

K- and L-theory of graph products of groups

Daniel Kasprowski, Kevin Li, and Wolfgang Lück

Abstract. We compute the group homology, the algebraic K - and L -groups, and the topological K -groups of right-angled Artin groups, right-angled Coxeter groups, and more generally, graph products.

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1. Introduction

1.1. Basic setup. Suppose that we are given the following data.

- A finite simplicial graph X on the vertex set V and a collection of groups $\mathcal{W} := \{W_v \mid v \in V\}$. Denote by $W = W(X, \mathcal{W})$ the associated graph product, see Section 3, and by Σ the flag complex associated to X . Examples of graph products are right-angled Artin groups and right-angled Coxeter groups.
- A commutative ring with unit Λ and an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ -modules, see Definition 2.1. Our main examples will be those associated to algebraic K - and L -theory or topological K -theory, which appear in the Farrell–Jones Conjecture or the Baum–Connes Conjecture.

- A non-empty class \mathcal{C} of finite groups which is closed under isomorphisms, passage to subgroups and passage to quotient groups. Our main example will be the class of all finite groups.
- A class \mathcal{E} of Λ -modules with the property that for an exact sequence $0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0$ the Λ -module V_1 belongs to \mathcal{E} if and only if both V_0 and V_2 belong to \mathcal{E} .

1.2. Main result. Fix an integer n . We obtain a covariant functor

$$\mathcal{H}_n^{\mathcal{C}}: \text{Groups} \longrightarrow \Lambda\text{-Modules}, \quad G \longmapsto \mathcal{H}_n^{\mathcal{C}}(E_{\mathcal{C}}(G)),$$

where $E_{\mathcal{C}}(G)$ is the classifying space of the family of subgroups of G which belong to \mathcal{C} , see Section 2 and (2.4).

Let \mathcal{S} be the poset of full subcomplexes of Σ and let \mathcal{P} be the poset of simplices of Σ , both ordered by inclusion, where the empty subcomplex and the empty simplex are allowed. For an element L in \mathcal{S} , we can consider the subgraph $X \cap L$ of X . Let $W(L)$ be the graph product associated to $X \cap L$ and the collection of groups $\mathcal{W}|_{V \cap L} = \{W_v \mid v \text{ is a vertex of } L\}$. With this notation $W(\Sigma)$ is the graph product $W(X, \mathcal{W})$ and $W(\emptyset) = \{1\}$. We obtain a covariant functor

$$W_*: \mathcal{S} \longrightarrow \text{Groups}, \quad L \longmapsto W(L).$$

Let

$$I: \mathcal{P} \longrightarrow \mathcal{S}$$

be the inclusion which sends a simplex σ of Σ to the corresponding full subcomplex of Σ . Sometimes we identify σ in \mathcal{P} with $I(\sigma)$ in \mathcal{S} . For instance we will often write $W(\sigma)$ instead of $W(I(\sigma))$. Notice that the covariant functor $W_* \circ I: \mathcal{P} \rightarrow \text{Groups}$ sends a simplex σ of Σ to $\prod_{v \in V \cap \sigma} W_v$.

We obtain a covariant functor

$$\mathcal{H}_n^{\mathcal{C}} \circ W_*: \mathcal{S} \longrightarrow \Lambda\text{-Modules}, \quad L \longmapsto \mathcal{H}_n^{\mathcal{C}}(E_{\mathcal{C}}(W(L))).$$

We are interested in the value at Σ , i.e., in $\mathcal{H}_n^{\mathcal{C}}(E_{\mathcal{C}}(W))$ for $W = W(X, \mathcal{W})$.

The composite $\mathcal{H}_n^{\mathcal{C}} \circ W_* \circ I: \mathcal{P} \rightarrow \Lambda\text{-Modules}$ is given by

$$\sigma \longmapsto \mathcal{H}_n^{\mathcal{C}}\left(\prod_{v \in V \cap \sigma} W_v\right) = \mathcal{H}_n^{\mathcal{C}}\left(\prod_{v \in V \cap \sigma} E_{\mathcal{C}}(W_v)\right),$$

since $\prod_{v \in V \cap \sigma} E_{\mathcal{C}}(W_v)$ is a model for $E_{\mathcal{C}}(\prod_{v \in V \cap \sigma} W_v)$.

Define for a simplex σ the quotient Λ -module of $\mathcal{H}_n^c \circ W_* \circ I(\sigma)$ by

$$S_\sigma(\mathcal{H}_n^c \circ W_* \circ I) := \text{cok} \left(\bigoplus_{\tau} \mathcal{H}_n^c \circ W_* \circ I(\tau) \longrightarrow \mathcal{H}_n^c \circ W_* \circ I(\sigma) \right),$$

where τ runs through the simplices of $I(\sigma)$ which are different from σ . The idea is to kill everything in $\mathcal{H}_n^c \circ W_* \circ I(\sigma)$ which comes from a proper simplex τ of $I(\sigma)$.

For a simplex σ of Σ , let $\text{ch}_n(\Sigma, \sigma)$ be the set of n -chains $\sigma_0 < \sigma_1 < \dots < \sigma_n$ in \mathcal{P} with $\sigma_0 = \sigma$. Define the integer

$$\text{ch}(\Sigma, \sigma) := \sum_{n \geq 0} (-1)^n \cdot |\text{ch}_n(\Sigma, \sigma)|.$$

Denote by $G_0(\mathcal{E})$ the Grothendieck group of elements in \mathcal{E} , i.e., the abelian group with the isomorphism classes of elements in \mathcal{E} as generators and relations $[V_1] = [V_0] + [V_2]$ for every short exact sequence $0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0$ of Λ -modules belonging to \mathcal{E} .

Theorem 1.1 (main theorem). (i) *The canonical Λ -homomorphism*

$$T: \text{colim}_{\sigma \in \mathcal{P}} \mathcal{H}_n^{W(\sigma)}(E_c(W(\sigma))) \xrightarrow{\cong} \mathcal{H}_n^{W(\Sigma)}(E_c(W(\Sigma)))$$

is an isomorphism.

(ii) *For every $\sigma \in \mathcal{P}$, the canonical projection*

$$p_\sigma: \mathcal{H}_n^c \circ W_* \circ I(\sigma) \longrightarrow S_\sigma(\mathcal{H}_n^c \circ W_* \circ I)$$

has a section

$$s_\sigma: S_\sigma(\mathcal{H}_n^c \circ W_* \circ I) \longrightarrow \mathcal{H}_n^c \circ W_* \circ I(\sigma).$$

Any collection of such sections and the canonical maps

$$\mathcal{H}_n^c \circ W_* \circ I(\sigma) \longrightarrow \text{colim}_{\sigma \in \mathcal{P}} \mathcal{H}_n^c \circ W_* \circ I$$

induce an isomorphism

$$\bigoplus_{\sigma \in \mathcal{P}} S_\sigma(\mathcal{H}_n^c \circ W_* \circ I) \xrightarrow{\cong} \text{colim}_{\sigma \in \mathcal{P}} \mathcal{H}_n^{W(\sigma)}(E_c(W(\sigma))).$$

Moreover, there is an explicit section s_σ (Lemma 4.4).

(iii) *Suppose that each Λ -module $\mathcal{H}_n^{W(\sigma)}(E_c(W(\sigma)))$ belongs to \mathcal{E} . Then we get in $G_0(\mathcal{E})$*

$$[\mathcal{H}_n^{W(\Sigma)}(E_c(W(\Sigma)))] = \sum_{\sigma \in \mathcal{P}} \text{ch}(\Sigma, \sigma) \cdot [\mathcal{H}_n^{W(\sigma)}(E_c(W(\sigma)))].$$

We mention that it is both unusual and fortunate that in assertion (i) the source and target involve the same degree. In general one would expect that in the source all degrees $m \leq n$ occur. The reason for this simplification is that for each simplex σ the inclusion $W(\sigma) \rightarrow W$ is split injective. This leads also to the explicit splitting in assertion (ii).

Note that the number $\text{ch}(\Sigma, \sigma)$ appearing in assertion (iii) depends only on Σ and σ . It is given by $1 - \chi(\Sigma)$ for $\sigma = \emptyset$. It has the following geometric interpretation if σ is non-empty. Let Σ' be the barycentric subdivision of Σ . Then $\text{ch}_n(\Sigma, \sigma)$ can be interpreted as a collection of n -simplices in Σ' . Each simplex in $\text{ch}_n(\Sigma, \sigma)$ contains the vertex given by σ . The collection of the faces of all these simplices of $\text{ch}_n(\Sigma, \sigma)$ determines a simplicial subcomplex D_σ of Σ' which can be contracted to the vertex given by σ . Its boundary ∂D_σ consists of all those faces of simplices of $\text{ch}_n(\Sigma, \sigma)$ which do not contain σ . One easily checks

$$\text{ch}(\Sigma, \sigma) = \chi(D_\sigma) - \chi(\partial D_\sigma) = 1 - \chi(\partial D_\sigma).$$

If Σ is the triangulation of a closed manifold of dimension d , then ∂D_σ is homeomorphic to $S^{d-1-\dim(\sigma)}$ and hence $\text{ch}(\Sigma, \sigma) = (-1)^{d-\dim(\sigma)}$.

1.3. Computations. We will illustrate the potential of Theorem 1.1 by computing the group homology of G , the algebraic K - and L -theory of the group ring of G , and the topological K -theory of the group C^* -algebra of G in Sections 6 and 7 if G is a right-angled Artin group or a right-angled Coxeter group. These computations are based on the Baum–Connes Conjecture and the Farrell–Jones Conjecture which we will briefly recall in Section 5 and which hold for these groups. The situation in the Farrell–Jones setting is more complicated since we have to deal with the family \mathcal{VCY} of virtually cyclic subgroups, whereas in Theorem 1.1 the family \mathcal{FJN} of finite subgroups is considered. The passage from \mathcal{FJN} to \mathcal{VCY} is discussed in Subsection 5.3. This is different in the Baum–Connes setting since there the family \mathcal{FJN} is used. In order to get full functoriality we need to consider the maximal group C^* -algebra instead of the reduced C^* -algebra which makes no difference for right-angled Artin groups and right-angled Coxeter groups.

Computation in this context means not only that we identify the corresponding K - and L -groups of G as abelian groups but we give explicit isomorphisms identifying them with K - and L -groups of the ground ring. For instance, we show for a right-angled Coxeter group W associated to the finite flag complex Σ that there is for every $n \in \mathbb{Z}$ an isomorphism

$$\bigoplus_{\sigma} K_n(f_{\sigma}): \bigoplus_{\sigma} K_n(\mathbb{C}) \xrightarrow{\cong} K_n(C_r^*(W)),$$

where σ runs through the simplices of Σ including the empty simplex and $f_{\sigma}: \mathbb{C} \rightarrow C_r^*(W)$ is an explicit homomorphism of C^* -algebras depending on σ . If σ is

empty, it is given by the obvious inclusion $\mathbb{C} \rightarrow C_r^*(W)$. If $k = \dim(\sigma) \geq 0$, then σ determines a subgroup $W(\sigma) = \prod_{i=1}^{k+1} \mathbb{Z}/2$ of W and f_σ is the composite of the homomorphism $\mathbb{C} \rightarrow \mathbb{C}[\prod_{i=1}^{k+1} \mathbb{Z}/2]$ sending λ to $2^{-k-1} \cdot \lambda \cdot \prod_{i=1}^{k+1} (1 - t_i)$ for t_i the generator of the i -th factor $\mathbb{Z}/2$ and the homomorphism $\mathbb{C}[W(\sigma)] \rightarrow C_r^*(W)$ coming from the inclusion $W(\sigma) \rightarrow W$, see Remark 7.17. This implies

$$K_n(C_r^*(W)) \cong \begin{cases} \mathbb{Z}^r & \text{if } n \text{ is even,} \\ \{0\} & \text{otherwise,} \end{cases}$$

where r is the number of simplices of Σ including the empty simplex. Moreover, we can write down an explicit basis $B = \{b_\sigma \mid \sigma \in \mathcal{P}\}$ for the finitely generated free \mathbb{Z} -module $K_0(C_r^*(W))$, namely, for $\sigma = \emptyset$ we take the class of the idempotent 1 in $C_r^*(W)$ and for $\sigma \neq \emptyset$ we take the class of the idempotent in $C_r^*(W)$ given by the image of the idempotent $2^{-k-1} \cdot \prod_{i=1}^{k+1} (1 - t_i) \in \mathbb{C}[W(\sigma)]$ under the inclusion $\mathbb{C}[W(\sigma)] \subset C_r^*(W)$. These computations for right-angled Coxeter groups were carried out in the second author’s master’s thesis [22].

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2. Equivariant homology theories and classifying spaces of families

In this section we recall the axioms of an equivariant homology theory and the notion of a classifying space of a family of subgroups. For an amalgamated product of groups, we deduce a Mayer–Vietoris type sequence for the values of an equivariant homology theory on classifying spaces.

Fix a discrete group G and a commutative ring Λ with unit. A G -homology theory \mathcal{H}_*^G with values in Λ -modules is a collection of covariant functors \mathcal{H}_n^G from the category of G -CW-pairs to the category of Λ -modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G(X, A): \mathcal{H}_n^G(X, A) \longrightarrow \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$$

for $n \in \mathbb{Z}$ such that the axioms concerning G -homotopy invariance, the long exact sequence of a pair, excision, and disjoint unions are satisfied, see [24, Section 1].

Let $\alpha: H \rightarrow G$ be a group homomorphism. Given an H -space X , define the induction of X with α to be the G -space $\alpha_*X = G \times_\alpha X$ which is the quotient of $G \times X$ by the right H -action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and

$(g, x) \in G \times X$. The following definition is taken from [24, Section 1] except that the induction structure in this paper is defined for every group homomorphism α .

Definition 2.1. An equivariant homology theory $\mathcal{H}_*^?$ with values in Λ -modules assigns to each group G a G -homology theory \mathcal{H}_*^G with values in Λ -modules together with the following so called *induction structure*.

Given a group homomorphism $\alpha: H \rightarrow G$ and an H - CW -pair (X, A) , there are for every $n \in \mathbb{Z}$ natural homomorphisms

$$\text{ind}_\alpha: \mathcal{H}_n^H(X, A) \longrightarrow \mathcal{H}_n^G(\alpha_*(X, A))$$

satisfying the following axioms.

- *Compatibility with the boundary homomorphisms.* $\partial_n^G \circ \text{ind}_\alpha = \text{ind}_\alpha \circ \partial_n^H$.
- *Functoriality.* Let $\beta: G \rightarrow K$ be another group homomorphism. Then we have for $n \in \mathbb{Z}$

$$\text{ind}_{\beta \circ \alpha} = \mathcal{H}_n^K(f_1) \circ \text{ind}_\beta \circ \text{ind}_\alpha: \mathcal{H}_n^H(X, A) \rightarrow \mathcal{H}_n^K((\beta \circ \alpha)_*(X, A)),$$

where

$$f_1: \beta_*(\alpha_*(X, A)) \xrightarrow{\cong} (\beta \circ \alpha)_*(X, A), \quad (k, g, x) \mapsto (k\beta(g), x),$$

is the natural K -homeomorphism.

- *Compatibility with conjugation.* For $n \in \mathbb{Z}$, $g \in G$ and a G - CW -pair (X, A) the map

$$\text{ind}_{c(g):G \rightarrow G}: \mathcal{H}_n^G(X, A) \longrightarrow \mathcal{H}_n^G(c(g)_*(X, A))$$

agrees with $\mathcal{H}_n^G(f_2)$ for the G -homeomorphism $f_2: (X, A) \rightarrow c(g)_*(X, A)$ which sends x to $(1, g^{-1}x)$ in $G \times_{c(g)}(X, A)$.

- *Bijectivity.* If $\ker(\alpha)$ acts freely on $X \setminus A$, then

$$\text{ind}_\alpha: \mathcal{H}_n^H(X, A) \longrightarrow \mathcal{H}_n^G(\alpha_*(X, A))$$

is bijective for all $n \in \mathbb{Z}$.

