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Confined subgroups of Thompson's group F and its embeddings into wobbling groups

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Abstract. We obtain a characterisation of confined subgroups of Thompson's group F. As a result, we deduce that the orbital graph of a point under an action of F has uniformly subexponential growth if and only if this point is fixed by the commutator subgroup. This allows us to prove non-embeddability of F into wobbling groups of graphs with uniformly subexponential growth.

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1. Introduction

Discovered in 1965, Richard Thompson's group F became an important object of study for geometric group theory and measured group theory. This group (see §2.1 for the definition of F) has an unusual combination of properties – it is a finitely presented, torsion-free group, which has a free subsemigroup but no free subgroups, and its commutator subgroup is simple. Furthermore, amenability of F remains a major open question. Recall that an action of a group G on a set X is amenable if X carries a G-invariant finitely additive probability measure, and a group is called *amenable* if its action on itself by left multiplication is amenable. We refer the reader to [1] and [2] for more details about Thompson's group F.

Even though the Cayley graph of F with its standard finite generating set is hard to visualize, a family of its orbital Schreier graphs associated to the action on the unit segment was explicitly described by Dmytro Savchuk in [16]. This description was later used in the study of Poisson–Furstenberg boundary of F for various classes of symmetric generating measures, see [10], [4], [11], [15], and [17].

The main result of this article is also related to the study of Schreier graphs of F. In Theorem 3.1 we give a characterisation of the confined subgroups of F in terms of the action of F on the unit interval. It is equivalent to a characterisation of the Schreier graphs of F which do not contain arbitrarily large balls isomorphic to a ball in the Cayley graph of F. Notice that the complicated structure of the standard Cayley graph of F implies that these are the only Schreier graphs of F that one could hope to describe explicitly. As a corollary, we show that F has no faithful actions with uniformly subexponential growth of orbital Schreier graphs and that F does not embed into the wobbling group of a graph with uniformly subexponential growth. Recall that the wobbling group $W(X)$ of a metric space X is defined as the group of all bijections from X to itself which keep distances between points of X and their images bounded by some constant (see §2.4). If G is a finitely generated group acting on a set X, and Γ is the orbital Schreier graph of a point $p \in X$ with standard graph distance, then G also acts on Γ by elements of its wobbling group. The statement of our main result is similar to the classification of the confined subgroups for the full groups of minimal étale groupoids of germs obtained by Nicolas Matte Bon in [14]. We also use generalizations of the commutator lemma obtained by Adrien Le Boudec and Nicolas Matte Bon in [13] and [14].

Our interest in embeddings of F into wobbling groups of "small" graphs is inspired by the proof of amenability of the full topological group of a Cantor minimal system in [7]. One of the steps in this proof is to embed a finitely generated subgroup of the full topological group into $W(\mathbb{Z})$ and later use the fact that action of $W(\mathbb{Z})$ on $\mathbb{Z}$ is extensively amenable (notion first defined in [8]), see [5] or [7]. Furthermore, amenability of interval exchange transformation group, or shortly the IET, is equivalent to a question about amenability of subgroups of the wobbling group $W(\mathbb{Z}^d)$ for $d > 3$, we refer the reader to [6] for more details. In this context a "small" graph could mean a recurrent graph (the action of the wobbling group of such a graph is always extensively amenable) or a graph of subexponential growth.

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2. Preliminaries

2.1. Properties of Thompson's group F. This subsection contains a brief overview of properties of Thompson's group F which are used in this article. Proofs of majority of these facts could be found in [1] or [2].

Definition 2.1. *Thompson's group F* is the group formed by all orientation-preserving piecewise linear homeomorphisms of the segment $[0, 1]$ which have breakpoints in dyadic rationals and slopes equal to powers of 2. Here a breakpoint of a piecewise linear function is a point where it is not differentiable, and a piecewise linear function on a segment must have a finite number of breakpoints.

