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Iterated monodromy groups of Chebyshev-like maps on $\mathbb{C}^n$

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Abstract. Every affine Weyl group appears as the iterated monodromy group of a Chebyshev-like polynomial self-map of $\mathbb{C}^n$.

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Introduction

The Chebyshev polynomials $T_d : \mathbb{C} \to \mathbb{C}$, defined for $d \geq 2$ by the equation $T_d(\cos\theta) = \cos d\theta$, are important in many areas of mathematics. In the study of single-variable complex dynamics, they are especially notable for being post-critically finite, and for having a "smooth" Julia set, namely the interval $[-1, 1]$. Moreover, their restrictions to $[-1, 1]$ act as "folding maps," whose dynamics can be completely described.

In the 1980s, Veselov [18] and Hoffman and Withers [9] defined, for each root system Φ in $\mathbb{R}^n$, a family of "Chebyshev-like" polynomial maps $T_{\Phi,d} : \mathbb{C}^n \to \mathbb{C}^n$. (Certain special cases, particularly in dimension 2, had been studied since the 1970s [3, 10], although not in a dynamical context.) These maps $T_{\Phi,d}$ are also post-critically finite (in an appropriate sense), and each of them acts as a "folding map" on a certain compact subset of $\mathbb{C}^n$, which depends only on Φ.

Any post-critically finite map $f : \mathbb{C}^n \to \mathbb{C}^n$ has an associated iterated monodromy group $\mathrm{IMG}(f)$, which encodes the dynamics of f algebraically, especially on (the boundary of) the set of points in $\mathbb{C}^n$ that do not escape to infinity under iteration of f. Iterated monodromy groups were introduced by Nekrashevych [12, 13, 14] and have proved to be a powerful tool in both dynamics and group theory. However, very few such groups have been calculated for post-critically finite maps of $\mathbb{C}^n$ where $n > 1$; see [1, 15] for the only examples known to the author. (A special case of the present article's main result, obtained by different methods, is shown in [2].)

It is known that the iterated monodromy group of a Chebyshev polynomial T_d is the infinite dihedral group $\langle a, b \mid a^2 = b^2 = \mathrm{id}\rangle$, which may be realized as the group of transformations of $\theta \in \mathbb{C}$ that leave the cosine $\frac{1}{2}(e^{i\theta} + e^{-i\theta})$ invariant. In this article, we generalize this result to Chebyshev-like maps in every dimension $n \geq 1$. Given a root system $\Phi \subset \mathbb{R}^n$, we let $\widetilde{W}_\Phi$ denote the associated affine Weyl group.

Theorem. *Let $T_{\Phi,d} \colon \mathbb{C}^n \to \mathbb{C}^n$ be a Chebyshev-like map associated to the root system Φ. Then $\mathrm{IMG}(T_{\Phi,d})$ is isomorphic to $\widetilde{W}_\Phi$.*

Our approach is somewhat indirect. Because $T_{\Phi,d}$ is post-critically finite, its post-critical locus is a (not necessarily irreducible) hypersurface $\mathcal{D}_\Phi \subset \mathbb{C}^n$, which we show depends only on Φ. The iterated monodromy group $\mathrm{IMG}(T_{\Phi,d})$ is defined as a quotient of the fundamental group of the complement of $\mathcal{D}_\Phi$. However, we do not compute this fundamental group at the start; instead, we relate $\mathrm{IMG}(T_{\Phi,d})$ and $\pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi)$ to the fundamental group of a certain complement of hyperplanes (the "Cartan–Stiefel diagram," see §2). Along the way, we uncover $\pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi)$.

Corollary. *For every root system Φ, there exists an infinite hyperplane arrangement $\mathcal{H}_\Phi \subset \mathbb{C}^n$, invariant under $\widetilde{W}_\Phi$, such that $\pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi)$ is isomorphic to an extension of $\widetilde{W}_\Phi$ by $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_\Phi)$.*

Iterated monodromy groups are examples of self-similar groups acting on trees. Thus we also have the following consequence.

Corollary. *Any affine Weyl group of rank n acts faithfully as a self-similar group on a rooted d^n-ary tree for any $d \geq 2$.*

It is natural to conjecture that the property of having an iterated monodromy group isomorphic to an affine Weyl group characterizes the Chebyshev-like maps.

Conjecture. *Let $f \colon \mathbb{C}^n \to \mathbb{C}^n$ be post-critically finite. If the iterated monodromy group of f is an affine Weyl group, then (some iterate of) f is a Chebyshev-like map.*

Here is the structure of the paper. In §1, we recall the necessary definitions from the theory of root systems and review the definition of the Chebyshev-like maps $T_{\Phi,d}$. In §2, we study the post-critical locus of each map $T_{\Phi,d}$. In §3, we recall the definition of iterated monodromy groups and establish a key lemma. Finally, in §4 we prove the main result.

1. Root systems and Chebyshev-like maps

First we review some of the theory of root systems. References are [7, 16]. The notation used here is similar but not identical to that of [9].

Throughout, we endow $\mathbb{R}^n$ with the standard inner product $\langle \cdot, \cdot \rangle$, which we extend to a Hermitian form on $\mathbb{C}^n$, also written $\langle \cdot, \cdot \rangle$, that is antilinear in the first variable and complex linear in the second variable—that is, for all $\lambda \in \mathbb{C}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$, we have $\langle \mathbf{v}, \lambda \mathbf{w} \rangle = \lambda \langle \mathbf{v}, \mathbf{w} \rangle = \langle \bar{\lambda} \mathbf{v}, \mathbf{w} \rangle$.