We briefly fix some conventions concerning spectra. If X is a space, denote by X_+ the pointed space obtained from X by adding a disjoint base point. Let Spectra be the category of spectra in the following naive sense. A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n): E(n) \wedge S^1 \rightarrow E(n+1)$. A *map of spectra* $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{E}'$ is a sequence of maps $f(n): E(n) \rightarrow E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \text{id}_{S^1})$ for all $n \in \mathbb{Z}$. Given a spectrum \mathbf{E} and a pointed space X , we can define their smash product $X \wedge \mathbf{E}$ by $(X \wedge \mathbf{E})(n) := X \wedge E(n)$ with the obvious structure maps.

It is a classical result that a spectrum \mathbf{E} defines a homology theory by setting

$$H_n(X, A; \mathbf{E}) = \pi_n((X_+ \cup_{A_+} \text{cone}(A_+)) \wedge \mathbf{E}),$$

where cone denotes the reduced cone. We want to extend this to equivariant homology theories.

Let Groupoids be the category of small connected groupoids with covariant functors as morphisms. Notice that a group can be considered as a groupoid with one object in the obvious way.

For the proof of the following result we refer to [27, Proposition 157 on p. 796].

Theorem 2.2. *Consider a covariant Groupoids-spectrum*

$$\mathbf{E}: \text{Groupoids} \longrightarrow \text{Spectra.}$$

Suppose that \mathbf{E} respects equivalences, i.e., it sends an equivalence of groupoids to a weak equivalence of spectra.

Then \mathbf{E} defines an equivariant homology theory $H_^?(-; \mathbf{E})$ such that we have*

$$H_n^G(G/H; \mathbf{E}) \cong H_n^H(\text{pt}; \mathbf{E}) \cong \pi_n(\mathbf{E}(H))$$

for every group G , subgroup $H \subseteq G$ and $n \in \mathbb{Z}$. The construction is natural in \mathbf{E} .

Example 2.3 (Borel homology). Let \mathbf{E} be a spectrum. Let $H_*(-; \mathbf{E})$ be the (non-equivariant) homology theory associated to \mathbf{E} . Given a groupoid \mathcal{G} , denote by $E\mathcal{G}$ its classifying space. If \mathcal{G} has only one object and the automorphism group of this object is G , then $E\mathcal{G}$ is a model for EG . We obtain two covariant functors

$$\begin{aligned} c_{\mathbf{E}}: \text{Groupoids} &\longrightarrow \text{Spectra}, & \mathcal{G} &\longmapsto \mathbf{E}, \\ b_{\mathbf{E}}: \text{Groupoids} &\longrightarrow \text{Spectra}, & \mathcal{G} &\longmapsto E\mathcal{G}_+ \wedge \mathbf{E}. \end{aligned}$$

Thus we obtain two equivariant homology theories $H_*^?(c_{\mathbf{E}})$ and $H_*^?(b_{\mathbf{E}})$ from Theorem 2.2. The second one is called the *equivariant Borel homology associated to $H_*(-; \mathbf{E})$* . We get for any group G and any G -CW-complex X natural isomorphisms

$$\begin{aligned} H_n^G(X; c_{\mathbf{E}}) &\cong H_n(G \setminus X; \mathbf{E}), \\ H_n^G(X; b_{\mathbf{E}}) &\cong H_n(EG \times_G X; \mathbf{E}). \end{aligned}$$

Let \mathcal{C} be a non-empty class of groups which is closed under isomorphisms, passage to subgroups and passage to quotient groups. Our main examples will be the class $\mathcal{F}\mathcal{N}$ of finite groups and the class $\mathcal{V}\mathcal{C}\mathcal{Y}$ of virtually cyclic groups.

Given a group G , denote by $E_{\mathcal{C}}(G)$ the *classifying space of G with respect to the family of subgroups $\mathcal{F}_{\mathcal{C}}(G) = \{H \subseteq G \mid H \in \mathcal{C}\}$* . It is defined to be a terminal object in the G -homotopy category of G -CW-complexes, whose isotropy groups

belong to $\mathcal{F}_c(G)$. A model for $E_c(G)$ is a G - CW -complex whose H -fixed point set is contractible if $H \in \mathcal{C}$ and is empty if $H \notin \mathcal{C}$. With this notation $E_{\mathcal{F}\mathcal{J}\mathcal{N}}(G)$ is the *classifying space of proper actions*, sometimes also denoted by $\underline{E}G$. We sometimes denote $E_{\text{vey}}(G)$ by $\underline{\underline{E}}G$. For more information about classifying spaces of families we refer for instance to [26].

Given a group homomorphism $f: G \rightarrow H$, we denote by

$$E_c(f): f_*E_c(G) \longrightarrow E_c(H)$$

the up to H -homotopy unique H -map coming from the universal property of $E_c(H)$ and the fact that $f_*E_c(G)$ is an H - CW -complex whose isotropy groups are of the shape $f(K)$ for $K \in \mathcal{C}$ and hence all belong to \mathcal{C} again. Given an equivariant homology theory \mathcal{H}_*^H with values in Λ -modules, it induces homomorphisms of \mathbb{Z} -graded Λ -modules

$$f_*: \mathcal{H}_*^G(E_c(G)) \xrightarrow{\text{ind}_f} \mathcal{H}_*^H(f_*E_c(G)) \xrightarrow{\mathcal{H}_*^H(E_c(f))} \mathcal{H}_*^H(E_c(H)).$$

One easily checks that thus we obtain a covariant functor

$$\mathcal{H}_n^c: \text{Groups} \longrightarrow \Lambda\text{-Modules}, \quad G \longmapsto \mathcal{H}_n^G(E_c(G)). \tag{2.4}$$

Lemma 2.5. *Let G_0, G_1 and G_2 be subgroups of G satisfying $G_0 \subseteq G_1, G_2$. Suppose that the inclusions $i_k: G_k \rightarrow G$ for $k = 0, 1, 2$ induce an isomorphism $G_1 *_{G_0} G_2 \xrightarrow{\cong} G$. Let $j_k: G_0 \rightarrow G_k$ be the inclusion for $k = 1, 2$. Suppose that each element in \mathcal{C} is a finite group.*

Then we obtain a long exact Mayer–Vietoris sequence

$$\begin{aligned} \dots &\xrightarrow{\partial_{n+1}} \mathcal{H}_n^{G_0}(E_c(G_0)) \xrightarrow{(j_1)_n \times -(j_2)_n} \mathcal{H}_n^{G_1}(E_c(G_1)) \oplus \mathcal{H}_n^{G_2}(E_c(G_2)) \\ &\xrightarrow{(i_1)_n \oplus (i_2)_n} \mathcal{H}_n^G(E_c(G)) \xrightarrow{\partial_n} \mathcal{H}_{n-1}^{G_0}(E_c(G_0)) \xrightarrow{(j_1)_{n-1} \times -(j_2)_{n-1}} \dots \end{aligned}$$

Proof. There is a 1-dimensional G - CW -complex T whose underlying space is a tree such that the 1-skeleton is obtained from the 0-skeleton by the G -pushout

$$\begin{array}{ccc} G/G_0 \times S^0 & \xrightarrow{q} & G/G_1 \amalg G/G_2 \\ \downarrow & & \downarrow \\ G/G_0 \times D^1 & \longrightarrow & T \end{array}$$

where q is the disjoint union of the canonical projections $G/G_0 \rightarrow G/G_1$ and $G/G_0 \rightarrow G/G_2$, see [35, Theorem 7 in §4.1 on p. 32]. If we take the cartesian product with $E_c(G)$ we obtain another cellular G -pushout. Its associated Mayer–Vietoris sequence yields the long exact sequence

$$\begin{aligned} \dots &\longrightarrow \mathcal{H}_n^G(G/G_0 \times E_c(G)) \longrightarrow \mathcal{H}_n^G(G/G_1 \times E_c(G)) \oplus \mathcal{H}_n^G(G/G_2 \times E_c(G)) \\ &\longrightarrow \mathcal{H}_n^G(T \times E_c(G)) \longrightarrow \mathcal{H}_{n-1}^G(G/G_0 \times E_c(G)) \longrightarrow \dots \end{aligned}$$

There is a G -homeomorphism $(j_k)_* j_k^* E_{\mathcal{C}}(G) \xrightarrow{\cong} G/G_k \times E_{\mathcal{C}}(G)$, where $j_k^* E_{\mathcal{C}}(G)$ is the restriction of $E_{\mathcal{C}}(G)$ to G_k by j_k . Obviously $j_k^* E_{\mathcal{C}}(G)$ is a model for $E_{\mathcal{C}}(G_k)$. Using the induction structure of the equivariant homology theory $\mathcal{H}_*^?$ we obtain identifications for $k = 0, 1, 2$

$$\mathcal{H}_*^{G_k}(E_{\mathcal{C}}(G_k)) \xrightarrow{\cong} \mathcal{H}_*^G(G/G_k \times E_{\mathcal{C}}(G)).$$

The K -fixed point set T^K is a non-empty subtree and hence contractible for every finite subgroup $K \subseteq G$, see [35, Theorem 15 in 6.1 on p. 58 and 6.3.1 on p. 60]. Hence the projection $T \times E_{\mathcal{C}}(G) \rightarrow E_{\mathcal{C}}(G)$ is a G -homotopy equivalence since every element in \mathcal{C} is finite by assumption. Hence we get an identification

$$\mathcal{H}_*^G(E_{\mathcal{C}}(G)) \xrightarrow{\cong} \mathcal{H}_*^G(T \times E_{\mathcal{C}}(G)).$$

Now we obtain the desired long exact sequence from the last long exact sequence and the identifications above. □

3. Graph products of groups

In this section we give the definition of a graph product of groups. We show that the value of an equivariant homology theory on the classifying space of a graph product is the colimit over a certain system of subgroups.

Let X be a finite simplicial graph on the vertex set V and suppose that we are given a collection of groups $\mathcal{W} := \{W_v \mid v \in V\}$. Then the *graph product* $W(X, \mathcal{W})$ is defined as the quotient of the free product $*_{v \in V} W_v$ of the collection of groups \mathcal{W} by introducing the relations

$$\{[g, g'] = 1 \mid v, v' \in V, \text{ there is an edge joining } v \text{ and } v', g \in W_v, g' \in W_{v'}\}.$$

In other words, elements of subgroups W_v and $W_{v'}$ commute if there is an edge joining v and v' . This notion is due to Green [18].

Let Σ be the flag complex associated to X . Denote by $\mathcal{S} = \mathcal{S}(\Sigma)$ the poset of full subcomplexes of Σ , ordered by inclusion, where we also allow the empty subcomplex. Then we can assign to $L \in \mathcal{S}$ the graph product group $W(X \cap L, \mathcal{W}|_{V \cap L})$, where $X \cap L$ agrees with the 1-skeleton of L and $\mathcal{W}|_{V \cap L} = \{W_v \mid v \in V \cap L\}$ is the restriction of \mathcal{W} to the vertices in L . Consider $L_0, L_1 \in \mathcal{S}$ with $L_0 \leq L_1$. Then we obtain group homomorphisms

$$\begin{aligned} W_*(L_0 \leq L_1): W(L_0, \mathcal{W}|_{V \cap L_0}) &\longrightarrow W(L_1, \mathcal{W}|_{V \cap L_1}), \\ W^*(L_0 \leq L_1): W(L_1, \mathcal{W}|_{V \cap L_1}) &\longrightarrow W(L_0, \mathcal{W}|_{V \cap L_0}) \end{aligned}$$

as follows. The morphism $W_*(L_0 \leq L_1)$ is induced by the obvious inclusion $*_{v \in V \cap L_0} W_v \rightarrow *_{v \in V \cap L_1} W_v$, whereas the second one is induced by the projection

$*_{v \in V \cap L_1} W_v \rightarrow *_{v \in V \cap L_0} W_v$ which is given on W_v for $v \in V \cap L_1$ by the inclusion $W_v \rightarrow *_{v \in V \cap L_0} W_v$ if $v \in L_0$, and by the trivial homomorphism if $v \notin L_0$. One easily checks that thus we obtain a covariant functor

$$W_*: \mathcal{S} \longrightarrow \text{Groups}$$

and a contravariant functor

$$W^*: \mathcal{S} \longrightarrow \text{Groups}.$$

By construction W_* and W^* agree on objects and we write

$$W(L) := W_*(L) = W^*(L) := W(X \cap L, \mathcal{W}|_{V \cap L})$$

for an object $L \in \mathcal{S}$.

The elementary proof of the following lemma is left to the reader.

Lemma 3.1. (i) *Let $L_0, L_1, L_2, L \in \mathcal{S}$ be elements such that $L = L_1 \cup L_2$ and $L_0 = L_1 \cap L_2$. Then we obtain a group isomorphism*

$$W_*(L_1 \leq L) *_{W_*(L_0 \leq L)} W_*(L_2 \leq L): W(L_1) *_{W(L_0)} W(L_2) \xrightarrow{\cong} W(L).$$

(ii) *Let $L_1, L_2, L \in \mathcal{S}$ be elements satisfying $L_1 \leq L$ and $L_2 \leq L$. Then we get an equality of group homomorphisms $W(L_1) \rightarrow W(L_2)$*

$$W^*(L_2 \leq L) \circ W_*(L_1 \leq L) = W_*((L_1 \cap L_2) \leq L_2) \circ W^*((L_1 \cap L_2) \leq L_1).$$

Remark 3.2. Notice that in particular we get from Lemma 3.1 (ii) that for any two elements L_0 and L_1 in \mathcal{S} with $L_0 \leq L_1$ the composite

$$W^*(L_0 \leq L_1) \circ W_*(L_0 \leq L_1)$$

is the identity on $W(L_0)$. Hence $W_*(L_0 \leq L_1)$ is split injective and $W^*(L_0 \leq L_1)$ is split surjective.

Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ -modules. Let \mathcal{C} be a non-empty class of groups which is closed under isomorphisms, passage to subgroups and passage to quotient groups. We have defined a covariant functor $\mathcal{H}_*^{\mathcal{C}}$ and studied its main properties in Section 2.

We want to study the covariant $\Lambda\mathcal{S}$ -module

$$\mathcal{H}_n^{\mathcal{C}} \circ W_*: \mathcal{S} \longrightarrow \Lambda\text{-Modules}, \quad L \longmapsto \mathcal{H}_n^{W(L)}(E_{\mathcal{C}}(W(L))),$$

and are in particular interested in its value at Σ itself.

