')$ so

$$d(x, z') + d(z', y') \geq d(x, y') \geq d(x, y) = d(x, z) + d(z, y),$$

which implies $d(x, z') \geq d(x, z)$. Thus, $\Pi_A(x) \subseteq N_C(\Pi_{N_C(A)}(x))$.

For any $z \in \Pi_{N_C(A)}(x)$, there exists $y \in A$ so that $d(y, z) \leq C$. Now for any $y' \in \Pi_A(x)$, there exists $z' \in \Pi_{N_C(A)}(x)$ so that $d(x, y') = d(x, z') + C$ by the above discussion. Then

$$d(x, y) \leq d(x, z) + d(y, z) \leq d(x, z) + C = d(x, z') + C = d(x, y').$$

This implies $y \in \Pi_A(x)$, and so $\Pi_{N_C(A)}(x) \subseteq N_C(\Pi_A(x))$. □

Lemma 2.8. *Let $C > 0$ be the contraction constant of A and α, β be two geodesics with the same initial endpoint. If x is the entry point of α into $N_C(A)$ and $\beta \cap N_{4C}(x) = \emptyset$, then $\beta \cap N_C(A) = \emptyset$.*

Proof. If $\beta \cap N_C(A) \neq \emptyset$, then let $y \in \beta$ be the entry point of β in $N_C(A)$. We have

$$d(x, y) \leq C + \|\Pi_A([\alpha_-, x]_\alpha)\| + \|\Pi_A([\beta_-, y]_\beta)\| + C \leq 4C,$$

which proves the lemma. □

Since $gN_C(A) = N_C(gA)$ for every $g \in G$, the following lemma is a consequence of Lemmas 2.6, 2.7, and 2.5.

Lemma 2.9. *For any $C \geq 0$, the collection $\mathbb{X} = \{gN_C(A) : g \in G\}$ is a $3C$-contracting system with bounded projection.*

2.3. Admissible path. Let $\mathbb{X}$ be a contracting system with the bounded intersection property. The following notion of an admissible path will be used to obtain a quasi-geodesic path.

Definition 2.10 (admissible path). Given $D, \tau \geq 0$ and a function $\mathcal{R}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a path γ is called (D,τ)-*admissible* in Y, if γ is a concatenation of geodesic subpaths $p_0 q_1 p_1 \dots q_n p_n$ ($n \in \mathbb{N}$), the two endpoints of p_i lie in some $X_i \in \mathbb{X}$ for each i, and satisfies the following called *Long Local* and *Bounded Projection* properties.

(LL1) Each p_i for $1 \leq i < n$ has length bigger than D, and p_0, p_n could be trivial.

(BP) For each X_i, we have $\max\{\|\Pi_{X_i}(q_i)\|, \|\Pi_{X_i}(q_{i+1})\|\} \leq \tau$, where $q_0 := \gamma_-$ and $q_{n+1} := \gamma_+$ by convention.

(LL2) Either $X_i \neq X_{i+1}$ have $\mathcal{R}$-bounded intersection or q_{i+1} has length bigger than D.

Saturation. The collection of $X_i \in \mathbb{X}$ indexed as above, denoted by $\mathbb{X}(\gamma)$, will be referred to as contracting subsets for γ. The union of all $X_i \in \mathbb{X}(\gamma)$ is called the *saturation* of γ.

The set of endpoints of p_i shall be referred to as the vertex set of γ. We call $(p_i)_-$ and $(p_i)_+$ the corresponding entry vertex and exit vertex of γ in X_i. (compare with entry and exit points in Section 2.1)

The basic fact is that a "long" admissible path is a quasi-geodesic.

Proposition 2.11 ([35, Corollary 3.2]). *Let κ be the contraction constant of $\mathbb{X}$. For any $\tau > 0$, there are constants $D_0 = D_0(\kappa,\tau) > 0, \Lambda = \Lambda(\kappa,\tau) > 0$ such that given $D > D_0$ any (D,τ)-admissible path is a (Λ,Λ)-quasi-geodesic.*

Remark. We note that the admissible path γ in [35, Corollary 3.2] was originally claimed to be a $(\Lambda, 0)$-quasi-geodesic, i.e. a bi-Lipschitz path. This is certainly wrong when the concatenated admissible path is not simple. However the quasi-geodesicity does follow from Proposition 3.1 there, which says that the endpoints of each p_i stay uniformly close to the geodesic with same endpoints of γ. We thank the referee for bringing our attention to this mistake.

We refer the reader to [35, 37] for further discussions about admissible paths.

2.4. SCC actions and barrier-free elements. We recall the notion of a barrier-free element from [37].

Definition 2.12. Fix constants $\nu, M > 0$.

(1) Given $\nu > 0$ and $g \in G$, we say that a geodesic γ contains an (ν, g)-*barrier* if there exists an element $z \in G$ so that

$$\max\{d(z \cdot o, \gamma), d(z \cdot go, \gamma)\} \leq \nu. \tag{2}$$

If no such $z \in G$ exists so that (2) holds, then γ is called (ν, g)-*barrier-free*.

(2) An element $f \in G$ is (ν, M, g)-*barrier-free* if there exists an (ν, g)-barrier-free geodesic between $N_M(o)$ and $N_M(fo)$.

We have chosen two parameters M_1, M_2 so that the definition of a statistically convex-cocompact action 1.2 is flexible and easy to verify. It is enough to take $M_1 = M_2 = M$ in our use. Henceforth, we set $\mathcal{O}_M := \mathcal{O}_{M,M}$ for ease of notation. When the SCC action contains a contracting element, the definition is independent of the basepoint (see [37, Lemma 6.2]).

Given $\nu, M > 0$ and any $g \in G$, let $\mathcal{V}_{\nu,M,g}$ be the collection of all (ν, M, g)-barrier-free elements of G. The following results will be key in next sections.

Proposition 2.13 ([37, Theorem B, C]). *If G admits a SCC action on a proper geodesic space (Y, d) with a contracting element, then*

(1) *G has purely exponential growth.*

(2) *Let M_0 be the constant in the definition of SCC action, then for any $M > M_0$, there exists $\nu = \nu(M) > 0$ such that $\mathcal{V}_{\nu,M,g}$ is exponentially negligible for any $g \in G$.*

It is easy to see from the proof of [37, Corollary 4.5] that the following conclusion holds in a general proper action.

Proposition 2.14. *Suppose that a group G acts properly on a proper geodesic space (Y, d) with a contracting element, then for any $M > 0$, there exists $\nu = \nu(M) > 0$, $\Delta_0 = \Delta_0(M)$ so that*

$$\sum_{n=1}^{+\infty} |\mathcal{V}_{\nu,M,g} \cap A(n, \Delta)| \exp(-n\delta_G) < +\infty$$

for any $g \in G$ and $\Delta > \Delta_0$.

2.5. The DOP condition. This subsection collects several useful consequences of the Dal'bo-Otal-Peigné condition. For any $0 \leq n_1 \leq n_2$, we consider the following annulus-like set

$$A([n_1, n_2], \Delta) := \{g \in G : n_1 - \Delta \leq d(o, go) \leq n_2 + \Delta\}.$$

Usually, we consider the (ρ, Δ)-*annulus* $A([\rho n, n], \Delta)$ for $\rho \in [0, 1]$. For simplicity, we write $A([\rho n, n])$ if $\Delta = 0$, and assume that ρn are integers.

Observe that

$$\sum_{g \in \mathcal{O}_{M_1, M_2}} |g| \exp(-\delta_G |g|) \asymp_\Delta \sum_{n=1}^{+\infty} n |\mathcal{O}_{M_1, M_2} \cap A(n, \Delta)| \exp(-n\delta_G), \tag{3}$$

for any $\Delta > 0$. Indeed, this follows from the fact that any $g \in \mathcal{O}_{M_1, M_2}$ is contained in a uniform number of annular sets $A(n, \Delta)$ where $n \geq 1$. Consequently,

$$\sum_{g \in \mathcal{O}_{M_1, M_2}} \exp(-\delta_G |g|) < \infty. \tag{4}$$

Thus, if G admits a SCC action on Y, then the action satisfies the DOP condition. We remark that the formula (4) turns out to be true for any proper action of G on (Y, d) with a contracting element: the methods in [37] can be invoked to prove (4). This generality is not used here and so the details are left to the interested reader.

For any $\Delta > 0$, let

$$\mathcal{O}_M(n, \Delta) := \mathcal{O}_M \cap A(n, \Delta) \cup \{1\}, \quad \mathcal{V}_{\nu,h}(n, \Delta) := \mathcal{V}_{\nu,h} \cap A(n, \Delta).$$

The following elementary lemma will be needed in the next section.

Lemma 2.15. *Assume that the proper group action satisfies the DOP condition. For any* $\varepsilon \in (0, 1)$ *and any* $\Delta > 0$*, we have*

(1)
$$\lim_{n \to \infty} \sum_{\varepsilon n \leq l \leq n} n |\mathcal{O}_M(l, \Delta)| \exp(-l\delta_G) = 0$$

and

(2)
$$\lim_{n \to \infty} \sum_{\varepsilon n \leq l \leq n} \sum_{l_1, l_2, l_3 \geq 0}^{l_1 + l_2 + l_3 = l} |\mathcal{O}_M(l_1, \Delta)| \cdot |\mathcal{V}_{\nu,h}(l_2, \Delta)| \cdot |\mathcal{O}_M(l_3, \Delta)| \cdot (l_1 + 1) \exp(-n\delta_G) = 0.$$

When the action is SCC, the convergence is exponentially fast.