Definition 1.1 (complex reflection). A nonzero vector $\mathbf{v} \in \mathbb{C}^n$ and a real number $\ell \in \mathbb{R}$ together determine a *complex reflection* $\rho_{\mathbf{v},\ell} \colon \mathbb{C}^n \to \mathbb{C}^n$, given algebraically by

$$\rho_{\mathbf{v},\ell}(\mathbf{x}) = \mathbf{x} - 2\frac{\langle \mathbf{v}, \mathbf{x} \rangle - \ell}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}.$$

Note that $\rho_{\mathbf{v},\ell}$ is complex-affine in $\mathbf{x}$, and its derivative is $D\rho_{\mathbf{v},\ell} = \rho_{\mathbf{v},0}$. The fixed-point set of $\rho_{\mathbf{v},\ell}$ is the complex hyperplane $H_{\mathbf{v},\ell}$ defined by the equation $\langle \mathbf{v}, \mathbf{x} \rangle = \ell$. If $\mathbf{v} \in \mathbb{R}^n$, then $\rho_{\mathbf{v},\ell}$ restricts to an ordinary reflection $\mathbb{R}^n \to \mathbb{R}^n$ across the real hyperplane $H_{\mathbf{v},\ell} \cap \mathbb{R}^n$.

Definition 1.2 (root system, root, coroot). A *root system* (with *rank n*) is a finite set of vectors $\Phi \subset \mathbb{R}^n$ such that the following conditions are satisfied:

- Φ spans $\mathbb{R}^n$;
- if $\mathbf{v} \in \Phi$ and $\lambda \in \mathbb{R}$, then $\lambda \mathbf{v} \in \Phi \iff \lambda = \pm 1$;
- if $\mathbf{v} \in \Phi$, then $\rho_{\mathbf{v},0}(\mathbf{w}) \in \Phi$ for all $\mathbf{w} \in \Phi$;
- if $\mathbf{v} \in \Phi$ and $\mathbf{w} \in \Phi$, then $2\dfrac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \in \mathbb{Z}$.

Elements of Φ are called *roots*. For $\mathbf{v} \in \Phi$, the *coroot* of $\mathbf{v}$ is $\mathbf{v}^\vee = \dfrac{2\,\mathbf{v}}{\langle \mathbf{v}, \mathbf{v} \rangle}$.

A root system Φ is *irreducible* if it cannot be partitioned into root systems of lower rank, which are contained in orthogonal subspaces.

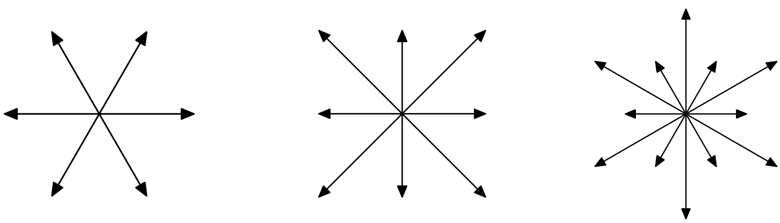

Figure 1. Irreducible root systems of rank 2: respectively, A_2, B_2, and G_2.

When the type of Φ is known (such as A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, or G_2 in the cases that Φ is irreducible), then in notation we may replace Φ with its type.

Definition 1.3 (Weyl group, affine Weyl group). Given a root system Φ, the *Weyl group* of Φ is the group W_Φ generated by all reflections of the form $\rho_{\mathbf{v},0}$, $\mathbf{v} \in \Phi$. The *affine Weyl group* of Φ is the group $\widetilde{W}_\Phi$ generated by all reflections of the form $\rho_{\mathbf{v},\ell}$, $\mathbf{v} \in \Phi$, $\ell \in \mathbb{Z}$.

Equivalently, the affine Weyl group $\widetilde{W}_\Phi$ can be defined as the semidirect product $Q_\Phi^\vee \rtimes W_\Phi$, where $Q_\Phi^\vee$ is the lattice in $\mathbb{R}^n$ generated by the coroots of Φ. Both W_Φ and $\widetilde{W}_\Phi$ may be thought of as acting on either $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 1.4 (simple roots, fundamental weights). Given a root system $\Phi \subset \mathbb{R}^n$, let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional that does not vanish on any elements of Φ. The elements $\mathbf{v}$ of Φ such that $\phi(\mathbf{v}) > 0$ are called *positive roots* (relative to ϕ); a positive root α is called *simple* if it cannot be written as a sum $\alpha = \mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ and $\mathbf{w}$ are distinct positive roots. The simple roots form a basis $\{\alpha_1, \dots, \alpha_n\}$ of $\mathbb{R}^n$; the *fundamental weights* $\omega_1, \dots, \omega_n$ form the dual basis to the coroots of the simple roots: that is,

$$\langle \omega_j, \alpha_k^\vee \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Now we reach our main definition.

Definition 1.5 (generalized cosine, Chebyshev-like map). Let $\Phi \subset \mathbb{R}^n$ be a root system and W_Φ its Weyl group. Let $\omega_1, \dots, \omega_n$ be a choice of fundamental weights for Φ. For each $1 \le k \le n$, define ψ_k from $\mathbb{C}^n$ to $\mathbb{C}$ by

$$\begin{aligned} \psi_k(\mathbf{x}) &:= \sum_{\mathbf{r} \in W_\Phi \omega_k} \exp(2\pi i \langle \mathbf{r}, \mathbf{x} \rangle) \\ &= \frac{1}{|\operatorname{Stab}_{W_\Phi}(\omega_k)|} \sum_{w \in W_\Phi} \exp(2\pi i \langle w\omega_k, \mathbf{x} \rangle), \end{aligned}$$

where $W_\Phi \omega_k$ is the orbit of ω_k under W_Φ, and $\operatorname{Stab}_{W_\Phi}(\omega_k)$ is the stabilizer of ω_k in W_Φ. Then define the *generalized cosine* $\Psi_\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Psi_\Phi := (\psi_1, \dots, \psi_n).$$