Viewing a simplex as a full subcomplex yields for every $L \in \mathcal{S}$ a map of posets

$$I_L: \mathcal{P}(L) \longrightarrow \mathcal{S}(L).$$

For two elements L and L' in \mathcal{S} with $L \leq L'$ let

$$\begin{aligned} J^{\mathcal{P}}(L \leq L') &: \mathcal{P}(L) \longrightarrow \mathcal{P}(L'), \\ J^{\mathcal{S}}(L \leq L') &: \mathcal{S}(L) \longrightarrow \mathcal{S}(L') \end{aligned}$$

be the maps of posets induced by the inclusion $L \subseteq L'$. Define the Λ -module

$$C_n(L) := \operatorname{colim}_{\mathcal{P}(L)} \mathcal{H}_n^{\mathcal{C}} \circ W_* \circ J^{\mathcal{S}}(L \leq \Sigma) \circ I_L$$

to be the colimit of the covariant functor

$$\mathcal{H}_n^{\mathcal{C}} \circ W_* \circ J^{\mathcal{S}}(L \leq \Sigma) \circ I_L : \mathcal{P}(L) \longrightarrow \Lambda\text{-Modules}.$$

Given elements L and L' in \mathcal{S} with $L \leq L'$, we obtain a map of Λ -modules

$$C_n(L \leq L') : C_n(L) \longrightarrow C_n(L')$$

from $J^{\mathcal{P}}(L \leq L')$ because of $J^{\mathcal{S}}(L' \leq \Sigma) \circ I_{L'} \circ J^{\mathcal{P}}(L \leq L') = J^{\mathcal{S}}(L \leq \Sigma) \circ I_L$. One easily checks that thus we obtain a covariant $\Lambda\mathcal{S}$ -module

$$C_n : \mathcal{S} \longrightarrow \Lambda\text{-Modules}, \quad L \longmapsto C_n(L).$$

For every object $L \in \mathcal{S}$ there is an obvious Λ -homomorphism

$$T_n(L) : C_n(L) = \operatorname{colim}_{\mathcal{P}(L)} \mathcal{H}_n^{\mathcal{C}} \circ W_* \circ J^{\mathcal{S}}(L \leq \Sigma) \circ I_L \longrightarrow \mathcal{H}_n^{\mathcal{C}} \circ W_*(L)$$

coming from the various Λ -maps $\mathcal{H}_n^{\mathcal{C}} \circ W_*(\sigma) \rightarrow \mathcal{H}_n^{\mathcal{C}} \circ W_*(L)$ induced by the inclusions $I_L(\sigma) \subseteq L$ for σ running through the simplices of L . One easily checks that the collection of the Λ -homomorphisms $T_n(L)$ fits together to a map of covariant $\Lambda\mathcal{S}$ -modules

$$T_n : C_n \longrightarrow \mathcal{H}_n^{\mathcal{C}} \circ W_*. \tag{3.3}$$

Theorem 3.4. *Suppose that each element in \mathcal{C} is a finite group.*

Then the map of $\Lambda\mathcal{S}$ -modules T_n of (3.3) is an isomorphism. In particular its evaluation at Σ yields a Λ -isomorphism

$$T_n(\Sigma) : \operatorname{colim}_{\sigma \in \mathcal{P}} \mathcal{H}_n^{W(\sigma)}(E_{\mathcal{C}}(W(\sigma))) \xrightarrow{\cong} \mathcal{H}_n^{W(\Sigma)}(E_{\mathcal{C}}(W(\Sigma))).$$

Proof. Notice that $T_n(L)$ is obviously an isomorphism if L lies in the image of $I : \mathcal{P} \rightarrow \mathcal{S}$, since then L is a terminal object in $\mathcal{P}(L)$ and hence under the obvious identification $C_n(L) = \operatorname{colim}_{\mathcal{P}(L)} \mathcal{H}_n^{\mathcal{C}} \circ W_* \circ J^{\mathcal{S}}(L \leq \Sigma) \circ I_L \cong \mathcal{H}_n^{\mathcal{C}} \circ W_*(L)$ the Λ -homomorphism $T_n(L)$ becomes the identity.

We show for any $L \in \mathcal{S}$ that $T_n(L)$ is an isomorphism by induction over the number of vertices of L . If L is the empty subcomplex, then L is in the image of $I : \mathcal{P} \rightarrow \mathcal{S}$ and the claim has already been proved. The induction step is done

as follows. We only have to consider the case, where L is not in the image of $I: \mathcal{P} \rightarrow \mathcal{S}$. Since L is itself a flag complex, there must be two vertices v_1 and v_2 in L which are not connected by an edge. Let L_1 be the full subcomplex of L spanned by v_1 and all vertices in L which are connected to v_1 by an edge. In particular v_2 is not a vertex of L_1 . Let L_0 be the full subcomplex of L which is spanned by all vertices v for which there exists an edge whose terminal points are v and v_1 . Notice that v_1 does not belong to L_0 and L_1 is the cone over L_0 with cone point v_1 . Let L_2 be the full subcomplex of L spanned by all vertices except v_1 . Then $L = L_1 \cup L_2$ and $L_0 = L_1 \cap L_2$ and the number of vertices of L_0 , L_1 and L_2 is smaller than the number of vertices of L . The induction hypothesis applies to L_k and hence $T_n(L_k)$ is an isomorphism for $k = 0, 1, 2$. Since $\mathcal{P}(L) = \mathcal{P}(L_1) \cup \mathcal{P}(L_2)$ and $\mathcal{P}(L_0) = \mathcal{P}(L_1) \cap \mathcal{P}(L_2)$, $\mathcal{P}(L)$ is the pushout of $\mathcal{P}(L_1) \leftarrow \mathcal{P}(L_0) \rightarrow \mathcal{P}(L_2)$. Hence $C_n(L)$ is the pushout of $C_n(L_1) \leftarrow C_n(L_0) \rightarrow C_n(L_2)$ along the maps induced by the inclusions $\mathcal{P}(L_0) \subseteq \mathcal{P}(L_k)$ and $\mathcal{P}(L_k) \subseteq \mathcal{P}(L)$ for $k = 1, 2$. This can be seen as follows. A morphism out of the pushout to a Λ -module X is an element in the pullback of $\text{Hom}_\Lambda(C_n(L_1), X) \rightarrow \text{Hom}_\Lambda(C_n(L_0), X) \leftarrow \text{Hom}_\Lambda(C_n(L_2), X)$. These are homomorphisms of Λ -modules indexed over $\mathcal{P}(L_1)$ and $\mathcal{P}(L_2)$ that agree on $\mathcal{P}(L_0)$. Hence this is the same as a Λ -module homomorphism from $C_n(L)$ to X . In particular,

$$C_n(L_0) \longrightarrow C_n(L_1) \oplus C_n(L_2) \longrightarrow C_n(L) \longrightarrow 0$$

is exact. Since the group homomorphism $W_*(L_0) \rightarrow W_*(L_1)$ is split injective, a retraction is given by $W^*(L_1) \rightarrow W^*(L_0)$, the Λ -homomorphism

$$\mathcal{H}_n^c \circ W_*(L_0 \leq L_1): \mathcal{H}_n^c \circ W_*(L_0) \longrightarrow \mathcal{H}_n^c \circ W_*(L_1)$$

is split injective by functoriality. We get from Lemma 2.5 and Lemma 3.1 (i) an exact sequence

$$\mathcal{H}_n^c \circ W_*(L_0) \longrightarrow \mathcal{H}_n^c \circ W_*(L_1) \oplus \mathcal{H}_n^c \circ W_*(L_2) \longrightarrow \mathcal{H}_n^c \circ W_*(L) \longrightarrow 0.$$

One easily checks that we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} C_n(L_0) & \longrightarrow & C_n(L_1) \oplus C_n(L_2) & \longrightarrow & C_n(L) & \longrightarrow & 0 \\ \cong \downarrow T_n(L_0) & & \cong \downarrow T_n(L_1) \oplus T_n(L_2) & & \downarrow T_n(L) & & \\ \mathcal{H}_n^c \circ W_*(L_0) & \longrightarrow & \mathcal{H}_n^c \circ W_*(L_1) \oplus \mathcal{H}_n^c \circ W_*(L_2) & \longrightarrow & \mathcal{H}_n^c \circ W_*(L) & \longrightarrow & 0 \end{array}$$

Now the induction step follows from the Five-Lemma. □

We have proven part (i) of the main Theorem 1.1.

4. Mackey modules

In this section we prove the remaining parts of the main Theorem 1.1. More generally, we show that the colimit of any Mackey module splits as a direct sum over its index category.

Let Σ be a finite simplicial complex. Denote by $\mathcal{P} = \mathcal{P}(\Sigma)$ the poset of its simplices ordered by inclusion, where we allow the empty simplex as well. The dimension of the empty simplex is defined to be -1 . Notice that for two elements σ and τ in \mathcal{P} the intersection $\sigma \cap \tau$ is again an element in \mathcal{P} which is uniquely determined by the property that it is maximal among all those elements μ in \mathcal{P} satisfying both $\mu \leq \sigma$ and $\mu \leq \tau$.

Let Λ be a commutative ring with unit. A Mackey $\Lambda\mathcal{P}$ -module $M = (M_*, M^*)$ is a bifunctor $\mathcal{P} \rightarrow \Lambda\text{-Modules}$, i.e., a covariant functor M_* and a contravariant functor M^* from \mathcal{P} to $\Lambda\text{-Modules}$ such that M_* and M^* agree on objects and for objects σ_1, σ_2, τ of \mathcal{P} satisfying $\sigma_k \leq \tau$ for $k = 1, 2$, we get

$$M^*(\sigma_2 \leq \tau) \circ M_*(\sigma_1 \leq \tau) = M_*(\sigma_1 \cap \sigma_2 \leq \sigma_2) \circ M^*(\sigma_1 \cap \sigma_2 \leq \sigma_1). \tag{4.1}$$

The name Mackey $\Lambda\mathcal{P}$ -module comes from the analogy to the classical notion of a Mackey functor, where (4.1) replaces the double coset formula, see [37, Section 6.1].

Example 4.2 (Mackey modules coming from graph products). Our main example comes from Section 3. Let X be a finite simplicial graph on the vertex set V and suppose that we are given a collection of groups $\mathcal{W} := \{W_v \mid v \in V\}$. Let $F: \text{Groups} \rightarrow \Lambda\text{-Modules}$ be a covariant functor, e.g. the functor \mathcal{H}_*^c defined in (2.4). Define $M_* = F \circ W_* \circ I$ and $M^* = F \circ W^* \circ I$. Then the pair (M_*, M^*) defines a Mackey $\Lambda\mathcal{P}$ -module by Lemma 3.1 (ii).

Fix elements τ in \mathcal{P} and $d \in \{-2, -1, 0, 1, 2, \dots\}$. Consider a covariant $\Lambda\mathcal{P}$ -module N , i.e., a covariant functor $N: \mathcal{P} \rightarrow \Lambda\text{-Modules}$. Define the Λ -submodules $L_\tau^d N$ and $L_\tau N$ of $N(\tau)$ to be the images of the maps

$$\bigoplus_{\substack{\sigma \in \mathcal{P}, \sigma < \tau \\ \dim(\sigma) \leq d}} N(\sigma \leq \tau): \bigoplus_{\substack{\sigma \in \mathcal{P}, \sigma < \tau \\ \dim(\sigma) \leq d}} N(\sigma) \longrightarrow N(\tau)$$

and

$$\bigoplus_{\sigma \in \mathcal{P}, \sigma < \tau} N(\sigma \leq \tau): \bigoplus_{\sigma \in \mathcal{P}, \sigma < \tau} N(\sigma) \longrightarrow N(\tau),$$

respectively. Define Λ -quotient modules of $N(\tau)$ by

$$S_\tau^d N = N(\tau)/L_\tau^d(N), \quad S_\tau N = N(\tau)/L_\tau(N).$$

Then we obtain a sequence of inclusions of Λ -modules

$$\{0\} = L_\tau^{-2}N \subseteq L_\tau^{-1}N \subseteq L_\tau^0N \subseteq L_\tau^1N \subseteq \dots \subseteq L_\tau^{\dim(\tau)-1}N = L_\tau^{\dim(\tau)}N = L_\tau N,$$

and a sequence of epimorphisms of Λ -modules

$$\begin{aligned} N(\tau) &= S_\tau^{-2}N \longrightarrow S_\tau^{-1}N \longrightarrow S_\tau^0N \longrightarrow S_\tau^1N \longrightarrow \dots \longrightarrow S_\tau^{\dim(\tau)-1}N \\ &= S_\tau^{\dim(\tau)}N = S_\tau N. \end{aligned}$$

Note that

$$L_\tau^{-1}N = \text{im}(N(\emptyset) \longrightarrow N(\tau)) \quad \text{and} \quad S_\tau^{-1}N = \text{cok}(N(\emptyset) \longrightarrow N(\tau))$$

for $\tau \neq \emptyset$ and $L_\emptyset^{-1}N = \{0\}$ and $S_\emptyset^{-1}N = N(\emptyset)$.

Consider a Mackey $\Lambda\mathcal{P}$ -module $M = (M_*, M^*)$. Define a Λ -homomorphism

$$s_\tau^d : M_*(\tau) \longrightarrow M_*(\tau)$$

by

$$s_\tau^d := \text{id}_{M_*(\tau)} - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu)=d}} M_*(\mu \leq \tau) \circ M^*(\mu \leq \tau).$$

Lemma 4.3. For $d \in \{-1, 0, 1, 2, \dots\}$,

$$s_\tau^d(L_\tau^d M_*) \subseteq L_\tau^{d-1} M_*.$$

Proof. If $d \geq \dim(\tau)$ we have $s_\tau^d = \text{id}_{M_*(\tau)}$ and $L_\tau^{d-1} M_* = L_\tau^d M_* = L_\tau M_*$. Therefore the claim is true in this case. Assume that $d < \dim(\tau)$. We compute for $\sigma \in \mathcal{P}$ satisfying $\sigma < \tau$ and $\dim(\sigma) \leq d$

$$\begin{aligned} & s_\tau^d \circ M_*(\sigma \leq \tau) \\ &= M_*(\sigma \leq \tau) - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu)=d}} M_*(\mu \leq \tau) \circ M^*(\mu \leq \tau) \circ M_*(\sigma \leq \tau) \\ &\stackrel{(4.1)}{=} M_*(\sigma \leq \tau) - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu)=d}} M_*(\mu \leq \tau) \circ M_*(\mu \cap \sigma \leq \mu) \circ M^*(\mu \cap \sigma \leq \sigma) \\ &= M_*(\sigma \leq \tau) - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu)=d}} M_*(\mu \cap \sigma \leq \tau) \circ M^*(\mu \cap \sigma \leq \sigma) \\ &= M_*(\sigma \leq \tau) - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu)=d, \dim(\mu \cap \sigma)=d}} M_*(\mu \cap \sigma \leq \tau) \circ M^*(\mu \cap \sigma \leq \sigma) \\ &\quad - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu)=d, \dim(\mu \cap \sigma) \leq d-1}} M_*(\mu \cap \sigma \leq \tau) \circ M^*(\mu \cap \sigma \leq \sigma). \end{aligned}$$

Suppose that $\dim(\mu \cap \sigma) = d$. Since $\dim(\mu) = d$ and $\dim(\sigma) \leq d$, we conclude $\mu = \mu \cap \sigma = \sigma$ and hence because of $\sigma < \tau$

$$\{\mu \in \mathcal{P} \mid \mu < \tau, \dim(\mu) = d, \dim(\mu \cap \sigma) = d\} = \{\sigma\}.$$

This implies

$$M_*(\sigma \leq \tau) - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu) = d, \dim(\mu \cap \sigma) = d}} M_*(\mu \cap \sigma \leq \tau) \circ M^*(\mu \cap \sigma \leq \sigma) = 0,$$

and hence

$$s_\tau^d \circ M_*(\sigma \leq \tau) = - \sum_{\substack{\mu \in \mathcal{P}, \mu < \tau \\ \dim(\mu) = d, \dim(\mu \cap \sigma) \leq d-1}} M_*(\mu \cap \sigma \leq \tau) \circ M^*(\mu \cap \sigma \leq \sigma).$$

We conclude for every $\sigma \in \mathcal{P}$ satisfying $\sigma < \tau$ and $\dim(\sigma) \leq d$

$$\text{im}(s_\tau^d \circ M_*(\sigma \leq \tau)) \subseteq L_\tau^{d-1} M_*.$$

Now the assertion follows. □

Define a map

$$s_\tau: M_*(\tau) \longrightarrow M_*(\tau)$$

to be the composite $s_\tau^{-1} \circ s_\tau^0 \circ \dots \circ s_\tau^{\dim(\tau)-1}$. Then we conclude by induction from Lemma 4.3 that s_τ restricted to $L_\tau M_*$ is trivial and hence induces a Λ -homomorphism

$$\bar{s}_\tau: S_\tau M_* \longrightarrow M_*(\tau).$$

Lemma 4.4. *Let $M = (M_*, M^*)$ be a Mackey $\Lambda\mathcal{P}$ -module. Consider an element $\tau \in \mathcal{P}$. Let $p_\tau: M_*(\tau) \rightarrow S_\tau M_*$ be the projection. Then*

$$p_\tau \circ \bar{s}_\tau = \text{id}_{S_\tau M_*}.$$

Proof. Obviously each map s_τ^d satisfies $p_\tau \circ s_\tau^d = p_\tau$. □

For every element $\tau \in \mathcal{P}$ the restriction

$$\text{RES}_\tau: \Lambda\mathcal{P}\text{-Modules} \longrightarrow \Lambda\text{-Modules}$$

has a left adjoint E_τ by [23, Lemma 9.31 on p. 171]. Explicitly, the functor $E_\tau: \Lambda\text{-Modules} \rightarrow \Lambda\mathcal{P}\text{-Modules}$ is given by $E_\tau(N) = N \otimes_\Lambda \Lambda \text{mor}_\mathcal{P}(\tau, ?)$, where $?$ runs through the objects of \mathcal{P} and $\Lambda \text{mor}_\mathcal{P}(\tau, ?)$ is the free Λ -module with the set $\text{mor}_\mathcal{P}(\tau, ?)$ as basis. Equivalently, $N \otimes_\Lambda \Lambda \text{mor}_\mathcal{P}(\tau, ?)$ assigns to an object $?$ the Λ -module N if $\tau \leq ?$, and $\{0\}$ otherwise. Functoriality in $?$ is given by the identity on $\{0\}$ or N , or by the inclusion $\{0\} \rightarrow N$.