Proof. By definition of the DOP condition, we obtain

$$\sum_{n=0}^{+\infty} n |\mathcal{O}_M(n, \Delta)| \exp(-n\delta_G) < \infty.$$

from the formulae (3) and (4). By the Cauchy criterion of series, we know

$$\lim_{n \to \infty} \sum_{\varepsilon n \leq l \leq n} l |\mathcal{O}_M(l, \Delta)| \exp(-l\delta_G) = 0$$

where the convergence is exponential fast when the action is SCC. The first statement (1) thus follows from the following

$$\sum_{\varepsilon n \le l \le n} \varepsilon n |\mathcal{O}_M(l, \Delta)| \exp(-l\delta_G) \le \sum_{\varepsilon n \le l \le n} l |\mathcal{O}_M(l, \Delta)| \exp(-l\delta_G).$$

By Proposition 2.14,

$$\sum_{n=1}^{+\infty} |\mathcal{V}_{v,h^m}(n, \Delta)| \exp(-n\delta_G) < \infty,$$

where the partial sum converges exponentially fast when the action is SCC. The second statement then follows from the convergence of the Cauchy product of three convergent series. The proof is finished. □

At last, we introduce a slightly more general notion of negligibility using (ρ, Δ)-annuli. Fix a number $\rho \in (0, 1]$ and $\Delta > 0$. We say that a set $K \subset G$ is *negligible* in the (ρ, Δ)-annuli if the following holds:

$$\frac{|K \cap A([\rho n, n], \Delta)|}{|A([\rho n, n], \Delta)|} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{5}$$

If the convergence is exponentially fast, the set K is called *exponentially negligible* in the (ρ, Δ)-annuli.

The following lemma clarifies its role in proving the genericity in the next sections. It follows immediately from the purely exponential growth.

Lemma 2.16. *Assume that the proper group action has purely exponential growth. For any $\rho \in (0, 1)$, we have $|A([\rho n, n])| \asymp_\rho \exp(\delta_G n)$ and*

$$\frac{|A([\rho n, n]) \times A([\rho n, n])|}{|B_n \times B_n|} \xrightarrow{\exp} 1, \qquad \frac{|A([\rho n, n])|}{|B_n|} \xrightarrow{\exp} 1.$$

Hence, in order to prove that a set K is (exponentially) negligible in G, we can assume that $K \subset A([\rho n, n], \Delta)$ for a certain choice of $\rho \in (0, 1)$ to simplify the discussion. That is to say, we only need to prove that K is (exponentially) negligible in (ρ, Δ)-annulus. And, it turns out that the proof of (5) for $\rho = 1$ is much simpler than that for $\rho \in (0, 1)$. Therefore, we shall consider the big annulus instead of the usual one in next sections.

The same consideration applies to the case of $G^{(2)}$ where K can be assumed to lie in $A([\rho n, n]) \times A([\rho n, n])$.

3. Negligible subsets

Throughout this section, let G admit a proper action on a proper geodesic metric space (Y, d) with a contracting element. If the group action satisfies the DOP condition, then we take $\nu, M, \Delta > 0$ to satisfy the definition of the DOP condition and Proposition 2.14. When the action is SCC, the constants $\nu, M > 0$ are given by Proposition 2.13. We denote $\mathcal{O}_M = \mathcal{O}_{M,M}, \mathcal{V}_{\nu,h} = \mathcal{V}_{\nu,M,h}$ for simplicity.

The goal of this section is to provide some negligible sets under the above assumptions. Moreover, these are exponentially negligible when the group action is SCC. We suggest that the reader only reads the definition of these sets first and then reads the proof of the theorems in the next section, finally returns to the proof that these sets are negligible.

In all results obtained in what follows, we assume in the DOP case, which have already held in the SCC case by Proposition 2.13, that G has purely exponential growth:

$$|B_n| \asymp \exp(\delta_G n) \asymp_\Delta |A(n, \Delta)|$$

for any $\Delta \gg 0$. We fix such a constant Δ. This estimate will be used implicitly several times.

3.1. Elements with definite barrier-free proportion. This subsection defines three negligible subsets of elements with definite proportion with(out) certain properties.

For any $\varepsilon \in (0, 1)$, let $U(\varepsilon)$ be the set of elements $u \in G$ such that some geodesic $\alpha = [o, uo]$ contains a (connected) subsegment α^ε of length $\varepsilon|u|$ outside $N_M(Go)$. That is to say,

$$\begin{aligned} U(\varepsilon) = \{u \in G\colon &\text{there exist } \alpha = [o, uo], \alpha^\varepsilon \subset \alpha \text{ such that} \\ &|\alpha^\varepsilon| \geq \varepsilon|\alpha|,\ \alpha^\varepsilon \cap N_M(Go) = \emptyset.\} \end{aligned} \tag{6}$$

Lemma 3.1. *If the action has PEG and satisfies the DOP condition, then for any $\varepsilon \in (0, 1)$ and $\rho \in (\varepsilon, 1]$, we have that $U(\varepsilon)$ is negligible in (ρ, Δ)-annuli for $\Delta \geq 2M$.*

Moreover, if the action is SCC, then $U(\varepsilon)$ is exponentially negligible.

Proof. Assume first that the group action satisfies the PEG and DOP condition.

Fix any $1 \geq \rho > \varepsilon$. By Lemma 2.16, we only need to show that

$$\frac{|U(\varepsilon) \cap A([\rho n, n])|}{|A([\rho n, n])|} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Consider any $g \in U(\varepsilon) \cap A([\rho n, n])$ and denote $|g| = k$, then $\rho n \leq k \leq n$. By definition of $U(\varepsilon)$, there exists a geodesic $\alpha = [o, go]$ such that

$$\alpha \text{ contains a subsegment of length } \varepsilon k \text{ which lies outside } N_M(Go). \tag{7}$$

Among those, we consider the first maximal open segment $(x, y)_\alpha$ of α which lies outside $N_M(Go)$ and whose length is bigger than $\varepsilon k \in [\varepsilon\rho n, n]$.

According to the length and the position of $(x, y)_\alpha$, we subdivide the intersection $U(\varepsilon) \cap A([\rho n, n])$ into a sequence of subsets as follows.

For $0 \le i \le (1-\varepsilon)n, \varepsilon\rho n \le l \le n$, define U_i^l to be the set of element $g \in U(\varepsilon) \cap A([\rho n, n])$ such that the segment $(x, y)_\alpha \subseteq \alpha$ defined as above satisfies $d(o, x) = i$ and $d(x, y) = l$. Then we have the following decomposition,

$$U(\varepsilon) \cap A([\rho n, n]) = \bigcup_{\substack{0 \le i \le (1-\varepsilon)n \\ \varepsilon\rho n \le l \le n}} U_i^l.$$

For any $g \in U_i^l$, there exists a geodesic $\alpha = [o, go]$ such that $\alpha_{(i,i+l)}$ lies outside $N_M(Go)$ and $\max\{d(\alpha(i), uo), d(\alpha(i+l), vo)\} \le M$ for some $u, v \in G$. Now we can write $g = u(u^{-1}v)(v^{-1}g)$, where

$$u \in A(i, M), \quad u^{-1}v \in \mathcal{O}_M(l, 2M),$$
$$v^{-1}g \in A([\rho n - l - i, n - l - i], M) \subseteq B_{n-l-i+M}.$$

We assumed that G has purely exponential growth, so

$$|A(n, \Delta)| \asymp_\Delta \exp(\delta_G n) \asymp |B_n|.$$

We thus obtain

$$\begin{aligned} |U(\varepsilon) \cap A([\rho n, n])| &\le \sum_{\substack{0 \le i \le (1-\varepsilon)n \\ \varepsilon\rho n \le l \le n}} |U_i^l| \\ &\le \sum_{\substack{0 \le i \le (1-\varepsilon)n \\ \varepsilon\rho n \le l \le n}} |A(i, \Delta)| \cdot |\mathcal{O}(l, \Delta)| \cdot |B_{n-l-i+\Delta}| \\ &\prec_\Delta \sum_{\substack{0 \le i \le (1-\varepsilon)n \\ \varepsilon\rho n \le l \le n}} \exp(i\delta_G) \cdot |\mathcal{O}_M(l, \Delta)| \cdot \exp((n-l-i)\delta_G) \\ &\prec_\Delta \sum_{\varepsilon\rho n \le l \le n} n |\mathcal{O}_M(l, \Delta)| \exp((n-l)\delta_G). \end{aligned}$$

Therefore, the negligibility of $U(\varepsilon)$ follows from Lemma 2.15.

If the group action is SCC, then there exists $0 < \delta_{\mathcal{O}} < \delta_G$ such that $|\mathcal{O}_M(l, \Delta)| \prec_\Delta \exp(l\delta_{\mathcal{O}})$. The above computation goes through without changes, and so we get

$$\begin{aligned} |U(\varepsilon) \cap A([\rho n, n])| &\prec_\Delta \sum_{\varepsilon\rho n \le l \le n} n |\mathcal{O}_M(l, \Delta)| \exp((n-l)\delta_G) \\ &\prec_\Delta \sum_{\varepsilon\rho n \le l \le n} n \exp(l\delta_{\mathcal{O}_M}) \cdot \exp((n-l)\delta_G) \\ &\prec_\Delta n^2 \exp(-(\delta_G - \delta_{\mathcal{O}})\varepsilon\rho n) \exp(n\delta_G). \end{aligned}$$

Hence, in this case, $U(\varepsilon)$ is exponentially negligible. □

Let $h \in G$ be a contracting element with the axis $\mathbf{Ax}(h) = E(h) \cdot o$, where $E(h)$ is the maximal elementary subgroup given in Lemma 2.6.

