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Subgroups of word hyperbolic groups in rational dimension 2

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Abstract. A result of Gersten states that if G is a hyperbolic group with integral cohomological dimension $\mathrm{cd}_{\mathbb{Z}}(G) = 2$ then every finitely presented subgroup is hyperbolic. We generalize this result for the rational case $\mathrm{cd}_{\mathbb{Q}}(G) = 2$. In particular, our result applies to the class of torsion-free hyperbolic groups G with $\mathrm{cd}_{\mathbb{Z}}(G) = 3$ and $\mathrm{cd}_{\mathbb{Q}}(G) = 2$ discovered by Bestvina and Mess.

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1. Introduction

The cohomological dimension $\mathrm{cd}_R(G)$ of a group G with respect to a ring R is less than or equal to n if the trivial RG-module R has a projective resolution of length n. Let $\mathbb{Q}$ denote the field of rational numbers. The main result of this note:

Theorem 1.1. *Let G be a hyperbolic group such that $\mathrm{cd}_{\mathbb{Q}}(G) \leq 2$. If H is a finitely presented subgroup, then H is hyperbolic.*

The analogous statement for $\mathrm{cd}_{\mathbb{Z}}(G)$ is a result of Steve Gersten that we recover as a consequence of the inequality

$$\mathrm{cd}_{\mathbb{Q}}(G) \leq \mathrm{cd}_{\mathbb{Z}}(G).$$

Corollary 1.2 (Gersten, [10, Theorem 5.4]). *Let G be a hyperbolic group such that $\mathrm{cd}_{\mathbb{Z}}(G) = 2$. If H is a finitely presented subgroup, then H is hyperbolic.*

The first motivation to generalize Gersten's result to the rational case is the existence of hyperbolic groups of integral cohomological dimension three and rational cohomological dimension two. The nature of finitely presented subgroups of groups in this class was not known. The first examples of such groups were discovered by Bestvina and Mess [3] based on methods by Davis and Januszkiewicz [6]. The class also contains finite index subgroups of hyperbolic Coxeter groups, examples that were discovered by Dranishnikov [7, Corollary 2.3]. We recall the nature of Bestvina-Mess examples in the following corollary.

Corollary 1.3 ([3]). *Let X be a finite polyhedral 3-complex such that*

- *X admits piecewise constant negative curvature cellular structure satisfying Gromov's link condition, and*
- *X is a 3-manifold (without boundary) in the complement of a single vertex whose link is a non-orientable closed surface.*

If $G = \pi_1 X$ then $\mathsf{cd}_{\mathbb{Q}}(G) = 2$, $\mathsf{cd}_{\mathbb{Z}}(G) = 3$ and any finitely presented subgroup of G is hyperbolic.

The statement of Corollary 1.2 is sharp in the sense that there exist hyperbolic groups of integral cohomological dimension three containing finitely presented subgroups that are not hyperbolic, the first example was found by Noel Brady [4]. More recently, infinite families of hyperbolic groups of integral cohomological dimension three containing non-hyperbolic finitely presented subgroups have been constructed, see for example [15].

Corollary 1.4. *If G is a hyperbolic group such that $\mathsf{cd}_{\mathbb{Z}}(G) = 3$ and it contains a non-hyperbolic finitely presented subgroup, then $\mathsf{cd}_{\mathbb{Q}}(G) = \mathsf{cd}_{\mathbb{Z}}(G)$.*

A second motivation of this project was to generalize Gersten's result to groups admitting torsion, specifically, to the class of hyperbolic groups G admitting a 2-dimensional classifying space for proper actions $\underline{E}G$. Recall that a model for $\underline{E}G$ is a G-CW-complex X with the property that for each subgroup H the subcomplex of fixed points is contractible if H is finite, and empty if H is infinite. The minimal dimension of a model for $\underline{E}G$ is denoted by $\underline{\mathsf{gd}}(G)$. Considering the cellular chain complex with rational coefficients of a model for $\underline{E}G$ with minimal dimension shows that

$$\mathsf{cd}_{\mathbb{Q}}(G) \leq \underline{\mathsf{gd}}(G).$$

This inequality implies the following corollary.

Corollary 1.5. *If G is a hyperbolic group such that $\underline{\mathsf{gd}}(G) \leq 2$, then any finitely presented subgroup is hyperbolic.*

The statement of Corollary 1.5 was known in the following cases:

- If G admits a $\mathsf{CAT}(-1)$ 2-dimensional model for $\underline{E}G$, see [12, Corollary 1.5].
- If G admits a 2-dimensional model for $\underline{E}G$, and H is finitely presented with finitely many conjugacy classes of finite groups, a consequence of [16, Theorem 1.3].
- If G is a hyperbolic small cancellation group of type $C(7)$, $C(5){-}T(4)$, $C(4){-}T(5)$, $C(3){-}T(7)$ or $C'(1/6)$, see [10, Theorem 7.6].

We remark that for a group G satisfying the hypothesis of Corollary 1.5, the conclusion follows from Gersten's result 1.2 if, in addition, G is assumed to be virtually torsion free. It is an outstanding question whether hyperbolic groups are virtually torsion free [14].

Remark 1.6. During the refereeing process of this manuscript, a generalization of Theorem 1.1 was proved in context of totally disconnected locally compact hyperbolic groups [1].

Homological filling functions and the proof of Theorem 1.1. Let R be a subgring of $\mathbb{Q}$. The $(n+1)$-dimensional homological Filling Volume function over R of a cellular complex X is a function $\mathsf{FV}^{n+1}_{X,R}\colon \mathbb{N} \to \mathbb{R}$ describing the minimal volume required to fill integral cellular n-cycles with cellular $(n+1)$-chains with coefficients in R.

