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Abelian subgroups of the fundamental group of a space with no conjugate points

James Dibble

Abstract. Each Abelian subgroup of the fundamental group of a compact and locally simply connected d-dimensional length space with no conjugate points is isomorphic to $\mathbb{Z}^k$ for some $0 \leq k \leq d$. It follows from this and previously known results that each solvable subgroup of the fundamental group is a Bieberbach group. In the Riemannian setting, this may be proved using a novel property of the asymptotic norm of each Abelian subgroup.

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1. Introduction

A locally simply connected length space X with universal cover $\pi: \widehat{X} \to X$ has *no conjugate points* if any two points in $\widehat{X}$ can be joined by a unique geodesic. Let X be a compact and locally simply connected length space with no conjugate points and finite Hausdorff dimension d. In the Riemannian case, it has been believed for some time that Abelian subgroups of $\pi_1(X)$ must be finitely generated; for example, this is stated in [2], although the argument there contains a gap. It will be shown here that each Abelian subgroup is isomorphic to $\mathbb{Z}^k$ for some $0 \leq k \leq d$.

Theorem 1. *Each Abelian subgroup of $\pi_1(X)$ is isomorphic to $\mathbb{Z}^k$ for some $0 \leq k \leq d$.*

For nonpositively curved manifolds, Theorem 1 is a consequence of the flat torus theorem of Gromoll and Wolf [3] and Lawson and Yau [6], which was generalized to manifolds with no focal points by O'Sullivan [10].

It was proved by Yau [11] in the case of nonpositive curvature, and O'Sullivan [10] for no focal points, that every solvable subgroup of the fundamental group is a Bieberbach group. Croke and Schroeder [2] mapped out a way to generalize this to spaces with no conjugate points: if a torsion-free solvable group has the property that its Abelian subgroups are all finitely generated and straight, then it must be a Bieberbach group. Lebedeva [7] showed that finitely generated Abelian subgroups of the fundamental group of a compact and locally simply connected length space with no conjugate points must be straight. Combining this with Theorem 1 completes the argument set out by Croke and Schroeder.

Theorem 2. *Each solvable subgroup of $\pi_1(X)$ is a Bieberbach group.*

This continues the theme, developed in [2], [7], and [4], as well as in unpublished work of Kleiner, that, at the level of fundamental group, spaces with non-positive curvature.

Since the exponential map at each point of its universal cover is a diffeomorphism, a Riemannian manifold with no conjugate points must be aspherical. It's worth pointing out that this condition isn't enough to guarantee the conclusion of Theorem 1, as Mess [9] showed that, for each $n \geq 4$, there exists a compact manifold with universal cover $\mathbb{R}^n$ whose fundamental group contains a divisible Abelian subgroup, which cannot be finitely generated. This is discussed further in [8].

The second section contains a short proof of Theorem 1. The third section gives a different proof in the Riemannian setting, based on a property of Riemannian norms satisfied by the asymptotic norm of each Abelian subgroup of the fundamental group.

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2. Proof of Theorem 1

Fix $\hat{p} \in \widehat{X}$ and a basepoint $p = \pi(\hat{p})$ for $\pi_1(X)$. Overloading notation, each $\gamma \in \pi_1(X)$ will be identified with the corresponding deck transformation of $\widehat{X}$. Let Γ be an Abelian subgroup of $\pi_1(X)$, in which the group operation is written additively, and suppose $\sigma_1, \ldots, \sigma_k \in \Gamma$ are linearly independent. Denote by G the subgroup generated by the σ_i. The following are proved in [7]: on $\pi_1(X)$, the function

$$|\gamma|_\infty = \lim_{m\to\infty} \frac{\hat{d}(m\gamma(\hat{p}), \hat{p})}{m}$$

is positively homogeneous over $\mathbb{Z}$; it is bounded below on $\pi_1(X) \setminus \{e\}$ by $\mathrm{sys}(X)$, the length of the shortest nontrivial geodesic loop in X, so $\pi_1(X)$ is torsion free; its restriction to Γ satisfies the triangle inequality; and, with respect to the isomorphism $G \cong \mathbb{Z}^k$ that takes each σ_i to the i-th standard basis vector, $|\cdot|_\infty$ extends to a norm $\|\cdot\|_\infty$ on $\mathbb{R}^k$.

Denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^k$. From the identifications

$$G(\hat{p}) \cong G \cong \mathbb{Z}^k,$$

$G(\hat{p})$ inherits the coordinate functions $\rho_1, \dots, \rho_k$ on $\mathbb{Z}^k$. Since $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^k$, there exists $C > 0$ such that

$$\frac{1}{C}\|u\|_\infty \le \|u\| \le C\|u\|_\infty$$

for all $u \in \mathbb{R}^k$. The number C is a Lipschitz constant for the ρ_i on $G(\hat{p})$, and, as in the proof of Kirszbraun's theorem [5], the functions

$$f_i(\hat{x}) = \min_{\gamma \in G}[\rho_i(\gamma(\hat{p})) + C\hat{d}(\hat{x}, \gamma(\hat{p}))]$$

are Lipschitz extensions of the ρ_i to $\widehat{N}$. Each f_i is $(G, \mathbb{Z})$-equivariant, in the sense that $f_i(\gamma(\hat{x})) - f_i(\hat{x}) \in \mathbb{Z}$ for all $\hat{x} \in \widehat{N}$ and all $\gamma \in G$.

The map $f = (f_1, \dots, f_k)\colon \widehat{N} \to \mathbb{R}^k$ is Lipschitz, and $f(\gamma(\hat{x})) - f(\hat{x}) \in \mathbb{Z}^k$ for all $\hat{x} \in \widehat{N}$ and all $\gamma \in G$. By construction, $f(G(\hat{p})) = \mathbb{Z}^k$. Since G is Abelian, there exists a map $\phi\colon \mathbb{T}^k \to X$ such that $\phi_*(\pi_1(\mathbb{T}^k)) \cong G$. Lift ϕ to a map $\hat{\phi}\colon \mathbb{R}^k \to \widehat{N}$. The composition $f \circ \hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^k$ descends to a map $\mathbb{T}^k \to \mathbb{T}^k$ with surjective induced homomorphism, so by degree theory it must be surjective. Thus, f is surjective. Since a Lipschitz map cannot increase Hausdorff dimension, $k \le d$.

It follows that Γ has rank at most d. If it has rank zero, then the result is trivial. Without loss of generality, suppose it has rank $k > 0$. For any $\gamma \in \Gamma$, there exist $n, a_1, \dots, a_k \in \mathbb{Z}$ such that $n\gamma = \sum_{i=1}^k a_i \sigma_i$. It is well known that the function $F\colon \Gamma \to \mathbb{Q}^k$ defined by $F(e) = (0, \dots, 0)$ and

$$F(\gamma) = (a_1/n, \dots, a_k/n)$$

for $\gamma \ne e$ is a well-defined and injective homomorphism, so F is an isomorphism onto its image Γ_0. This map satisfies

$$\begin{aligned}\|F(\gamma)\|_\infty &= \|(a_1/n, \dots, a_k/n)\|_\infty = \frac{1}{|n|}\|(a_1, \dots, a_k)\|_\infty \\ &= \frac{1}{|n|}\Big|\sum_{i=1}^k a_i\sigma_i\Big|_\infty = \frac{1}{|n|}|n\gamma|_\infty = |\gamma|_\infty\end{aligned}$$

for any $\gamma \neq e$. For any distinct $q_0, q_1 \in \Gamma_0$, there exist distinct $\gamma_0, \gamma_1 \in \Gamma$ such that $F(\gamma_i) = q_i$ for each i. For $c = 1/C$, one has that

$$\begin{aligned}\|q_0 - q_1\| \geq c\|q_0 - q_1\|_\infty &= c\|F(\gamma_0) - F(\gamma_1)\|_\infty = c\|F(\gamma_0 - \gamma_1)\|_\infty \\ &= c|\gamma_0 - \gamma_1|_\infty \geq c \cdot \mathrm{sys}(X) > 0.\end{aligned}$$

Thus, Γ_0 is a discrete subgroup of $\mathbb{R}^k$ and, consequently, $\Gamma \cong \mathbb{Z}^k$.