Given $\varepsilon \in (0, 1)$ and $C > 0$, consider the following set of elements $g \in G$ such that an ε-percentage α^ε of $\alpha = [o, go]$ is contained in some translate of $\mathbf{Ax}(h)$:

$$W(\varepsilon, h, C) = \{g \in G\colon \text{there exists } \alpha = [o, go], \alpha^\varepsilon \subset \alpha \cap N_C(f\,\mathbf{Ax}(h)) \text{ such that } |\alpha^\varepsilon| \geq \varepsilon|\alpha| \text{ for some } f \in G\}. \tag{8}$$

Lemma 3.2. *Assume that the action has PEG. For any $\varepsilon \in (0, 1)$, $\rho \in (\varepsilon, 1]$ and $C > 0$, we have that $W(\varepsilon, h, C)$ is exponentially negligible in (ρ, Δ)-annuli in G for $\Delta > 0$.*

Proof. Since $h \in G$ is contracting, and by definition, $i \mapsto h^i o$ is a quasi-isometric embedding, we have $|\langle h\rangle \cap B_n| \asymp n$. By Lemma 2.6, we have $[E(h)\colon \langle h\rangle] < \infty$, so the following holds

$$|E(h) \cap B_n| \asymp n.$$

As before, by Lemma 2.16, we want to show

$$\lim_{n\to\infty} \frac{|W(\varepsilon, h, C) \cap A([\rho n, n])|}{|A([\rho n, n])|} = 0.$$

Let $g \in W(\varepsilon, h, C) \cap A([\rho n, n])$, so $\rho n \leq j \leq n$, where $j := |g|$. By definition of $W(\varepsilon, h, C)$, there exists $\alpha = [o, go]$, $i \in [0, (1-\varepsilon)j]$ and $f \in G, k \in E(h)$ such that

$$d(\alpha(i), fo) \leq C, \quad d(\alpha(i + \varepsilon j), fko) \leq C.$$

Thus, we have $f \in A(i, C)$ and $d(o, ko) \leq \varepsilon j + 2C \leq \varepsilon n + 2C$, which yields that $k \in E(h) \cap B_{\varepsilon n+2C}$. Consequently, we can write $g = fk((fk)^{-1}g)$ where $(fk)^{-1}g \in B_{n-i-\varepsilon\rho n+C}$. This gives the following:

$$W(\varepsilon, h, C) \cap A([\rho n, n]) \subseteq \bigcup_{i=0}^{(1-\varepsilon)n} A(i, C) \cdot (E(h) \cap B_{\varepsilon n+2C}) \cdot B_{n-i-\varepsilon\rho n+C}.$$

Since G has purely exponential growth, we have the following estimate:

$$\begin{aligned} |W(\varepsilon, h, C) \cap A([\rho n, n])| &\leq \sum_{i=0}^{(1-\varepsilon)n} |A(i, C)| \cdot |E(h) \cap B_{\varepsilon n+2C}| \cdot |B_{n-i-\varepsilon\rho n+C}| \\ &\prec n \cdot n \cdot \exp((1-\varepsilon\rho)n\delta_G) \end{aligned}$$

which concludes the proof of the result. □

We now introduce the third negligible sets of elements whose certain percentage is barrier-free. To be precise, we need a bit more notation. Let α be a geodesic and $\varepsilon_1 \le \varepsilon_2 \in [0,1]$. We denote by $\alpha^{[\varepsilon_1,\varepsilon_2]}$ the subsegment $\alpha([\varepsilon_1 n, \varepsilon_2 n])$ of α, where $n = |\alpha|$.

Given $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $h \in G$, we define

$$V(\varepsilon_1,\varepsilon_2,h) = \{g \in G\colon \text{there exists } \alpha = [o, go] \text{ such that } \alpha^{[\varepsilon_1,\varepsilon_2]} \text{ is } (\nu,h)\text{-barrier-free}\}. \tag{9}$$

Lemma 3.3. *Fix $\rho \in (0,1]$, and choose any two numbers $\varepsilon_1 < \varepsilon_2 \in (0,\rho)$ so that $\varepsilon_2\rho \in (\varepsilon_1,\varepsilon_2)$. Let h be any element. If our group action satisfies the DOP condition and PEG, then $V(\varepsilon_1,\varepsilon_2,h)$ is negligible in (ρ,Δ)-annuli in G for $\Delta \ge 2M$.*

Moreover, if the action is SCC, then $V(\varepsilon_1,\varepsilon_2,h)$ is exponentially negligible in G.

Proof. By Lemma 2.16, it suffices to prove that

$$\frac{|V(\varepsilon_1,\varepsilon_2,h) \cap A([\rho n, n])|}{|A([\rho n, n])|} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Let $g \in V(\varepsilon_1,\varepsilon_2,h) \cap A([\rho n, n])$ and denote $|g| = k$, so $\rho n \le k \le n$. By definition of $V(\varepsilon_1,\varepsilon_2,h)$, there exists a geodesic $\alpha = [o, go]$, so that $\alpha([\varepsilon_1 k, \varepsilon_2 k])$ is (ν,h)-barrier-free. Set $x = \alpha(\varepsilon_1 n), y = \alpha(\varepsilon_2\rho n)$. By the choice of $\varepsilon_2\rho \in (\varepsilon_1,\varepsilon_2)$, we see that $[x,y]_\alpha = \alpha([\varepsilon_1 n, \varepsilon_2\rho n])$ is a subsegment of $\alpha([\varepsilon_1 k, \varepsilon_2 k])$, and thus is (ν,h)-barrier-free.

We now subdivide our discussion into three cases, the first two of which could be viewed as degenerate cases of the third one. However, we treat them separately in order to illustrate the idea of the latter one.

Case 1. Assume that $x, y \in N_M(Go)$ so there exists $u, v \in G$ such that

$$d(x, uo) \le M, \quad d(y, vo) \le M.$$

Thus, $[x,y]_\alpha$ is a (ν,h)-barrier-free geodesic between $N_M(uo)$ and $N_M(vo)$. So $u^{-1}v \in \mathcal{V}_{\nu,h}$.

Denote $\varepsilon = \varepsilon_2\rho - \varepsilon_1 > 0$. Since $d(x,y) = \varepsilon n$ and $|d(uo,vo) - d(x,y)| \le 2M$, we have

$$u^{-1}v \in \mathcal{V}_{\nu,h}(\varepsilon n, 2M).$$

Clearly,

$$u \in A(\varepsilon_1 n, M), \quad v^{-1}g \in A(k - \varepsilon_2\rho n, M).$$

Therefore, for $\Delta \ge 2M$, we obtain that $g = u(u^{-1}v)(v^{-1}g)$ lies the following set

$$A(\varepsilon_1 n, \Delta) \cdot \mathcal{V}_{\nu,h}(\varepsilon n, \Delta) \cdot A(k - \varepsilon_2\rho n, \Delta). \tag{10}$$

Case 2. Assume that one of $\{x, y\}$ lies outside $N_M(Go)$. Let's assume first that $x \in N_M(Go), y \notin N_M(Go)$, so there exists $u \in G$ such that $d(x, uo) \leq M$. Consider the maximal open segment (y_1, y_2) of α which contains y but lies outside $N_M(Go)$. Hence, there exists $v_1, v_2 \in G$ such that $d(y_i, v_i o) \leq M$ for $i = 1, 2$. By definition, $v_1^{-1} v_2 \in \mathcal{O}_M$.

Set $s = d(o, y_1) \in [\varepsilon_1 n, \varepsilon_2 \rho n], t = d(o, y_2) \in [\varepsilon_2 \rho n, k]$, where $n \geq k \geq \rho n$. Thus, $d(y_1, y_2) = t - s$, and $|d(v_1 o, v_2 o) - (t - s)| \leq 2M \leq \Delta$. This means that

$$v_1^{-1} v_2 \in \mathcal{O}_M(t - s, \Delta).$$

Similarly as above,

$$u^{-1} v_1 \in \mathcal{V}_{\nu,h}(s - \varepsilon_1 n, \Delta), \quad v_2^{-1} g \in A(k - t, \Delta).$$

Consequently, the element $g = u(u^{-1} v_1)(v_1^{-1} v_2)(v_2^{-1} g)$ lies in the following set

$$A(\varepsilon_1 n, \Delta) \cdot \mathcal{V}_{\nu,h}(s - \varepsilon_1 n, \Delta) \cdot \mathcal{O}_M(t - s, \Delta) \cdot A(k - t, \Delta) \tag{11}$$

where $s \in [\varepsilon_1 n, \varepsilon_2 \rho n]$ and $t \in [\varepsilon_2 \rho n, k]$.

Similarly, when $x \notin N_M(Go)$ and $y \in N_M(Go)$, we obtain

$$g \in A(i, \Delta) \cdot \mathcal{O}_M(j - i, \Delta) \cdot \mathcal{V}_{\nu,h}(\varepsilon_2 \rho n - j, \Delta) \cdot A(k - \varepsilon_2 \rho n, \Delta),$$

where $i \in [0, \varepsilon_1 n], j \in [\varepsilon_1 n, \varepsilon_2 \rho n]$.