For a group G with a $K(G,1)$ model X with finite $(n+1)$-skeleton, the $(n+1)$-dimensional homological Filling Volume function over R of G, denoted by $\mathsf{FV}^{n+1}_{G,R}$, is defined as $\mathsf{FV}^{n+1}_{\widetilde{X},R}$ where $\widetilde{X}$ is the universal cover of X. This function depends only of the group G up to the equivalence relation on the set of non-decreasing functions $\mathbb{N} \to \mathbb{R}$ defined as $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means there is $C > 0$ such that for all $k \in \mathbb{N}$,

$$f(k) \le Cg(Ck + C) + Ck + C.$$

Recall that a group G is of type R-FP_n if the trivial RG-module R admits a partial projective resolution

$$P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \to P_0 \longrightarrow R \to 0$$

where each P_i is a finitely generated RG-module. In [13], it is shown that to define $\mathsf{FV}^{n+1}_{G,\mathbb{Z}}$ is enough to assume that the group G is of type $\mathbb{Z}$-FP_{n+1}. We prove that the same statement holds for $\mathsf{FV}^{n+1}_{G,R}$ in Section 3. The main technical result of this note is the following.

Theorem 1.7. *Let R be a subring of $\mathbb{Q}$. Let G be a group of type R-FP_{n+1} and suppose $\mathsf{cd}_R(G) = n+1$. Let $H \le G$ be a subgroup of type R-FP_{n+1}. Then there is a constant $C > 0$ such that for all k*

$$FV^{n+1}_{H,R} \preceq FV^{n+1}_{G,R}.$$

This theorem generalizes the main result of [13], by considering an arbitrary subgring of the rational numbers instead of only the ring of integers, and by replacing the topological assumptions F_{n+1} on G and H with the weaker hypothesis R-FP_{n+1}.

The main result of this note, Theorem 1.1, is a consequence of Theorem 1.7 and the characterization of hyperbolic groups stated below, which is credited to Gersten [9]. This characterization was revised by Mineyev [17, Theorem 7, statements (0) and (2)], and it was also revisited by Groves and Manning in [11, Theorem 2.30].

Theorem 1.8 ([17, Theorem 7], [11, Theorem 2.30]). *A group G is hyperbolic if and only if G is finitely presented and the rational filling function $\mathsf{FV}^2_{G,\mathbb{Q}}$ is bounded by a linear function, i.e., $\mathsf{FV}^2_{G,\mathbb{Q}}(k) \preceq k$.*

Proof of Theorem 1.1. Let G be a hyperbolic group such that $\mathsf{cd}_{\mathbb{Q}}(G) = 2$, and let H be a finitely presented subgroup. Theorem 1.8 implies that $\mathsf{FV}^2_{G,\mathbb{Q}}$ is bounded by a linear function. By Theorem 1.7, $\mathsf{FV}^2_{H,\mathbb{Q}} \preceq \mathsf{FV}^2_{G,\mathbb{Q}}$. It follows $\mathsf{FV}^2_{H,\mathbb{Q}}$ is bounded by a linear function. Then Theorem 1.8 implies that H is a hyperbolic group. □

In view of Theorem 1.8, we raised the following question.

Question 1.9. *Let G be a $\mathbb{Q}$-FP_2 group and suppose $FV^2_{G,\mathbb{Q}}$ is bounded by a linear function. Is G a hyperbolic group*?

The analogous question obtained by replacing $\mathbb{Q}$ with $\mathbb{Z}$ is known to have a positive answer [10, Theorem 5.2]. One motivation behind this question is that a positive answer would imply that in Theorem1.1 H can be assumed to be $\mathbb{Q}$-FP_2 instead of being finitely presented. Recall that $\mathbb{Q}$-FP_2 condition is weaker than being finitely presented, see the examples in [2].

The rest of the note is devoted to the definition of homological filling function and the proof of Theorem 1.7. The argument is relatively self-contained, and uses and simplifies ideas from [13]. The main contributions of the article beside the results stated above are

(1) the definition of filling functions for arbitrary subdomains of the rationals, since the definition in [13] does not generalize directly, and

(2) the replacement of topological arguments in [13] by algebraic ones that allow us to prove certain statements under the weaker homological finiteness condition R-FP_{n+1} instead of the topological assumption F_{n+1}; see Proposition 4.1 which is a construction based on the homological mapping cylinders, and Remark 4.2.

Organization. Preliminary definitions are included in Section 2, specifically the notions of filling norms and bounded morphisms on modules over arbitrary normed rings. Section 3 discusses the generalization of homological filling functions defined over arbitrary subdomains of the rational numbers. The last section contains the proof of Theorem 1.7.

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2. Filling norms, bounded morphisms

All rings considered in this article have a multiplicative identity. Let R be a ring and let $\mathbb{R}$ denote the ordered field of real numbers. A norm on R is a function $|\cdot|\colon R \to \mathbb{R}$ such that for any $r, r' \in R$

- $|r| \geq 0$ with equality if and only if $r = 0$,
- $|r + r'| \leq |r| + |r'|$, and
- $|r_1 r_2| \leq |r_1||r_2|$ for $r_1, r_2 \in R$.

A *normed ring* is a ring equipped with a norm.