In this article we consider the right action of F on the unit segment, thus for any two homeomorphisms $h, g \in F$ their composition is defined as $hg(x) = g(h(x))$. It is well known that for any $n \geq 1$ the action of F on ordered $n-$tuples of dyadic rationals is transitive. The *support* of $g \in F$ is the closure of the set of points of the unit segment on which g acts nontrivially. Thompson's group F has the finite presentation

$$\langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0] = [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] = \mathrm{id} \rangle,$$

where the generators are defined as follows:

$$x_0(t) = \begin{cases} t/2, & 0 \leq t \leq 1/2, \\ t - 1/4, & 1/2 \leq t \leq 3/4, \\ 2t - 1, & 3/4 \leq t \leq 1, \end{cases}$$

and

$$x_1(t) = \begin{cases} t, & 0 \leq t \leq 1/2, \\ t/2 + 1/4, & 1/2 \leq t \leq 3/4, \\ t - 1/8, & 3/4 \leq t \leq 7/8, \\ 2t - 1, & 7/8 \leq t \leq 1. \end{cases}$$

Its commutator subgroup F' is a simple group which coincides with the subgroup of all elements with support contained in the unit interval $(0, 1)$. The quotient F/F' is isomorphic to $\mathbb{Z}^2$, and a subgroup $H < F$ is normal if and only if it contains the commutator subgroup of F.

For any segment $[a, b] \subseteq [0, 1]$ (or an interval (a, b)) denote $F[a, b]$ (respectively $F(a, b)$) the subgroup of all elements of F with support in $[a, b]$ (respectively (a, b)). We will need the following properties of these subgroups:

(1) if $a < b$ are dyadic rationals, then $F[a, b]$ is isomorphic to F. Furthermore, its commutator subgroup is exactly the subgroup $F(a, b)$;

(2) if $a < b < c < d$ are dyadic rationals then the group generated by elements of $F[a, c]$ and $F[b, d]$ contains $F[a, d]$.

2.2. Schreier graphs and orbital graphs

Definition 2.2. Assume that G is a group generated by a finite set S and let $H < G$ be its subgroup. The Schreier graph of G modulo H is an oriented labeled graph with the set of vertices equal to the set of right cosets $\{Hg, g \in G\}$ and the set of edges equal to $\{(Hg, Hgs), s \in S\}$. We denote this graph by $\Gamma(G, H)$.

If G acts on a set X and $p \in X$, the Schreier graph of G modulo the stabilizer of p is called the *orbital graph* of p. Its vertex set can be identified with the orbit $\mathcal{O}_G(p)$ of p, and two vertices $v, w \in \mathcal{O}_G(p)$ are connected by an edge with label s if and only if $s(v) = w$.

For a connected graph with bounded degree one can define its uniform growth function as follows:

Definition 2.3. Let $\Gamma = (V, E)$ be a connected graph with bounded degree. For $v \in \Gamma$ denote by $B_\Gamma(v, n)$ the ball of radius n centered at v in Γ. Then the uniform growth function of Γ is defined as

$$\bar{b}(n) = \sup_{v \in V} |B_\Gamma(v, n)|,$$

where $|B_\Gamma(v, n)|$ stands for the cardinality of the set of vertices.

If Γ is a Schreier graph of G, changing a finite generating set of G preserves the equivalence class of the uniform growth function of Γ under the following equivalence relation: for two functions $f, g\colon \mathbb{N} \to \mathbb{N}$, f is said *to grow asymptotically not slower than* g ($g \preceq f$) if there exists a constant $C > 0$ such that $g(n) < f(Cn)$ for all $n \in \mathbb{N}$ and $f \sim g$ if and only if $g \preceq f$ and $f \preceq g$.

In general, this equivalence class is preserved under bi-Lipschitz equivalence and we refer to this class when we talk about uniform growth rate.

One should note that although the uniform growth is always greater or equal to usual growth function of a graph, they might be completely different, and it is easy to construct an example of a graph with exponential uniform growth which has linear growth function.