Case 3. We now consider the general case that $x, y \notin N_M(Go)$. Recall that $\varepsilon = \varepsilon_2 \rho - \varepsilon_1$. By Lemma 3.1, the set $U(\varepsilon)$ is negligible. Without loss of generality, we can assume that $g \notin U(\varepsilon)$. This implies that $[x, y]_\alpha \cap N_M(Go) \neq \emptyset$. Indeed, if not, then the geodesic segment $[x, y]_\alpha$ lies outside $N_M(Go)$. Since $[x, y]_\alpha$ is a subsegment of $\alpha = [o, uo]$ of length $(\varepsilon_2 \rho - \varepsilon_1)n$ outside $N_M(Go)$, we obtain $g \in U(\varepsilon_2 \rho - \varepsilon_1)$, that is a contradiction.

Hence, consider the maximal open segments $(x_1, x_2)_\alpha, (y_1, y_2)_\alpha$ of α outside $N_M(Go)$ which contain x, y respectively. Since $[x, y]_\alpha \cap N_M(Go) \neq \emptyset$, these two intervals are disjoint.

Denote $i = d(o, x_1), j = d(o, x_2)$ and $s = d(o, y_1), t = d(o, y_2)$. Then $i \in [0, \varepsilon_1 n], j < s \in [\varepsilon_1 n, \varepsilon_2 \rho n], t \in [\varepsilon_2 \rho n, k]$, where $k \in [\rho n, n]$. By the same reasoning as in the previous two cases, we have

$$g \in A(i, \Delta) \cdot \mathcal{O}_M(j - i, \Delta) \cdot \mathcal{V}_{\nu,h}(s - j, \Delta) \cdot \mathcal{O}_M(t - s, \Delta) \cdot A(k - t, \Delta) \tag{12}$$

for each $g \in V(\varepsilon_1, \varepsilon_2) \cap A([\rho n, n])$ with $|g| = k$.

Note that $\varepsilon_2 \rho \in (\varepsilon_1, \varepsilon_2)$ and $k \in [\rho n, n]$. We look at the index set

$$\Lambda = \{(i, j, s, t) \in \mathbb{N}^4 \colon 0 \leq i \leq \varepsilon_1 n \leq j \leq s \leq \varepsilon_2 \rho n \leq t \leq n\},$$

over which we define

$$V_{(i,j),(s,t)} := A(i,\Delta)\cdot \mathcal{O}_M(j-i,\Delta)\cdot \mathcal{V}_{\nu,h}(s-j,\Delta)\cdot \mathcal{O}_M(t-s,\Delta)\cdot B_{n-t+\Delta}.$$

Combining (10), (11), and (12), we have the decomposition

$$V(\varepsilon_1,\varepsilon_2,h)\cap A([\rho n,n]) \subseteq \bigcup_{(i,j,s,t)\in\Lambda} V_{(i,j),(s,t)}, \tag{13}$$

up to a negligible set $U(\varepsilon)$.

To conclude the proof, it remains to show that the right-hand set in (13) is negligible. For that purpose, we consider a triple of lengths (l_1,l_2,l_3) with $l_1+l_2+l_3 = l \in [\varepsilon n, n]$. We observe that there are at most (l_1+1) indexes $(i,j,s,t)\in\Lambda$ satisfying $j-i=l_1, s-j=l_2, t-s=l_3$. In fact, we can choose some $i\in[0,\varepsilon_1 n]$ first, and once i is fixed, then j,s,t are all determined by the triple (l_1,l_2,l_3). However, the choice of i can only change from $\varepsilon_1 n - l_1$ to $\varepsilon_1 n$, so we have at most l_1+1 many $(i,j,s,t)\in\Lambda$ falling in the same triple (l_1,l_2,l_3).

For each $V_{(i,j),(s,t)}$ with $j-i=l_1, s-j=l_2, t-s=l_3$, we have the following estimate:

$$\begin{aligned}
|V_{(i,j),(s,t)}| &\le |A(i,\Delta)|\cdot|\mathcal{O}_M(j-i,\Delta)|\cdot|\mathcal{V}_{\nu,h}(s-j,\Delta)| \\
&\qquad \cdot|\mathcal{O}_M(t-s,\Delta)|\cdot|B_{n-t+\Delta}| \\
&\prec \exp(i\delta_G)\cdot|\mathcal{O}_M(l_1,\Delta)|\cdot|\mathcal{V}_{\nu,h}(l_2,\Delta)| \\
&\qquad \cdot|\mathcal{O}_M(l_3,\Delta)|\cdot\exp((n-l_1-l_2-l_3-i)\delta_G) \\
&\prec \exp(n\delta_G)\cdot|\mathcal{O}_M(l_1,\Delta)|\cdot|\mathcal{V}_{\nu,h}(l_2,\Delta)| \\
&\qquad \cdot|\mathcal{O}_M(l_3,\Delta)|\exp((-l_1-l_2-l_3)\delta_G),
\end{aligned}$$

where we used $|B_{n-t+\Delta}| \asymp \exp((n-t)\delta_G)$ since the action has purely exponential growth.

Since the indexes $(i,j,s,t)\in\Lambda$ can be grouped according to the triple (l_1,l_2,l_3), we obtain

$$\begin{aligned}
&\frac{\sum_{(i,j,s,t)\in\Lambda}|V_{(i,j),(s,t)}|}{\exp(n\delta_G)} \\
&\quad \le \sum_{\substack{l_1+l_2+l_3=l \\ \varepsilon n\le l\le n}} (l_1+1)|\mathcal{O}(l_1,\Delta)|\cdot|\mathcal{V}_{\nu,h}(l_2,\Delta)|\cdot|\mathcal{O}(l_3,\Delta)|\exp((-l_1-l_2-l_3)\delta_G).
\end{aligned}$$

This tends 0 as $n\to\infty$ by Lemma 2.15(2). We conclude that the intersection $V(\varepsilon_1,\varepsilon_2,h)\cap A([\rho n,n])$ is negligible. When the action is SCC, the above inequality tends to 0 exponentially fast. The proof of the result is complete. □

3.2. Negligible pairs of elements. The goal of Theorem 1.4 is to show a random pair $(u_1, u_2) \in G^{(2)}$ generates a free group of rank 2. We now define two negligible sets of 2-tuples $(u_1, u_2) \in G^{(2)}$ with big cancellation which shall prevent them to be being free bases.

For any $u \in G$, let $\alpha = [o, uo]$ be any geodesic with length parametrization $\alpha(t)$. Define $= \bar{\alpha} = [o, u^{-1}o]$ to be the geodesic with parametrization

$$\bar{\alpha}(t) := u^{-1}\alpha(|u| - t).$$

Given $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $C > 0$, let $Z(\varepsilon_1, \varepsilon_2, C)$ be the set of $u \in G$ such that for some $\alpha = [o, uo]$, one of the following two statements holds:

(1) $\bar{\alpha}$ intersects the C-neighborhood of the subsegment $\alpha^{[\varepsilon_1,\varepsilon_2]}$ of α;

(2) α intersects the C-neighborhood of the subsegment $\bar{\alpha}^{[\varepsilon_1,\varepsilon_2]}$ of $\bar{\alpha}$

(where $\alpha^{[\varepsilon_1,\varepsilon_2]}$ denotes the subsegment $\alpha([\varepsilon_1 n, \varepsilon_2 n])$ of α for $n = |\alpha|$).

In other words,

$$\begin{aligned} Z(\varepsilon_1, \varepsilon_2, C) = \{u \in G\colon & \text{there exists } \alpha = [o, uo] \text{ such that} \\ & \bar{\alpha} \cap N_C(\alpha^{[\varepsilon_1,\varepsilon_2]}) \neq \emptyset \text{ or } \alpha \cap N_C(\bar{\alpha}^{[\varepsilon_1,\varepsilon_2]}) \neq \emptyset\}. \end{aligned} \tag{14}$$

Lemma 3.4 (big cancellation I). *Let $0 < \varepsilon_1 < \varepsilon_2 \le 1 - \varepsilon_1 < \rho < 1$ and $C > 0$. If our group action satisfies the DOP condition and purely exponential growth, then $Z(\varepsilon_1, \varepsilon_2, C)$ is negligible in (ρ, Δ)-annuli in G, where $\Delta \ge C + 2M$.*

Moreover, if the action is SCC, then $Z(\varepsilon_1, \varepsilon_2, C)$ is exponentially negligible in G.

Proof. For any $u \in Z(\varepsilon_1, \varepsilon_2, C) \cap A([\rho n, n]) \setminus U\left(\frac{\varepsilon_1}{8}\right)$, there exists $\alpha = [o, uo]$ satisfying the condition in the definition of $Z(\varepsilon_1, \varepsilon_2, C)$. Denote $j = |u|$ and then $\rho n \le j \le n$.

Without loss of generality, assume that $\bar{\alpha} \cap N_C(\alpha^{[\varepsilon_1,\varepsilon_2]}) \neq \emptyset$. By definition, there exists $\varepsilon_1 j \le i \le \varepsilon_2 j$, so that $\bar{\alpha} \cap N_C(\alpha(i)) \neq \emptyset$. Thus, there exists $s \in [i - C, i + C]$ such that

$$d(\bar{\alpha}(s), \alpha(i)) \le C.$$

Set $x = \alpha(i)$, $y = \bar{\alpha}(s) = u^{-1}\alpha(j - s)$. Thus, $d(x, y) \le C$.