From here on, assume that R is a normed ring. A norm on an R-module M is a function $\|\cdot\|\colon M \to \mathbb{R}$ such that for any $m, m' \in M$ and $r \in R$

- $\|m\| \geq 0$ with equality if and only if $m = 0$,
- $\|m + m'\| \leq \|m\| + \|m'\|$, and
- $\|rm\| \leq |r|\|m\|$.

A function $M \to \mathbb{R}$ that satisfies the last two conditions and has only non-negative values is called a pseudo-norm.

The ℓ_1*-norm* on a free R-module F with fixed basis Λ is defined as

$$\Big\| \sum_{x \in \Lambda} r_x x \Big\|_1 = \sum_{x \in \Lambda} |r_x|.$$

A free R-module with fixed basis is called a *based free module*.

Definition 2.1 (filling norm). A *filling norm* on a finitely generated R-module M is defined as follows. Let $\rho\colon F \to M$ be a surjective morphism of R-modules where F is a finitely generated free R-module with fixed basis Λ and induced ℓ_1-norm $\|\cdot\|_1$. The *filling norm on M* induced by ρ and Λ is defined as

$$\|m\|_M = \inf\{\|x\|_1 \colon x \in F, \rho(x) = m\}.$$

Remark 2.2. The following statements can be easily verified.

(1) An ℓ_1-norm $\|\cdot\|_1$ on a finitely generated free R-module F is a filling norm.

(2) A filling norm $\|\cdot\|$ on a finitely generated R-module M is a pseudo-norm, and is regular in the sense that

$$\|rm\| = |r|\|m\|$$

for any $m \in M$ and $r \in R$ such that r is a unit and $|r||r^{-1}| = 1$.

Definition 2.3 (bounded morphism). A morphism $f: M \to N$ between R-modules with norms $\|\cdot\|_M$ and $\|\cdot\|_N$ respectively is called *bounded* (*with respect to these norms*) if there exists a fixed constant $C > 0$ such that $\|f(a)\|_N \leq C\|a\|_M$ for all $a \in M$.

The following lemma appears in [16] for the case that R is a group ring. The proof for an arbitrary ring is analogous, we have included the argument for the convenience of the reader.

Lemma 2.4 ([16, Lemma 4.6]). *Morphisms between finitely generated R-modules are bounded with respect to filling norms.*

Proof. First observe that if $\tilde{\varphi}: A \to B$ is a morphism between finitely generated based free R-modules, then for $a \in A$,

$$\|\tilde{\varphi}(a)\|_B \leq C\|a\|_A,$$

where $\|\cdot\|_A$ and $\|\cdot\|_B$ are the corresponding ℓ_1-norms, the constant C is defined as $\max\{\|\tilde{\varphi}(a)\|_B : a \in \Lambda\}$ where Λ is the fixed basis of A.

Now we prove the statement of the lemma. Let $\varphi: P \to Q$ be a morphism between finitely generated R-modules, and let $\|\cdot\|_P$ and $\|\cdot\|_Q$ denote filling norms on P and Q respectively. Suppose A is a finitely generated based free R-module and that $\rho: A \to P$ induces the filling norm $\|\cdot\|_P$, and analogously assume that $\rho': B \to Q$ induces the filling norm $\|\cdot\|_Q$. Then, since A is free, there is a morphism $\tilde{\varphi}: A \to B$ such that $\varphi \circ \rho = \rho' \circ \tilde{\varphi}$. Let C be the constant for $\tilde{\varphi}$ defined above. Let $p \in P$ and note that for any $a \in A$ such that $\rho(a) = p$,

$$\|\varphi(p)\|_Q \leq \|\tilde{\varphi}(a)\|_B \leq C\|a\|_A.$$

Hence $\|\varphi(p)\|_Q \leq C\|p\|_P$. □

Two norms $\|\cdot\|$ and $\|\cdot\|'$ on an R-module M are said to be *equivalent* if there exists a constant $C > 0$ such that for all $m \in M$

$$C^{-1}\|m\| \leq \|m\|' \leq C\|m\|.$$

By considering the identity function on a finitely generated module M, the previous lemma implies:

Corollary 2.5. *Any two filling norms on a finitely generated R-module M are equivalent.*

Remark 2.6. Let M be a free R-module with basis Λ, and let N be a free R-submodule generated by a finite subset $\Lambda' \subseteq \Lambda$. Consider the induced ℓ_1-norms $\|\cdot\|_\Lambda$ and $\|\cdot\|_{\Lambda'}$ on M and N respectively.

(1) The projection map $\pi\colon M \to N$ is bounded with respect to the induced ℓ_1-norms.

(2) The inclusion map $\iota\colon N \to M$ preserves the induced ℓ_1-norms, in particular, it is bounded.

Lemma 2.7. *Let N be a finitely generated module with filling norm $\|\cdot\|_N$. Suppose that N is an internal direct summand of a free module F with an ℓ_1-norm $\|\cdot\|_1$. Then $\|\cdot\|_N \sim \|\cdot\|_1$ on N.*

Proof. Since N is a finitely generated module contained in F, there exist a finitely generated free submodule I of F which is an internal summand, $F = I \oplus J$, such that $N \subseteq I$, and the restriction of $\|\cdot\|_1$ to I is an ℓ_1-norm on I. Let $\iota\colon N \to I$ denote the inclusion and $\phi\colon F \to N$ denote the projection. By Lemma 2.4, both $\phi|_I\colon I \to N$ and $\iota\colon N \to I$ are bounded morphisms with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_N$; let C_1 and C_2 be the corresponding constants. Then

$$\|n\|_N = \|\phi(\iota(n))\|_N \leq C_1\|\iota(n)\|_1 \leq C_2C_1\|n\|_N$$

for all $n \in N$, and hence $\|\cdot\|_N \sim \|\cdot\|_1$ on N. □

For the rest of this section, let G be a group, let H be a subgroup, and as above let R be a ring with norm $|\cdot|$.