We get for every element $\tau \in \mathcal{P}$ and every map of covariant Λ -modules $u: S_\tau M_* \rightarrow M_*(\tau)$ a map of covariant $\Lambda\mathcal{P}$ -modules

$$\text{ad}(u): E_\tau(S_\tau M_*) \longrightarrow M_*$$

by the adjoint of u under the adjunction $(E_\tau, \text{RES}_\tau)$. For $? \in \mathcal{P}$ the map $\text{ad}(u)(?)$ is given by the composite

$$S_\tau M_* \xrightarrow{u} M_*(\tau) \xrightarrow{M_*(\tau \leq ?)} M_*(?)$$

if $\tau \leq ?$ and by the inclusion $\{0\} \rightarrow M_*(?)$ otherwise.

Lemma 4.5. *Consider any collection of homomorphisms $\hat{s}_\tau: S_\tau M_* \rightarrow M_*(\tau)$ satisfying $p_\tau \circ \hat{s}_\tau = \text{id}_{S_\tau M_*}$, where τ runs through the elements in \mathcal{P} .*

Then the homomorphism of covariant $\Lambda\mathcal{P}$ -modules

$$\text{ad}(\hat{s}) := \bigoplus_{\tau \in \mathcal{P}} \text{ad}(\hat{s}_\tau): \bigoplus_{\tau \in \mathcal{P}} E_\tau(S_\tau M_*) \longrightarrow M_*$$

is an isomorphism.

Proof. We start with injectivity. Suppose that $\text{ad}(\hat{s})$ is not injective. Then there exists an element $\sigma \in \mathcal{P}$ and a non-trivial element $a = (a_\tau)_{\tau \in \mathcal{P}} \in \bigoplus_{\tau \in \mathcal{P}} E_\tau(S_\tau M_*)(\sigma)$ such that $\text{ad}(\hat{s})(\sigma)(a) = 0$. Choose $\tau_0 \in \mathcal{P}$ with $a_{\tau_0} \neq 0$ such that for all $\tau \in \mathcal{P}$ with $a_\tau \neq 0$ we have $\dim(\tau) \leq \dim(\tau_0)$. If $\tau \in \mathcal{P}$ satisfies $a_\tau \neq 0$ and hence $E_\tau(S_\tau M_*)(\sigma) \neq 0$, we use the explicit description of $E_\tau(S_\tau M_*)$ to conclude $\tau \leq \sigma$. The composite

$$\bigoplus_{\tau \in \mathcal{P}} E_\tau(S_\tau M_*)(\sigma) \xrightarrow{\text{ad}(\hat{s})(\sigma)} M_*(\sigma) \xrightarrow{M^*(\tau_0 \leq \sigma)} M_*(\tau_0) \xrightarrow{p_{\tau_0}} S_{\tau_0} M_* \quad (4.6)$$

sends a to 0 since $\text{ad}(\hat{s})(\sigma)(a) = 0$.

Consider an element $\tau \in \mathcal{P}$ with $a_\tau \neq 0$. Then $\dim(\tau) \leq \dim(\tau_0)$ and hence we get $\tau \cap \tau_0 < \tau_0$ if $\tau \neq \tau_0$ and $\tau \cap \tau_0 = \tau_0$ if $\tau = \tau_0$. The composite

$$M_*(\tau) \xrightarrow{M_*(\tau \leq \sigma)} M_*(\sigma) \xrightarrow{M^*(\tau_0 \leq \sigma)} M_*(\tau_0) \xrightarrow{p_{\tau_0}} S_{\tau_0} M_*$$

agrees because of (4.1) with the composite

$$M_*(\tau) \xrightarrow{M^*(\tau \cap \tau_0 \leq \tau)} M_*(\tau \cap \tau_0) \xrightarrow{M^*(\tau \cap \tau_0 \leq \tau_0)} M_*(\tau_0) \xrightarrow{p_{\tau_0}} S_{\tau_0} M_*$$

Hence it is zero if $\tau \neq \tau_0$ and it is p_{τ_0} if $\tau = \tau_0$. This implies that the restriction of the composite (4.6) to the summand associated to τ is trivial if $\tau \neq \tau_0$ and is the identity under the obvious identification $E_{\tau_0}(S_{\tau_0} M_*)(\sigma) = S_{\tau_0} M_*$ if $\tau = \tau_0$. We conclude that the composite (4.6) sends a to a_{τ_0} under the obvious identification

$E_{\tau_0}(S_{\tau_0}M_*)(\sigma) = S_{\tau_0}M_*$. Since this implies $a_{\tau_0} = 0$, we get a contradiction. This finishes the proof that $\text{ad}(\hat{s})$ is injective.

Next we show by induction for $d = -1, 0, 1, 2, \dots$ that $\text{ad}(\hat{s})(\sigma)$ is surjective for all $\sigma \in \mathcal{P}$ with $\dim(\sigma) \leq d$. The induction beginning is obvious since \emptyset is the unique initial object, hence $S_{\emptyset}M_* = M_*(\emptyset)$ holds and therefore $\text{ad}(\hat{s})_{\emptyset}(\emptyset)$ is bijective. The induction step from $(d - 1)$ to $d \geq 0$ is done as follows. The composite

$$E_{\sigma}(S_{\sigma}M_*)(\sigma) \xrightarrow{\text{ad}(\hat{s})_{\sigma}} M_*(\sigma) \longrightarrow S_{\sigma}M_*$$

is surjective. Hence it suffices to show that $L_{\sigma}M_*$ is contained in the restriction of $\text{ad}(\hat{s})(\sigma)$ to $\bigoplus_{\tau \in \mathcal{P}, \tau \neq \sigma} E_{\tau}(S_{\tau}M_*)(\sigma)$. It suffices to show that for every σ' with $\sigma' < \sigma$ the image of $M_*(\sigma') \rightarrow M_*(\sigma)$ is contained in the restriction of $\text{ad}(\hat{s})(\sigma)$ to $\bigoplus_{\tau \in \mathcal{P}, \tau \neq \sigma} E_{\tau}(S_{\tau}M_*)(\sigma)$. By induction hypothesis $\text{ad}(\hat{s})(\sigma')$ is surjective. Now Lemma 4.5 follows from naturality of $\text{ad}(\hat{s})$ and the fact that $E_{\sigma}(S_{\sigma}M_*)(\sigma')$ vanishes. \square

For a covariant $\Lambda\mathcal{P}$ -module N , denote by $H_n(\mathcal{P}; N)$ its homology. This is $H_n(P_* \otimes_{\Lambda\mathcal{P}} N)$ for the Λ -homology of the Λ -chain complex $P_* \otimes_{\Lambda\mathcal{P}} N$ for any projective $\Lambda\mathcal{P}$ -resolution P_* of the constant $\Lambda\mathcal{P}$ -module $\underline{\Lambda}$ with value Λ . In the notation of [23, Chapter 17] this is $\text{Tor}_n^{\Lambda\mathcal{P}}(\underline{\Lambda}, N)$.

Theorem 4.7. *Let $M = (M_*, M^*)$ be a Mackey $\Lambda\mathcal{P}$ -module. Then,*

- (i) $H_n(\mathcal{P}; M_*)$ vanishes for $n \geq 1$,
- (ii) we obtain an isomorphism

$$\text{colim}_{\mathcal{P}} M_* \cong H_0(\mathcal{P}; M_*) \cong \bigoplus_{\sigma \in \mathcal{P}} S_{\sigma}M_*$$

- (iii) $S_{\sigma}M_*$ is a direct summand in the Λ -module $M_*(\sigma)$.

Proof. (i) From Lemma 4.5, we obtain an isomorphism

$$\bigoplus_{\sigma \in \mathcal{P}} H_n(\mathcal{P}; E_{\sigma}(S_{\sigma}M_*)) \cong H_n\left(\mathcal{P}; \bigoplus_{\sigma \in \mathcal{P}} E_{\sigma}(S_{\sigma}M_*)\right) \cong H_n(\mathcal{P}; M_*).$$

Since the automorphism group of the object σ in \mathcal{P} is the trivial group $\{1\}$, we get for any Λ -module N an isomorphism

$$H_n(\mathcal{P}; E_{\sigma}(N)) \cong H_n(\{1\}; N) \cong \begin{cases} \{0\} & \text{if } n \geq 1, \\ N & \text{if } n = 0. \end{cases}$$

This follows from the adjunction $(E_{\sigma}, \text{RES}_{\sigma})$ of [23, Lemma 9.31 on p. 171] and the fact that $\text{RES}_{\sigma}(P_*)$ is a projective Λ -resolution of Λ .

(ii) For every covariant $\Lambda\mathcal{P}$ -module N , there are canonical Λ -isomorphisms

$$\operatorname{colim}_{\mathcal{P}} N \cong \underline{\Lambda} \otimes_{\Lambda\mathcal{P}} N \cong H_0(\mathcal{P}; N),$$

where $\underline{\Lambda}$ is the constant $\Lambda\mathcal{P}$ -module with value Λ . This follows from the adjunction between tensor product and the hom-functor and the fact that $- \otimes_{\Lambda\mathcal{P}} N$ is right-exact, see [23, 9.21 and 9.23 on p. 169].

(iii) This follows from Lemma 4.4. □

Let $\operatorname{ch}_n(\Sigma, \sigma)$ be the set of n -chains $\sigma_0 < \sigma_1 < \dots < \sigma_n$ in \mathcal{P} with $\sigma_0 = \sigma$. Define the integer

$$\operatorname{ch}(\Sigma, \sigma) := \sum_{n \geq 0} (-1)^n \cdot |\operatorname{ch}_n(\Sigma, \sigma)|.$$

Fix a class of Λ -modules \mathcal{E} with the property that for an exact sequence $0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0$ the Λ -module V_1 belongs to \mathcal{E} if and only if both V_0 and V_2 belong to \mathcal{E} . An example is the class \mathcal{E} of Λ -modules whose underlying set is finite, and for a Noetherian ring Λ the class \mathcal{E} of finitely generated Λ -modules. Denote by $G_0(\mathcal{E})$ the Grothendieck group of elements in \mathcal{E} , i.e., the abelian group with the isomorphism classes of elements in \mathcal{E} as generators and relations $[V_1] = [V_0] + [V_2]$ for every short exact sequence $0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0$ of Λ -modules belonging to \mathcal{E} .

Theorem 4.8. *Let $M = (M_*, M^*)$ be a Mackey $\Lambda\mathcal{P}$ -module. Suppose that $M_*(\sigma)$ lies in \mathcal{E} for all $\sigma \in \mathcal{P}$.*

Then we get in $G_0(\mathcal{E})$

$$[\operatorname{colim}_{\mathcal{P}} M_*] = \sum_{\sigma \in \mathcal{P}} \operatorname{ch}(\Sigma, \sigma) \cdot [M_*(\sigma)].$$

Proof. The bar-resolution yields a finite free $\Lambda\mathcal{P}$ -resolution C_* of the constant $\Lambda\mathcal{P}$ -module $\underline{\Lambda}$ with value Λ such that

$$C_n = \bigoplus_{\sigma \in \mathcal{P}} \bigoplus_{\operatorname{ch}_n(\Sigma, \sigma)} \Lambda \operatorname{mor}_{\mathcal{P}}(?, \sigma),$$

see [11, Section 3]. Since C_* is a finite free $\Lambda\mathcal{P}$ -chain complex, the Λ -chain complex $C_* \otimes_{\Lambda\mathcal{P}} M_*$ is a finite-dimensional Λ -complex whose Λ -chain modules belong to \mathcal{E} , and we get in $G_0(\mathcal{E})$

$$\begin{aligned} & \sum_{n \geq 0} (-1)^n \cdot [H_n(C_* \otimes_{\Lambda\mathcal{P}} M_*)] \\ &= \sum_{n \geq 0} (-1)^n \cdot [C_n \otimes_{\Lambda\mathcal{P}} M_*] = \sum_{n \geq 0} (-1)^n \cdot \left(\sum_{\sigma \in \mathcal{P}} \sum_{\operatorname{ch}_n(\Sigma, \sigma)} [M_*(\sigma)] \right) \\ &= \sum_{\sigma \in \mathcal{P}} \left(\sum_{n \geq 0} (-1)^n \cdot |\operatorname{ch}_n(\Sigma, \sigma)| \right) \cdot [M_*(\sigma)] = \sum_{\sigma \in \mathcal{P}} \operatorname{ch}(\Sigma, \sigma) \cdot [M_*(\sigma)]. \end{aligned}$$

Since $H_n(C_* \otimes_{\Lambda \mathcal{P}} M_*)$ agrees with $H_n(\mathcal{P}; M_*)$, the claim follows from Theorem 4.7. \square

4.1. Proof of Theorem 1.1. We defined the functor $\mathcal{H}_n^{\mathcal{C}}: \text{Groups} \rightarrow \Lambda\text{-Modules}$ in (2.4). We conclude from Example 4.2 that $(\mathcal{H}_n^{\mathcal{C}} \circ W_* \circ I, \mathcal{H}_n^{\mathcal{C}} \circ W^* \circ I)$ is a Mackey $\Lambda \mathcal{P}$ -module. Now Theorem 1.1 follows from Theorem 3.4, Theorem 4.7, and Theorem 4.8.

5. Isomorphism Conjectures in K- and L-theory

In this section we review the Isomorphism Conjectures of Baum–Connes and Farrell–Jones and recollect the most important results on the passage from \mathcal{FJN} to \mathcal{VCY} in the Farrell–Jones setting.

Let \mathcal{C} be a non-empty class of groups which is closed under isomorphisms, passage to subgroups and passage to quotient groups. Recall that given a group G , we denote by $E_{\mathcal{C}}(G)$ the classifying space of G with respect to the family of subgroups $\mathcal{F}_{\mathcal{C}}(G) = \{H \subseteq G \mid H \in \mathcal{C}\}$. Consider a covariant Groupoids-spectrum

$$\mathbf{E}: \text{Groupoids} \longrightarrow \text{Spectra}$$

which respects equivalences. We obtain an equivariant homology theory $H_*^?(-; \mathbf{E})$ associated to \mathbf{E} from Theorem 2.2.

Then the *Meta-Isomorphism Conjecture for \mathbf{E} and the class \mathcal{C}* predicts that the projection $\text{pr}: E_{\mathcal{C}}(G) \rightarrow G/G$ induces for all $n \in \mathbb{Z}$ an isomorphism

$$H_n^G(\text{pr}; \mathbf{E}): H_n^G(E_{\mathcal{C}}(G); \mathbf{E}) \xrightarrow{\cong} H_n^G(G/G; \mathbf{E}) = \pi_n(\mathbf{E}(G)).$$

If we make the appropriate choices for \mathbf{E} and \mathcal{C} , this becomes the Baum–Connes Conjecture or it becomes the Farrell–Jones Conjecture for algebraic K -theory, for algebraic L -theory, for Waldhausen’s A -theory, or for topological Hochschild homology.