We also consider Schreier graph $\Gamma(G, H)$ as a rooted graph with the root at H. Space of rooted labeled graphs could be naturally equipped with a distance d defined as follows. Let Γ_1 and Γ_2 be two rooted labeled graphs with roots v_1 and v_2. Then $d(\Gamma_1, \Gamma_2) = \frac{1}{n+1}$, where $n \geq 0$ is the least integer such that $B_{\Gamma_1}(v_1, n)$ and $B_{\Gamma_2}(v_2, n)$ are not isomorphic as rooted labeled graphs.

2.3. Chabauty space. Let G be a countable discrete group. The set of its subgroups $\mathrm{Sub}(G)$ is endowed with a topology (called the *Chabauty topology*) induced from the space 2^G of all subsets of G with usual product topology. The base of this topology is formed by sets

$$U_{A,B} = \{H \in \mathrm{Sub}(G)\colon A \subset H, H \cap B = \emptyset\},$$

where A, B are finite subsets of G. With the Chabauty topology $\mathrm{Sub}(G)$ becomes a compact metrizable space on which G acts by conjugation, and this is an action by homeomorphisms.

Assume that G is finitely generated and fix a finite generating set S. Convergence in the Chabauty topology has a reformulation in terms of Schreier graphs.

Namely, a sequence of subgroups H_n converges to $H < G$ if and only if, for any generating set S, the sequence $\Gamma(G, H_n)$ converges to $\Gamma(G, H)$ in the space of labeled rooted graphs.

Suppose that H and K are subgroups of G, then H is said *confined by* K if the closure of K-orbit of H does not contain the trivial subgroup. Since the base of neighborhoods of the trivial subgroup is formed by sets of form $U_{\{1\},P}$ with P finite, the last condition is equivalent to the existence of a finite set $P = \{g_1, g_2, \dots, g_r\} \subseteq G \setminus 1$, such that for all $k \in K$

$$kHk^{-1} \cap P \neq \emptyset,$$

we call such sets P *confining*. If $K = G$ we say that H is a *confined subgroup of* G.

In terms of Schreier graphs, H is not confined if and only if one can find a sequence of vertices $v_n, n \geq 1$, in the Schreier graph $\Gamma(G, H)$, such that versions of $\Gamma(G, H)$ rooted at v_n converge to the Cayley graph of G in the space of rooted labeled graphs.

Confined subgroups are also related to the study of *uniformly recurrent subgroups* – closed minimal G-invariant subsets of $\mathrm{Sub}(G)$ (it is easy to see that any nontrivial uniformly recurrent subgroup consists of confined subgroups). We refer the reader to [13] for a description of uniformly recurrent subgroups of Thompson's groups and its applications to C^*-simplicity.

2.4. Wobbling groups. The wobbling groups were first studied in [12] and applied to Tarski's circle-squaring problem.

Definition 2.4. Let $\Gamma = (V, E)$ be a locally finite connected graph equipped with standard graph metric d_Γ. The wobbling group $W(\Gamma)$ is defined as the group of all bijections $g : V \to V$ such that

$$\sup_{x \in V} d_\Gamma(x, g(x)) < \infty.$$

One can show that the wobbling group of any graph containing infinite path contains a free subgroup, but no property (T) group could be embedded into the wobbling group of a graph with uniformly subexponential growth, see [5] or [9]. However, as it was pointed out by Matte Bon, if one removes uniformity requirement, any residually finite group can be embedded into the wobbling group of a graph of linear growth. In particular, group $\mathrm{SL}_3(\mathbb{Z})$ which has property (T) embeds into the wobbling group of a graph with linear growth. Therefore, general embeddability questions would require some uniformity, although in case of Thompson's group F, which has few normal subgroups, a question without assumptions about uniformity still makes sense.

Finally, as we already mentioned in the introduction, if G is a finitely generated group with a generating set S, and H is its subgroup, then the action of G on right cosets of H defines a homomorphism from G to $W(\Gamma(G, H))$.

