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On spherical unitary representations of groups of spheromorphisms of Bruhat–Tits trees

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Abstract. Consider an infinite homogeneous tree $\mathcal{T}_n$ of valence $n+1$, its group $\mathrm{Aut}(\mathcal{T}_n)$ of automorphisms, and the group $\mathrm{Hier}(\mathcal{T}_n)$ of its spheromorphisms (hierarchomorphisms), i.e., the group of homeomorphisms of the boundary of $\mathcal{T}_n$ that locally coincide with transformations defined by automorphisms. We show that the subgroup $\mathrm{Aut}(\mathcal{T}_n)$ is spherical in $\mathrm{Hier}(\mathcal{T}_n)$, i.e., any irreducible unitary representation of $\mathrm{Hier}(\mathcal{T}_n)$ contains at most one $\mathrm{Aut}(\mathcal{T}_n)$-fixed vector. We present a combinatorial description of the space of double cosets of $\mathrm{Hier}(\mathcal{T}_n)$ with respect to $\mathrm{Aut}(\mathcal{T}_n)$ and construct a "new" family of spherical representations of $\mathrm{Hier}(\mathcal{T}_n)$. We also show that the Thompson group Th has $\mathrm{PSL}(2,\mathbb{Z})$-spherical unitary representations.

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1. Introduction

1.1. Groups of spheromorphisms of trees. Fix an integer $n \geqslant 2$. The *Bruhat–Tits tree* $\mathcal{T}_n$ is the infinite tree such that each vertex belongs to $n+1$ edges, see Figure 1. Denote by $\mathrm{Aut}(\mathcal{T}_n)$ the group of all automorphisms of $\mathcal{T}_n$. It is a totally disconnected locally compact group, its topology is defined by the condition: stabilizers of finite subtrees are open in $\mathrm{Aut}(\mathcal{T}_n)$.

Recall that Bruhat and Tits in 1966-1967 (see [3]) invented simplicial complexes (Bruhat–Tits buildings), which are p-adic counterparts of noncompact Riemannian symmetric spaces. Analogs of rank one noncompact symmetric spaces (as the Lobachevsky plane) are Bruhat–Tits trees with n being powers of prime p. In particular, p-adic $\mathrm{PSL}(2)$ acts on the tree $\mathcal{T}_p$. This fact became an initial point for investigations of groups acting on trees, see, e.g., Tits [58] and Serre [55]. Cartier [5] observed that the groups $\mathrm{Aut}(\mathcal{T}_n)$ are interesting objects from the point

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of view of representation theory and non-commutative harmonic analysis, and these groups are relatives of SL(2) over real and p-adic fields. G. Olshanski established that $\mathrm{Aut}(\mathcal{T}_n)$ are type I groups [46] and obtained a pleasant classification [47] of irreducible unitary representations of $\mathrm{Aut}(\mathcal{T}_n)$ (see an exposition in [11], see also the work [8] on tensor products).

The *boundary* (or *absolute*) $\mathrm{Abs}(\mathcal{T}_n)$ of $\mathcal{T}_n$ is a totally disconnected compact set, for a prime $n = p$ it can be identified with the p-adic projective line. The group $\mathrm{Aut}(\mathcal{T}_n)$ acts by homeomorphisms of the boundary. A *spheromorphism* (or *hierarchomorphism*) of $\mathcal{T}_n$ is a homeomorphism q of $\mathrm{Abs}(\mathcal{T}_n)$ such that for each point $y \in \mathrm{Abs}(\mathcal{T}_n)$ there is a neighborhood $\mathcal{N}(y)$, in which q coincides with some $r_y \in \mathrm{Aut}(\mathcal{T}_n)$. In other words, we cut a finite number of mid-edges of the tree and get a collection of finite pieces W_i and infinite pieces U_j. We forget finite pieces and choose embeddings $\theta_j : U_j \to \mathcal{T}_n$ such that images of θ_j are mutually disjoint and cover the whole tree (may be) without a finite subtree, see Figure 1.

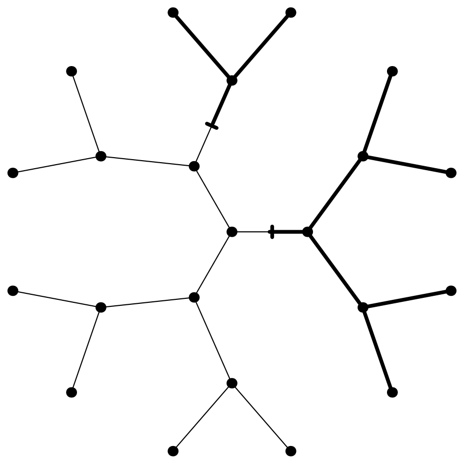


Figure 1. Cf. Subsection 1.1. A piece of the Bruhat–Tits tree $\mathcal{T}_2$. Transposing the thick branches we get an spheromorphism.