For each integer $d \ge 2$, let $m_d\colon \mathbb{C}^n \to \mathbb{C}^n$ denote multiplication by d. Then the *Chebyshev-like map* $T_{\Phi,d}\colon \mathbb{C}^n \to \mathbb{C}^n$ is defined by the functional equation

$$T_{\Phi,d} \circ \Psi_\Phi = \Psi_\Phi \circ m_d. \tag{1}$$

Theorem ([9, 18]). *Given any root system Φ of rank n and any integer $d \geq 2$, equation* (1) *defines a polynomial map $T_{\Phi,d} \colon \mathbb{C}^n \to \mathbb{C}^n$.*

The construction above is due independently to Veselov [18, 19] and Hoffman–Withers [9]. Up to permutation of coordinates, Ψ_Φ is independent of the choice of fundamental weights, because the Weyl group acts transitively on bases of simple roots. The terminology of "generalized cosine" comes from Hoffman and Withers, who described these Chebyshev-like maps in terms of folding figures in $\mathbb{R}^n$, which leads to the consideration of root systems. Veselov expressed the construction in terms of exponential invariants of semi-simple Lie algebras and noted, from Chevalley's theorem, that the coefficients of the polynomials defining $T_{\Phi,d}$ are integers.

Example 1.6 (Chebyshev polynomials). In the classical case, from which the Chebyshev-like maps get their name, $\Phi = \{\pm 1\}$ is the A_1 root system in $\mathbb{R}$. Then 1 is a simple root, $\frac{1}{2}$ is the corresponding fundamental weight, and the Weyl group is just the two-element group generated by multiplication by -1 (in either $\mathbb{R}$ or $\mathbb{C}$). Set $t = e^{i\pi x}$, so that $\Psi_{A_1}(x) = \psi_1(x) = t + t^{-1}$, and we have the dth Chebyshev polynomial $T_d := T_{A_1,d}$ defined by the equation $T_d(t + t^{-1}) = t^d + t^{-d}$. (Here we have followed the convention that $T_d(2\cos\theta) = 2\cos d\theta$, contra the equation $T_d(\cos\theta) = \cos d\theta$ stated in the introduction. These two conventions produce dynamically conjugate maps.)

Example 1.7 (a Chebyshev-like map in 2 dimensions). The A_2 root system is the simplest of the irreducible rank 2 root systems (see Figure 1). It can be realized in the plane in $\mathbb{R}^3$ having equation $x_1 + x_2 + x_3 = 0$ as the set of six vectors $\Phi = \{(\pm 1, \mp 1, 0), (0, \pm 1, \mp 1), (\pm 1, 0, \mp 1)\}$. One choice of simple roots is $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$. The corresponding fundamental weights are $\omega_1 = (2/3, -1/3, -1/3)$ and $\omega_2 = (1/3, 1/3, -2/3)$, and so when $\mathbf{x} = (x_1, x_2, x_3)$ satisfies $x_1 + x_2 + x_3 = 0$, we have $\langle \omega_1, \mathbf{x} \rangle = x_1$ and $\langle \omega_2, \mathbf{x} \rangle = x_1 + x_2$. The Weyl group in this case is the symmetric group on 3 elements, realized as the permutations of the coordinates in $\mathbb{R}^3$. If we set $t_j = \exp(i\,2\pi x_j)$ and $(X_1, X_2) = \Psi_{A_2}(\mathbf{x})$, then we have the equalities $t_1 t_2 t_3 = 1$, $X_1 = t_1 + t_2 + t_3$ and $X_2 = t_1 t_2 + t_2 t_3 + t_3 t_1$. In the coordinates (X_1, X_2), we may write $T_{A_2,2}$, for instance, as the map

$$T_{A_2,2}(X_1, X_2) = ({X_1}^2 - 2X_2, {X_2}^2 - 2X_1),$$

which has been independently studied, e.g., in [2, 11, 17].

2. Critical and post-critical loci

Definition 2.1 (critical point, critical value, post-critical locus, post-critically finite). Let $f \colon \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map. A *critical point* of f is a point

$\mathbf{c}$ such that the derivative $Df(\mathbf{c})\colon \mathbb{C}^n \to \mathbb{C}^n$ is singular. The *critical locus* of f is the set $\mathcal{C}_f$ containing all critical points of f. A *critical value* of f is a point of the form $f(\mathbf{c})$, where $\mathbf{c} \in \mathcal{C}_f$. The *post-critical locus* of f is the union $\mathcal{P}_f$ of all (strict) forward images of the critical locus of f, in symbols

$$\mathcal{P}_f := \bigcup_{k\geq 1} f^k(\mathcal{C}_f).$$

We say f is *post-critically finite* if $\mathcal{C}_f \neq \mathbb{C}^n$ and $\mathcal{P}_f$ is a closed, proper subvariety of $\mathbb{C}^n$.

The notion of a post-critically finite map of $\mathbb{C}^n$ was introduced by Fornæss and Sibony [4, 5] as a generalization of post-critically finite polynomials on $\mathbb{C}$. The post-critical locus of such a map f includes the critical values of f, but it may (and generally does) include more points of $\mathbb{C}^n$. The map f is locally a covering map away from its critical values. Thus, the restriction of f to the complement of $\mathcal{C}_f \cup \mathcal{P}_f$ is a covering of the complement of $\mathcal{P}_f$.

Although we will not treat the generalized cosine Ψ_Φ dynamically, we do need to know what its critical locus and critical values are.

Definition 2.2 (Cartan–Stiefel diagram). Given a root system Φ, let $\mathcal{H}_\Phi$ be the union of all complex hyperplanes fixed by some non-identity element of the affine Weyl group $\widetilde{W}_\Phi$. That is,

$$\mathcal{H}_\Phi := \bigcup_{\substack{\mathbf{v}\in\Phi \\ \ell\in\mathbb{Z}}} H_{\mathbf{v},\ell}\,.$$

This union of hyperplanes is the (complex) *Cartan–Stiefel diagram* of $\widetilde{W}_\Phi$.