We follow a similar analysis as in the proof of Lemma 3.3.

Case 1. Assume that $x, y \in N_M(Go)$, so there exist $v, w \in G$ such that

$$d(x, vo) \le M, \quad d(y, wo) \le M.$$

This implies $d(vo, wo) \le d(x, y) + 2M \le 2M + C$, so $v^{-1}w \in B_{2M+C}$. Since $\bar{\alpha}(0) = o$ and $s \in [i - C, i + C]$, we have $d(o, y) = s$ and then $|d(o, wo) - s| \le M + C$. Thus,

$$v \in A(i, M), \quad w \in A(i, M + C).$$

We can now write $u = v(v^{-1}uw)(w^{-1}v)v^{-1}$, where

$$\begin{aligned} d(uwo, vo) &\le d(o, uy) - d(o, x) + 2M \\ &\le d(o, \alpha(j-s)) - d(o, \alpha(i)) + 2M \\ &\le j - s - i + 2M \le j - 2i + C \end{aligned}$$

which implies $v^{-1}uw \in A(j-2i, 2M+C)$.

Noting that $v \in A(i, M)$, the set of elements u in this case belongs to the following set

$$A(i, M) \cdot A(j-2i, 2M+C) \cdot B_{2M+C} \cdot A(i, M).$$

Case 2. Assume that one of $\{x, y\}$ lies outside $N_M(Go)$. For definiteness, assume that $x \in N_M(Go), y \notin N_M(Go)$; the other case is symmetric. Then there exists $v \in G$ such that $d(x, vo) \le M$. Consider the maximal open segment (y_1, y_2) of $\bar{\alpha}$ which contains y but lies outside $N_M(Go)$. Hence, there exists $w \in G$ such that $d(y_1, wo) \le M$.

Since $u \notin U\left(\frac{\varepsilon_1}{8}\right)$ is assumed and then $u^{-1} \notin U\left(\frac{\varepsilon_1}{8}\right)$ by definition, we obtain that $d(y_1, y_2) < \frac{\varepsilon_1}{8} j$. Thus we have $d(y_1, y) \le d(y_1, y_2) < \frac{\varepsilon_1}{8} j$. This yields

$$d(vo, wo) \le d(vo, x) + d(x, y) + d(y, y_1) + d(y_1, wo) \le \frac{\varepsilon_1}{8} j + 2M + C.$$

Hence, we can also write $u = v(v^{-1}uw)(w^{-1}v)v^{-1}$, where

$$v \in A(i, M), \quad v^{-1}w \in B_{\frac{\varepsilon_1}{8} j + 2M + C}, \quad v^{-1}uw \in B_{j - 2i + \frac{\varepsilon_1}{8} j + 2M + C}.$$

Case 3. Assume $x, y \notin N_M(Go)$. Consider the maximal open segment $(x_1, x_2)_\alpha$ (resp. $(y_1, y_2)_\alpha$) of α (resp. $\bar{\alpha}$) which contains x (resp. y) but lies outside $N_M(Go)$. Then there exist $v, w \in G$ such that $d(x_1, vo) \le M, d(y_1, wo) \le M$. By a similar argument as above we have the following conclusion: we can write

$$u = v(v^{-1}uw)(w^{-1}v)v^{-1},$$

where

$$v \in B_{i+M+C}, \quad v^{-1}w \in B_{\frac{\varepsilon_1}{4} j + 2M + C}, \quad v^{-1}uw \in B_{j - 2i + \frac{\varepsilon_1}{4} j + 2M + C}.$$

Summarizing the above three cases, we have

$$\begin{aligned} &\left| Z(\varepsilon_1, \varepsilon_2, C) \cap A([\rho n, n]) \setminus U\left(\frac{\varepsilon_1}{8}\right) \right| \\ &\le 2 \sum_{j=\rho n}^{n} \sum_{i=\varepsilon_1 j}^{\varepsilon_2 j} |B_{i+\Delta}| \cdot |B_{j-2i+\frac{\varepsilon_1}{4} j + \Delta}| \cdot |B_{\frac{\varepsilon_1}{4} j + \Delta}| \\ &\prec (1-\rho)n \cdot (\varepsilon_2 - \varepsilon_1)n \cdot \exp\left(\left(1 - \frac{\varepsilon_1}{2}\right) n \delta_G\right) \end{aligned}$$

where the last inequality used

$$|B_{i+\Delta}| \cdot |B_{j-2i+\frac{\varepsilon_1}{4}j+\Delta}| \cdot |B_{\frac{\varepsilon_1}{4}j+\Delta}| \asymp \exp\left(\delta_G\left(j + \frac{\varepsilon_1}{2}j - i\right)\right)$$

which follows from the purely exponential growth.

This shows that $Z(\varepsilon_1, \varepsilon_2, C) \cap A([\rho n, n]) \setminus U\left(\frac{\varepsilon_1}{8}\right)$ is negligible. By Lemma 3.1, $U\left(\frac{\varepsilon_1}{8}\right)$ is negligible, thus the conclusion follows. □

Fix $0 < \varepsilon_1 < \varepsilon_2 < \rho < 1$ and $C > 0$. Let $T(\varepsilon_1, \varepsilon_2, C)$ be the set of $(u_1, u_2) \in G \times G$ with the following property: there exist two geodesics $\alpha := [o, u_1 o]$, $\beta := [o, u_2 o]$ such that neither of them is disjoint from the C-neighborhood of the $[\varepsilon_1, \varepsilon_2]$-interval of the other. In other words,

$$\begin{aligned} T(\varepsilon_1, \varepsilon_2, C) = \{(u_1, u_2) \in G \times G\colon & \\ & \text{there exists } \alpha := [o, u_1 o], \beta := [o, u_2 o] \text{ such that} \\ & \alpha \cap N_C(\beta^{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset \text{ or } \beta \cap N_C(\alpha^{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset\} \end{aligned} \tag{15}$$

where $\alpha^{[\varepsilon_1, \varepsilon_2]}$ denotes the subsegment $\alpha([\varepsilon_1 n, \varepsilon_2 n])$ of α for $n = |\alpha|$.

Lemma 3.5 (big cancellation II). *For any $0 < \varepsilon_1 < \varepsilon_2 \leq 1 - \varepsilon_1 < \rho < 1$ and $C > 0$, if our group action satisfies the DOP condition and PEG condition, then $T(\varepsilon_1, \varepsilon_2, C)$ is negligible in (ρ, Δ)-annuli in $G \times G$, where $\Delta \geq C + 2M$.*

Moreover, if the action is SCC, then $T(\varepsilon_1, \varepsilon_2, C)$ is exponentially negligible in $G \times G$.

Proof. Since the union of two (exponentially) negligible sets is (exponentially) negligible, without loss of generality, we can assume that for all $(u_1, u_2) \in T(\varepsilon_1, \varepsilon_2, C)$, we have

$$\beta \cap N_C(\alpha^{[\varepsilon_1, \varepsilon_2]}) \neq \emptyset.$$

Choose $1 - \varepsilon_1 < \rho < 1$. By Lemma 2.16, we can assume further that (u_1, u_2) belongs to $T(\varepsilon_1, \varepsilon_2, C) \cap (A([\rho n, n]) \times A([\rho n, n]))$.

Denote $n_1 = |u_1|$. By definition of $T(\varepsilon_1, \varepsilon_2, C)$, there exists $i \in [\varepsilon_1 n_1, \varepsilon_2 n_1]$ so that $\beta \cap N_C(\alpha(i)) \neq \emptyset$. Denote $x = \alpha(i)$. We proceed by a similar argument as before.

Case 1. Assume that $x \in N_M(Go)$ so there exists $v \in G$ such that $d(x, vo) \leq M$. Thus, $v \in A(i, M)$. Then (u_1, u_2) can be written as $(v(v^{-1}u_1), v(v^{-1}u_2))$, where

$$v^{-1}u_1 \in A(n_1 - i, M), \quad v^{-1}u_2 \in A(n_2 - i, C + M).$$

Note that $n_1 \in [\rho n, n]$. In this case, we bound by above the number of elements (u_1, u_2) as follows:

$$\begin{aligned} &\le \sum_{\substack{\rho n \le n_1 \le n \\ \varepsilon_1 n_1 \le i \le \varepsilon_2 n_1}} |A(i, \Delta)| \cdot |A(n_1 - i, \Delta)| \cdot |A([\rho n - i, n - i], \Delta)| \\ &\prec n \exp((2 - \varepsilon_1)\delta_G n) = o(\exp(2\delta_G n)), \end{aligned}$$

so these pairs $(u_1, u_2) \in T(\varepsilon_1, \varepsilon_2, C)$ are exponentially negligible.

Case 2. Otherwise, consider the maximal open segment $(x_1, x_2)_\alpha$ of α, which contains x but lies outside $N_M(Go)$. Denote $j := d(o, x_1), l := d(o, x_2)$. Thus $0 \le j \le i$ and $i < l \le n_1$.