3. Characterisation of confined subgroups of Thompson's group F

In this section we obtain a characterisation of confined subgroups of F in terms of the action of F on the unit interval. This characterisation is analogous to one obtained in Theorem 6.1 (Theorem 4.1 in version 1) of [14], although the proof of Theorem 6.1 does not directly apply to F, because in the present case the corresponding action of a confined subgroup on a Cantor space may have infinite fixed closed proper subsets, and as a result, the statement of Step 1 in the proof of Theorem 6.1 in [14] is false in this setting.

If a group G acts by homeomorphisms on a topological space X and $Y \subset X$, we call the *rigid stabilizer* of Y the subgroup of G consisting of all elements which fix the complement of Y pointwise. We denote this subgroup by $R_G(Y)$, and the stabilizer of Y is denoted by $\mathrm{St}_G(Y)$. Finally, the subgroup of all elements $g \in G$ which act trivially on some neighborhood of Y is denoted by $\mathrm{St}^0_G(Y)$ and is called the *germ stabilizer* of S.

Theorem 3.1. *A subgroup H of Thompson's group F is confined if and only if there exists a finite subset of the unit segment $S \subset [0,1]$ such that $\mathrm{St}^0_{F'}(S) \le H \le \mathrm{St}_F(S)$. In particular, H is confined if and only if it is confined by F'.*

Proof. We first prove that any subgroup satisfying inclusions from the theorem is confined. It suffices to show that for any finite S the subgroup $\mathrm{St}^0_{F'}(S)$ is confined by F. Put $r = |S| + 3$ and take any nontrivial $g_1, g_2, \dots, g_r \in F$ with pairwise disjoint supports. Then for any $h \in F$ the support of at least one of the elements $h^{-1}g_1h, h^{-1}g_2h, \dots, h^{-1}g_rh$ does not intersect $S \cup \{0,1\}$, and thus this element belongs to the germ stabilizer of S and to F'.

We will prove the reverse direction under a weaker assumption that the subgroup H is confined by F'. Consider a maximal open subset V of the interval $(0,1)$ such that H contains every $g \in F$, whose support belongs to V. It is easy to see that such a maximal set exists, since if H contains every element supported in one of a family of open sets, then it contains every element supported in their union. Our aim is to show that the complement of V is finite. We first check that V is non-empty. We are going to use the following theorem (a generalization of the commutator lemma, see also a similar Theorem 3.10 in [13]):

Theorem 3.2 (N. Matte Bon, [14]). *Let G be a countable group acting by homeomorphisms of a Hausdorff space X, and assume that $A \le G$ is a subgroup whose action on X is minimal and proximal. Let $H \in \mathrm{Sub}(G)$ be confined by A. Then there exists a non-empty open subset $U \subset X$ and a finite index subgroup Γ of the rigid stabilizer $R_A(U)$ such that H contains the derived subgroup $[\Gamma, \Gamma]$.*

Apply this theorem to $X = (0,1)$, $A = F'$ and H (the action of F' is transitive on ordered n-tuples of dyadic rationals for any n, so it is minimal and proximal on the unit interval). Let $a < b$ be dyadic rationals such that $[a, b]$ is contained in U. As we mentioned before, the subgroup $F[a, b]$ of elements of F supported on $[a, b]$ is isomorphic to Thompson's group F, and its commutator is simple and coincides with the group $F(a, b)$ of all elements of F supported on the interval (a, b). Then since $F'[a, b]$ is simple, it must be contained in Γ, and consequently, in its derived subgroup. Therefore, H contains $F(a, b)$ and V is non-empty.