The group $\mathrm{Hier}(\mathcal{T}_n)$ of all spheromorphisms of the tree $\mathcal{T}_n$ is a locally compact topological group (see [14]). The topology is defined by the condition: the subgroup $\mathrm{Aut}(\mathcal{T}_n)$ is open and closed (*clopen*) in $\mathrm{Hier}(\mathcal{T}_n)$. The (countable) space of cosets $\mathrm{Hier}(\mathcal{T}_n)/\,\mathrm{Aut}(\mathcal{T}_n)$ has a discrete topology.

Remark. So we have a group $G = \mathrm{Hier}(\mathcal{T}_n)$ and a subgroup $K = \mathrm{Aut}(\mathcal{T}_n)$ such that K is a non-discrete totally disconnected group and the homogeneous space G/K is discrete. Topologies of this kind arise in representation theory of infinite symmetric groups, see [42], Subsection 3.7; a group with such a topology is used below in Section 4.

We can imagine the Bruhat–Tits tree as drawn on the plane $\mathbb{R}^2$. Then we get a structure of a cyclically ordered set on the boundary $\mathrm{Abs}(\mathcal{T}_n)$. The *Thompson group* Th is the group of all spheromorphisms preserving the cyclic order on $\mathrm{Abs}(\mathcal{T}_2)$. Initially R. Thompson proposed the group Th as a counterexample, this countable group really has strange properties but also it is an interesting positive object (see, e.g., [17], [6], [21], [50], [26], [4], [31], and [12]).

The groups $\mathrm{Hier}(\mathcal{T}_n)$ were introduced in 1984 ([37] and [38]) by the following reasoning:

1. for prime $n = p$ the group $\mathrm{Hier}(\mathcal{T}_p)$ contains the group of locally analytic diffeomorphisms of the p-adic projective line;
2. part of constructions of unitary representations of the group of diffeomorphisms of the circle have twins for the groups $\mathrm{Hier}(\mathcal{T}_n)$;
3. the groups $\mathrm{Hier}(\mathcal{T}_n)$ have several families of unitary representations that are spherical (see below) with respect to (noncompact) subgroup $\mathrm{Aut}(\mathcal{T}_n)$; in Addendum we explain why this property seems to be distinguished.

The topic of the present paper are unitary representations, we list some references on a wider context. The groups $\mathrm{Hier}(\mathcal{T}_n)$ are simple as abstract groups (Kapoudjian [23]), uniformly simple (Gal and Gismatullin [13]), compactly generated (Caprace and De Medts [7]) compactly presentable (Le Boudec [30]), they have nontrivial $\mathbb{Z}_2$-central extensions constructed by Kapoudjian [24] (it is interesting to find faithful unitary representations of this extension). They don't have property (T) (Navas [36][1]). These groups are simple locally compact groups that do not admit a lattice (Bader, Caprace, Gelander, and Mozes [2], this is the first example of such kind). See Kapoudjian [25] and Sauer and Thumann [52] on the action of $\mathrm{Hier}(\mathcal{T}_n)$ on CW-complexes. These groups can be included to families of relatives [39], [32], and [52].

It seems to the author that these groups while locally compact have various properties of infinite-dimensional (or "large") groups. For instance, constructions of spherical representations both in [37] and [38] and below in Section 4 are distinctive construction for infinite-dimensional groups. On the other hand, a parallel with infinite-dimensional groups also is incomplete, apparently the groups $\mathrm{Hier}(\mathcal{T}_n)$ have no trains in the sense of [40].

1.2. Sphericity. Let G be a topological group, K a closed subgroup. Let ρ be an irreducible unitary representation of the group G in a Hilbert space H. We say that a *representation ρ is K-spherical* if H contains a unique up to a scalar factor nonzero K-fixed vector v (the *spherical vector*). Its matrix element

$$\Phi(g) = \langle \rho(g)v, v\rangle_H, \quad \text{where } \|v\|^2 = 1,$$

[1] Notice that families of spherical representations of $\mathrm{Hier}(\mathcal{T}_n)$ in the boson and fermion Fock spaces constructed in [38] approximate the trivial one-dimensional representation

is called a *spherical function.* This function is automatically K-biinvariant, i.e.,

$$\Phi(k_1 g k_2) = \Phi(g) \qquad \text{for } g \in G,\, h_1, h_2 \in K.$$

In other words, a spherical function is defined on the double coset space $K \setminus G/K$.