We also let $\mathcal{D}_\Phi$ be the image of $\mathcal{H}_\Phi$ by Ψ_Φ; that is,

$$\mathcal{D}_\Phi := \Psi_\Phi(\mathcal{H}_\Phi).$$

Example 2.3. When Φ is the A_1 root system as in Example 1.6, we have $\mathcal{H}_\Phi = \mathbb{Z}$ and $\mathcal{D}_\Phi = \{\pm 2\}$.

The next example provides part of the motivation for the notation $\mathcal{D}_\Phi$.

Example 2.4. When Φ is the A_2 root system as in Example 1.7, the points of $\mathcal{H}_\Phi$ with real coordinates form the edges of a planar tiling by equilateral triangles. $\mathcal{D}_\Phi$ is the complex version of the deltoid (a.k.a. three-cusped hypocycloid) with equation ${X_1}^2{X_2}^2 + 18X_1X_2 = 4({X_1}^3 + {X_2}^3) + 27$.

The next three lemmas demonstrate the importance of $\mathcal{H}_\Phi$ and $\mathcal{D}_\Phi$ to our study.

Lemma 2.5. *For any root system Φ, the critical locus of Ψ_Φ is $\mathcal{H}_\Phi$.*

Proof. It is evident from the definition of Ψ_Φ that $\Psi_\Phi(\tilde{w}\mathbf{x}) = \Psi_\Phi(\mathbf{x})$ for all $\tilde{w} \in \widetilde{W}_\Phi$, and in particular that the set of critical points of Ψ_Φ is invariant under the action of $\widetilde{W}_\Phi$. Thus it is sufficient to show that $\mathbf{c}$ is a critical point for Ψ_Φ if and only if it is equivalent under $\widetilde{W}_\Phi$ to some point fixed by a non-identity element of W_Φ.

First, note that if $\rho_{\mathbf{v},0}(\mathbf{c}) = \mathbf{c}$ for some $\mathbf{v} \in \Phi$, then for all $\lambda \in \mathbb{C}$ we have $\rho_{\mathbf{v},0}(\mathbf{c} + \lambda\mathbf{v}) = \mathbf{c} - \lambda\mathbf{v}$, and thus Ψ_Φ is not locally injective at $\mathbf{c}$; in other words, $\mathbf{c}$ is a critical point of Ψ_Φ. Therefore, all of $\mathcal{H}_\Phi$ is contained in the critical locus of Ψ_Φ.

To see that Ψ_Φ has no other critical points, first observe that the natural projection $\mathbb{C}^n \to \mathbb{C}^n/Q_\Phi^\vee$ is a covering map, having no critical points. Next, Ψ_Φ is the composition of this projection and the quotient map $\mathbb{C}^n/Q_\Phi^\vee \to \mathbb{C}^n$ given by the induced action of W_Φ on $\mathbb{C}^n/Q_\Phi^\vee$. The critical points of this latter action are precisely the points that are fixed by some non-identity element of W_Φ, which is to say, the image of $\mathcal{H}_\Phi$ in $\mathbb{C}^n/Q_\Phi^\vee$. □

Lemma 2.6. *Given a root system Φ and an integer $d \geq 2$, the Chebyshev-like map $T_{\Phi,d}$ is post-critically finite, with $\mathcal{D}_\Phi$ as its post-critical locus.*

Proof. Differentiating both sides of (1) at a variable point $\mathbf{x}$ and applying the chain rule yields

$$[DT_{\Phi,d}(\Psi_\Phi(\mathbf{x}))] \circ [D\Psi_\Phi(\mathbf{x})] = [D\Psi_\Phi(d\mathbf{x})] \circ m_d$$

(using the fact that m_d is already linear). Therefore we shall determine when $\Psi_\Phi(\mathbf{x})$ is a critical point of $T_{\Phi,d}$.

First suppose that $\mathbf{x}$ is not a critical point of Ψ_Φ, i.e., $\mathbf{x} \notin \mathcal{H}_\Phi$. Then we can rewrite the above equation as

$$[DT_{\Phi,d}(\Psi_\Phi(\mathbf{x}))] = [D\Psi_\Phi(d\mathbf{x})] \circ m_d \circ [D\Psi_\Phi(\mathbf{x})]^{-1}.$$

This equation implies that $\Psi_\Phi(\mathbf{x})$ is a critical point of $T_{\Phi,d}$ whenever $d\mathbf{x}$ is a critical point of Ψ_Φ, i.e., when $d\mathbf{x} \in \mathcal{H}_\Phi$.

Now the set of critical points is closed, and so every point in the closure of ${m_d}^{-1}(\mathcal{H}_\Phi) \setminus \mathcal{H}_\Phi$ (which is to say, the union of all hyperplanes which are strict preimages of hyperplanes in $\mathcal{H}_\Phi$) also yields a critical point of $T_{\Phi,d}$. The critical values of $T_{\Phi,d}$ are therefore the images of $\mathcal{H}_\Phi$ by Ψ_Φ, which is to say $\mathcal{D}_\Phi$.

Finally, note that $\mathcal{H}_\Phi$ is invariant under m_d, because $d \cdot H_{\mathbf{v},\ell} = H_{\mathbf{v},d\ell}$ and d is an integer. Therefore all critical values of $T_{\Phi,d}$ lie in $\mathcal{D}_\Phi$, every point of $\mathcal{D}_\Phi$ is a critical value, and $\mathcal{D}_\Phi$ is invariant under $T_{\Phi,d}$. □

Recall that a covering map $p\colon \mathcal{Y} \to \mathcal{X}$ of path-connected topological spaces is called *regular* when the group of deck transformations $\mathrm{Gal}(\mathcal{Y}/\mathcal{X})$ acts transitively on each fiber of p.