3. Busemann functions in the Riemannian setting

For simplicity, it will be assumed in this section that X is a smooth d-dimensional Riemannian manifold, although what follows holds when X is C^r for some r depending on d. As before, let G be an Abelian subgroup of $\pi_1(X)$ generated by linearly independent $\gamma_1, \dots, \gamma_k$. The key step in the proof of Theorem 1 is the construction of a $(G, \mathbb{Z}^k)$-equivariant map $f: \widehat{X} \to \mathbb{R}^k$ such that $f(G(\hat{p})) = \mathbb{Z}^k$. When X is Riemannian, another such map may be constructed using a nondegenerate collection of Busemann functions.

An important theorem of Ivanov and Kapovitch [4] states that, whenever $\alpha_1, \alpha_2 \in \pi_1(X)$ commute, the change in the Busemann functions of axes of α_2 under the action of α_1 is constant on $\widehat{X}$. This was previously proved by Croke and Schroeder [2] for analytic X. Thus one may define a function $B: G \times G \to \mathbb{R}$ by setting $B(\alpha_1, \alpha_2)$ equal to that change.

Because $B(\alpha, \alpha) = |\alpha|_\infty^2$ for all $\alpha \in G$, one might hope to show that B extends to an inner product and, consequently, that $\|\cdot\|_\infty$ is Riemannian. In fact, B satisfies a number of the properties of an inner product: it is linear over $\mathbb{Z}$ in the first slot (see Corollary 4.2 of [4]), $B(\alpha_1, n\alpha_2) = nB(\alpha_1, \alpha_2)$ for all $n \in \mathbb{Z}$, and it satisfies a version of the Cauchy–Schwarz inequality,

$$|B(\alpha_1, \alpha_2)| \leq |\alpha_1|_\infty |\alpha_2|_\infty, \tag{1}$$

with equality if and only if α_1 and α_2 are rationally related. It follows that B extends to an inner product if and only if it is symmetric, but it's far from clear that symmetry holds in general (cf. [1]). Regardless, B also resembles an inner product in the following way.

Lemma 3. *For each $1 \leq m \leq k$, there exist $\alpha_1, \dots, \alpha_m \in \mathrm{span}\{\gamma_1, \dots, \gamma_m\}$ such that the $m \times m$ matrix $[B(\alpha_i, \alpha_j)]$ is nonsingular.*

If $\alpha_1, \dots, \alpha_k$ are as in Lemma 3 and $b_1, \dots, b_k$ are Busemann functions of respective axes, then, up to composition with an affine isomorphism, the map $F = (b_1, \dots, b_k): \widehat{X} \to \mathbb{R}^k$ is $(G, \mathbb{Z}^k)$-equivariant and satisfies $F(G(\hat{p})) = \mathbb{Z}^k$. The Riemannian version of Theorem 1 follows.

The proof of Lemma 3 is by induction. When $m = 1$, the conclusion holds with $\alpha_1 = \gamma_1$. Suppose the conclusion holds for some $1 \le m < k$. If the conclusion fails when $\alpha_{m+1} = \gamma_{m+1}$, then there exists a nonzero $c = (c_1, \ldots, c_{m+1})$ in the null space of the $(m+1) \times (m+1)$ matrix $[B(\alpha_j, \alpha_i)]$. The following lemma then completes the inductive step.

Lemma 4. *There exists a solid cone C centered around the ray $\{rc \mid r \ge 0\}$ such that, if $x = (x_1, \ldots, x_{m+1}) \in C \cap \mathbb{Z}^{m+1}$, $\tilde{\alpha}_i = \alpha_i$ for $1 \le i \le m$, and $\tilde{\alpha}_{m+1} = \sum_{i=1}^{m+1} x_i \alpha_i$, then the $(m+1) \times (m+1)$ matrix $[B(\tilde{\alpha}_i, \tilde{\alpha}_j)]$ is nonsingular.*

The proof of Lemma 4 uses the following elementary fact.