Definition 1.1. Let G be a topological group, K a closed subgroup. The subgroup K is spherical if

A. for any irreducible unitary representation of G the subspace of K-fixed vectors has dimension $\leqslant 1$;
B. There is a faithful unitary representation[2] of G and a vector v such that the stabilizer of v is K.

Remark. a) The second condition is necessary for the following reason. Quite often (if K is not compact or "heavy" in the sense of [40]) a restriction of any nontrivial irreducible unitary representation of G to K has no nonzero K-fixed vector. More generally, if a vector v is fixed by K, then quite often v is automatically fixed by a certain larger group $\widetilde{K} \supset K$. Such phenomena were widely used in classical ergodic theory after Gelfand, Fomin [16] and Mautner [34]. A detailed investigation of such phenomena for Lie groups were done by Moore [35] and Wang [60], for p-adic groups by Wang [60] and [61]. Kaniuth and Lau [22] and Losert [33] discussed stabilizers of vectors in unitary representations of general locally compact groups (in their terminology subgroups that can be stabilizers of vectors "satisfy separation property").

b) For Lie groups there are weak analogs of sphericity for noncompact subgroups. One variant is "generalized Gelfand pairs" $G \supset K$, see [10]; in this case one considers spaces H^∞ of G-smooth vectors and fixed vectors in spaces $(H^\infty)'$ dual to H^∞. Another variant is "commutative spaces" (see, e.g., [59]), in this case one considers pairs $G \supset K$, for which algebras of K-invariant differential operators on G are commutative. Both definition do not require existence of K-invariant vectors in spaces of unitary representations.

1.3. The purposes of the paper. We prove the following statements.

Theorem 1.2. *The subgroup* $\mathrm{Aut}(\mathcal{T}_n)$ *is spherical in* $\mathrm{Hier}(\mathcal{T}_n)$.

Proposition 1.3. *Let* $\Phi_1(g)$, $\Phi_2(g)$ *be* $\mathrm{Aut}(\mathcal{T}_n)$*-spherical functions on* $\mathrm{Hier}(\mathcal{T}_n)$. *Then* $\Phi_1(g)\,\Phi_2(g)$ *is a spherical function.*

For known spherical pairs $G \supset K$ (finite-dimensional and infinite-dimensional) double coset spaces $K \setminus G/K$ admit explicit descriptions. In Section 3, we present such a description for the double coset space

$$\mathrm{Aut}(\mathcal{T}_n) \setminus \mathrm{Hier}(\mathcal{T}_n)/\,\mathrm{Aut}(\mathcal{T}_n).$$

[2] It can be reducible.

Double cosets correspond to $(n+1)$-valent graphs Γ consisting of two disjoint trees T_+ and T_- and a collection of edges connecting vertices of T_+ with vertices of T_- (cf. "tree pairs diagrams" in [4]).

In Section 4 we apply Nessonov's construction [45] of representations of infinite symmetric group to obtain a "new" family of spherical representations of $\mathrm{Hier}(\mathcal{T}_n)$.

The Addendum contains some comments on problem of sphericity for locally compact groups. We also show that the Thompson group Th has $\mathrm{PSL}(2,\mathbb{Z})$-spherical representations.

1.4. Some questions. Theorem 1.2 implies the following questions.

1. Is it possible to classify $\mathrm{Aut}(\mathcal{T}_n)$-spherical functions on $\mathrm{Hier}(\mathcal{T}_n)$?

2. Is $\mathrm{Hier}(\mathcal{T}_n)$ a type I group?

3. Is it possible a harmonic analysis on the space $\mathrm{Hier}(\mathcal{T}_n)/\mathrm{Aut}(\mathcal{T}_n)$ in some sense? This is not a question about the decomposition of ℓ^2 on this space, see the Addendum, Proposition A.3.

4. Let ρ be a spherical representation of $\mathrm{Hier}(\mathcal{T}_n)$, let P be the operator of orthogonal projection to the $\mathrm{Aut}(\mathcal{T}_n)$-fixed line. Consider the closure Γ_ρ of $\rho(g)$, where g ranges in $\mathrm{Hier}(\mathcal{T}_n)$, in the weak operator topology. Obviously (see Lemma 2.4) the semigroup Γ_ρ contains P, therefore $\Gamma_\rho \setminus \mathrm{Hier}(\mathcal{T}_n)$ contains operators of the form $\rho(g_1)P\rho(g_2)$ with $g_1, g_2 \in \mathrm{Hier}(\mathcal{T}_n)$. Does it contain something else?