Lemma 2.7. *Let $\Phi \subset \mathbb{R}^n$ be a root system having affine Weyl group $\widetilde{W}_\Phi$. Set $\mathcal{X} = \mathbb{C}^n \setminus \mathcal{D}_\Phi$ and $\mathcal{Y} = \mathbb{C}^n \setminus \mathcal{H}_\Phi$. Then the restriction of the generalized cosine Ψ_Φ to $\mathcal{Y}$ is a regular covering of $\mathcal{X}$, with* $\mathrm{Gal}(\mathcal{Y}/\mathcal{X}) = \widetilde{W}_\Phi$.

Proof. By Lemma 2.5, no points of $\mathcal{Y}$ are critical for Ψ_Φ, and therefore Ψ_Φ is locally a homeomorphism when restricted to $\mathcal{Y}$; i.e., $\Psi_\Phi|_\mathcal{Y}$ is a covering map. By definition, we have $\mathcal{D}_\Phi = \Psi_\Phi(\mathcal{H}_\Phi)$, so $\Psi_\Phi(\mathcal{Y}) = \mathcal{X}$. As observed in the proof of Lemma 2.5, $\Psi_\Phi(\tilde{w}\mathbf{x}) = \Psi_\Phi(\mathbf{x})$ for all $\tilde{w} \in \widetilde{W}_\Phi$, so $\widetilde{W}_\Phi$ is contained in $\mathrm{Gal}(\mathcal{Y}/\mathcal{X})$. Moreover, the same proof shows that the fiber over each point of $\mathcal{X}$ can be identified with $\widetilde{W}_\Phi$, which implies $\mathrm{Gal}(\mathcal{Y}/\mathcal{X}) = \widetilde{W}_\Phi$. □

An immediate consequence of Lemma 2.7 is an expression for the fundamental group $\pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi)$ as an extension of $\widetilde{W}_\Phi$. Recall that any covering map $p: \mathcal{Y} \to \mathcal{X}$ induces an injective group homomorphism $p_*: \pi_1(\mathcal{Y}) \to \pi_1(\mathcal{X})$, defined by $p_*([\eta]) = [p \circ \eta]$. The subgroup $p_*(\pi_1(\mathcal{Y}))$ is normal in $\pi_1(\mathcal{X})$ precisely when p is a regular covering, and in this situation the quotient $\pi_1(\mathcal{X})/p_*(\pi_1(\mathcal{Y}))$ is isomorphic to the deck transformation group $\mathrm{Gal}(\mathcal{Y}/\mathcal{X})$. (See [8] for details.)

Corollary 2.8. *Given a root system Φ with affine Weyl group $\widetilde{W}_\Phi$, let Ψ_Φ, $\mathcal{H}_\Phi$, and $\mathcal{D}_\Phi$ be defined as above. Then we have the following exact sequence*:

$$0 \longrightarrow \pi_1(\mathbb{C}^n \setminus \mathcal{H}_\Phi) \longrightarrow \pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi) \longrightarrow \widetilde{W}_\Phi \longrightarrow 0,$$

where the map $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_\Phi) \to \pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi)$ is the injection $(\Psi_\Phi)_*$, and the map $\pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi) \to \widetilde{W}_\Phi$ is the induced canonical projection.

3. Iterated monodromy groups

Iterated monodromy groups of dynamical systems (and of topological automata more generally) were introduced by V. Nekrashevych [12, 13, 14]. We recall the definition, using slightly different notation.

Definition 3.1 (partial self-covering, monodromy action, iterated monodromy group). Let $\mathcal{X}$ be a path-connected, locally path-connected topological space. A *partial self-covering* of $\mathcal{X}$ is a covering map $f: \mathcal{X}_1 \to \mathcal{X}$, where $\mathcal{X}_1$ is an open, path-connected subset of $\mathcal{X}$. Each iterate f^k of a partial self-covering of $\mathcal{X}$ is again a partial self-covering, with domain $\mathcal{X}_k = f^{-k}(\mathcal{X})$. We will label a partial self-covering by the pair $(\mathcal{X}, f)$.

Given a partial self-covering $(\mathcal{X}, f)$ and a point $x_0 \in \mathcal{X}$, let $\mathbf{T}_f$ be the *tree of preimages of* x_0, namely, the vertex set of $\mathbf{T}_f$ is the disjoint union

$$\mathbf{T}_f = \bigsqcup_{k \geq 0} f^{-k}(x_0),$$

and $\mathbf{T}_f$ has an edge from $x' \in f^{-k}(x_0)$ to $x'' \in f^{-(k-1)}(x_0)$ if $x'' = f(x')$. If f has topological degree δ, then $\mathbf{T}_f$ is a rooted δ-ary tree with root x_0.

The fundamental group $\pi_1(\mathcal{X}, x_0)$ acts on $\mathbf{T}_f$ as follows: given a loop γ based at x_0 and $x' \in f^{-k}(x_0)$, use f^k to lift γ to a path $\tilde{\gamma}$ starting at x', and let $[\gamma] \cdot x'$ be the endpoint of $\tilde{\gamma}$. This is the *monodromy action*, which induces the *monodromy homomorphism* $\mu_f : \pi_1(\mathcal{X}, x_0) \to \mathrm{Aut}(\mathbf{T}_f)$,

$$\mu_f([\gamma]) : x' \longmapsto [\gamma] \cdot x'.$$

The image of $\pi_1(\mathcal{X}, x_0)$ via μ_f is the *iterated monodromy group* of f, denoted $\mathrm{IMG}(f)$.