5.1. The Baum–Connes Conjecture. Given a discrete group G , denote by $C_r^*(G)$ and $C_r^*(G; \mathbb{R})$ its *reduced complex and reduced real group C^* -algebra* and by $C_m^*(G)$ and $C_m^*(G; \mathbb{R})$ its *maximal complex and maximal real group C^* -algebra*. There are covariant functors

$$\mathbf{K}_{C_m^*}: \text{Groupoids} \longrightarrow \text{Spectra}, \tag{5.1}$$

$$\mathbf{K}_{C_m^*; \mathbb{R}}: \text{Groupoids} \longrightarrow \text{Spectra}, \tag{5.2}$$

which send equivalences of groupoids to weak equivalences of spectra and satisfy $\pi_n(\mathbf{K}_{C_m^*}(G)) = K_n(C_m^*(G))$ and $\pi_n(\mathbf{K}_{C_m^*; \mathbb{R}}(G)) = KO_n(C_m^*(G; \mathbb{R}))$ for $n \in \mathbb{Z}$. Here K_* and KO_* denote topological K -theory. If we consider the class of

finite groups, the Meta-Isomorphism Conjecture reduces to the *Baum–Connes Conjecture for the maximal group C^* -algebra*. It predicts the bijectivity of the assembly maps for $n \in \mathbb{Z}$

$$\begin{aligned} K_n^G(\underline{E}G) &\longrightarrow K_n(C_m^*(G)), \\ KO_n^G(\underline{E}G) &\longrightarrow KO_n(C_m^*(G; \mathbb{R})), \end{aligned}$$

where the source is given by equivariant K -homology for which we have the identifications $K_n^G(\underline{E}G) = H_n^G(\underline{E}G; \mathbf{K}_{C_m^*})$ and $KO_n^G(\underline{E}G) = H_n^G(\underline{E}G; \mathbf{K}_{C_m^*; \mathbb{R}})$.

We can apply K -theory to the natural maps of C^* -algebras $C_m^*(G) \rightarrow C_r^*(G)$ and $C_m^*(G; \mathbb{R}) \rightarrow C_r^*(G; \mathbb{R})$ to obtain maps $f_n: K_n(C_m^*(G)) \rightarrow K_n(C_r^*(G))$ and $f_n^{\mathbb{R}}: KO_n(C_m^*(G; \mathbb{R})) \rightarrow KO_n(C_r^*(G; \mathbb{R}))$. The *Baum–Connes Conjecture* predicts that the composites

$$\begin{aligned} K_n^G(\underline{E}G) &\longrightarrow K_n(C_m^*(G)) \xrightarrow{f_n} K_n(C_r^*(G)), \\ KO_n^G(\underline{E}G) &\longrightarrow KO_n(C_m^*(G; \mathbb{R})) \xrightarrow{f_n^{\mathbb{R}}} KO_n(C_r^*(G; \mathbb{R})) \end{aligned}$$

are bijective for all $n \in \mathbb{Z}$.

There are counterexamples to the Baum–Connes Conjecture for the maximal group C^* -algebra, but no counterexamples to the Baum–Connes Conjecture are known. We want to consider the Baum–Connes Conjecture for the maximal group C^* -algebra since $K_n(C_m^*(G))$ and $KO_n(C_m^*(G; \mathbb{R}))$ are functorial in G for all group homomorphisms, whereas $K_n(C_r^*(G))$ and $KO_n(C_r^*(G; \mathbb{R}))$ are functorial for injective group homomorphisms, but not in general for any group homomorphism. Moreover the covariant functors (5.1) and (5.2) are defined on Groupoids. This ensures that the induction structure is available for all group homomorphisms and not only for injective group homomorphisms as it is the case if we replace (5.1) and (5.2) by their versions for the reduced C^* -algebras. We later want to apply the induction structure also to certain split surjective group homomorphisms, see Remark 3.2.

There is a more general *Baum–Connes Conjecture with coefficients*, which is known to be true for a large class of groups and which has good inheritance properties. In particular, the class of groups satisfying the Baum–Connes Conjecture with coefficients is closed under taking graph products, since it is stable under finite direct products and amalgamated products, see [31] and [32].

5.2. The Farrell–Jones Conjecture. Given a ring R (with involution), there are covariant functors

$$\begin{aligned} \mathbf{K}_R &: \text{Groupoids} \longrightarrow \text{Spectra}, \\ \mathbf{L}_R^{(-\infty)} &: \text{Groupoids} \longrightarrow \text{Spectra}, \end{aligned}$$

which send equivalences of groupoids to weak equivalences of spectra and satisfy $\pi_n(\mathbf{K}_R(G)) = K_n(RG)$ and $\pi_n(\mathbf{L}_R^{(-\infty)}(G)) = L_n^{(-\infty)}(RG)$. Here K_* denotes non-connective algebraic K -theory and $L_*^{(-\infty)}$ denotes algebraic L -theory with decoration $\langle -\infty \rangle$. If we consider the class of virtually cyclic groups, the Meta-Isomorphism Conjecture reduces to the *K-theoretic or the L-theoretic Farrell–Jones Conjecture* which predicts that for all $n \in \mathbb{Z}$ the corresponding map

$$\begin{aligned} H_n^G(\underline{EG}; \mathbf{K}_R) &\longrightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(RG), \\ H_n^G(\underline{EG}; \mathbf{L}_R^{(-\infty)}) &\longrightarrow H_n^G(G/G; \mathbf{L}_R^{(-\infty)}) = L_n^{(-\infty)}(RG) \end{aligned}$$

is bijective.

There is a more general *Full Farrell–Jones Conjecture* which allows additive G -categories as coefficients. It is known to be true for a large class of groups and has good inheritance properties. In particular, the class of groups satisfying the Full Farrell–Jones Conjecture is closed under taking graph products, which is a result of Gandini–Rüping [17]. There also is a version of the Farrell–Jones Conjecture for Waldhausen’s A -theory which we will not discuss here. It satisfies similar inheritance properties as the Full Farrell–Jones Conjecture, see [15] and [38]. Also the following Theorem 5.3 (i) holds in this setting, see [6].

5.3. The passage from \mathcal{FJN} to \mathcal{VCY} . The Farrell–Jones Conjecture is more complicated than the Baum–Connes Conjecture since for the Farrell–Jones Conjecture the class of virtually cyclic groups has to be considered, whereas for the Baum–Connes Conjecture the class of finite groups suffices. Hence one has to understand the passage from \underline{EG} to \underline{EG} in the Farrell–Jones setting.

Theorem 5.3. (i) *Let G be any group and R be a ring. Then the relative assembly maps*

$$\begin{aligned} H_n^G(\underline{EG}; \mathbf{K}_R) &\longrightarrow H_n^G(\underline{EG}; \mathbf{K}_R), \\ H_n^G(\underline{EG}; \mathbf{L}_R^{(-\infty)}) &\longrightarrow H_n^G(\underline{EG}; \mathbf{L}_R^{(-\infty)}) \end{aligned}$$

are split injective for all $n \in \mathbb{Z}$.

(ii) *Let G be any group and R be a regular ring. Then the relative assembly map*

$$H_n^G(\underline{EG}; \mathbf{K}_R) \longrightarrow H_n^G(\underline{EG}; \mathbf{K}_R)$$

is rationally bijective for all $n \in \mathbb{Z}$.

(iii) *Let G be any group and R be a regular ring. Suppose that for any finite subgroup $H \subseteq G$ its order $|H|$ is invertible in R . Then the relative assembly map*

$$H_n^G(\underline{EG}; \mathbf{K}_R) \longrightarrow H_n^G(\underline{EG}; \mathbf{K}_R)$$

is bijective for all $n \in \mathbb{Z}$.

(iv) Let R be a regular ring. Let $W = W(X, \mathcal{W})$ be a graph product and d be a natural number. Suppose that for any vertex $v \in V$ the group W_v is either torsionfree or a finite group whose order divides d . Then the relative assembly map

$$H_n^W(\underline{E}W; \mathbf{K}_R) \longrightarrow H_n^W(\underline{\underline{E}}W; \mathbf{K}_R)$$

is bijective after inverting d for all $n \in \mathbb{Z}$.

(v) Let G be any group and R be a ring with involution. Then the relative assembly map

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \longrightarrow H_n^G(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle})$$

is bijective after inverting 2 for all $n \in \mathbb{Z}$.

(vi) Let G be any group and R be a ring with involution such that 2 is invertible in R . Then the relative assembly map

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \longrightarrow H_n^G(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty \rangle})$$

is bijective for all $n \in \mathbb{Z}$.

(vii) Let R be a ring with involution. Let $W = W(X, \mathcal{W})$ be a graph product. Suppose that for any vertex $v \in V$ the group W_v is either torsionfree or a finite group of odd order. Then the relative assembly map

$$H_n^W(\underline{E}W; \mathbf{L}_R^{\langle -\infty \rangle}) \longrightarrow H_n^W(\underline{\underline{E}}W; \mathbf{L}_R^{\langle -\infty \rangle})$$

is bijective for all $n \in \mathbb{Z}$.

Proof. (i) See [3] and [28].

(ii) This is proved in [28, Theorem 0.3].

(iii) See [27, Proposition 70 on p. 744].

(iv) Let $\mathcal{V}\mathcal{C}\mathcal{Y}_I$ be the class of virtually cyclic groups of type I, i.e., groups which admit a homomorphism to \mathbb{Z} with finite kernel. Then the relative assembly map

$$H_n^W(E_{\mathcal{V}\mathcal{C}\mathcal{Y}_I}(W); \mathbf{K}_R) \longrightarrow H_n^W(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(W); \mathbf{K}_R)$$

is bijective for all $n \in \mathbb{Z}$ by [12, Theorem 1.1]. Hence it suffices to show that

$$H_n^W(E_{\mathcal{F}\mathcal{J}\mathcal{N}}(W); \mathbf{K}_R) \longrightarrow H_n^W(E_{\mathcal{V}\mathcal{C}\mathcal{Y}_I}(W); \mathbf{K}_R)$$

is bijective for all $n \in \mathbb{Z}$. By the same argument as it appears in [28, Proof of Theorem 0.3 on p. 370] the claim is reduced to showing that for any infinite virtually cyclic subgroup $H \subseteq W$ of type I we get

$$NK_n(RK_H; \phi)[1/d] = 0,$$

where $K_H \subseteq H$ is a finite subgroup such that H/K_H is infinite cyclic and $\phi: K_H \rightarrow K_H$ is given by conjugation with an element $h \in H$ which is sent

under the projection $H \rightarrow H/K_H$ to a generator. Any finite subgroup of W is conjugated into a group $W(\sigma)$ for some simplex σ of Σ such that W_v is finite for every $v \in V \cap \sigma$. (Indeed, this is obvious if Σ is a simplex. Otherwise, we can express W as an amalgamated product and use the fact that a finite subgroup of an amalgamated product is conjugated into one of the factors, see [35, Theorem 8 in 4.3 on p. 36]. Then conclude via induction on the number of vertices.) Hence we can assume that there exists a simplex σ of Σ such that $K_H \subseteq W(\sigma)$ and $W(\sigma)$ is finite. There is a group homomorphism $r: W \rightarrow W(\sigma)$ whose restriction to $W(\sigma)$ is the identity, see Remark 3.2. Consider $w \in K_H$. Then hwh^{-1} belongs to K_H again. We compute

$$\phi(w) = hwh^{-1} = r(hwh^{-1}) = r(h)r(w)r(h)^{-1} = r(h)wr(h)^{-1}.$$

Hence ϕ is given by conjugation with $r(h) \in W(\sigma)$. The order of $r(h)$ and thus also of ϕ divides d . We conclude from [28, Theorem 9.4] that

$$NK_n(RK_H; \phi)[1/d] = 0.$$

(v) This is proved in [25, Lemma 4.2].

(vi) The proof is analogous to the first part of the proof of (iv) using the fact that UNil-groups vanish, if $1/2$ is contained in R , see [7, Corollary 3] and that the map

$$H_n^G(E_{\mathcal{FJN}}(G); \mathbf{L}_R^{(-\infty)}) \longrightarrow H_n^G(E_{\mathcal{Vey}_I}(G); \mathbf{L}_R^{(-\infty)})$$

is bijective for all $n \in \mathbb{Z}$, see [25, Lemma 4.2].

(vii) Any finite subgroup of W is conjugated into a group $W(\sigma)$ for some simplex σ of Σ such that W_v is finite for every $v \in V \cap \sigma$. Hence every finite subgroup has odd order and thus every infinite virtually cyclic subgroup of W is of type I. Now the claim follows from [25, Lemma 4.2]. This finishes the proof of Theorem 5.3. □

6. Right-angled Artin groups

In this section we want to compute the group homology, the algebraic K - and L -theory, and the topological K -theory of a right-angled Artin group W . Recall that a *right-angled Artin group* is a graph product $W = W(X, \mathcal{W})$ for which each of the groups W_v is infinite cyclic. Right-angled Artin groups satisfy the Baum–Connes Conjecture and the Baum–Connes Conjecture for the maximal group C^* -algebra, which follows from [13] and [19]. Both the K -theoretic and the L -theoretic Farrell–Jones Conjecture are satisfied for right-angled Artin groups, see [2] and [40]. For general information about right-angled Artin groups we refer for instance to Charney [9].

In the sequel we denote by r_k the number of k -simplices in $\mathcal{P} = \mathcal{P}(\Sigma)$ and put $r = |\mathcal{P}| = \sum_{k=-1}^{\dim(\Sigma)} r_k$. Recall that the empty simplex is allowed in \mathcal{P} and has dimension -1 .

Note that W is torsionfree and that therefore a model for $\underline{E}W$ is given by EW . For an equivariant homology theory $\mathcal{H}_*^?$ and any group G the induction structure along the trivial homomorphism $G \rightarrow \{1\}$ yields an isomorphism

$$\mathcal{H}_*^G(EG) \xrightarrow{\cong} \mathcal{H}_*^{\{1\}}(BG).$$

Let \mathcal{K}_* be a (non-equivariant) generalized homology theory with values in Λ -modules. Let X be a CW -complex. It follows from the axioms of a generalized homology theory that there is an isomorphism, natural in X

$$B_n(X) := \mathcal{K}_n(\text{pr}) \times (s_n \circ \mathcal{K}_n(\text{id}_X \times i)): \mathcal{K}_n(X \times S^1) \xrightarrow{\cong} \mathcal{K}_n(X) \times \mathcal{K}_{n-1}(X),$$

where we denote by

$$\text{pr}: X \times S^1 \longrightarrow X$$

the projection, by

$$i: S^1 = (S^1, \emptyset) \longrightarrow (S^1, \text{pt})$$

the inclusion, and by

$$s_n: \mathcal{K}_n(X \times (S^1, \{\text{pt}\})) \xrightarrow{\cong} \mathcal{K}_{n-1}(X)$$

the suspension isomorphism.

By induction over $k \geq 0$ we obtain an isomorphism

$$B_n^k: \mathcal{K}_n(T^k) \xrightarrow{\cong} \prod_{i=0}^k \prod_{j=1}^{\binom{k}{i}} \mathcal{K}_{n-i}(\text{pt}), \tag{6.1}$$

where we denote by T^k the k -dimensional torus $\prod_{i=1}^k S^1$. Note that $T^0 = \text{pt}$. For $i = 1, 2, \dots, k$, let $T_i^k \subseteq T^k$ be the subspace consisting of elements of the form (z_1, z_2, \dots, z_k) with $z_i = *$, where $*$ is a fixed base point in S^1 . Let $j_i^k: T_i^k \rightarrow T^k$ be the inclusion. We will identify $T_k^k = T^{k-1}$.