Remark. The analog of the group of spheromorphisms for $n=\infty$ and its unitary representations are topics of a separate paper [41].

2. Sphericity

2.1. Notation. A *ray* in the Bruhat–Tits tree is a sequence of vertices a_j such that a_i and a_{i+1} are adjacent and $a_{i+2} \neq a_i$ for all i. We say that rays a_i and b_j are *equivalent* if $a_i = b_{i+k}$ starting from some i. The *boundary* (the notation: $\mathrm{Abs}(\mathcal{T}_n)$) of $\mathcal{T}_n$ is the space of classes of equivalent ways.

Let us cut the tree $\mathcal{T}_n$ at the middle of an edge. We call *branches* the two pieces of the tree. Each branch U determines a subset $B = \mathrm{Ba}[U]$ in the boundary corresponding to rays lying in U. Such subsets are called *balls*. For a given ball B denote by $\mathrm{Br}[B]$ the corresponding branch of the tree. In particular, each mid-edge determines a partition of $\mathrm{Abs}(\mathcal{T}_n)$ into two disjoint balls. We define the *topology* on $\mathrm{Abs}(\mathcal{T}_n)$ assuming that balls are clopen subsets in $\mathrm{Abs}(\mathcal{T}_n)$, this defines on $\mathrm{Abs}(\mathcal{T}_n)$ a structure of a totally disconnected compact set.

If B_1, B_2 are two balls, then

$$B_1 \supset B_2, \quad \text{or} \quad B_2 \supset B_1, \quad \text{or} \quad B_1 \cap B_2 = \varnothing. \tag{2.1}$$

Lemma 2.1. *Let $B_1 \subset B_2 \subset \dots$ be an increasing sequence of balls. Then it has a maximal element or* $\mathrm{Abs}(\mathcal{T}_n) \setminus \cup_j B_j$ *is one point.*

Proof. Let a sequence of balls $B_j = \mathrm{Ba}[U_j]$ strictly decrease. Let u_j be the corresponding mid-edges, $[p_j q_j]$ the corresponding edges and assume $p_j \notin U_j$, $q_j \in U_j$. Then the points q_1, p_1, q_2, p_2, … lye on a ray. Let $a \in \mathrm{Abs}(\mathcal{T}_n)$ be the limit of this ray. Then $\cup B_j = \mathrm{Abs}(\mathcal{T}_n) \setminus a$. □

We say that $h \in \mathrm{Aut}(\mathcal{T}_n)$ is *hyperbolic* if it has two fixed points a, b on $\mathrm{Abs}(\mathcal{T}_n)$ and induces a nontrivial shift on the bi-infinite ray $\dots x_{-1}, x_0, x_1, \dots$ connecting a and b. Let c be a point of the boundary. The *parabolic subgroup* $P_c \subset \mathrm{Aut}(\mathcal{T}_n)$ is the group of transformations g such that g fixes c, and for any ray $x_1, x_2, \dots$ going to c we have $g x_N = x_N$ for sufficiently large N.

2.2. Proof of Theorem 1.2. The group $\mathrm{Aut}(\mathcal{T}_n)$ has a normal subgroup $\mathrm{Aut}_+(\mathcal{T}_n)$ of index 2 defined as follows. Let us color vertices of $\mathcal{T}_n$ black and white in such a way that each edge has edges of different colors. Then $\mathrm{Aut}_+(\mathcal{T}_n)$ is the subgroup of those automorphisms of $\mathcal{T}_n$ which preserve the coloring. This defines a homomorphism of $\mathrm{Aut}(\mathcal{T}_n)$ to the group $\mathbb{Z}_2$ and therefore a one-dimensional representation of $\mathrm{Aut}(\mathcal{T}_n)$. Other nontrivial irreducible representations of $\mathrm{Aut}(\mathcal{T}_n)$ are infinite-dimensional. It is sufficient to prove the following statement:

Proposition 2.2. *Consider an irreducible unitary representation ρ of* $\mathrm{Hier}(\mathcal{T}_n)$ *in a Hilbert space H. Denote by $H^{\mathrm{Aut}+}$ the subspace of all* $\mathrm{Aut}_+(\mathcal{T}_n)$*-fixed vectors. Then* $\dim H^{\mathrm{Aut}+}$ *is* $\leqslant 1$.