It is not hard to check that, up to isomorphism, $\mathrm{IMG}(f)$ is independent of the choice of basepoint x_0. However, in what follows we will occasionally need to be attentive to basepoints for other reasons.

Example 3.2. Suppose $f : \mathbb{C}^n \to \mathbb{C}^n$ is post-critically finite, with critical locus $\mathcal{C}$ and post-critical locus $\mathcal{P}$. Then the restriction of f to $\mathbb{C}^n \setminus (\mathcal{C} \cup \mathcal{P})$ is a partial self-covering of $\mathcal{X} = \mathbb{C}^n \setminus \mathcal{P}$. In this situation, we define $\mathrm{IMG}(f)$ to be the iterated monodromy group of $(\mathcal{X}, f)$.

Given a partial self-covering $(\mathcal{X}, f)$, it follows from the definitions that $[\gamma] \in \pi_1(\mathcal{X}, x_0)$ is in the kernel of the monodromy homomorphism μ_f if and only if every lift of γ by every iterate of f is a loop (i.e., closed). This observation will be useful at several points.

Example 3.3 (cf. [12]). Let $T_d : \mathbb{C} \to \mathbb{C}$ be the dth Chebyshev polynomial (as in Example 1.6), and set $\mathcal{X} = \mathbb{C} \setminus \{\pm 2\}$. The $d-1$ critical points of T_d are $2\cos(j\pi/d)$, $1 \le j \le d-1$, and the images of these points lie in $\{\pm 2\}$; moreover, $\{\pm 2\}$ is forward invariant under T_d. Thus the restriction of T_d to $\mathcal{X}_1 = \mathbb{C} \setminus \{2\cos(j\pi/d) \mid 0 \le j \le d\}$ is a partial self-covering of $\mathcal{X}$. The fundamental group of $\mathcal{X}$ with basepoint 0 is generated by $[\gamma_+]$ and $[\gamma_-]$, where $\gamma_\pm$ are the loops defined by $\gamma_\pm(s) = \pm 2(1 - e^{2\pi i s})$ (Figure 2, left). Using the relation $T_d(t + t^{-1}) = t^d + t^{-d}$, it can be seen that $[\gamma_+]$ and $[\gamma_-]$ both act on the tree $\mathbf{T}_{T_d}$ as order 2 automorphisms (Figure 2, right). On the other hand, the product $[\gamma_-][\gamma_+]$ acts on the kth level of $\mathbf{T}_{T_d}$ as a permutation of order d^k; therefore the order of $\mu_{T_d}([\gamma_-][\gamma_+])$ is infinite. Thus the iterated monodromy group of T_d is isomorphic to the infinite dihedral group, or in other words the affine Weyl group of the A_1 root system.

Definition 3.4 (semiconjugacy). Two partial self-coverings $g : \mathcal{Y}_1 \to \mathcal{Y}$ and $f : \mathcal{X}_1 \to \mathcal{X}$ are *semiconjugate* if there exists a continuous map $p : \mathcal{Y} \to \mathcal{X}$ such that $p(\mathcal{Y}_1) = \mathcal{X}_1$ and $p \circ g = f \circ p$ on $\mathcal{Y}_1$. The map p is then called a *semiconjugacy* from g to f, and we write $p : (\mathcal{Y}, g) \to (\mathcal{X}, f)$.

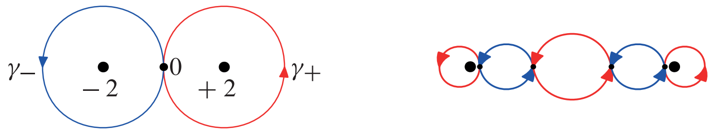

Figure 2. LEFT. Generators $\gamma_\pm$ of the fundamental group of $\mathbb{C} \setminus \{+2, -2\}$. RIGHT. Lifts of $\gamma_\pm$ by the Chebyshev polynomial $T_4 = T_2^2$. The large dots represent -2 and $+2$. Red curves are lifts of γ_+, and blue curves are lifts of γ_-. Each curve begins and ends at a point of $T_4^{-1}(0)$.

We are particularly interested in certain cases where two partial self-coverings $(\mathcal{X}, f)$ and $(\mathcal{Y}, g)$ are semiconjugate by a covering map $p\colon \mathcal{Y} \to \mathcal{X}$.

Lemma 3.5. *Let $(\mathcal{X}, f)$ and $(\mathcal{Y}, g)$ be partial self-coverings, with $p\colon (\mathcal{Y}, g) \to (\mathcal{X}, f)$ a semiconjugacy. Suppose that p is a regular covering map such that $p_*(\ker \mu_g) \subseteq \ker \mu_f$. Choose a basepoint $y_0 \in \mathcal{Y}$, and set $x_0 = p(y_0)$. Then the diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker \mu_g & \longrightarrow & \ker \mu_f & \longrightarrow & \ker \mu_f / \ker \mu_g & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_1(\mathcal{Y}, y_0) & \xrightarrow{p_*} & \pi_1(\mathcal{X}, x_0) & \longrightarrow & \mathrm{Gal}(\mathcal{Y}/\mathcal{X}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_g} & & \downarrow{\scriptstyle \mu_f} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{IMG}(g) & \longrightarrow & \mathrm{IMG}(f) & \longrightarrow & \mathrm{IMG}(f) / \mathrm{IMG}(g) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2}$$

is commutative, with exact rows and columns.

Proof. The vertical maps $\ker \mu_g \to \pi_1(\mathcal{Y}, y_0)$ and $\ker \mu_f \to \pi_1(\mathcal{X}, x_0)$ are inclusions, and so the upper left square of the diagram commutes by assumption. Outside of this square, the maps are all canonically determined, and the rest of the diagram commutes by standard theorems of group theory. Exactness is by construction. □

We close this section with a set of sufficient conditions for the inclusion $p_*(\ker \mu_g) \subseteq \ker \mu_f$ in the hypothesis of Lemma 3.5 to hold, which will allow us to employ the diagram (2) in our arguments of the next section.