Lemma 6.2. *For every $k \geq 0$ and $n \in \mathbb{Z}$ there is an isomorphism*

$$c_n^k: \text{cok} \left(\bigoplus_{i=1}^k \mathcal{K}_n(j_i^k): \bigoplus_{i=1}^k \mathcal{K}_n(T_i^k) \longrightarrow \mathcal{K}_n(T^k) \right) \xrightarrow{\cong} \mathcal{K}_{n-k}(\text{pt}).$$

Its inverse is induced by the restriction of the inverse of the isomorphism B_n^k of (6.1) to the factor $\mathcal{K}_{n-k}(\text{pt})$ for the index $i = k$.

Proof. We use induction over $k = 0, 1, 2, \dots$. If $k = 0$, take $c_n^0 = \text{id}_{\mathcal{K}_n(\text{pt})}$. The induction step from $(k - 1)$ to $k \geq 1$ is done as follows. We have the following commutative diagram of Λ -modules

$$\begin{array}{ccc}
 \bigoplus_{i=1}^k \mathcal{K}_n(T_i^k) & \xrightarrow{\bigoplus_{i=1}^k \mathcal{K}_n(j_i^k)} & \mathcal{K}_n(T^k) \\
 \text{id} \downarrow \cong & & \downarrow \cong \text{id} \\
 \mathcal{K}_n(T^k) \oplus \bigoplus_{i=1}^{k-1} \mathcal{K}_n(T_i^k) & & \mathcal{K}_n(T^{k-1} \times S^1) \\
 \text{id} \oplus \bigoplus_{i=1}^{k-1} \text{id} \downarrow \cong & & \downarrow \cong \\
 \mathcal{K}_n(T^{k-1}) \oplus \bigoplus_{i=1}^{k-1} \mathcal{K}_n(T_i^{k-1} \times S^1) & & \mathcal{K}_n(T^{k-1} \times S^1) \\
 \text{id} \oplus \bigoplus_{i=1}^{k-1} B_n(T_i^{k-1}) \downarrow \cong & & \downarrow \cong \\
 \mathcal{K}_n(T^{k-1}) \oplus \bigoplus_{i=1}^{k-1} (\mathcal{K}_n(T_i^{k-1}) \oplus \mathcal{K}_{n-1}(T_i^{k-1})) & & \mathcal{K}_n(T^{k-1}) \\
 f \downarrow \cong & & \downarrow \cong \\
 (\mathcal{K}_n(T^{k-1}) \oplus \bigoplus_{i=1}^{k-1} \mathcal{K}_n(T_i^{k-1})) & & \mathcal{K}_n(T^{k-1}) \\
 \bigoplus_{i=1}^{k-1} \mathcal{K}_{n-1}(T_i^{k-1}) & \xrightarrow{\begin{pmatrix} f_0 & 0 \\ 0 & f_1 \end{pmatrix}} & \mathcal{K}_{n-1}(T^{k-1})
 \end{array}$$

where f is the obvious isomorphism, $f_0 := \text{id}_{\mathcal{K}_n(T^{k-1})} \oplus \bigoplus_{i=1}^{k-1} \mathcal{K}_n(j_i^{k-1})$, and $f_1 := \bigoplus_{i=1}^{k-1} \mathcal{K}_{n-1}(j_i^{k-1})$. Since f_0 is surjective, the diagram above induces an isomorphism

$$\text{cok} \left(\bigoplus_{i=1}^k \mathcal{K}_n(j_i^k) \right) \xrightarrow{\cong} \text{cok}(f_0) \oplus \text{cok}(f_1) = \{0\} \oplus \text{cok} \left(\bigoplus_{i=1}^{k-1} \mathcal{K}_{n-1}(j_i^{k-1}) \right).$$

By induction hypothesis we have an isomorphism

$$c_{n-1}^{k-1} : \text{cok} \left(\bigoplus_{i=1}^{k-1} \mathcal{K}_{n-1}(j_i^{k-1}) \right) \xrightarrow{\cong} \mathcal{K}_{n-k}(\text{pt}).$$

This finishes the induction step and hence the proof of Lemma 6.2. □

6.1. Group homology. Let \mathcal{K}_* be any generalized homology theory with values in Λ -modules. Notice that for any group G the CW -complex $EG \times_G \underline{E}G$ is a model for BG since $\underline{E}G$ is contractible after forgetting the G -action. We have introduced the equivariant homology theory given by the Borel construction and \mathcal{K}_* in Example 2.3. We conclude from Theorem 1.1 and Lemma 6.2 that there is

an explicit Λ -isomorphism

$$\bigoplus_{\sigma \in \mathcal{P}} \mathcal{K}_{n-\dim(\sigma)-1}(\text{pt}) \xrightarrow{\cong} \mathcal{K}_n(BW).$$

If we take for \mathcal{K}_* singular homology $H_*(-; \Lambda)$ with coefficients in Λ , this boils down to the well-known, see for example [21, Corollary 11], isomorphism of Λ -modules

$$\Lambda^{r_{n-1}} \xrightarrow{\cong} H_n(BW; \Lambda).$$

In particular we get the following relation for the Euler characteristics

$$\chi(BW) = 1 - \chi(\Sigma).$$

6.2. Algebraic K -theory. Let R be a regular ring. We conclude from Theorem 1.1, Theorem 5.3 (iv), Lemma 6.2, and the validity of the K -theoretic Farrell–Jones Conjecture for right-angled Artin groups that there is an explicit isomorphism of abelian groups

$$\bigoplus_{\sigma \in \mathcal{P}} K_{n-\dim(\sigma)-1}(R) \xrightarrow{\cong} K_n(RW).$$

Its restriction to the summand belonging to σ is the composite of the map $K_n(RW(\sigma)) \rightarrow K_n(RW)$ coming from the inclusion $\mathbb{Z}^{\dim(\sigma)+1} = W(\sigma) \rightarrow W$ with the restriction of the inverse of the iterated Bass–Heller–Swan isomorphism

$$\bigoplus_{i=0}^{\dim(\sigma)+1} \bigoplus_{j=1}^{\binom{\dim(\sigma)+1}{i}} K_{n-i}(R) \xrightarrow{\cong} K_n(R[\mathbb{Z}^{\dim(\sigma)+1}])$$

to the summand $K_{n-\dim(\sigma)-1}(R)$ belonging to $i = \dim(\sigma) + 1$.

Since for a regular ring R its negative K -theory vanishes, we conclude $K_n(RW) = 0$ for $n \leq -1$. If we take $R = \mathbb{Z}$, we conclude that $K_n(\mathbb{Z}W)$ for $n \leq -1$, $\tilde{K}_0(\mathbb{Z}W)$, and $\text{Wh}(W)$ vanish what is actually true if we replace W by any torsionfree group satisfying the Farrell–Jones Conjecture.

6.3. Algebraic L -theory. Let R be a ring with involution. We conclude from Theorem 1.1, Theorem 5.3 (vii), Lemma 6.2, and the validity of the L -theoretic Farrell–Jones Conjecture for right-angled Artin groups that there is an explicit isomorphism of abelian groups

$$\bigoplus_{\sigma \in \mathcal{P}} L_{n-\dim(\sigma)-1}^{(-\infty)}(R) \xrightarrow{\cong} L_n^{(-\infty)}(RW).$$

Its restriction to a summand comes from the Shaneson splitting.

6.4. Topological K-theory. We conclude from Theorem 1.1, Lemma 6.2, and the validity of the Baum–Connes Conjecture for right-angled Artin groups that there are explicit isomorphisms of abelian groups

$$\bigoplus_{\sigma \in \mathcal{P}} K_{n-\dim(\sigma)-1}(\mathbb{C}) \xrightarrow{\cong} K_n(C_m^*(W)) \cong K_n(C_r^*(W)),$$

$$\bigoplus_{\sigma \in \mathcal{P}} KO_{n-\dim(\sigma)-1}(\mathbb{R}) \xrightarrow{\cong} KO_n(C_m^*(W; \mathbb{R})) \cong KO_n(C_r^*(W; \mathbb{R})).$$

In particular we get an isomorphism of abelian groups

$$K_n(C_m^*(W)) \cong K_n(C_r^*(W)) \cong \mathbb{Z}^{t_n},$$

if we put

$$t_n = \sum_{\substack{k \in \{-1, 0, 1, 2, \dots, \dim(\Sigma)\} \\ (n-k) \text{ odd}}} r_k.$$

7. Right-angled Coxeter groups

In this section we want to compute the group homology, the algebraic K- and L-theory, and the topological K-theory of a right-angled Coxeter group W . Recall that a *right-angled Coxeter group* is a graph product $W = W(X, \mathcal{W})$ for which each of the groups W_v is cyclic of order two. Right-angled Coxeter groups satisfy the Baum–Connes Conjecture and the Baum–Connes Conjecture for the maximal group C^* -algebra, which follows from [13] and [19]. Both the K-theoretic and the L-theoretic Farrell–Jones Conjecture are satisfied for right-angled Coxeter groups, see [2] and [40].

In the sequel we denote by r_k the number of k -simplices in $\mathcal{P} = \mathcal{P}(\Sigma)$ and put $r = |\mathcal{P}| = \sum_{k=-1}^{\dim(\Sigma)} r_k$. Recall that the empty simplex is allowed in \mathcal{P} and has dimension -1 .

During this section we denote by C_2 the cyclic group of order two. Fix an integer $k \geq 1$. We will identify $C_2^k = C_2^{k-1} \times C_2$ and put $C_2^0 = \{1\}$. For $i = 1, 2, \dots, k$, let $(C_2^k)_i$ be the subgroup of C_2^k consisting of those elements (a_1, a_2, \dots, a_k) satisfying $a_i = 0$ and denote by $j_i^k: (C_2^k)_i \rightarrow C_2^k$ the inclusion.

7.1. Group homology. Define for $n \geq 0$ and $k \geq 0$ an integer

$$\rho_{n,k} := \sum_{j=k}^{n-1} (-1)^{n-1-j} \cdot \binom{j}{k},$$

where here and in the sequel we use the convention $\sum_{j=a}^b c_j = 0$ for $a > b$.

Theorem 7.1. *We have for $n \geq 1$*

$$H_n(W; \mathbb{Z}) \cong \bigoplus_{\substack{\sigma \in \mathcal{P} \\ \sigma \neq \emptyset}} \bigoplus_{j=1}^{\rho_{n, \dim(\sigma)}} C_2.$$

Its proof needs some preparation. Firstly, the numbers $\rho_{n,k}$ satisfy the following:

$$\rho_{n,0} = \begin{cases} 1 & \text{for } n \geq 1, n \text{ odd,} \\ 0 & \text{for } n \geq 0, n \text{ even,} \end{cases} \tag{7.2}$$

$$\rho_{n,k} = \rho_{n-1,k-1} + \rho_{n-1,k} \quad \text{for } k, n \geq 1, \tag{7.3}$$

$$\rho_{n,k} = \sum_{i=1}^{n-1} \rho_{i,k-1} \quad \text{for } n \geq 0 \text{ and } k \geq 1. \tag{7.4}$$

Equation (7.2) follows directly from the definition and equation (7.3) follows from an easy calculation. Then equation (7.4) follows by induction from equation (7.3).

Lemma 7.5. *We have for $n \geq 1$ and $k \geq 0$*

$$H_n(C_2^k; \mathbb{Z}) \cong \bigoplus_{i=1}^{t_{n,k}} C_2$$

with $t_{n,k} = \sum_{j=1}^k \binom{k}{j} \cdot \rho_{n,j-1}$.

Proof. The assertion is obviously true for $k = 0$. The induction step from $k - 1$ to $k \geq 1$ is done as follows. Recall that $H_n(C_2; \mathbb{Z})$ is \mathbb{Z} if $n = 0$, C_2 if $n \geq 1$ and n is odd, and $\{0\}$ otherwise. The Künneth formula gives the following short exact sequence of \mathbb{Z} -modules, which is natural in C_2^{k-1}

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i+j=n} H_i(C_2^{k-1}; \mathbb{Z}) \otimes_{\mathbb{Z}} H_j(C_2; \mathbb{Z}) &\longrightarrow H_n(C_2^{k-1} \times C_2; \mathbb{Z}) \\ &\longrightarrow \bigoplus_{i+j=n-1} \text{Tor}_1^{\mathbb{Z}}(H_i(C_2^{k-1}; \mathbb{Z}), H_j(C_2; \mathbb{Z})) \longrightarrow 0. \end{aligned}$$

It splits but the splitting is not natural in C_2^{k-1} . Computing the Tor- and tensor-products and rearranging the summands we obtain an isomorphism of \mathbb{Z} -modules

$$H_n(C_2^k; \mathbb{Z}) \cong H_n(C_2; \mathbb{Z}) \oplus \bigoplus_{i=1}^n H_i(C_2^{k-1}; \mathbb{Z}).$$

Using the induction hypothesis we calculate

$$\begin{aligned}
 t_{n,k} &= t_{n,1} + \sum_{i=1}^n t_{i,k-1} \\
 &\stackrel{(7.2)}{=} \rho_{n,0} + \sum_{i=1}^n \sum_{j=1}^{k-1} \binom{k-1}{j} \cdot \rho_{i,j-1} \\
 &= \rho_{n,0} + \sum_{j=1}^{k-1} \binom{k-1}{j} \cdot \sum_{i=1}^{n-1} \rho_{i,j-1} + \sum_{j=1}^{k-1} \binom{k-1}{j} \cdot \rho_{n,j-1} \\
 &\stackrel{(7.4)}{=} \rho_{n,0} + \sum_{j=1}^{k-1} \binom{k-1}{j} \cdot \rho_{n,j} + \sum_{j=1}^{k-1} \binom{k-1}{j} \cdot \rho_{n,j-1} \\
 &= \rho_{n,0} + \sum_{j=2}^k \binom{k-1}{j-1} \cdot \rho_{n,j-1} + \sum_{j=1}^{k-1} \binom{k-1}{j} \cdot \rho_{n,j-1} \\
 &= \sum_{j=1}^k \binom{k-1}{j-1} \cdot \rho_{n,j-1} + \sum_{j=1}^k \binom{k-1}{j} \cdot \rho_{n,j-1} \\
 &= \sum_{j=1}^k \left(\binom{k-1}{j-1} + \binom{k-1}{j} \right) \cdot \rho_{n,j-1} \\
 &= \sum_{j=1}^k \binom{k}{j} \cdot \rho_{n,j-1}.
 \end{aligned}$$

This finishes the proof of Lemma 7.5. □

For $n \geq 1$ and $k \geq 0$ define

$$S_k H_n := \text{cok} \left(\bigoplus_{i=1}^k H_n(j_i^k; \mathbb{Z}) : \bigoplus_{i=1}^k H_n((C_2^k)_i; \mathbb{Z}) \longrightarrow H_n(C_2^k; \mathbb{Z}) \right).$$

Note that in the notation of Subsection 1.2 the group $S_k H_n$ coincides with $S_\sigma(H_n(-; \mathbb{Z}) \circ W_* \circ I)$ for σ a $(k-1)$ -simplex. Let the integer $s_{n,k} \geq 0$ be defined by

$$S_k H_n \cong \bigoplus_{j=1}^{s_{n,k}} C_2.$$

Lemma 7.6. *For $n \geq 1$ and $k \geq 1$ we have*

$$s_{n,k} = \rho_{n,k-1}$$

and $s_{n,0} = 0$.

Proof. Since $H_n(\{1\}; \mathbb{Z}) = \{0\}$ for $n \geq 1$, we have $s_{n,0} = 0$. The induction step from $k - 1$ to $k \geq 1$ is done as follows. Theorem 1.1 yields an isomorphism

$$\bigoplus_{j=0}^k \bigoplus_{i=1}^{\binom{k}{j}} S_j H_n \cong H_n(C_2^k; \mathbb{Z}).$$

Using the induction hypothesis and Lemma 7.5 we conclude

$$\begin{aligned} s_{n,k} &= t_{n,k} - \sum_{j=0}^{k-1} \binom{k}{j} \cdot s_{n,j} \\ &= \sum_{j=1}^k \binom{k}{j} \rho_{n,j-1} - \sum_{j=1}^{k-1} \binom{k}{j} \cdot \rho_{n,j-1} \\ &= \rho_{n,k-1}. \end{aligned}$$

This finishes the proof of Lemma 7.6. □

Now Theorem 7.1 follows from Theorem 1.1 applied to the equivariant homology theory given by Borel homology and singular homology with \mathbb{Z} -coefficients, see Example 2.3, and from Lemma 7.6. Here we use the fact that for any group G the space $EG \times_G \underline{E}G$ is a model for BG .