Denote by P the operator of orthogonal projection to $H^{\mathrm{Aut}+}$. Clearly,

$$P\rho(h) = \rho(h)P = P \quad \text{for all } h \in \mathrm{Aut}_+(\mathcal{T}_n). \tag{2.2}$$

For $g \in \mathrm{Hier}(\mathcal{T}_n)$ we define the operator $\hat\rho(g)\colon H^{\mathrm{Aut}+} \to H^{\mathrm{Aut}+}$ by

$$\hat\rho(g) := P\rho(g)P.$$

Clearly, $\hat\rho(g)$ depends only on the double coset $\mathrm{Aut}_+(\mathcal{T}_n) \cdot g \cdot \mathrm{Aut}_+(\mathcal{T}_n)$.

Lemma 2.3. *The operators $\hat\rho(g)$ commute, i.e., for any g_1, $g_2 \in$* $\mathrm{Hier}(\mathcal{T}_n)$

$$\hat\rho(g_1)\hat\rho(g_2) = \hat\rho(g_2)\hat\rho(g_1). \tag{2.3}$$

Reduction of Theorem 1.2 to Lemma 2.3. Let the conclusion of the lemma hold. Assume that $\dim H^{\mathrm{Aut}_+} > 1$. Notice that $\hat{\rho}(g^{-1}) = \hat{\rho}(g)^*$, therefore commuting bounded operators

$$\hat{\rho}(g) + \hat{\rho}(g^{-1}), \quad i(\hat{\rho}(g) - \hat{\rho}(g^{-1})),$$

are self-adjoint. Therefore all operators $\hat{\rho}(g)$ have a proper common invariant subspace $V \subset H^{\mathrm{Aut}_+}$. Then $\mathrm{Aut}_+(\mathcal{T}_n)$-cyclic span of V is a proper $\mathrm{Aut}_+(\mathcal{T}_n)$-subspace in H. Indeed, let $v \in V$. Then

$$P\rho(g)v = P\rho(g)Pv = \hat{\rho}(g)v \in V,$$

and the projection of the cyclic span to H^{Aut_+} is contained in V. This contradicts to the irreducibility of ρ.

Lemma 2.4. *Let $h_j \in \mathrm{Aut}_+(\mathcal{T}_n)$ tend to infinity*[3]. *Then for any unitary representation ρ of $\mathrm{Aut}_+(\mathcal{T}_n)$ the sequence $\rho(h_j)$ converges to P in the weak operator topology.*

Indeed, by [29], for any nontrivial irreducible representation of $\mathrm{Aut}_+(\mathcal{T}_n)$ the sequence $\rho(h_j)$ weakly converges to 0.

On the other hand this can be easily verified case-by-case starting Olshanski's classification theorem [47]. Notice also that this is a counterpart of the well-known Howe–Moore theorem [20] about real Lie groups.

In fact, we need the following special case of Lemma 2.4.

Corollary 2.5. *Let $h \in \mathrm{Aut}_+(\mathcal{T}_n)$ be a hyperbolic element. Then for any unitary representation ρ of $\mathrm{Aut}_+(\mathcal{T}_n)$ the sequence $\rho(h^m)$ weakly converges to P.*

Proof of Lemma 2.3. Fix a ball $B \subset \mathrm{Abs}(\mathcal{T}_n)$. Denote by $G(B)$ the subgroup in $\mathrm{Hier}(\mathcal{T}_n)$ consisting of spheromorphisms trivial outside B. Clearly,

$$\mathrm{Aut}_+(\mathcal{T}_n) \cdot G(B) \cdot \mathrm{Aut}_+(\mathcal{T}_n) = \mathrm{Hier}(\mathcal{T}_n),$$

i.e., any double coset has a representative in $G(B)$. Choose two disjoint balls B_1 and B_2. For a verification of (2.3) we can assume $g_1 \in G(B_1)$, $g_2 \in G(B_2)$. Choose a hyperbolic element $U \in \mathrm{Aut}_+(\mathcal{T}_n)$ with an attractive fixed point $a \in B_2$. For $k > 0$ we have

$$U^k g_2 U^{-k} \in G(U^k B_2) \subset G(B_2).$$

Hence g_1 and $U^k g_2 U^{-k}$ have disjoint supports, therefore they commute. Thus,

$$\rho(g_1)\,\rho(U^k)\,\rho(g_2)\,\rho(U^{-k}) = \rho(U^k)\,\rho(g_2)\,\rho(U^{-k})\,\rho(g_1).$$

[3] We say that h_j *tends to* ∞ if any compact subset of $\mathrm{Aut}_+(\mathcal{T}_n)$ contains only a finite number of elements h_j. In other words h_j tends to infinity in the Alexandroff compactification of a locally compact space $\mathrm{Aut}_+(\mathcal{T}_n)$.