Subcase 2.1 $l - j \ge \frac{\varepsilon_1}{2} n_1$, then $u_1 \in U\left(\frac{\varepsilon_1}{2}\right)$. Since $U\left(\frac{\varepsilon_1}{2}\right)$ is negligible in G by Lemma 3.1, we have that $U\left(\frac{\varepsilon_1}{2}\right) \times G$ is negligible as well in $G \times G$.

Subcase 2.2 $l - j < \frac{\varepsilon_1}{2} n_1$. As before, there exist $v_1, v_2 \in G$ such that $d(x_1, v_1 o) \le M, d(x_2, v_2 o) \le M$. Thus, $v_1 \in A(j, M)$.

Then (u_1, u_2) can be written as

$$(v_1(v_1^{-1}v_2)(v_2^{-1}u_1), v_1(v_1^{-1}v_2)(v_2^{-1}u_2)),$$

where

$$\begin{aligned} &v_1^{-1}v_2 \in A(l - j, 2M), v_2^{-1}u_1 \in A(n_1 - l, M), \\ &v_2^{-1}u_2 \in A((n_2 - i) + (l - i), C + M) \end{aligned}$$

We consider the index set

$$\Lambda = \Big\{(n_1, i, j, l) \in \mathbb{Z}^4 \colon \rho n \le n_1 \le n, \varepsilon_1 n_1 \le i \le \varepsilon_2 n_1, \\ 0 \le j \le i \le l \le j + \frac{\varepsilon_1}{2} n_1\Big\}.$$

Hence, we have the upper bound on pairs (u_1, u_2) of the second case as follows

$$\begin{aligned} &\le \sum_{(n_1, i, j, l) \in \Lambda} |A(j, \Delta)| \cdot |A(l - j, \Delta)| \cdot |A(n_1 - l, \Delta)| \\ &\qquad \cdot |A([\rho n + l - 2i, n + l - 2i], \Delta)| \\ &\prec \sum_{(n_1, i, j, l) \in \Lambda} \exp((n + n_1 + l - 2i)\delta_G) \\ &\prec n^4 \exp\left(\left(2 - \frac{\varepsilon_1}{2}\right) n\delta_G\right) = o(\exp(2\delta_G n)). \end{aligned}$$

Therefore, in this case, we have proved the negligibility of $T(\varepsilon_1, \varepsilon_2, C)$. The proof is complete. □

4. The proof of the theorems

This section is devoted to the proof of the theorems of this paper.

4.1. Generically free subgroups. Let $\Lambda > 0$ and $k \geq 2$. Denote by $\mathcal{F}^{(k)}$ the set of k-tuples

$$\{u_1, \ldots, u_k\} \in G^{(k)}$$

such that

(1) $\langle u_1, u_2, \ldots, u_k \rangle$ is a free subgroup of rank k consisting of contracting elements except the identity;

(2) the orbital map $W \in \langle u_1, u_2, \ldots, u_k \rangle \mapsto Wo \in Y$ is a (Λ, Λ)-quasi-isometric embedding with contracting image.

Let $\mathbb{F}(u_1, \ldots, u_k)$ be the free group generated by the k-tuples $\{u_1, \ldots, u_k\}$. In order to prove that $\mathcal{F}^{(k)}$ is generic in $G^{(k)}$, the idea is to construct a generic subset $E \subseteq G^{(k)}$, such that for any $\vec{u} = (u_1, \ldots, u_k) \in E$ and any nontrivial freely reduced word $W \in \mathbb{F}(u_1, \ldots, u_k)$, we can construct an admissible path from o to Wo that satisfies the conditions of Proposition 2.11 and thus the path is a quasi-geodesic by the same proposition. This then concludes the proof of Theorem 1.4.

To be clear, we fix some notations and constants at the beginning (the reader is encouraged to read the proof first and return here until the constant appears).

Setup (1) If the group action satisfies the DOP and PEG conditions, then constants $\nu, M, \Delta > 0$ are given by definition of DOP condition and Proposition 2.14. Moreover if the group action is SCC, then $\nu, M > 0$ are given by Proposition 2.13.

(2) We fix a contracting element h so that by Lemma 2.9, the system $\mathbb{X} = \{g\mathbf{Ax}(h) \colon g \in G\}$ is C-contracting for a constant $C > 0$. Assume that C satisfies Lemma 2.5 as well.

(3) By taking the maximum, we can assume $C > \nu$ and $\Delta \geq 2M + C$.

(4) Let $D = D(3C, 9C) > 16C$ be the constant given by Proposition 2.11 for $(D, 9C)$-admissible paths.

(5) Take $m > 0$ so that $|h^m| > D + 2\nu$. This can be done since h is a contracting element, so that we have that $n \in \mathbb{Z} \mapsto h^n \in G$ is a quasi-isometric embedding of $\mathbb{Z} \to G$.

We refer the reader to the definitions of the set $V(2\varepsilon, 1-2\varepsilon, h^m)$ in (9), the set $W(\varepsilon, h, C)$ in (8), the set $Z(\varepsilon, 1-\varepsilon, C)$ in (14) and the set $T(\varepsilon, 1-\varepsilon, C)$ in (15).

Lemma 4.1. *Fix* $\rho \in (\frac{8}{9}, 1)$ *and* $\varepsilon \in \left(1-\rho, \frac{1}{4}\right)$. *The subset* E *of all* $\vec{u} = (u_1, \ldots, u_k) \in G^{(k)}$ *satisfying the following conditions is generic in* (ρ, Δ)*-annuli.*

(1) $|u_i| \ge \rho|\vec{u}|$ *for* $1 \le i \le k$.

(2) $u_i^{\pm 1} \notin V(2\varepsilon, 1-2\varepsilon, h^m) \cup W(\varepsilon, h, \sigma(C)) \cup Z(\varepsilon, 1-\varepsilon, 4C)$ *for* $1 \le i \le k$, *where* $\sigma(\cdot)$ *is the quasiconvexity function given by Lemma* 2.5.

(3) $(u_i^{\pm 1}, u_j^{\pm 1}) \notin T(\varepsilon, 1-\varepsilon, 4C)$ *for* $i \ne j \in \{1, 2, \dots, k\}$.

When the action is SCC, the set E *is exponentially generic.*

Proof. It suffices to show that the set of $\vec{u} \in G^{(k)}$ in each statement as above is generic. It is clear that our choice of ρ, ε satisfy all the condition of the lemmas in Section 3. Hence assertion (1) is given by Lemma 2.16. Assertion (2) is a consequence of Lemmas 3.3, 3.2, and 3.4 together. Assertion (3) follows from Lemma 3.5 by taking a finite intersection of (exponential) generic sets. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. For notational simplicity, we give the proof for $k = 2$. Let E be the subset of $G^{(2)}$ provided by Lemma 4.1. It suffices to show that E in contained in $\mathcal{F}^{(2)}$.

First of all, for each $(u_1, u_2) \in E \cap A([\rho N, N])$, we shall prove that $\langle u_1, u_2\rangle$ generates a free group of rank 2. Namely, let W be a non-trivial reduced word in $\mathbb{F}(u_1, u_2)$. The goal is to prove that the evaluation of the word W in G gives a non-trivial contracting element. Write $W = x_1 x_2 \dots x_i \dots x_n$ where $x_i \in \{u_1, u_2, u_1^{-1}, u_2^{-1}\}$.

We choose a geodesic $p = [o, u_1 o]$, and denote by $\bar{p} := [o, u_1^{-1} o]$ the u_1^{-1}-translate of p with reverse orientation (as in Section 3.2). Similarly, we define $q = [o, u_2 o]$ and its reverse $\bar{q} := [o, u_2^{-1} o]$.

For the word W, define a path γ to be a concatenation of geodesic segments γ_i for $1 \le i \le n$ as follows:

$$\gamma = \gamma_1 \cdot \gamma_2 \cdot \dots \cdot \gamma_n,$$

where γ_i is the $x_1 \dots x_{i-1}$-translate of one of $p, q, \bar{p}$ or $\bar{q}$ depending on x_i.

We now describe a procedure to truncate γ to get an admissible path. See Figure 1.

By the definition of the set $V(2\varepsilon, 1-2\varepsilon, h^m)$ in (9), for $u_1 \notin V(2\varepsilon, 1-2\varepsilon, h^m)$, we have that the geodesic p contains a (ν, h^m)-barrier in $p^{[2\varepsilon,(1-2\varepsilon)]}$. Then there exists $P \in \mathbb{X}$ such that $p \cap N_\nu(P) \subset p \cap N_C(P)$ is of diameter at least $d(o, h^m o) - 2\nu$, where the assumption $\sigma < C$ is used. On the other hand, since $u_1 \notin W(\varepsilon, h, \sigma(C))$ defined in (8), the diameter of $p \cap N_{\sigma(C)}(P)$ is at most $\varepsilon|u_1| \le \varepsilon\ell(p)$, and so $p \cap N_C(P)$ is contained in $p^{[\varepsilon,(1-\varepsilon)]}$. Similarly, for $u_2 \notin V(2\varepsilon, 1-2\varepsilon, h^m)$, we find $Q \in \mathbb{X}$ so that $q \cap N_C(Q) \subset q^{[\varepsilon,(1-\varepsilon)]}$ has diameter at least $d(o, h^m o) - 2\nu$.