Remark 7.7. If we replace in this subsection $C_2 = \mathbb{Z}/2$ everywhere by \mathbb{Z}/p^l for some prime number p and some natural number l , then Theorem 7.1 remains true. This follows from two facts. Since \mathbb{Z}/p^l is a local ring, we conclude from [29, Lemma 1.2 on p. 5] that for every natural number a , every summand of the abelian group $\bigoplus_{i=1}^a \mathbb{Z}/p^l$ is isomorphic to $\bigoplus_{j=1}^b \mathbb{Z}/p^l$ for some natural number b . The group homology $H_n(\mathbb{Z}/p^l; \mathbb{Z})$ is isomorphic to \mathbb{Z}/p^l if n is odd and vanishes for even n with $n \geq 2$.

7.2. Negative K -groups for $R = \mathbb{Z}$

Theorem 7.8. *We have $K_n(\mathbb{Z}W) = \{0\}$ for $n \leq -1$.*

Proof. Since right-angled Coxeter groups satisfy the Farrell–Jones Conjecture, we get $K_n(\mathbb{Z}W) = 0$ for $n \leq -2$ and an isomorphism

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FJN}}(W)} K_{-1}(\mathbb{Z}H) \longrightarrow K_{-1}(\mathbb{Z}W)$$

from [27, p. 749].

Since any finite subgroup of W is isomorphic to $(\mathbb{Z}/2)^k$ for some natural number k , and $K_{-1}(\mathbb{Z}A) = 0$ holds for a finite abelian group whose order is a prime power, see [4, Theorem 10.6 on p. 695] or [8], the claim follows. □

7.3. Projective class group for $R = \mathbb{Z}$

Theorem 7.9. (i) *There is an isomorphism*

$$K_0(\mathbb{Z}) \oplus \bigoplus_{\sigma \in \mathcal{P}} \mathbb{Z}/(2^{\dim \sigma - 1}) \xrightarrow{\cong} H_0^W(\underline{EW}; \mathbf{K}_{\mathbb{Z}}).$$

(ii) *The map*

$$H_0^W(\underline{EW}; \mathbf{K}_{\mathbb{Z}}) \longrightarrow H_0^W(\underline{\underline{EW}}; \mathbf{K}_{\mathbb{Z}})$$

is an isomorphism after inverting 2.

(iii) *The canonical map*

$$H_0^W(\underline{\underline{EW}}; \mathbf{K}_{\mathbb{Z}}) \longrightarrow K_0(\mathbb{Z}W)$$

is an isomorphism.

(iv) *We have $\tilde{K}_0(\mathbb{Z}W) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = \{0\}$.*

Proof. (i) We have for every group G the obvious splitting

$$K_0(\mathbb{Z}G) \cong K_0(\mathbb{Z}) \oplus \tilde{K}_0(\mathbb{Z}G).$$

By [39, Theorem 12.9], $\tilde{K}_0(\mathbb{Z}[C_2^k]) \cong \bigoplus_{i=3}^k \binom{k}{i} \mathbb{Z}/(2^{i-2})$. Using naturality of this isomorphism, we conclude that the inclusions $(C_2^k)_i \subset C_2^k$ collectively hit everything except the top summand $\mathbb{Z}/(2^{k-2})$. This implies that for $H_*^?(-; \mathbf{K}_{\mathbb{Z}})$ the groups S_{σ} in Theorem 1.1 are given by $\mathbb{Z}/(2^{\dim \sigma - 1})$. Now the assertion follows from Theorem 1.1 applied to the equivariant homology theory $H_*^?(-; \mathbf{K}_{\mathbb{Z}})$.

(ii) This follows from Theorem 5.3 (iv).

(iii) This follows from the fact that a right-angled Coxeter group satisfies the Farrell–Jones Conjecture.

(iv) This follows from assertions (i)–(iii). □

7.4. Whitehead group

Theorem 7.10. (i) *The canonical map*

$$H_1^W(EW; \mathbf{K}_{\mathbb{Z}}) \longrightarrow H_1^W(\underline{EW}; \mathbf{K}_{\mathbb{Z}})$$

is an isomorphism and we have an isomorphism

$$H_1^W(EW; \mathbf{K}_{\mathbb{Z}}) \cong H_1(W; \mathbb{Z}) \oplus K_1(\mathbb{Z}).$$

(ii) *The map*

$$H_1^W(\underline{\underline{EW}}; \mathbf{K}_{\mathbb{Z}}) \longrightarrow H_1^W(\underline{\underline{EW}}; \mathbf{K}_{\mathbb{Z}})$$

is an isomorphism after inverting 2.

(iii) *The canonical map*

$$H_1^W(\underline{E}W; \mathbf{K}_{\mathbb{Z}}) \longrightarrow K_1(\mathbb{Z}W)$$

is an isomorphism.

(iv) *We have $K_1(\mathbb{Z}W) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = \{0\}$.*

Proof. (i) Notice that we have isomorphisms

$$\begin{aligned} H_1^W(EW; \mathbf{K}_{\mathbb{Z}}) &\cong H_1(BW; \mathbf{K}(\mathbb{Z})) \\ &\cong H_1(BW, K_0(\mathbb{Z})) \oplus H_0(BW, K_1(\mathbb{Z})) \\ &\cong H_1(W; \mathbb{Z}) \oplus K_1(\mathbb{Z}) \\ &\cong W/[W, W] \oplus \{\pm 1\}. \end{aligned}$$

Hence it remains to show that the canonical map $H_1^W(EW; \mathbf{K}_{\mathbb{Z}}) \rightarrow H_1^W(\underline{E}W; \mathbf{K}_{\mathbb{Z}})$ is bijective. The Whitehead group $\text{Wh}(C_2^k)$ vanishes for all natural numbers k by [30, Theorem 14.1 on p. 328]. Hence the obvious map $H_1(C_2^k; \mathbb{Z}) \times K_1(\mathbb{Z}) \rightarrow K_1(\mathbb{Z}[C_2^k])$ is an isomorphism. Now apply Theorem 1.1 to the equivariant homology theories given by the Borel construction, see Example 2.3, and to $H_*^?(-; \mathbf{K}_{\mathbb{Z}})$.

(ii) This follows from Theorem 5.3 (iv).

(iii) This follows from the fact that a right-angled Coxeter group satisfies the Farrell–Jones Conjecture.

(iv) This follows from Theorem 7.1 and assertions (i)–(iii). \square

7.5. Rationalized K -groups. Let R be a ring. For any non-empty simplex σ of Σ we have the diagonal embedding

$$\Delta_{\sigma}: C_2 \rightarrow W(\sigma) = \prod_{v \in V \cap \sigma} W_v = \prod_{v \in V \cap \sigma} C_2.$$

Let $j_{\sigma}: W(\sigma) \rightarrow W$ be the inclusion. Then $j_{\sigma} \circ \Delta_{\sigma}: C_2 \rightarrow W$ induces a homomorphism $(j_{\sigma} \circ \Delta_{\sigma})_*: K_n(R[C_2]) \rightarrow K_n(RW)$. Denote by

$$i_{\sigma, n}: \ker(K_n(R[C_2]) \rightarrow K_n(R)) \longrightarrow K_n(RW)$$

its composite with the inclusion $\ker(K_n(R[C_2]) \rightarrow K_n(R)) \rightarrow K_n(R[C_2])$, where $K_n(R[C_2]) \rightarrow K_n(R)$ is the homomorphism induced by the projection $C_2 \rightarrow \{1\}$. Let $i_{\emptyset, n}: K_n(R) \rightarrow K_n(RW)$ be the map induced by the inclusion $\{1\} \rightarrow W$.

Theorem 7.11. *Let R be a regular ring.*

(i) *The map*

$$i_{\emptyset,n} \oplus \bigoplus_{\substack{\sigma \in \mathcal{P} \\ \sigma \neq \emptyset}} i_{\sigma,n}: K_n(R) \oplus \bigoplus_{\substack{\sigma \in \mathcal{P} \\ \sigma \neq \emptyset}} \ker(K_n(R[C_2])) \longrightarrow K_n(R) \longrightarrow K_n(RW)$$

is rationally an isomorphism for all $n \in \mathbb{Z}$.

(ii) *For $R = \mathbb{Z}$,*

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{Z}W) \cong \begin{cases} \mathbb{Q}^r & \text{if } n = 4k + 1 \text{ with } k \geq 1, \\ \mathbb{Q} & \text{if } n = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. (i) Notice that any non-trivial finite cyclic subgroup C of C_2^k is isomorphic to C_2 and that the obvious composite

$$\begin{aligned} \ker(\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R)) &\longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \\ &\longrightarrow \text{cok}(\mathbb{Q} \otimes_{\mathbb{Z}} K_n(R) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)) \end{aligned}$$

is an isomorphism. The isomorphism appearing in [1, (2.11)], which exists for $\Lambda = \mathbb{Q}$ and the equivariant homology theory $H_*^?(-; \mathbf{K}_R)$ because of [1, Lemma 4.1 (e)], which in turn follows from [36, Corollary 4.2], boils down to an isomorphism

$$\begin{aligned} j_{\{1\}} \oplus \bigoplus_{\substack{C \subseteq C_2^k \\ C \cong C_2}} j_C: \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R) \oplus \bigoplus_{\substack{C \subseteq C_2^k \\ C \cong C_2}} \ker(\varepsilon_C^{\mathbb{Q}}: \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R)) \\ \cong \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R[C_2^k]), \end{aligned}$$

where $\varepsilon_C^{\mathbb{Q}}: \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R)$ is induced by the projection $C \rightarrow \{1\}$, the map $j_{\{1\}}$ is induced by the inclusion $\{1\} \rightarrow C$, and j_C is the composite of the inclusion $\ker(\varepsilon_C^{\mathbb{Q}}) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC)$ with the map $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RC) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R[C_2^k])$ coming from the inclusion $C \rightarrow C_2^k$. By naturality we get a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^k \left(\mathbb{Q} \otimes_{\mathbb{Z}} K_n(R) \oplus \bigoplus_{\substack{C \subseteq (C_2^k)_i \\ C \cong C_2}} \ker(\varepsilon_C^{\mathbb{Q}}) \right) & \xrightarrow{\cong} & \bigoplus_{i=1}^k \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R[(C_2^k)_i]) \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R) \oplus \bigoplus_{\substack{C \subseteq C_2^k \\ C \cong C_2}} \ker(\varepsilon_C^{\mathbb{Q}}) & \xrightarrow{\cong} & \mathbb{Q} \otimes_{\mathbb{Z}} K_n(R[C_2^k]) \end{array}$$

where the vertical arrows come from the inclusions $(C_2^k)_i \rightarrow C_2^k$. Notice that a cyclic subgroup of C_2^k belongs to $(C_2^k)_i$ for some $i \in \{1, 2, \dots, k\}$ if and only if it is different from the diagonal subgroup $(C_2^k)_\Delta := \{(a, a, \dots, a) \mid a \in C_2\} \subseteq C_2^k$. Hence the composite

$$\begin{aligned} \ker(K_n(R[(C_2^k)_\Delta]) &\longrightarrow K_n(R)) \longrightarrow K_n(R[(C_2^k)_\Delta]) \longrightarrow K_n(R[C_2^k]) \\ &\longrightarrow \operatorname{cok}\left(\bigoplus_{i=1}^k K_n(R[(C_2^k)_i]) \longrightarrow K_n(R[C_2^k])\right) \end{aligned}$$

is rationally bijective.

Now assertion (i) follows from Theorem 1.1, Theorem 5.3 (iv), and the validity of the K -theoretic Farrell–Jones Conjecture for right-angled Coxeter groups.

(ii) Due to Borel [5] we know for $R = \mathbb{Z}$ that

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Q} & \text{if } n = 4k + 1 \text{ with } k \geq 1, \\ \mathbb{Q} & \text{if } n = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

We get from [20, Theorem 2.2] for $C \cong C_2$

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{Z}[C]) \cong \begin{cases} \mathbb{Q}^2 & \text{if } n = 4k + 1 \text{ with } k \geq 1, \\ \mathbb{Q} & \text{if } n = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

Hence we get

$$\ker(\mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{Z}[C]) \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{Z})) \cong \begin{cases} \mathbb{Q} & \text{if } n = 4k + 1 \text{ with } k \geq 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Now assertion (ii) follows from assertion (i). □

7.6. L -groups after inverting 2. The maps appearing in the result below are defined analogously to the maps appearing in Theorem 7.11.

Theorem 7.12. *Let R be a ring with involution.*

(i) *The map*

$$\begin{aligned} i_{\emptyset, n} \oplus \bigoplus_{\substack{\sigma \in \mathcal{P} \\ \sigma \neq \emptyset}} i_{\sigma, n}: L_n^{(-\infty)}(R) \oplus \bigoplus_{\substack{\sigma \in \mathcal{P} \\ \sigma \neq \emptyset}} \ker(L_n^{(-\infty)}(R[C_2])) &\longrightarrow L_n^{(-\infty)}(R) \\ &\longrightarrow L_n^{(-\infty)}(RW) \end{aligned}$$

is an isomorphism after inverting 2 for all $n \in \mathbb{Z}$.

(ii) We have for $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$

$$L_n^{(-\infty)}(RW)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^r & \text{if } n = 4k \text{ for } k \in \mathbb{Z}, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. (i) Note that any non-trivial subgroup of the form $C \times P$ of C_2^k for a cyclic group C and a p -group P for an odd prime number p is isomorphic to C_2 . The isomorphism appearing in [1, (2.11)] which exists for $\Lambda = \mathbb{Z}[1/2]$ and the equivariant homology theory $H_*^{?}(-; \mathbf{L}_R^{(-\infty)})$ because of [14, Theorem 2], boils down to an isomorphism

$$\begin{aligned} j_{\{1\}} \oplus \bigoplus_{\substack{C \subseteq C_2^k \\ C \cong C_2}} j_C : L_n^{(-\infty)}(R)[1/2] \\ \oplus \bigoplus_{\substack{C \subseteq C_2^k \\ C \cong C_2}} \ker(L_n^{(-\infty)}(RC)[1/2]) \longrightarrow L_n^{(-\infty)}(R)[1/2] \\ \xrightarrow{\cong} L_n^{(-\infty)}(R[C_2^k])[1/2], \end{aligned}$$

where the map $j_{\{1\}}$ is induced by the inclusion $\{1\} \rightarrow C$, and j_C is the composite of the inclusion of $\ker(L_n^{(-\infty)}(RC)[1/2]) \rightarrow L_n^{(-\infty)}(R)[1/2]$ into $L_n^{(-\infty)}(RC)[1/2]$ with the map $L_n^{(-\infty)}(RC)[1/2] \rightarrow L_n^{(-\infty)}(R[C_2^k])[1/2]$ coming from the inclusion $C \rightarrow C_2^k$. Now Theorem 7.12 follows completely analogous to the argument appearing in the proof of Theorem 7.11 (i) and the validity of the L-theoretic Farrell–Jones Conjecture for right-angled Coxeter groups.

(ii) This follows from assertion (i) using [33, Proposition 22.34 on p. 254]. \square

7.7. K- and L-groups for R containing 1/2

Theorem 7.13. *For all $n \in \mathbb{Z}$ there are explicit isomorphisms*

- (i) $\bigoplus_{\sigma \in \mathcal{P}} K_n(R) \xrightarrow{\cong} K_n(RW)$ if R is regular and contains $1/2$,
- (ii) $\bigoplus_{\sigma \in \mathcal{P}} L_n^{(-\infty)}(R) \xrightarrow{\cong} L_n^{(-\infty)}(RW)$ if R contains $1/2$.