Multiplying this from the left and the right by P and keeping in mind (2.2), we get

$$P\rho(g_1)\rho(U^k)\rho(g_2)P = P\rho(g_2)\rho(U^{-k})\rho(g_1)P.$$

Passing to the weak limit as $k \to +\infty$ and applying Lemma 2.4 we arrive to

$$P\,\rho(g_1)\,P\,\rho(g_2)\,P = P\,\rho(g_2)\,P\,\rho(g_1)\,P.$$

This is the equality (2.3). □

Proof of Proposition 1.3

Proposition 2.6. *Let $G \supset K$ be topological groups. Assume that K does not admit nontrivial finite-dimensional unitary representations. Let $\Phi_1(g)$, $\Phi_2(g)$ be K-spherical functions on G. Then $\Phi_1(g)\,\Phi_2(g)$ is a spherical function.*

Recall the following lemma (see [53], Sublemma 1):

Lemma 2.7. *Let ν_1, ν_2 be unitary representations of a group Γ. If the tensor product $\nu_1 \otimes \nu_2$ contains a nonzero Γ-invariant vector, then both ν_1 and ν_2 have finite-dimensional subrepresentations.*

Proof of Proposition 2.6. Let ρ_1 and ρ_2 be K-spherical representations of G in H_1 and H_2. Let v_1, v_2 be fixed vectors. By the lemma, $v_1 \otimes v_2$ is a unique K-fixed vector in $H_1 \otimes H_2$. The cyclic span W of $v_1 \otimes v_2$ is an irreducible subrepresentation. Indeed, assume that $W = W_1 \oplus W_2$ is a sum of invariant subspaces. Then both projections of $v_1 \otimes v_2$ to W_1, W_2 are K-fixed, therefore $v_1 \otimes v_2$ must be contained in one of summands, say W_1, and thus the cyclic span of $v_1 \otimes v_2$ is contained in W_1, i.e., $W = W_1$.

Now we consider the representation of G in W,

$$\begin{aligned}\langle(\rho_1(g) \otimes \rho_2(g))v_1 \otimes v_2,\ v_1 \otimes v_2\rangle_W &= \langle\rho_1(g)v_1, v_1\rangle_{H_1} \cdot \langle\rho_2(g)v_2, v_2\rangle_{H_2}\\ &= \Phi_1(g)\,\Phi_2(g).\end{aligned}$$ □

Proof of Proposition 1.3. Consider $\mathrm{Aut}(\mathcal{T}_n)$-spherical representations ρ_1, ρ_2 of $\mathrm{Hier}(\mathcal{T}_n)$. They also are $\mathrm{Aut}_+(\mathcal{T}_n)$-spherical. Therefore their tensor product has a unique $\mathrm{Aut}_+(\mathcal{T}_n)$-fixed vector. This vector also is $\mathrm{Aut}(\mathcal{T}_n)$-fixed. □

3. The space of double cosets

3.1. Terminology. Let T be a tree, $A_1, \dots, A_N$ a collection of vertices. The *subtree spanned by* $A_1, \dots, A_N$ is the minimal subtree containing these points.

Let S be a finite tree. The *boundary* ∂S of S is the set of vertices of valence 1.

We regard the Bruhat–Tits trees as 1-dimensional complexes with 0-cells located at vertices of the tree and mid-edges. Respectively, 1-cells are half-edges, see Figure 2.

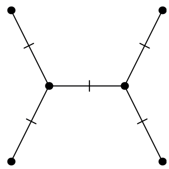


Figure 2. Cf. to Subsection 3.1. The subdivision of the Bruhat–Tits tree.

Let R be a tree such that valences of all vertices are $\leqslant (n+1)$ and number of vertices is $\geqslant 3$. A *thorn* R is such a tree equipped with the following structure of an 1-dimensional simplicial complex. Consider the subtree R° (the *skeleton of the thorn*) of R spanned by all vertices that are not contained in the boundary ∂R. Then 0-cells of the thorn are vertices of R and mid-edges of R°. Respectively, 1-cells are half-edges of R° and edges of $R \setminus R^\circ$. We call vertices of R° by *vertices of thorn*, and points of ∂R by *spikes of the thorn*, see Figure 3(a).