Recall that γ_i is a translate of one of $p, q, \bar{p}, \bar{q}$ by $x_1 \dots x_{i-1}$. We thus have a sequence of contracting subsets X_i which are corresponding translates of either $N_C(P)$ or $N_C(Q)$. Let y_i, z_i be the corresponding entry and exit points of $\gamma_i \cap X_i$ for each $1 \le i \le n$. Thus, by the above discussion, we have $y_i, z_i \in \gamma_i^{[\varepsilon,(1-\varepsilon)]}$ and $d(y_i, z_i) \ge |h^m| - 2\nu$.

We now truncate the subpath $[z_{i-1}, (\gamma_i)_-]\cdot[(\gamma_i)_-, y_i]$ from γ for each $1 \le i \le j$ and replace it with a geodesic $[z_{i-1}, y_i]$. The resulting path β with same endpoints with γ is given as follows

$$\beta = [1, y_1]_{\gamma_1} \cdot [y_1, z_1]_{\gamma_1} \cdot [z_1, y_2] \cdot \dots \cdot [z_{n-1}, y_n] \cdot [y_n, z_n]_{\gamma_n} \cdot [z_n, \gamma_-]_{\gamma_n}.$$

By Lemma 2.5, the subsegment $[y_i, z_i]_{\gamma_i}$ is contained in $N_{\sigma(C)}(X_i)$. By Lemma 2.9, the system $\tilde{\mathbb{X}} = \{gN_C(\mathbf{Ax}(h)) \colon g \in G\}$ is a $3C$-contracting system with bounded projection. In the following lemma, we shall consider the admissible path β associated with $X_i \in \tilde{\mathbb{X}}$.

Lemma 4.2. *There exists $K = K(N, D, C) > 0$ such that the path β is a $(D, 9C)$-admissible path which is K-contracting.*

By construction, the truncated path β has a Hausdorff distance at most N to γ. By Lemma 2.5, γ is also contracting.

Proof of Lemma 4.2. First of all, $y_i, z_i \in X_i$ and $d(y_i, z_i) \ge |h^m| - 2\nu \ge D$ by the choice of a high power h^m, thus the condition (**LL1**) is satisfied.

Recall that W is a reduced word over $\{u_1, u_2, u_1^{-1}, u_2^{-1}\}$, so the pair of any two adjacent letters (x_j, x_{j+1}) does not belong to $Z(\varepsilon, 1-\varepsilon, 4C)$ and $T(\varepsilon, 1-\varepsilon, 4C)$. By their definitions (14) (15), since $y_i, z_i \in \gamma_i^{[\varepsilon,(1-\varepsilon)]}$, we obtain that both γ_{i-1} and γ_{i+1} are disjoint with the $4C$-neighborhood of $[y_i, z_i]_{\gamma_i}$. By Lemma 2.8, γ_{i-1} and γ_{i+1} are disjoint from X_i, so by Lemma 2.7,

$$\max\{\|\Pi_{X_i}(\gamma_{i-1})\|, \|\Pi_{X_i}(\gamma_{i+1})\|\} \le 3C. \tag{16}$$

Consequently, for any $1 < i \le n$, we have

$$\begin{aligned}
\|\Pi_{X_i}([z_i, y_{i+1}])\| &\overset{(1)}{\le} \|\Pi_{X_i}(\{z_i, y_{i+1}\})\| + C \\
&\le \|\Pi_{X_i}(\{z_i, \gamma_+^i\})\| + \|\Pi_{X_i}(\{(\gamma_i)_+, y_{i+1}\})\| + C \\
&\overset{(1)}{\le} \|\Pi_{X_i}([z_i, (\gamma_i)_+]_{\gamma_i})\| + \|\Pi_{X_i}(\gamma_{i+1})\| + 3C \\
&\overset{(16)}{\le} 6C + 3C \le 9C.
\end{aligned}$$

For any $1 \le i < n$, a similar estimate as above shows

$$\|\Pi_{X_i}([z_{i-1}, y_i])\| \le 9C,$$

and $\|\Pi_{X_1}([1, y_1]_{\gamma_1})\| \le 9C$, $\|\Pi_{X_n}([z_n, z]_{\gamma_n})\| \le 9C$. Thus the condition (**BP**) is satisfied.

Since γ_{i+1} is disjoint from X_i, we have $X_i \neq X_{i+1}$ for all $1 \le i < n$. Then all conditions in Definition 2.10 of admissible paths are verified. Thus, β is a $(D, 9C)$-admissible path.

Finally, let us note that β is contracting. By [37, Proposition 2.9], if the geodesics q_i in Definition 2.10 are all bounded above (resp. blow) by a constant $L > 0$ (resp. $l > 0$), then under the assumption of Proposition 2.11, the corresponding (D, τ)-admissible path is K-contracting for some $K > 0$ depending on L, l, D, τ. Since $4C =: l < d(y_i, z_{i+1}) < L := 2N$ for each i, there exists $K = K(N, D, C) > 0$ such that β is K-contracting. △

By Proposition 2.11, we know that β is a (Λ_0, Λ_0)-quasi-geodesic for some uniform $\Lambda_0 > 1$ depending only on D and C. Noting that $u_1, u_2 \in A([\rho N, N])$, we have $\rho N \le |u_1|, |u_2| \le N$. Since $\ell(\beta) \ge nD$, we obtain from the (Λ_0, Λ_0)-quasi-geodesicity of β that

$$\frac{D}{\Lambda_0}|W| - \Lambda_0 \le \frac{D}{\Lambda_0}n - \Lambda_0 \le d(o, Wo) \le n \max\{|u_1|, |u_2|\} \le N|W|.$$

This holds for each nontrivial reduced word W, so $H = \langle u_1, u_2 \rangle$ is a free group of rank 2.

Set $\Lambda := \max\left\{N, \frac{D}{\Lambda_0}, \Lambda_0\right\}$, which equals N for $N \gg 0$. If the subgroup H is equipped with the word metric coming from the free basis $\{u_1, u_2\}$, then the orbital map $W \in H \mapsto Wo \in Y$ is a (Λ, Λ)-quasi-isometric embedding.

Note that H is a free group and so the image Ho has a quasi-geodesic tree structure: any two points can be connected by a piecewise geodesic γ labeled by a reduced word W. Since the truncation β lies in the N-beighborhood of γ and is contracting by Lemma 4.2, we obtain that γ is K-contracting for a constant K independent of W. By an elementary argument, it is straightforward to verify that the image Ho is contracting by definition of the contracting property.

Now, it remains to prove that an element represented by a nontrivial reduced word W is contracting. Since contracting elements are preserved under conjugation, we can assume that the word W is cyclically reduced. We now form the bi-infinite word $W^\infty = \ldots W \cdot W \ldots$, which is reduced. We then define a bi-infinite path γ by concatenating geodesics and truncate it to get a contracting $(D, 9C)$-admissible path β by the same argument in Lemma 4.2. Since the element represented by W acts by translation on the contracting quasi-geodesic β, this implies by definition that W is contracting. This concludes the proof of Theorem 1.4. □

Proof of Theorem 1.5. If a non-elementary group G admits a proper SCC action on (Y, d) with a contracting element, then the corresponding set defined in Lemma 4.1 is exponentially generic since these sets provided in Section 3 are exponentially negligible. Therefore, $\mathcal{F}^{(k)}$ is exponentially generic in $G^{(k)}$. □

4.2. Statistical hyperbolicity

Proof of Theorem 1.8 for the annuli case. Choose any $0 < \varepsilon < \frac{1}{4}$. Let

$$E = U\left(\frac{\varepsilon}{2}\right) \cup V(2\varepsilon, 3\varepsilon, h^m) \cap W(\varepsilon, h, \sigma(C)),$$

where $\sigma(\cdot)$ is given by Lemma 2.5.

Then by Lemmas 3.1, 3.3, and 3.2 together, we have

$$\lim_{n\to+\infty} \frac{|E \cap A(n,\Delta)|}{|A(n,\Delta)|} = 0 \tag{17}$$

for some $\Delta > 0$. Below we fix such a constant Δ.

For any $x \in A(n,\Delta) \setminus E$, we fix a geodesic $\alpha = [o, xo]$, and consider the following set

$$K_x = \{z \in A(n,\Delta)\colon \text{there exists } \beta = [o, zo] \text{ such that } \beta \cap N_{4C}(\alpha^{[\varepsilon,4\varepsilon]}) \neq \emptyset\}.$$

We shall show that:

Lemma 4.3. *The set K_x is negligible*:

$$\lim_{n\to\infty} \frac{|K_x|}{|A(n,\Delta)|} = 0 \tag{18}$$

Proof of Lemma 4.3. If $n_1 = |x|$, then $n - \Delta \le n_1 \le n + \Delta$. We carry out the same analysis as in the proof of Lemma 3.5 to bound $|K_x|$.

Let $z \in K_x$ be an element of length $n_2 = |z|$. If there exists some $\varepsilon n_1 \le i \le 4\varepsilon n_1$ such that $\alpha(i)$ lies in $N_M(Go)$, then there is some $v_1 \in A(i, M)$ such that the pair (x, z) can be written separately as $(v_1(v_1^{-1}x), v_1(v_1^{-1}z))$ where $v_1^{-1}z \in A(n_2 - i, 2\Delta)$.

Otherwise, we can write (x, z) separately as $(v_2(v_2^{-1}x), v_2(v_2^{-1}z))$ for some $i \le l \le i + \frac{\varepsilon}{2}n_1$ and some $v_2 \in A(l, M)$, such that $v_2^{-1}z \in A((n_2-i)+(l-i), 2\Delta)$ (by our choice of $x \notin U\left(\frac{\varepsilon}{2}\right)$, Subcase 2.1 of Lemma 3.5 can not happen).