Lemma 3.6. *Let $p: (\mathcal{Y}, g) \to (\mathcal{X}, f)$ be a semiconjugacy of partial self-coverings. If g^k is a regular covering map for all k, and p is a regular covering map such that $p^{-1}(\mathcal{X}_k) = \mathcal{Y}_k$ for all k, then $p_*(\ker \mu_g) \subseteq \ker \mu_f$.*

Proof. Choose a basepoint $y_0 \in \mathcal{Y}$, and set $x_0 = p(y_0)$. Let $[\eta] \in \ker \mu_g$; then every lift of η by every iterate g^k is a loop. Set $\gamma = p \circ \eta$, and let $\tilde{\gamma}$ be a lift of γ by some iterate f^k. We want to show that $\tilde{\gamma}$ is a loop. By definition, $\tilde{\gamma}$ starts at some $x' \in f^{-k}(x_0) \subseteq \mathcal{X}_k$. Choose $y' \in p^{-1}(x') \subseteq \mathcal{Y}_k$, and lift $\tilde{\gamma}$ to a path $\tilde{\eta}$ in $\mathcal{Y}_k$ starting at y'. Now, $\tilde{\eta}$ is not necessarily a lift of η by g^k. However, if we apply p to $g^k(y')$, we find $p(g^k(y')) = f^k(p(y')) = f^k(x') = x_0$, and so there exists $\tau \in \mathrm{Gal}(\mathcal{Y}/\mathcal{X})$ such that $y_0 = \tau(g^k(y'))$, and $\tilde{\eta}$ is a lift of η by $\tau \circ g^k$. Because g^k is a regular covering map, the lifting property implies that there exists a homeomorphism $\tau': \mathcal{Y}_k \to \mathcal{Y}_k$ such that $\tau \circ g^k = g^k \circ \tau'$. Thus $\tau' \circ \tilde{\eta}$ is a lift of η by g^k, which means that $\tau' \circ \tilde{\eta}$ must be a loop. Because τ' is a homeomorphism, $\tilde{\eta}$ must also be a loop, and thus $p \circ \tilde{\eta} = \tilde{\gamma}$ is a loop. Therefore $[\gamma] = p_*([\eta]) \in \ker \mu_f$. □

It is worth making a couple of remarks on the condition $p^{-1}(\mathcal{X}_k) = \mathcal{Y}_k$ in the statement of Lemma 3.6. First, for a general semiconjugacy $p: (\mathcal{Y}, g) \to (\mathcal{X}, f)$, we have only the inclusion $\mathcal{Y}_k \subseteq p^{-1}(\mathcal{X}_k)$. Second, when p is a regular covering map, the equation $p^{-1}(\mathcal{X}_k) = \mathcal{Y}_k$ is equivalent to the statement that $\mathrm{Gal}(\mathcal{Y}/\mathcal{X})$ preserves $\mathcal{Y}_k$, in the sense that $\tau(\mathcal{Y}_k) = \mathcal{Y}_k$ for all $\tau \in \mathrm{Gal}(\mathcal{Y}/\mathcal{X})$.

4. Proof of main theorem

In Section 2 we saw that all Chebyshev-like maps $T_{\Phi,d}$ are post-critically finite. Thus they have iterated monodromy groups, which we compute in this section.

Theorem 4.1. *Let Φ be a root system with affine Weyl group $\widetilde{W}_\Phi$. For any $d \geq 2$, the iterated monodromy group of $T_{\Phi,d}$ is isomorphic to $\widetilde{W}_\Phi$.*

Before completing the proof of Theorem 4.1, we make one more general observation.

Lemma 4.2. *Let $(\mathcal{X}, f)$ be a partial self-covering. If f is injective, then its iterated monodromy group is* $\mathrm{IMG}(f) = 0$.

Proof. If f is injective, then it is a homeomorphism. In this case, any lift of any loop by any iterate of f remains a loop, and therefore all of $\pi_1(\mathcal{X})$ lies in the kernel of μ_f.

Alternatively, observe that when f is injective, every level of $\mathbf{T}_f$ has only one vertex, and therefore $\pi_1(\mathcal{X})$ must act trivially at every level. □

We can apply Lemma 4.2 to the partial self-covering $(\mathbb{C}^n \setminus \mathcal{H}_\Phi, m_d)$, because m_d is evidently injective. Thus we have $\ker \mu_{m_d} = \pi_1(\mathbb{C}^n \setminus \mathcal{H}_\Phi)$. The following lemma is now the primary piece that remains to be established.

Lemma 4.3. *Let Φ be a root system, and let Ψ_Φ be its associated generalized cosine. Given $d \geq 2$, let $T_{\Phi,d}$ be the associated Chebyshev-like map. Then $(\Psi_\Phi)_*$ is a surjective map from $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_\Phi)$ to $\ker \mu_{T_{\Phi,d}}$.*

Proof. Let $\mathcal{Y} = \mathbb{C}^n \setminus \mathcal{H}_\Phi$, and choose a point $y_0 \in \mathbb{R}^n \cap \mathcal{Y}$. Set $x_0 = \Psi_\Phi(y_0)$.