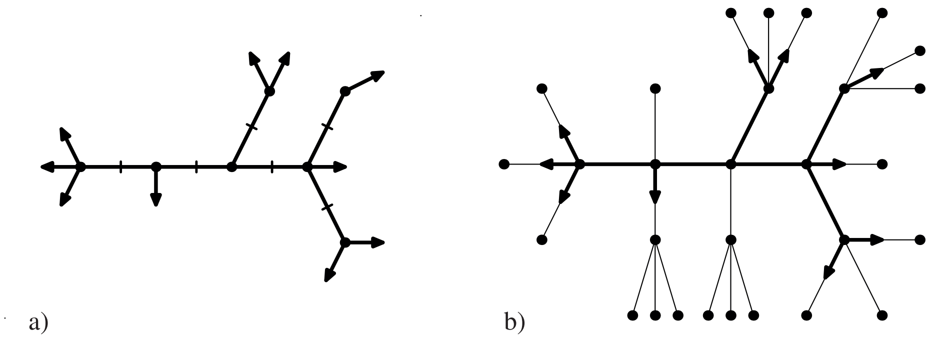


Figure 3. Cf. Subsection 3.1. a) A thorn ($n = 3$). The left vertex is perfect. Cutting the adjacent mid-edge off we get a reduced thorn. b) A sub-thorn of the Bruhat–Tits tree $\mathcal{T}_3$. On b) and figures below we omit mid-edges.

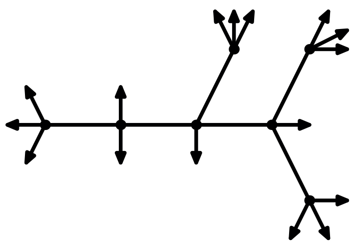


Figure 4. Cf. Subsection 3.1. A perfect thorn ($n = 3$).

a) b) 

Figure 5. a) The empty thorn. b) The thorn with one vertex and one spike.

Additionally, we allow an *empty thorn* and a *thorn having 1 vertex and one spike*, see Figure 5.

Denote by $\mathrm{spike}(R)$ the set of spikes of a thorn R, $\mathrm{vert}(R)$ the set of vertices of R. Two thorns R_1, R_2 are *isomorphic* if there is an isomorphism $R \to R'$ of complexes sending vertices to vertices and spikes to spikes. Cutting a thorn in a mid-edge we get two *branches*.

We embed thorns R to the Bruhat–Tits tree $\mathfrak{T}_n$ isomorphically sending vertices to vertices and spikes to mid-edges. We call images of such embeddings by *sub-thorns* of the Bruhat–Tits tree, see Figure 3(b).

Let R be a thorn. We say a *thorn is perfect* if all its vertices have valence $(n+1)$, see Figure 4. We say that a *vertex is perfect* if it is contained in ∂R° and its valence is $(n+1)$, see Figure 3(b). More generally, a *branch of a thorn is perfect* if all its vertices have valences $(n+1)$.

A thorn is *reduced* if it has no perfect vertices. Let R be an arbitrary thorn. Cutting of all perfect branches off we come to a reduced thorn (in particular, if R is perfect, then the corresponding reduced thorn is empty.)

3.2. Clopen sets. Denote by $\mathrm{Clop}(\mathfrak{T}_n)$ the set of all nonempty clopen subsets of $\mathrm{Abs}(\mathfrak{T}_n)$, by $\mathrm{Clop}^\star(\mathfrak{T}_n)$ the subset consisting of proper clopen subsets (i.e., we remove the point of $\mathrm{Clop}(\mathfrak{T}_n)$ corresponding the whole $\mathrm{Abs}(\mathfrak{T}_n)$).

Clearly, any clopen subset Ω can be represented as a union of a finite number of disjoint balls

$$\Omega := B_1 \sqcup \dots \sqcup B_\iota .$$

This representation is not unique, since any ball B can be canonically represented as a disjoint union of n smaller balls. It is easy to observe (see [54], Addendum "Structure of p-adic varieties", or [38]), that the remainder $\upsilon(\Omega)$ of ι modulo $n-1$ is uniquely defined by Ω. According this, $\mathrm{Clop}^\star(\mathfrak{T}_n)$ splits as a disjoint union

$$\mathrm{Clop}^\star(\mathfrak{T}_n) = \coprod_{\iota=0}^{n-2} \mathrm{Clop}^\star_\iota(\mathfrak{T}_n). \tag{3.1}$$