If we introduce the index sets

$$\Lambda_1 = \{(n_2, i) \in \mathbb{Z}^2\colon n - \Delta \le n_2 \le n + \Delta, \varepsilon n_1 \le i \le 4\varepsilon n_1\},$$
$$\Lambda_2 = \left\{(n_2, i, l) \in \mathbb{Z}^3\colon \rho n \le n_2 \le n, \varepsilon n_1 \le i \le 4\varepsilon n_1, i \le l \le i + \frac{\varepsilon}{2}n_1\right\},$$

then we have

$$\begin{aligned}|K_x| &\le \sum_{(n_2,i)\in\Lambda_1} |A(n_2 - i, 2\Delta)| + \sum_{(n_2,i,l)\in\Lambda_2} |A(n_2 + l - 2i, 2\Delta)| \\ &\prec \exp((1-\varepsilon)n\delta_G) + n^3 \exp\left(\left(1 - \frac{\varepsilon}{2}\right)n\delta_G\right),\end{aligned}$$

which implies (18) from the purely exponential growth $|A(n,\Delta)| \asymp \exp(\delta_G n)$. $\triangle$

The next step is to bound the distance between xo with the orbit point yo outside K_x. (See figure 2)

Lemma 4.4. *For any $y \in A(n, \Delta) \setminus K_x$,*

$$d(xo, yo) \geq 2(n - 4\varepsilon n - 4\varepsilon\Delta - \Delta - 4C).$$

Proof of Lemma 4.4. Since $x \notin V(2\varepsilon, 3\varepsilon, h^m)$ in (9), α contains a (ν, h^m)-barrier in $\alpha^{[2\varepsilon,3\varepsilon]}$, so there exists an element $g \in G$ such that

$$\max\{d(go, \alpha^{[2\varepsilon,3\varepsilon]}),\ d(gh^m o, \alpha^{[2\varepsilon,3\varepsilon]})\} \leq \nu \leq C.$$

We denote $X = N_C(\mathbf{Ax}(h))$. Let v, w be the entry and exit point of α into gX respectively, so that

$$d(v, w) \geq d(o, h^m o) - 2\nu > D.$$

For any $y \in A(n, \Delta) \setminus K_x$, we know from the definition of K_x that for any geodesic $\beta = [o, yo]$, $\beta \cap N_{4C}(\alpha^{[\varepsilon,4\varepsilon]}) = \emptyset$. Thus, we have $\beta \cap gX = \emptyset$ by Lemma 2.8.

If we choose $d(o, h^m o) - 2\nu > D \geq 16C$ as in the setup, then for any $\gamma = [xo, yo]$, we have $\gamma \cap gX \neq \emptyset$. Indeed, if $\gamma \cap gX = \emptyset$, then $\|\Pi_{gX}(\gamma)\| \leq 3C$ by the contracting property of gX. We will then obtain a contradiction:

$$\begin{aligned} d(v, w) &\leq \|\Pi_{gX}(\{v, o\})\| + \|\Pi_{gX}(\{o, yo\})\| \\ &\quad + \|\Pi_{gX}(\{yo, xo\})\| + \|\Pi_{gX}(\{xo, w\})\| \\ &\leq \|\Pi_{gX}([o, v]_\alpha)\| + \|\Pi_{gX}(\beta)\| + \|\Pi_{gX}(\gamma)\| + \|\Pi_{gX}([w, xo]_\alpha)\| + 4C \\ &\leq 12C + 4C < D, \end{aligned}$$

where $\|\Pi_{gX}(\beta)\| \leq 3C$ follows from the fact $\beta \cap gX = \emptyset$.

Since $x \notin W(\varepsilon, h, \sigma(C))$ in (8), this implies that $v, w \in \alpha^{[\varepsilon,4\varepsilon]}$. Thus, $d(o, w) \geq \varepsilon n_1$ and $d(w, xo) \geq (1 - 4\varepsilon)n_1$.

Let u be the entry point of γ in gX. Then $d(u, w) \leq 4C$ by the contracting property of X. Hence,

$$d(xo, u) \geq d(xo, w) - d(u, w) \geq n_1 - 4\varepsilon n_1 - 4C.$$

Since $y \in A(n, \Delta)$,

$$n - \Delta \leq d(o, yo) \leq d(o, w) + d(w, u) + d(u, yo) \leq \varepsilon n_1 + 4C + d(u, yo)$$

which yields

$$d(u, yo) \geq n - 4\varepsilon n_1 - \Delta - C.$$

We finally obtain

$$d(xo, yo) = d(xo, u) + d(u, yo) \geq 2(n - 4\varepsilon n - 4\varepsilon\Delta - \Delta - 4C)$$

concluding the proof of the claim. △

Let us return to the proof of the theorem. By Lemma 4.4,

$$\sum_{x,y\in A(n,\Delta)} d(xo, yo) \geq 2(n-4\varepsilon n-4\varepsilon\Delta-\Delta-4C)\cdot(|A(n,\Delta)|-|E|)\cdot(|A(n,\Delta)|-|K_x|).$$

Since

$$\lim_{n\to\infty} \frac{|K_x|}{|A(n,\Delta)|} = 0$$

in (18) and

$$\lim_{n\to\infty} \frac{|E|}{|A(n,\Delta)|} = 0$$

in (17), we obtain

$$\liminf_{n\to\infty} \frac{1}{|A(n,\Delta)|^2} \sum_{x,y\in A(n,\Delta)} \frac{d(xo,yo)}{n} \geq 2(1-4\varepsilon).$$

Since ε is arbitrary, we have $E_A(G,\Delta) = 2$. □

Proof of Theorem 1.8 for the ball case. The proof is almost identical to that in the annuli case. We only spell out the difference in the proof.

Choose any $\frac{1}{2} < \rho < 1$ and any $0 < \varepsilon < \frac{\rho}{8}$. Let

$$E = U\left(\frac{\varepsilon}{2}\right) \cup V(2\varepsilon, 3\varepsilon, h^m) \cap W(\varepsilon, \sigma(C)).$$

Then by Lemmas 3.1, 3.3, and 3.2 together, we have

$$\lim_{n\to+\infty} \frac{|E\cap B_n|}{|B_n|} = 0.$$

For any $x \in A([\rho n, n]) \setminus E$, set $n_1 = |x|$, then $\rho n \leq n_1 \leq n$. We fix a geodesic $\alpha = [o, xo]$ and consider

$$K_x = \{z \in A([\rho n, n])\colon \text{there exists } \beta = [o, zo] \text{ such that } \beta \cap N_{4C}(\alpha^{[\varepsilon,4\varepsilon]}) \neq \emptyset\}.$$

By the same argument as in the annuli case, we have

$$\lim_{n\to\infty} \frac{|K_x|}{|A([\rho n, n])|} = 0.$$

Claim. *For any $y \in A([\rho n, n]) \setminus K_x$, we have $d(xo, yo) \geq 2\rho n - 8\varepsilon n - 8C$.*

Proof of the claim. The proof is the same as that in the annuli case, except that we now use the big annulus $A([\rho n, n])$. Note that

$$d(xo, u) \geq d(xo, w) - d(u, w) \geq n_1 - 4\varepsilon n_1 - 4C,$$

where $n_1 \in [\rho n, n]$. Since $y \in A([\rho n, n])$,

$$\rho n \leq d(o, yo) \leq d(o, w) + d(w, u) + d(u, yo) \leq 4\varepsilon n_1 + C + d(u, yo),$$

from which $d(u, yo) \geq \rho n - 4\varepsilon n_1 - 4C$. So

$$d(xo, yo) = d(xo, u) + d(u, yo) \geq 2\rho n - 8\varepsilon n - 8C. \qquad \triangle$$

The same computation as above in the annuli case gives

$$\liminf_{n\to\infty} \frac{1}{|B_n|^2} \sum_{x,y\in B_n} \frac{d(xo, yo)}{n} \geq 2\rho - 8\varepsilon.$$

Since ε can be made arbitrary small and ρ can be arbitrary close to 1, we obtain $E_B(G) = 2$. ☐

Example 4.5. We carry out a concrete example to explain the convergence speed of $E_A(G, \Delta) = 2$ of a statistically hyperbolic group is at most of order $O(n^{-1})$. Consider the free group $\mathbb{F}(a, b)$ and its Cayley graph with respect to the free generators $\{a, b\}$. It is easy to calculate

$$\frac{1}{|A(n,0)|^2} \sum_{x,y\in A(n,0)} d(x, y) = \frac{3}{4} \cdot 2n + \frac{1}{4}\frac{2}{3} \cdot (2n - 2) + \frac{1}{4}\frac{1}{3}\frac{2}{3} \cdot (2n - 4) + \cdots + 0.$$

Thus we obtain

$$\begin{aligned}\left|\frac{1}{|A(n,0)|^2} \sum_{x,y\in A(n,0)} d(x, y) - 2\right| &= \left|\frac{n-1}{2n} - \frac{3}{4n} \cdot \left(\frac{1}{3} - \frac{1}{3^n}\right) - \frac{1}{2}\right| \\ &= \frac{1}{2n} + \frac{3}{4n} \cdot \left(\frac{1}{3} - \frac{1}{3^n}\right) \\ &= O\left(\frac{1}{n}\right).\end{aligned}$$

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