Remark 2.8. Let M be a free RG-module with ℓ_1-norm $\|\cdot\|_\Lambda$ induced by a free basis set Λ. Then M is a free RH-module and there exist a free RH-basis Λ_H of M such that the induced ℓ_1-norms $\|\cdot\|_\Lambda$ and $\|\cdot\|_{\Lambda_H}$ are equal.

Indeed, if S is a right transversal of the subgroup H in G, then

$$\Lambda_H = \{gx\colon x \in \Lambda, g \in S\}$$

is a free RH-basis of M as an H-module, and the statement about the ℓ_1-norms holds.

Lemma 2.9. *Let M be a finitely generated and projective RG-module with filling norm $\|\cdot\|_M$ and let N be a finitely generated RH-module with filling norm $\|\cdot\|_N$. Suppose that N is a internal direct summand of M as an RH-module. Then $\|\cdot\|_N \sim \|\cdot\|_M$ on N.*

Proof. Let F be a finitely generated free based module with ℓ_1-norm $\|\cdot\|_1$, and let $\phi\colon F \to M$ be a surjective RG-morphism inducing filling norm $\|\cdot\|_M$. Since M is projective, there exist an RG-morphism $j\colon M \to F$ such that $j \circ \phi = \mathrm{id}_M$. Lemma 2.4 implies that j and ϕ are bounded RG-morphisms. Therefore $\|\cdot\|_M \sim \|\cdot\|_1$. Now consider F as an RH-module with same ℓ_1-norm $\|\cdot\|_1$, see Remark 2.8. Since N is a direct summand of M as an RH-module, it is a direct summand of F as an RH-module. Then Lemma 2.7 implies $\|\cdot\|_N \sim \|\cdot\|_1$ on N. ☐

3. Definition of homological filling functions

In this section R denotes a subring of the rational numbers with the absolute value as a norm. Let G be a group. The group ring RG is a free abelian module over R, observe that RG is a normed ring with ℓ_1-norm induced by the free R-basis G. From now on, we consider RG as a normed ring with this norm.

Definition 3.1 (integral part). Let P be a finitely generated RG-module. An integral part of P is a $\mathbb{Z}G$-submodule A which is finitely generated as a $\mathbb{Z}G$-module, and A generates P as an RG-module.

From here on, $[0,\infty]$ denotes the set of non-negative real numbers and infinity. The order relation as well as the addition operations are extended in the natural way.

Definition 3.2. The n^{th}-*filling function* of a group G of type R-FP_{n+1},

$$FV^{n+1}_{G,R}\colon \mathbb{N} \longrightarrow [0,\infty],$$

is defined as follows. Let

$$P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow R \to 0, \tag{1}$$

be a partial projective resolution of finitely generated RG-modules of the trivial RG-module R. Let K_n be an integral part for $\ker(\partial_n)$, let $\|\cdot\|_{P_n}$ and $\|\cdot\|_{P_{n+1}}$ be filling norms for P_n and P_{n+1} respectively. Then

$$FV^{n+1}_{G,R}(k) = \sup\{\|\gamma\|_{\partial_{n+1}}\colon \gamma \in K_n,\ \|\gamma\|_{P_n} \le k\},$$

where

$$\|\gamma\|_{\partial_{n+1}} = \inf\{\|\mu\|_{P_{n+1}}\colon \mu \in P_{n+1},\ \partial_{n+1}(\mu) = \gamma\}.$$

By convention, define the supremum of the empty set as zero.

See Remark 3.8 on finiteness of $FV^{n+1}_{G,R}$. The rest of this section discusses the proof of the following theorem, which generalizes [13, Theorem 3.5]. Consider the equivalence relation on the set of non-decreasing functions $\mathbb{N} \to [0, \infty]$ defined as $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means there is $C > 0$ such that for all $k \in \mathbb{N}$,

$$f(k) \leq Cg(Ck + C) + Ck + C.$$

Theorem 3.3. *Let G be a group of type R-FP_{n+1}. Then the n^{th}-filling function $FV^{n+1}_{G,R}$ of G is well defined up to the equivalence relation $\sim$.*

The proof of Theorem 3.3 relies on the following basic structure theorem for subgrings of $\mathbb{Q}$.

Proposition 3.4. *Let R be a subring of $\mathbb{Q}$. Then there is a set S of prime numbers in $\mathbb{Z}_+$ such that R consists of all fractions $\frac{a}{b}$ where $a \in \mathbb{Z}$ and b is product of powers of elements of S.*

In the following proposition, which is a consequence of Proposition 3.4, we use the convention that for an element a of an RG-module A, and any $r \in R$, ra denotes the element $(re)a \in A$ where e is the identity element of G; moreover, the ring of integers $\mathbb{Z}$ is naturally identified with the subring of RG via $m \mapsto me$.

Proposition 3.5. *Let P and Q be finitely generated RG-modules. Then*

(1) *if A is an integral part, then for all units $r \in R$, $rA = \{ra : a \in A\}$ is an integral part*;

(2) *if $f: P \to Q$ is a morphism of RG-modules, and A and B are integral parts of P and Q respectively, then there exists a positive integer m which is a unit in RG and such that $f(mA) \subseteq B$.*

Proof. The first statement is immediate from the definition. For the second statement. Let S be a finite generating set of A as a $\mathbb{Z}G$-module, and observe that S generates P as an RG-module. Let $F(S)$ be the free RG-module on S, let $\phi: F(S) \to P$, and let C be the $\mathbb{Z}G$-submodule of $F(S)$ generated by S, and observe that $\phi(C) = A$. Analogously, let T be a finite generating set of B as a $\mathbb{Z}G$-module, let $\psi: F(T) \to Q$, and let C' be the $\mathbb{Z}G$-submodule of $F(T)$ generated by T, and note that $\psi(C') = B$.