Next, we show that V has only finitely many connected components. Suppose that $P = \{g_1, g_2, \ldots, g_r\}$ is confining for H. Note that if we replace some elements of P with their inverses and reorder elements of P, we will still have a confining set. We need the following fact:

Lemma 3.3. *For any nontrivial $g_1, g_2, \ldots, g_r \in F$, possibly after permuting and taking inverses, one can find intervals $U_1, \ldots, U_r$ with dyadic endpoints such that $g_1(U_1) < U_1 < g_2(U_2) < U(2) < \cdots < g_r(U_r) < U_r$, where an interval (a, b) is less than (c, d) if $b < c$.*

Proof. We induct on r. Case $r = 1$ is obvious. For the inductive step, choose an element with maximal supremum of support. Take this element as g_r, and let s be the supremum of its support. Note that s must be a fixed point of g_r. If g_r is greater than the identity on $(s - \epsilon, s]$ for any sufficiently small ϵ, take its inverse. It remains to apply inductive hypothesis to elements $g_1, g_2, \ldots, g_{r-1}$ and then choose U_r sufficiently close to s to ensure that desired inequalities hold for U_r. △

Suppose that one can find r connected components $(x_1, y_1), \ldots, (x_r, y_r)$ of V such that $0 < x_1$ and $y_i \le x_{i+1}$ for $i = 1, \ldots, r-1$. Using Lemma 3.3, we can show that at least one of these intervals could be extended over its left endpoint contradicting the definition of connected component. Indeed, let U_i, $i = 1, \ldots, r$, be as in Lemma 3.3 and let $g_1, g_2, \ldots, g_r$ be the corresponding modification of P, then one can choose dyadic intervals $V_i \subset (x_i, y_i)$ and sufficiently small dyadic intervals $W_i, x_i \in W_i$, $i = 1, \ldots, r$, such that $W_1 < V_1 < W_2 < V_2 < \cdots < W_r < V_r$. Since F' acts transitively on increasing $4r$-tuples of dyadic rationals, there exists $h \in F'$ such that $h(V_i) = U_i, h(W_i) = g_i(U_i), i = 1, \ldots, r$. Then for some i an element $k = h^{-1} g_i h$ belongs to H and $k(V_i) = W_i$. Consequently, $F'[W_i] = k^{-1} F'[V_i] k < H$. Therefore, any element of F with support in W_i belongs to H. Now, since elements with support in W_i together with elements with support in (x_i, y_i) generate group of all elements supported on $W_i \cup (x_i, y_i)$, we obtain an interval in V which contains (x_i, y_i) as a strictly smaller subinterval, a contradiction. As a result, V can not have more than $r+1$ connected components.

Since V consists of finitely many intervals, endpoints of these intervals must be fixed by H, and $[0, 1] \setminus V$ is a disjoint union of finitely many segments and points. Assume that a segment $[x, y], x < y$, is a connected component of the complement of V, then its endpoints must be fixed by H. Notice that Lemma 3.3

together with transitivity on r-tuples imply that H can not have more than r fixed points in the interval $(0,1)$, so H can not act on $[x,y]$ trivially. Consider element $h \in F'$ which maps all break points (except for 0 and 1) of each of $g_1, g_2, \ldots, g_r$ inside (x,y). Then $h^{-1}Ph = \{h^{-1}g_1h, h^{-1}g_2h, \ldots, h^{-1}g_rh\}$ is still a confining set for H. Furthermore, each of its elements either has support in (x,y) or moves at least one endpoint of $[x,y]$. Since $h^{-1}Ph$ is confining, its conjugation by any element of F' must hit H. Thus, if we conjugate $h^{-1}Ph$ by elements with support in (x,y), we will still be hitting H, but elements of $h^{-1}Ph$ which move endpoint of this segment will still move the endpoint, so they can not belong to H. Therefore, we can only consider those elements of $h^{-1}Ph$ which are supported on $[x,y]$. But this implies that the restriction of H to $[x,y]$ is confined by the subgroup of all elements of F with support on (x,y). Then we can repeat the argument in the beginning of the proof to obtain a subinterval $(z,t) \subset (x,y)$ that must belong to V, which contradicts the definition of $[x,y]$. Consequently, the complement of V is some finite set of points S which are fixed by H. Thus, by the definition of V, $\mathrm{St}^0_{F'}(S) \leq H \leq \mathrm{St}_F(S)$, which completes the proof of the theorem. □