Lemma 5. *Let $A, C > 0$. Suppose M_ℓ is a sequence of $(p+1) \times q$ matrices of the form*

$$\begin{bmatrix} M \\ b_\ell \end{bmatrix}$$

for a fixed $p \times q$ matrix M and a sequence $b_\ell \in \mathbb{R}^q$ such that $\|b_\ell\| \to 0$. Suppose also that w_ℓ is a sequence of vectors in $\mathbb{R}^{p+1}$ of the form

$$\begin{bmatrix} a_\ell \\ C_\ell \end{bmatrix}$$

for $a_\ell \in \mathbb{R}^p$ satisfying $\|a_\ell\| \le A$ and $|C_\ell| \ge C$. If $v_\ell \in \mathbb{R}^q$ satisfy $M_\ell v_\ell = w_\ell$, then $\|M(v_\ell/\|v_\ell\|)\| \to 0$. Consequently, M has nontrivial null space.

Proof of Lemma 4. Without loss of generality, one may suppose that $\max |c_i| = 1$. Assume for the sake of contradiction that the result is false. Then, for each i and any fixed sequence $\varepsilon_\ell \searrow 0$, there exists a sequence of rational numbers p_i^ℓ / q_i^ℓ such that $|c_i - p_i^\ell / q_i^\ell| < \varepsilon_\ell$ and, when $\tilde{\alpha}_i^\ell = \alpha_i$ for $1 \le i \le m$ and $\tilde{\alpha}_{m+1}^\ell = \sum_{i=1}^{m+1} \left(\prod_{j \ne i} q_j^\ell\right) p_i^\ell \alpha_i$, each $(m+1) \times (m+1)$ matrix $M_\ell = [B(\tilde{\alpha}_i^\ell, \tilde{\alpha}_j^\ell)]$ is singular.

Let $W = [B(\alpha_j, \alpha_i)]$ for $1 \le i, j \le m+1$, and write

$$\begin{aligned} w_\ell &= W\Big(\Big(\prod_{j \ne 1} q_j^\ell\Big) p_1^\ell, \ldots, \Big(\prod_{j \ne m+1} q_j^\ell\Big) p_{m+1}^\ell\Big) \\ &= \Big(\sum_{i=1}^{m+1} \Big(\prod_{j \ne i} q_j^\ell\Big) p_i^\ell B(\alpha_i, \alpha_1), \ldots, \sum_{i=1}^{m+1} \Big(\prod_{j \ne i} q_j^\ell\Big) p_i^\ell B(\alpha_i, \alpha_{m+1})\Big) \\ &= (B(\tilde{\alpha}_{m+1}^\ell, \tilde{\alpha}_1^\ell), \ldots, B(\tilde{\alpha}_{m+1}^\ell, \tilde{\alpha}_m^\ell), B(\tilde{\alpha}_{m+1}^\ell, \alpha_{m+1})). \end{aligned} \tag{2}$$

Let $K = \max_{1\leq i,j\leq m+1} |B(\alpha_i,\alpha_j)|$. Then,

$$\begin{aligned}\|w_\ell\| &= \Big\|\Big(\prod_j q_j^\ell\Big) W(p_1^\ell/q_1^\ell,\ldots,p_{m+1}^\ell/q_{m+1}^\ell)\Big\| \\ &\leq \Big|\prod_j q_j^\ell\Big| K\varepsilon_\ell\sqrt{m+1}.\end{aligned} \tag{3}$$

The inductive hypothesis and the linearity of B in the first slot imply that $\alpha_1,\ldots,\alpha_{m+1}$ are linearly independent. The word norm of $\tilde{\alpha}_{m+1}^\ell$ with respect to the subgroup of H generated by $\alpha_1,\ldots,\alpha_{m+1}$ is

$$|\tilde{\alpha}_{m+1}^\ell|_{\text{word}} = \sum_{i=1}^{m+1}\Big|\prod_{j\neq i} q_j^\ell\Big||p_i^\ell|.$$