Since $F(S)$ is free, there is an RG-morphism $\eta: F(S) \to F(T)$ such that the following diagram commutes:

$$\begin{array}{ccc} F(S) & \xrightarrow{\eta} & F(T) \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ P & \xrightarrow[f]{} & Q \end{array} \tag{2}$$

Note that $\eta\colon F(S) \to F(T)$ is described by a finite matrix with entries in RG. By Proposition 3.4, there is an integer m, which is a unit in R, such that the morphism $m.\eta\colon F(S) \to F(T)$ given by $\alpha \mapsto m\alpha$ has the property that $\eta(C) \subseteq C'$. By commutativity of the diagram $f \circ (m\phi) = \psi \circ (m\eta)$ and therefore $f(mA) \subseteq B$. □

The following lemma is a strengthening of Proposition 3.5 that will be used in the last section.

Lemma 3.6. *Let $H \leqslant G$ be a subgroup and let P and Q be finitely generated RH and RG modules respectively. If $f: P \to Q$ is an RH-morphism, and A and B are integral parts of P and Q respectively, then there exists a positive integer m, which is a unit in R, such that $f(mA) \subseteq B$.*

Proof. Considering Q as an RH-module, the proof proceeds similar to 3.5 except that here $F(T)$ is infinitely generated and so the matrix is infinite. But observe that only finitely many entries are non-zero, so the same argument holds. □

Proof of Theorem 3.3. The proof is divided into two steps. The second step is a small variation of the argument in [13, Proof of Theorem 3.5] for which we only remark the changes.

Step 1. FV_G^{n+1} *(up to equivalence) does not depend on the choice of the integral part K_n.*

Let A and B be two integral parts of K_n, and let FV_A and FV_B denote the corresponding n^{th}-filling functions of G. By Proposition 3.5, there exists a positive integer m, that is a unit in RG, such that $m.A \subseteq B$. Let $\gamma \in A$ such that $\|\gamma\|_{P_n} \leq k$. Then, since m is a unit and $|m||m^{-1}| = 1$, $\|\gamma\|_{\delta_{n+1}} = \frac{1}{m}\|m\gamma\|_{\delta_{n+1}}$ and $\|m\gamma\|_{P_n} = m\|\gamma\|_{P_n} \leq mk$; see Remark 2.2. Observe that $m\gamma \in B$ therefore $\|\gamma\|_{\delta_{n+1}} \leq \frac{1}{m}FV_B(mk)$. Since γ was arbitrary, $FV_A(k) \leq \frac{1}{m}FV_B(mk)$. By symmetry we get the other inequality.

Step 2. FV_G^{n+1} *(up to equivalence) does not depend on the choice of the resolution* (1).

Let (P_*, ∂_*) and (Q_*, δ_*) be a pair of resolutions as in (1). Since any two projective resolutions of R are chain homotopy equivalent, there exist chain maps $f_i\colon P_i \to Q_i$, $g_i\colon Q_i \to P_i$, and a map $h_i\colon P_i \to P_{i+1}$ such that

$$\partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i = g_i \circ f_i - Id.$$

By Proposition 3.5, there exist integral parts K_n and K'_n of $\ker(\partial_n)$ and $\ker(\delta_n)$ respectively, such that $f_n(K_n) \subseteq K'_n$. This ensures that the same argument in [13, Proof of Theorem 3.5] works except for a minor change in the choice of the element named β in the cited proof. Replace it by the following: "for $\epsilon > 0$, choose $\beta \in Q_{n+1}$ such that $\delta_{n+1}(\beta) = f_n(\alpha)$ and $\|\beta\|_{Q_{n+1}} < \|f_n(\alpha)\|_{\delta_{n+1}} + \epsilon$." The rest of the proof proceeds in the same manner. □

Remark 3.7 (Topological interpretation of filling functions). Assume G admits a $K(G,1)$ model X with finite $(n+1)$-skeleton. The augmented cellular chain complex $C_*(\widetilde{X}, R)$ of the universal cover $\widetilde{X}$ of X is a projective resolution of the trivial RG-module R by free modules. By considering the ℓ_1-norm of $C_i(\widetilde{X}, R)$ induced by the basis consisting of i-dimensional cells of $\widetilde{X}$, the definition of $\mathsf{FV}^{n+1}_{G,R}$ using this resolution provides the interpretation $\mathsf{FV}^{n+1}_{G,R}$ as the minimal volume required to fill integral n-cycles with $(n+1)$-cellular chains with coefficients in R. Observe that

$$FV^{n+1}_{G,R} \leq FV^{n+1}_{G,\mathbb{Z}} \tag{3}$$

Remark 3.8 (finiteness of $FV^{n+1}_{G,R}$). Assume that G admits a $K(G,1)$ model X with finite $(n+1)$-skeleton. By the main result of [8], for every positive integer k, $FV^{n+1}_{G,\mathbb{Z}}(k) < \infty$. Then equation (3) implies that $FV^{n+1}_{G,R}(k) < \infty$ for any $k \geq 0$.

A positive answer to the following question in the case that $R = \mathbb{Z}$ is given in [8].