Its proof needs some preparations. In the sequel we will write C_2^k multiplicatively and we denote by t_i the generator of the i -th factor C_2 viewed as an element in C_2^k for $i = 1, 2, \dots, k$.

Let R be a ring in which 2 is invertible. We get a decomposition of rings, natural in R ,

$$R[C_2] \xrightarrow{\cong} R \times R, \quad a + bt \mapsto (a + b, a - b).$$

Its inverse sends (c, d) to $\frac{1}{2} \cdot ((c + d) + (c - d) \cdot t)$. Since algebraic K -theory is compatible with products, we obtain an isomorphism, natural in R ,

$$S_n(R): K_n(R[C_2]) \xrightarrow{\cong} K_n(R) \times K_n(R).$$

One can iterate this using the obvious ring isomorphism $(R[C_2^{k-1}])[C_2] \cong R[C_2^k]$ and thus obtains an isomorphism

$$S_n^k(R): K_n(R[C_2^k]) \xrightarrow{\cong} \prod_{\epsilon \in \text{hom}_{\mathbb{Z}}(C_2^k, \{\pm 1\})} K_n(R), \quad (7.14)$$

which comes from the isomorphism of rings

$$R[C_2^k] \xrightarrow{\cong} \prod_{\epsilon \in \text{hom}_{\mathbb{Z}}(C_2^k, \{\pm 1\})} R, \quad \sum_{g \in C_2^k} \lambda_g \cdot g \mapsto \left(\sum_{g \in C_2^k} \lambda_g \cdot \epsilon(g) \right)_{\epsilon}.$$

Its inverse is given by

$$(\mu_{\epsilon})_{\epsilon} \mapsto 2^{-k} \cdot \sum_{g \in C_2^k} \left(\sum_{\epsilon} \epsilon(g) \cdot \mu_{\epsilon} \right) \cdot g.$$

Lemma 7.15. (i) *Suppose that 2 is invertible in R . Then there is an isomorphism*

$$d_n^k: \text{cok} \left(\bigoplus_{i=1}^k K_n(R[j_i^k]): \bigoplus_{i=1}^k K_n(R[(C_2^k)_i]) \longrightarrow K_n(R[C_2^k]) \right) \xrightarrow{\cong} K_n(R).$$

Its inverse is the composite of the homomorphism

$$\beta: K_n(R) \longrightarrow K_n(R[C_2^k])$$

coming from the ring homomorphism $R \rightarrow R[C_2^k]$ sending λ to $2^{-k} \cdot \lambda \cdot \prod_{i=1}^k (1 - t_i)$ with the projection $K_n(R[C_2^k]) \rightarrow \text{cok} \left(\bigoplus_{i=1}^k K_n(R[j_i^k]) \right)$. The homomorphism β agrees with the restriction of the inverse of the isomorphism $S_n^k(R)$ of (7.14) to the factor $K_n(R)$ which belongs to ϵ given by $\epsilon(t_i) = -1$ for $i = 1, 2, \dots, k$.

(ii) *The same assertion holds if we replace algebraic K -theory by algebraic L -theory with the decoration $\langle -\infty \rangle$.*

(iii) *The same assertion is true if we take R to be \mathbb{R} or \mathbb{C} and we replace algebraic K -theory by topological K -theory.*

Proof. We give the proof for algebraic K -theory only, the one for the other cases is completely analogous.

We use induction over k . If $k = 0$, the map d_n^0 comes from the identification

$$K_n(R[C_2^0]) = K_n(R[\{1\}]) = K_n(R).$$

The induction step from $(k - 1)$ to $k \geq 1$ is done as follows. We have the following commutative diagram of \mathbb{Z} -modules

$$\begin{array}{ccc}
 \bigoplus_{i=1}^k K_n(R[(C_2^k)_i]) & \xrightarrow{\bigoplus_{i=1}^k K_n(j_i^k)} & K_n(R[C_2^k]) \\
 \text{id} \downarrow \cong & & \downarrow \cong \text{id} \\
 K_n(R[(C_2^k)_k]) \oplus \bigoplus_{i=1}^{k-1} K_n(R[(C_2^k)_i]) & & \\
 \text{id} \oplus \bigoplus_{i=1}^{k-1} \text{id} \downarrow \cong & & \\
 K_n(R[C_2^{k-1}]) \oplus \bigoplus_{i=1}^{k-1} K_n(R[(C_2^{k-1})_i][C_2]) & & K_n(R[C_2^{k-1}][C_2]) \\
 \cong \downarrow \text{id} \oplus \bigoplus_{i=1}^{k-1} S_n(R[(C_2^{k-1})_i]) & & \\
 K_n(R[C_2^{k-1}]) & & \\
 \oplus & & \\
 \bigoplus_{i=1}^{k-1} (K_n(R[(C_2^{k-1})_i]) \oplus K_n(R[(C_2^{k-1})_i])) & \xrightarrow{S_n(R[C_2^{k-1}]) \cong} & \\
 f \downarrow \cong & & \\
 K_n(R[C_2^{k-1}]) & & K_n(R[C_2^{k-1}]) \\
 \oplus & & \oplus \\
 \bigoplus_{i=1}^{k-1} K_n(R[(C_2^{k-1})_i]) & \xrightarrow{\begin{pmatrix} \text{id} & u & 0 \\ \text{id} & 0 & u \end{pmatrix}} & K_n(R[C_2^{k-1}]) \\
 \oplus & & \\
 \bigoplus_{i=1}^{k-1} K_n(R[(C_2^{k-1})_i]) & &
 \end{array}$$

where f is the obvious isomorphism and $u := \bigoplus_{i=1}^{k-1} K_n(j_i^{k-1})$. The diagram above induces an isomorphism

$$\text{cok} \left(\bigoplus_{i=1}^k K_n(j_i^k) \right) \xrightarrow{\cong} \text{cok} \begin{pmatrix} \text{id} & u & 0 \\ \text{id} & 0 & u \end{pmatrix}.$$

If $k = 1$, then $\begin{pmatrix} \text{id} & u & 0 \\ \text{id} & 0 & u \end{pmatrix}$ reduces to $K_n(R) \rightarrow K_n(R) \oplus K_n(R)$, $x \mapsto (x, x)$ and the desired isomorphism d_n^1 is induced by $K_n(R) \oplus K_n(R) \rightarrow K_n(R)$, $(x, y) \mapsto x - y$. Suppose $k \geq 2$. Since the first and third map in the composite

$$\begin{pmatrix} \text{id} & 0 \\ -\text{id} & \text{id} \end{pmatrix} \cdot \begin{pmatrix} \text{id} & u & 0 \\ \text{id} & 0 & u \end{pmatrix} \cdot \begin{pmatrix} \text{id} & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & \text{id} & \text{id} \end{pmatrix}$$

are isomorphisms and the composite is given by $\begin{pmatrix} \text{id} & u & 0 \\ 0 & 0 & u \end{pmatrix}$, we obtain an isomorphism

$$\text{cok} \begin{pmatrix} \text{id} & u & 0 \\ \text{id} & 0 & u \end{pmatrix} \xrightarrow{\cong} \text{cok} \begin{pmatrix} \text{id} & u & 0 \\ 0 & 0 & u \end{pmatrix}.$$

Since $(\text{id } u): K_n(R[C_2^{k-1}]) \oplus \bigoplus_{i=1}^{k-1} K_n(R[(C_2^{k-1})_i]) \rightarrow K_n(R[C_2^k])$ is surjective, we obtain an isomorphism

$$\text{cok} \begin{pmatrix} \text{id} & u & 0 \\ 0 & 0 & u \end{pmatrix} \xrightarrow{\cong} \text{cok}(u).$$

Its inverse is induced by the composite

$$K_n(R[C_2^k]) \xrightarrow{\begin{pmatrix} 0 \\ \text{id} \end{pmatrix}} K_n(R[C_2^{k-1}]) \oplus K_n(R[C_2^{k-1}]) \xrightarrow{S_n(R[C_2^{k-1}])^{-1}} K_n(R[C_2^k]),$$

which is the homomorphism $K_n(R[C_2^{k-1}]) \rightarrow K_n(R[C_2^k])$ induced by the ring homomorphism $R[C_2^{k-1}] \rightarrow R[C_2^k]$ sending x to $\frac{1}{2} \cdot x \cdot (1 - t_k)$. Since the induction hypothesis applies to u , Lemma 7.15 follows. \square

Now Theorem 7.13 follows from Theorem 1.1, Theorem 5.3 (iii, vi), and Lemma 7.15, and the validity of the Farrell–Jones Conjecture for right-angled Coxeter groups.

7.8. Topological K -theory

Theorem 7.16. *There are for every $n \in \mathbb{Z}$ isomorphisms*

$$\bigoplus_{\sigma \in \mathcal{P}} K_n(\mathbb{C}) \xrightarrow{\cong} K_n(C_m^*(W)) \cong K_n(C_r^*(W)),$$

$$\bigoplus_{\sigma \in \mathcal{P}} KO_n(\mathbb{R}) \xrightarrow{\cong} KO_n(C_m^*(W; \mathbb{R})) \cong KO_n(C_r^*(W; \mathbb{R})).$$

In particular there are isomorphisms of abelian groups

$$K_n(C_m^*(W)) \cong K_n(C_r^*(W)) \cong \begin{cases} \mathbb{Z}^r & \text{if } n \text{ is even,} \\ \{0\} & \text{otherwise,} \end{cases}$$

$$KO_n(C_m^*(W; \mathbb{R})) \cong KO_n(C_r^*(W; \mathbb{R})) \cong \begin{cases} \mathbb{Z}^r & \text{if } n \equiv 0 \pmod{4}, \\ (\mathbb{Z}/2)^r & \text{if } n \equiv 1, 2 \pmod{8}, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. This follows from Theorem 1.1, Lemma 7.15, and the validity of the Baum–Connes Conjecture for right-angled Coxeter groups. \square

The result for complex coefficients was already obtained by Sánchez-García using the Davis complex as a model for $\underline{E}W$ in [34]. In special cases, the topological K -theory of $\underline{E}W$ was computed in Fuentes Rumí’s masters’s thesis [16].

Remark 7.17. In Subsection 1.3 we have given an explicit description of the isomorphism

$$\bigoplus_{\sigma \in \mathcal{P}} K_n(\mathbb{C}) \xrightarrow{\cong} K_n(C_r^*(G))$$

above which actually carries over to many of the other situations. In order to prove the description, one has to go through the construction of the isomorphism and to make in the application of Lemma 4.5 the right choice for \hat{s}_τ . Namely, one takes for \hat{s}_τ the composite of the homomorphism β with the isomorphism d_n^k appearing in assertion (i) of Lemma 7.15.

8. An example

In this section, we want to apply the computations from the previous sections to a concrete example. For this we picked the group $W := \mathbb{Z}/2 \times \mathbb{Z}/2 \times D_\infty \times \mathbb{Z}$. Note that it is a graph product with vertex groups $\mathbb{Z}/2$ and \mathbb{Z} . In [10, Example 3.28] Davis, Khan and Ranicki showed that the Whitehead group of W is infinitely generated due to Nil elements.

It will be useful to consider W as $W_0 \times \mathbb{Z}$, where $W_0 = \mathbb{Z}/2 \times \mathbb{Z}/2 \times D_\infty$ is the right-angled Coxeter group associated to the simplicial graph X with vertex set $V = \{1, 2, 3, 4\}$ whose edges are $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}$, and $\{1, 3\}$. Then the flag complex Σ associated to X is the suspension of a one-simplex so that in the notation of Section 7 we have $r_{-1} = 1, r_0 = 4, r_1 = 5, r_2 = 2$, and $r = 12$.

We conclude from Theorem 7.1 for $n \geq 2$

$$H_n(W; \mathbb{Z}) \cong H_n(W_0; \mathbb{Z}) \oplus H_{n-1}(W_0; \mathbb{Z}) \cong \bigoplus_{i=1}^{u_n} C_2,$$

$$H_1(W; \mathbb{Z}) \cong \mathbb{Z} \oplus \bigoplus_{i=1}^4 C_2,$$

where

$$\begin{aligned} u_n &= \sum_{k=0}^2 r_k \cdot \rho_{n,k} + \sum_{k=0}^2 r_k \cdot \rho_{n-1,k} \\ &= 4 \cdot (\rho_{n,0} + \rho_{n-1,0}) + 5 \cdot (\rho_{n,1} + \rho_{n-1,1}) + 2 \cdot (\rho_{n,2} + \rho_{n-1,2}) \\ &= \begin{cases} 4 + 5(k + k - 1) + 2(k(k - 1) + (k - 1)^2) & \text{if } n = 2k \text{ for } k \geq 1, \\ 4 + 5(k + k) + 2(k^2 + k(k - 1)) & \text{if } n = 2k + 1 \text{ for } k \geq 1, \end{cases} \\ &= \begin{cases} 4k^2 + 4k + 1 & \text{if } n = 2k \text{ for } k \geq 1, \\ 4k^2 + 8k + 4 & \text{if } n = 2k + 1 \text{ for } k \geq 1. \end{cases} \end{aligned}$$

Note that the group W satisfies the Baum–Connes Conjecture and the Farrell–Jones Conjecture since it is a graph product of abelian groups. Hence for every regular ring R the assembly map

$$H_n^W(\underline{E}W; \mathbf{K}_R) \longrightarrow K_n(R[W])$$

is bijective after inverting 2 by Theorem 5.3 (iv).

The proof of Theorem 7.8 applies verbatim to the group W so that we obtain

$$K_n(\mathbb{Z}W) = \{0\} \quad \text{for } n \leq -1.$$

For any equivariant homology theory we have

$$\mathcal{H}_n^W(\underline{E}W) \cong \mathcal{H}_n^{W_0}(\underline{E}W_0 \times S^1) \cong \mathcal{H}_n^{W_0}(\underline{E}W_0) \oplus \mathcal{H}_{n-1}^{W_0}(\underline{E}W_0). \quad (8.1)$$

Using (8.1), we have

$$\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} \tilde{K}_0(\mathbb{Z}W) = \{0\}$$

by Theorem 7.9 and Theorem 7.8.

By (8.1), Theorem 7.9 and Theorem 7.10, we have

$$\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} K_1(\mathbb{Z}W) \cong \mathbb{Z}[1/2] \quad \text{and} \quad \mathbb{Z}[1/2] \otimes_{\mathbb{Z}} \text{Wh}(W) = \{0\}.$$

Note that without inverting two, the Whitehead group $\text{Wh}(W)$ contains a non-trivial Nil term by [10, Example 3.28] as mentioned above.

By (8.1) and Theorem 7.11

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_n(\mathbb{Z}W) \cong \begin{cases} \mathbb{Q}^{12} & \text{if } n = 4k + 1 \text{ or } n = 4k + 2 \text{ for } k \geq 1, \\ \mathbb{Q} & \text{if } n = 0, 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

The Shaneson splitting yields for all $n \in \mathbb{Z}$ an isomorphism

$$L_n^{(-\infty)}(\mathbb{Z}W) \cong L_n^{(-\infty)}(\mathbb{Z}W_0) \oplus L_{n-1}^{(-\infty)}(\mathbb{Z}W_0).$$

Hence by Theorem 7.12 we find

$$\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} L_n^{(-\infty)}(\mathbb{Z}W) \cong \begin{cases} \mathbb{Z}[1/2]^{12} & \text{if } n = 4k \text{ or } n = 4k + 1 \text{ for } k \in \mathbb{Z}, \\ \{0\} & \text{otherwise.} \end{cases}$$

By (8.1), we get from Theorem 7.16 for all $n \in \mathbb{Z}$

$$K_n(C_r^*(W)) \cong \mathbb{Z}^{12}.$$

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Daniel Kasprowski, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

e-mail: kasprowski@uni-bonn.de

Kevin Li, School of Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom

e-mail: kevin.li@soton.ac.uk

Wolfgang Lück, Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

e-mail: wolfgang.lueck@him.uni-bonn.de