Because the corresponding norms on $\mathbb{R}^{m+1}$ are equivalent, there exists $D > 0$, depending only on $\alpha_1,\ldots,\alpha_{m+1}$, such that

$$\frac{1}{D}\sum_{i=1}^{m+1}\Big|\prod_{j\neq i} q_j^\ell\Big||p_i^\ell| \leq |\tilde{\alpha}_{m+1}^\ell|_\infty \leq D\sum_{i=1}^{m+1}\Big|\prod_{j\neq i} q_j^\ell\Big||p_i^\ell|.$$

By the Cauchy–Schwarz inequality (1), for each $1 \leq i \leq m$,

$$|B(\tilde{\alpha}_i^\ell,\tilde{\alpha}_{m+1}^\ell)| \leq |\tilde{\alpha}_i^\ell|_\infty|\tilde{\alpha}_{m+1}^\ell|_\infty \leq D\sqrt{K}\sum_{i=1}^{m+1}\Big|\prod_{j\neq i} q_j^\ell\Big||p_i^\ell|. \tag{4}$$

Similarly,

$$B(\tilde{\alpha}_{m+1}^\ell,\tilde{\alpha}_{m+1}^\ell) = |\tilde{\alpha}_{m+1}^\ell|_\infty^2 \geq (1/D^2)\Big[\sum_{i=1}^{m+1}\Big|\prod_{j\neq i} q_j^\ell\Big||p_i^\ell|\Big]^2. \tag{5}$$

Let

$$\begin{aligned} a_\ell &= (B(\tilde{\alpha}_1^\ell,\tilde{\alpha}_{m+1}^\ell),\ldots,B(\tilde{\alpha}_m^\ell,\tilde{\alpha}_{m+1}^\ell)),\\ b_\ell &= (B(\tilde{\alpha}_{m+1}^\ell,\tilde{\alpha}_1^\ell),\ldots,B(\tilde{\alpha}_{m+1}^\ell,\tilde{\alpha}_m^\ell)),\\ c_\ell &= B(\tilde{\alpha}_{m+1}^\ell,\tilde{\alpha}_{m+1}^\ell),\end{aligned}$$

and $M = [B(\alpha_i,\alpha_j)]$ for $1 \leq i, j \leq m$. Write

$$\tilde{a}_\ell = a_\ell/\Big[\sum_{i=1}^{m+1}\Big|\prod_{j\neq i} q_j^\ell\Big||p_i^\ell|\Big],$$

$$\tilde{b}_\ell = b_\ell \Big/ \Big| \prod_j q_j^\ell \Big|,$$

and

$$\tilde{c}_\ell = c_\ell \Big/ \Big[\Big| \prod_j q_j^\ell \Big| \sum_{i=1}^{m+1} \Big| \prod_{j \neq i} q_j^\ell \Big| |p_i^\ell| \Big].$$

By (2) and (3), $\|\tilde{b}_\ell\| \leq \|w_\ell\| / |\prod_j q_j^\ell| \leq K \varepsilon_\ell \sqrt{m+1}$; by (4), $\|\tilde{a}_\ell\| \leq D\sqrt{mK}$; and, by (5), $\tilde{c}_\ell \geq 1/(2D^2)$ for all sufficiently large ℓ. Since M is nonsingular, it follows from Lemma 5 that the matrices

$$\begin{bmatrix} M & \tilde{a}_\ell \\ \tilde{b}_\ell & \tilde{c}_\ell \end{bmatrix}$$

are nonsingular for all sufficiently large ℓ. The corresponding M_ℓ must also be nonsingular, which is a contradiction. □

When $m = 2$ in Lemma 3, inequality (1) implies that one may take $\alpha_1 = \gamma_1$ and $\alpha_2 = \gamma_2$. When X has no focal points, one may, by the flat torus theorem, take $\alpha_i = \gamma_i$ for all i. However, in the general case for $m \geq 3$, there is no apparent local structure that forces the Busemann functions of the axes of the γ_i to have linearly independent gradients, and it is not clear that the conclusion of Lemma 3 holds with $\alpha_i = \gamma_i$ for all i.

Question 6. *Must the $k \times k$ matrix $[B(\gamma_i, \gamma_j)]$ be nonsingular?*

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James Dibble, Department of Mathematics and Statistics, University of Southern Maine, Portland, ME 04103, USA

e-mail: james.dibble@maine.edu