Question 3.9. *Suppose that G is of type R-FP_{n+1}. Is $FV^{n+1}_{G,R}(k) < \infty$ for all $k \in \mathbb{N}$?*

Remark 3.10 (on the use integral part in Definition 3.2). We note that the filling function $\mathsf{FV}^{n+1}_{G,\mathbb{Z}}$ was defined in [13] by considering $\ker(\partial_n)$ in lieu of its integral part. This approach does not work to define $\mathsf{FV}^{n+1}_{G,\mathbb{Q}}$ as the following example illustrates. Consider the group presentation $G = \langle x, y | [x, y] \rangle$ and let X be the universal cover of the presentation complex, i.e., the Cayley complex. In X consider the following cycles with rational coefficients $a_n = \frac{1}{4n}[x^n y^n]$ for $n \in \mathbb{N}$. Then $\|a_n\|_1 = 1$ and by regularity $\|a_n\|_\partial = \frac{1}{4}n$, in particular

$$\max\{\|\gamma\|_{\partial_2} : \gamma \in Z_n(\widetilde{X}, \mathbb{Q}), \|\gamma\|_1 \leq 1\} = \infty,$$

and hence the approach in [13] does not yield a well defined $FV^2_{G,\mathbb{Q}}(k)$. In contrast, using Definition 3.2, $FV^2_{G,\mathbb{Q}} \preceq FV^2_{G,\mathbb{Z}} \sim k^2$.

4. Proof of Theorem 1.7

The proof of Theorem 1.7 is discussed after the proof of the following proposition.

Proposition 4.1. *Suppose that $cd_R(G) = n+1$, G is of type R-FP_{n+1}, and H is a subgroup of G of type R-FP_{n+1}. Then for any partial projective resolution of the trivial RH-module R of finite type*

$$Q_{n+1} \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow R \longrightarrow 0, \tag{4}$$

there is a projective resolution of the trivial RG-module R of finite type

$$0 \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow R \longrightarrow 0, \tag{5}$$

an injective morphisms $\imath_i\colon Q_i \to M_i$ of RH-modules, $0 \leq i \leq n$, such that

$$\begin{array}{ccccccc} Q_n & \longrightarrow \cdots \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & R \\ \downarrow{\scriptstyle \imath_n} & & \downarrow{\scriptstyle \imath_1} & & \downarrow{\scriptstyle \imath_0} & & \downarrow{\scriptstyle Id} \\ M_n & \longrightarrow \cdots \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & R. \end{array} \tag{6}$$

is a commutative diagram of RH-modules, and the short exact sequences of RH-modules

$$0 \longrightarrow Q_i \xrightarrow{\imath_i} M_i \longrightarrow S_i \longrightarrow 0 \tag{7}$$

split. In particular each S_i is a projective RH-module.

Remark 4.2. Proposition 4.1 replaces topological arguments in [13], based on work of Gersten [10], that use topological mapping cylinders. The arguments there are relatively less involved. In the generality that we are working, it is not possible to rely on this type of topological constructions. We would need free cocompact actions on $(n+1)$-acyclic complexes for G and H, they are not known to exist under our hypothesis. Specifically, recall that a group G is of type FH_n, if G admits a cocompact action on an n-acyclic space X; in this case the action of G on the cellular chain complex of X induces a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module. Hence FH_n implies FP_n. It is an open question whether groups of type FP_n are of type FH_n for $n \geq 3$, see [2].

The proof of the Proposition 4.1 is an application of the *mapping cylinder* of chain complexes from basic homological algebra that we recall below.

Let $B_* = \{B_i, d_i\}$ and $C_* = \{C_i, d'_i\}$ be two chain complexes of modules over some fixed ring, and let $f\colon B_* \to C_*$ be a chain map. Then the mapping cylinder $M_* = \{M_i, d''_i\}$ is a chain complex where $M_i = C_i \oplus B_i \oplus B_{i-1}$ with

$$d''_i = \begin{pmatrix} d'_i & 0 & -f_i \\ 0 & d_i & Id_B \\ 0 & 0 & -d_i \end{pmatrix}$$

Observe that, if both B_* and C_* consists of only finitely generated projective modules then the the same holds for M_*. The natural inclusion $C_* \hookrightarrow M_*$ given by $c \mapsto (c, 0, 0)$ is a chain homotopy equivalence. The chain homotopy inverse map $\kappa_*\colon M_* \to C_*$ is given by $(c, b, b') \mapsto c + f(b)$. Let $\jmath_*\colon B_* \to M_*$ be the inclusion given by $b \mapsto (0, b, 0)$. It is an observation that the triangle

$$\begin{array}{ccc} B_* & \xrightarrow{f_*} & C_* \\ & {\scriptstyle \jmath_*}\searrow \quad \nearrow{\scriptstyle \kappa_*} & \\ & M_* & \end{array} \tag{8}$$

commutes. For background on mapping cylinders see [18].

Proof of Proposition 4.1. We split the proof into four steps.

Step 1. *Definition of the resolution* (5) *as a mapping cylinder*

Since $cd_R(G) = n + 1$ and G is of type $R\text{-}FP_{n+1}$, there is a projective resolution of RG-modules of finite type

$$0 \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow R \longrightarrow 0, \tag{9}$$

see [5, p. 199, Proposition 6.1].