First we check that the conditions of Lemma 3.6 are met, in order to see that $(\Psi_\Phi)_*(\pi_1(\mathcal{Y}, y_0)) \subseteq \ker \mu_{T_{\Phi,d}}$. Because $g = m_d$ is a homeomorphism, g^k is a regular covering for all k. The restriction of Ψ_Φ to $\mathcal{Y}$ is a regular covering (by Lemma 2.7), and so it suffices to check that $\mathcal{Y}_k = \mathbb{C}^n \setminus \frac{1}{d^k}\mathcal{H}_\Phi$ is invariant under $\mathrm{Gal}(\mathcal{Y}/\mathcal{X}) = \widetilde{W}_\Phi$, which is true because $\frac{1}{d^k}\mathcal{H}_\Phi$ is even invariant under $\frac{1}{d^k}\widetilde{W}_\Phi$, which contains $\widetilde{W}_\Phi$.

Given $[\gamma] \in \pi_1(\mathbb{C}^n \setminus \mathcal{D}_\Phi, x_0)$, let η be a lift of γ by Ψ_Φ to a path in $\mathcal{Y}$. Then the endpoints of η join points that differ by an element of $\widetilde{W}_\Phi$. Suppose that $\tilde{\gamma}$ is a lift of γ by $(T_{\Phi,d})^k$. Lift $\tilde{\gamma}$ to a path $\tilde{\eta}$ by Ψ_Φ. Using the relation $(T_{\Phi,d})^k \circ \Psi_\Phi = \Psi_\Phi \circ m_{d^k}$, we see that $\tilde{\eta} = \frac{1}{d^k}\eta$, up to an element of $\widetilde{W}_\Phi$. If $\tilde{\gamma}$ is also a loop, then the endpoints of $\tilde{\eta}$ must again differ by an element of $\widetilde{W}_\Phi$. Thus, if $[\gamma] \in \ker \mu_{T_{\Phi,d}}$, it must be true that, for all k, the path $\frac{1}{d^k}\eta$ joins points that differ by some element of $\widetilde{W}_\Phi$. We wish to show that this condition implies that η is a closed loop.

Let $Q_\Phi^\vee \subset \mathbb{R}^n$ be the lattice generated by the coroots of Φ. For each $a \in Q_\Phi^\vee$, the path $a + \eta$ also projects to γ. We can choose an element of $Q_\Phi^\vee$ that sends the endpoints of η to a single Weyl chamber of Φ. (For instance, we can assume that the endpoints of η are linear combinations of simple roots with positive coefficients.) The elements of $\mathcal{H}_\Phi$ partition this Weyl chamber into regions of finite area, each of which is a fundamental domain for $\Psi_\Phi(\mathbb{R}^n)$. Now we can find k sufficiently large that both endpoints of $\frac{1}{d^k}\eta$ are in a single fundamental domain. This is impossible unless η is a loop, which proves the result. □

Proof of Theorem 4.1. Consider the diagram (2), with $\mathcal{X} = \mathbb{C}^n \setminus \mathcal{D}_\Phi$, $\mathcal{Y} = \mathbb{C}^n \setminus \mathcal{H}_\Phi$, $f = T_{\Phi,d}$, $g = m_d$, and $p = \Psi_\Phi$. By Lemma 4.2, $\mathrm{IMG}(m_d)$ is trivial, which implies that $\ker \mu_{m_d} = \pi_1(\mathcal{Y}, y_0)$, and also that $\mathrm{IMG}(T_{\Phi,d})/\,\mathrm{IMG}(m_d) = \mathrm{IMG}(T_{\Phi,d})$. Then Lemma 4.3 implies that $(\Psi_\Phi)_*\colon \ker \mu_{m_d} \to \ker \mu_{T_{\Phi,d}}$ is an isomorphism, so $\ker \mu_{T_{\Phi,d}}/\ker \mu_{m_d} = 0$. The exactness of the rows and columns of (2) now shows that $\mathrm{IMG}(T_{\Phi,d}) \cong \mathrm{Gal}(\mathcal{Y}/\mathcal{X})$, which by Lemma 2.7 is precisely the affine Weyl group of Φ. □

Finally, we state a consequence for the structure of affine Weyl groups, for which we need one more set of definitions (cf. [6, 12, 13]).

Definition 4.4 (δ-ary tree, self-similar group). Given a positive integer δ, the *δ-ary tree* is the graph $\mathbf{T}_\delta$ whose vertex set consists of all finite words in the alphabet $[\delta] = \{1, 2, \ldots, \delta\}$, with an edge between each pair of vertices w and wk, where $k \in [\delta]$. The *root* of $\mathbf{T}_\delta$ is the empty word $\varnothing$. For each $k \in [\delta]$, the subtree $\mathbf{T}_{\delta,k}$ is the induced graph on the set of vertices that begin with k. The map $\sigma_k : w \mapsto kw$ is an isomorphism from $\mathbf{T}_\delta$ to $\mathbf{T}_{\delta,k}$. Given an automorphism g of $\mathbf{T}_\delta$ and $k \in [\delta]$, the *renormalization of g at k* is the induced automorphism g_k of $\mathbf{T}_\delta$ defined by $g_k = \sigma_{g(k)}^{-1} \circ g \circ \sigma_k$. We say that a group G of automorphisms of $\mathbf{T}_\delta$ is *self-similar* if $g_k \in G$ for all $g \in G$ and for all $k \in [\delta]$.

If $(\mathcal{X}, f)$ is any partial self-covering having topological degree δ, then the tree of preimages $\mathbf{T}_f$ can be identified with $\mathcal{T}_\delta$ in a canonical (but non-unique) way, and under this identification $\mathrm{IMG}(f)$ is a self-similar group acting faithfully on $\mathcal{T}_\delta$. The construction of the Chebyshev-like map $T_{\Phi,d}$ from a root system of rank n implies that the topological degree of $T_{\Phi,d}$ is d^n, which leads to the following result.

Corollary 4.5. *Let Φ be a root system of rank n. Then, for any $d \geq 2$, $\widetilde{W}_\Phi$ acts faithfully as a self-similar group on the d^n-ary tree as the iterated monodromy group* $\mathrm{IMG}(T_{\Phi,d})$.

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