Remark 3.4. Theorem 3.1 implies that the graph $\Gamma(F,H)$, with H being a confined subgroup of F, must be amenable. Indeed, it suffices to prove this for $H = \mathrm{St}^0_{F'}(S) = \mathrm{St}^0_F(S \cup \{0,1\})$ with $S \cap (0,1) \neq \emptyset$. Notice that $x_0(t) < t$ for all $t \in (0,1)$, so for any finite $S \subset (0,1)$ there exists $k \in \mathbb{N}$ such that $x_0^k(S) \subset (0,1/2)$. Then, since $x_1(t) = t$ and $x_0(t) = t/2$ on $(0,1/2)$ and they coincide in a neighbourhood of 1, $\Gamma(F,H)$ contains arbitrarily large parts of square grid and thus it is amenable.

Remark 3.5. The theorem above also implies that the subgroups constructed in [3] are examples of maximal non-confined subgroups of F.

4. Growth of orbital graphs of Thompson's group F

Results from the previous section allow us to deduce lower bounds on the uniform growth of orbital graphs of F in the same fashion as in §8 of [14].

Theorem 4.1. *Assume that F acts on a set X and let p be any point in X. Then either the orbital graph of p has exponential uniform growth or it is fixed by the commutator subgroup of F.*

Proof. The orbital graph of p is isomorphic to the Schreier graph of $\mathrm{St}_F(p)$. Assume that $\mathrm{St}_F(p)$ is not confined by F. Then there exists a sequence of points $p_n, n \geq 1$, in the orbit of p such that orbital graphs of p_n converge to the Cayley graph of F in the space of rooted labeled graphs. Then, as F has exponential growth, the orbital graph of p has exponential uniform growth.

If $\mathrm{St}_F(p)$ is confined, then it either contains the commutator subgroup F' or it fixes a point x in the unit interval $(0, 1)$. In the latter case the orbital graph of p grows not slower than the orbital Schreier graph of x. But according to the classification of orbital Schreier graphs of points from the unit interval obtained by Savchuk in [16], all these graphs have exponential uniform growth, which completes the proof. □

Corollary 4.2. *F and F' do not embed into wobbling groups of graphs with uniformly subexponential growth.*

Proof. It suffices to notice that the orbital graph of a point under the action of a finitely generated subgroup of a wobbling group has uniform growth not exceeding the uniform growth of the initial graph. Then, if F acts on a graph by elements of its wobbling group, either every point of the graph is fixed by the commutator subgroup or it has exponential uniform growth. Conclusion for F' follows from the fact that F embeds in F'. □

5. Final remarks and questions

We still do not know whether F could be embedded into a wobbling group of a recurrent graph with bounded degree or into a wobbling group of a graph with subexponential growth. In the latter case, arguments used in §4 might fail only for a point p with non-confined stabilizer if a sequence $p_n, n \geq 1$, is sufficiently sparse in the orbital graph of p. Similarly, for recurrent graphs the case of a confined subgroup follows from results of Savchuk and Kaimanovich or Mishchenko. For non-confined subgroup H the fact that arbitrarily large balls from the Cayley graph appear in the Schreier graph modulo H does not imply that the Schreier graph is not recurrent, since one can modify a recurrent graph by inserting large components of a Cayley graph of F without affecting its recurrence. The following question still remains open:

Question 1. *Is there any non-confined maximal subgroup of F which corresponds to a recurrent Schreier graph? In particular, are the Schreier graphs modulo subgroups defined in [3] always transient?*

It also would be natural to try to estimate the density of fragments of the Cayley graph of F in a Schreier graph modulo its non-confined subgroup, possibly with additional assumptions concerning amenability. Affirmative answer to the following question would obviously settle the general case of graphs with subexponential growth.

Question 2. *Is it true that for any non-confined subgroup $H < F$ one can find constant K such that for infinitely many $n \in \mathbb{N}$ there is a copy of a ball $B_F(n)$ in a ball of the Schreier graph with radius Kn centered at the root*?

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