The group ring RG is a free right RH-module. It follows that the extension of scalars functor from left RH-modules to left RG-modules $A \mapsto RG \otimes_{RH} A$ is exact. This functor also preserves finite generation and projectiveness. From the given resolution (4), we obtain a partial projective resolution of the RG-module $RG \otimes_{RH} R$ of finite type

$$RG \otimes_{RH} Q_n \longrightarrow \cdots \longrightarrow RG \otimes_{RH} Q_0 \longrightarrow RG \otimes_{RH} R \longrightarrow 0. \tag{10}$$

Consider the RG-morphism $\phi\colon RG \otimes_{RH} R \to R$ induced by

$$\phi\colon RG \times R \longrightarrow R, \quad (s, r) \longmapsto \epsilon(s)r, \tag{11}$$

where $\epsilon\colon RG \to R$ is the augmentation map, $\epsilon(\sum r_i g_i) = \sum r_i$. Since each of the RG-modules $RG \otimes_{RH} Q_i$ is projective, there are RG-morphisms $f_i\colon RG \otimes_{RH} Q_i \to P_i$ such that

$$\begin{array}{ccccc} RG \otimes_{RH} Q_n & \to \cdots \to & RG \otimes_{RH} Q_0 & \to & RG \otimes_{RH} R \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \phi} \\ P_n & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & R. \end{array} \tag{12}$$

is a commutative diagram, see [5, p. 22, Lemma 7.4].

Let $M_* = (M_i)$ be the mapping cylinder of the chain map $f = (f_i)$ where f_i is the RG-morphism defined above for $0 \le i \le n$, f_{n+1} is the morphism $0 \to P_{n+1}$, and f_i is the morphism $0 \to 0$ for any other value of i.

Observe that

$$M_i = P_i \oplus (RG \otimes_{RH} Q_i) \oplus (RG \otimes_{RH} Q_{i-1})$$

for $1 \le i \le n$, $M_0 = P_0 \oplus (RG \otimes_{RH} Q_0) \oplus 0$, $M_{n+1} = P_{n+1} \oplus 0 \oplus (RG \otimes_{RH} Q_n)$, and $M_i = 0$ for any other value of i. Hence all M_i are finitely generated and projective.

Let $P_* = (P_i)$ be the chain complex induced by (9), where $P_i = 0$ for $i > n+1$ and $i < 0$. Observe that P_* is the target of the chain map f. Since P_* and M_* are chain homotopic,

$$0 \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow R \longrightarrow 0,$$

is a projective resolution of finite type of the trivial RG-module R.

Step 2. *Definition of the injective RH-morphisms $\imath_i \colon Q_i \to M_i$.*

We have the following commutative diagram of RH-modules

$$\begin{array}{ccccccc} Q_n & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 \\ \downarrow{\scriptstyle \tau_n} & & & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_0} \\ RG \otimes_{RH} Q_n & \longrightarrow & \cdots & \longrightarrow & RG \otimes_{RH} Q_1 & \longrightarrow & RG \otimes_{RH} Q_0 \\ \downarrow{\scriptstyle \jmath_n} & & & & \downarrow{\scriptstyle \jmath_1} & & \downarrow{\scriptstyle \jmath_0} \\ M_n & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0. \end{array} \tag{13}$$

where $\tau_k \colon Q_k \to RG \otimes_{RH} Q_k$ is the natural inclusion given by $q \mapsto e \otimes q$ (here e denotes the identity element of G), and the vertical arrows $\jmath_i \colon RG \otimes_{RH} Q_i \to M_i$ are the natural inclusions. Then define

$$\imath_i = \jmath_i \circ \tau_i$$

for $0 \le i \le n$, and observe that they are injective RH-morphisms.

Step 3. *Verifying commutative diagram* (6).

In view of the commutative diagram (13), we only need to verify that if $H_0(Q)$ and $H_0(M)$ denote the cokernels of $Q_1 \to Q_0$ and $M_1 \to M_0$ respectively, then the RH-morphism $H(\imath_0) \colon H_0(Q) \to H_0(M)$ induced by $\imath_0$ is an isomorphism.

Before the argument, we remark that this is not immediate, it depends on the choice of the RG-morphism f_0; the available choices for f_0 depend on the choice of the RG-morphism $\phi \colon RG \otimes_{RH} R \to R$; our choice is defined by (11).

Let $H_0(P)$ denote the cokernel of $P_1 \to P_0$. Let $\tau_{-1}\colon R \to RG \otimes_{RH} R$ be defined by $r \mapsto e \otimes r$ where e denotes the identity element of G. Then $\phi \circ \tau_{-1}$ is the identity map on R. It follows that the induced RH-morphism $H_0(f_0 \circ \tau_0)\colon H_0(Q) \to H_0(P)$ is an isomorphism. Since $\kappa\colon M_* \to P_*$ given by $(p, q, q') \mapsto p + f(q)$ is a chain homotopy equivalence, $H(\kappa_0)\colon H_0(M) \to H_0(P)$ is an isomorphism. Observe that $H(f_0 \circ \tau_0)$ equals $H(\kappa_0) \circ H(\iota_0)$ and hence $H(\iota_0)$ is an isomorphism.

Step 4. *The exact sequence* (7) *splits, and each S_i is a projective RH-module.*

This is immediate since $\iota_i\colon Q_i \to M_i$ is the inclusion of a direct summand of M_i as an RH-module. Since restriction of scalars preserves projectiveness, M_i is projective as an RH-module and hence S_i is projective as well. □

Proof of Theorem 1.7. Consider projective resolutions as (4) and (5) as well as RH-morphisms $\iota_i\colon Q_i \to M_i$ as described in Proposition 4.1.

Let $M_* = (M_i, \delta_i^M)$ denote the chain complex induced by (5), with the assumption that $M_i = 0$ for $i > n$ and $i < 0$. Analogously, let $Q_* = (Q_i, \delta_i^Q)$ be the chain complex induced by (4), with the assumption that $Q_i = 0$ for $i > n$ and $i < 0$. Observe that we are not using the modules Q_{n+1} and M_{n+1} in the definition of Q_* and M_*. Let S_* be the quotient chain complex M_*/Q_*. Consider the induced chain map $\iota = (\iota_i)\colon Q_* \to M_*$.

We use the following notation. The kernel of δ_n^Q is denoted by $Z_n(Q)$. The n-homology group of the complex Q_* is denoted by $H_n(Q)$. Analogous notation is used for the other chain complexes.

Step 1. *The induced sequence*

$$0 \longrightarrow Z_n(Q) \xrightarrow{\iota_n} Z_n(M) \longrightarrow Z_n(S) \longrightarrow 0 \tag{14}$$

is exact and satisfies:

- *$Z_n(Q)$ is a finitely generated RH-module*;
- *$Z_n(M)$ is a finitely generated and projective RG-module*;
- *$Z_n(Q)$ is a direct summand of $Z_n(M)$ as an RH-module.*

Observe that $H_{n+1}(Q)$ and $H_{n-1}(Q)$ are both trivial. The short exact sequence of chain complexes of RH-modules

$$0 \longrightarrow Q_* \xrightarrow{\iota} M_* \longrightarrow S_* \longrightarrow 0 \tag{15}$$

induces a long exact sequence

$$0 \longrightarrow H_n(Q) \xrightarrow{\iota_n} H_n(M) \longrightarrow H_n(S) \longrightarrow 0 \tag{16}$$

which is precisely (14).

The RH-module $Z_n(Q)$ is finitely generated since Q_{n+1} is a finitely generated RH-module and δ^Q_{n+1} maps Q_{n+1} onto $Z_n(Q)$.

That $Z_n(M)$ is a finitely generated and projective RG-module follows from a direct application of Schanuel's lemma [5, p. 193, Lemma 4.4] to the exact sequences (5) and

$$0 \longrightarrow Z_n(M) \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow R \longrightarrow 0. \tag{17}$$

Finally, to show that $Z_n(Q)$ is a direct summand of $Z_n(M)$ as an RH-module, we argue that that $Z_n(S)$ is projective RH-module. Consider the sequence of RH-modules induced by S_*

$$0 \longrightarrow Z_n(S) \longrightarrow S_n \longrightarrow \cdots \longrightarrow S_0 \longrightarrow 0. \tag{18}$$

Note that this sequence is exact by observing the long exact sequence of homologies induced by (15). Indeed, $H_i(Q)$ and $H_i(M)$ are trivial for $0 < i < n$, and $H(\imath)\colon H_0(Q) \to H_0(M)$ is an isomorphism by (6). Since each S_i is projective, exactness of (18) implies that $Z_n(S)$ is projective.

Step 2. $FV^{n+1}_{H,R} \preceq FV^{n+1}_{G,R}$.

Let $\|\cdot\|_{M_n}$ and $\|\cdot\|_{Z_n(M)}$ denote filling norms on the RG-modules M_n and $Z_n(M)$ respectively. Similarly, let $\|\cdot\|_{Q_n}$ and $\|\cdot\|_{Z_n(Q)}$ denote filling norms on RH-modules Q_n and $Z_n(Q)$. For the map $Z_n(Q) \xrightarrow{\imath_n} Z_n(M)$, by Lemma 3.6 there exist integral parts K and K' of $Z_n(Q)$ and $Z_n(M)$ respectively, such that K maps into K' by the morphism $\imath$.

Since $\imath\colon Q_n \to M_n$ is the inclusion of a direct summand of M_n as an RH-module, and M_n is a projective RH-module, Lemma 2.9 implies that $\|\cdot\|_{M_n} \sim \|\cdot\|_{Q_n}$ on Q_n. In particular, there is a constant C_0 such that

$$\|\imath_n(\gamma)\|_{M_n} \leq C_0 \|\gamma\|_{Q_n}$$

for every $\gamma \in Q_n$.

By Step 1, $\imath_n\colon Z_n(Q) \to Z_n(M)$ is the inclusion of a direct summand of $Z_n(M)$ as an RH-module, and $Z_n(M)$ is a projective RH-module. Lemma 2.9 implies $\|\cdot\|_{Z_n(M)} \sim \|\cdot\|_{Z_n(Q)}$ on $Z_n(Q)$. Hence there is $C_1 > 0$ such that

$$\|\gamma\|_{Z_n(Q)} \leq C_1 \|\imath_n(\gamma)\|_{Z_n(M)}$$

for every $\gamma \in Z_n(M)$, and $\rho \circ \imath$ is identity on $Z_n(Q)$.

Let $k \in \mathbb{N}$ and $\gamma \in K \subseteq Z(Q_n)$ such that $\|\gamma\|_{Q_n} \leq k$. Then

$$\|\gamma\|_{Z_n(Q)} \leq C_1 \|\imath_n(\gamma)\|_{Z_n(M)} \leq C_1\, \mathsf{FV}^{n+1}_{G,R}(\|\imath_n(\gamma)\|_{M_n}) \leq C_1\, \mathsf{FV}^{n+1}_{G,R}(C_0\|\gamma\|_{Q_n})$$

Therefore $\mathsf{FV}^{n+1}_{H,R}(k) \leq C_1\, \mathsf{FV}^{n+1}_{G,R}(C_0 k)$ for every k. □

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