Proposition 3.1. a) *Disjoint unions of balls* $B_1 \sqcup \dots \sqcup B_\iota$ *are in one-to-one correspondence with sub-thorns of* $\mathfrak{T}_n$.

b) *Partitions* $\mathrm{Abs}(\mathfrak{T}_n) = B_1 \sqcup \dots \sqcup B_N$ *are in one-to-one correspondence with perfect sub-thorns of* $\mathfrak{T}_n$.

c) *Nonempty clopen sets in* $\mathrm{Abs}(\mathfrak{T}_n)$ *are in one-to-one correspondence with reduced sub-thorns of* $\mathfrak{T}_n$.

d) *Orbits of* $\mathrm{Aut}(\mathfrak{T}_n)$ *on* $\mathrm{Clop}(\mathfrak{T}_n)$ *are numerated by equivalence classes of reduced thorns.*

Description of the correspondence. Let p, q be adjacent vertices of $\mathcal{T}_n$. Denote by $\overrightarrow{pq}$ the thorn having one vertex p and one spike in the mid-edge pq. Cutting the edge pq at the mid-point we get two branches. We choose the branch U containing q and the corresponding ball $B[\overrightarrow{pq}]$, see Figure 6.

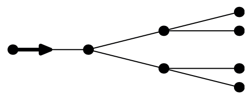


Figure 6. A spike and the corresponding branch ($n = 2$).

A sub-thorn $\longrightarrow$ a union of balls. Consider a sub-thorn in $\mathcal{T}_n$. Then each spike corresponds to a ball. Taking a union of these balls we get a clopen subset with a given partition into balls.

Notice, that starting a perfect thorn we get the whole boundary $\mathrm{Abs}(\mathcal{T}_n)$.

A union of balls $\longrightarrow$ a sub-thorn. Conversely, fix a representation of Ω as a disjoint union of balls $B_1 \sqcup \cdots \sqcup B_m$. Let $U_1, \ldots U_m$ be the corresponding branches of $\mathcal{T}_n$. Let $u_1, \ldots, u_m$ be mid-edges that cut these branches off. We consider the minimal sub-thorn R of $\mathcal{T}_n$ containing $u_1, \ldots, u_m$.

A clopen set $\longrightarrow$ a reduced sub-thorn. Let Ω be a proper clopen set. By Lemma 2.1, any sub-ball $B \subset \Omega$ is contained in a unique maximal sub-ball $\widetilde{B} \subset \Omega$. We take the partition of Ω into maximal sub-balls and take the corresponding thorn. Clearly, it is reduced.

3.3. Double cosets and bi-thorns. A *bi-thorn* is the following collection of data $\mathfrak{Bt}(R, Q; \theta)$:

- an ordered pair of perfect thorns R, Q with the same number of vertices;
- a bijection θ: spike$(R) \to$ spike(Q).

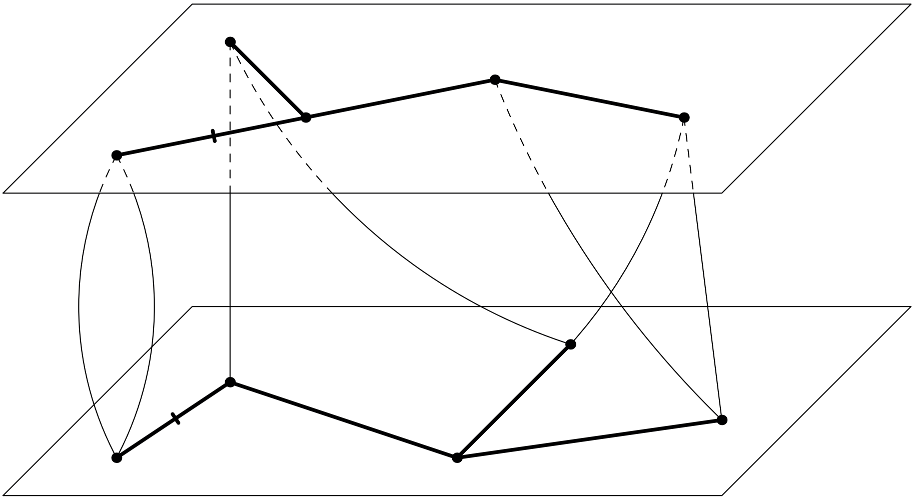
We admit an *empty bi-thorn*.

Equivalently, we have an $(n+1)$-valent graph, which contains a pair of disjoint subtrees R°, Q° and the remaining edges connect vertices of R° and vertices of Q° (we admit several edges between two vertices), see Figure 7.

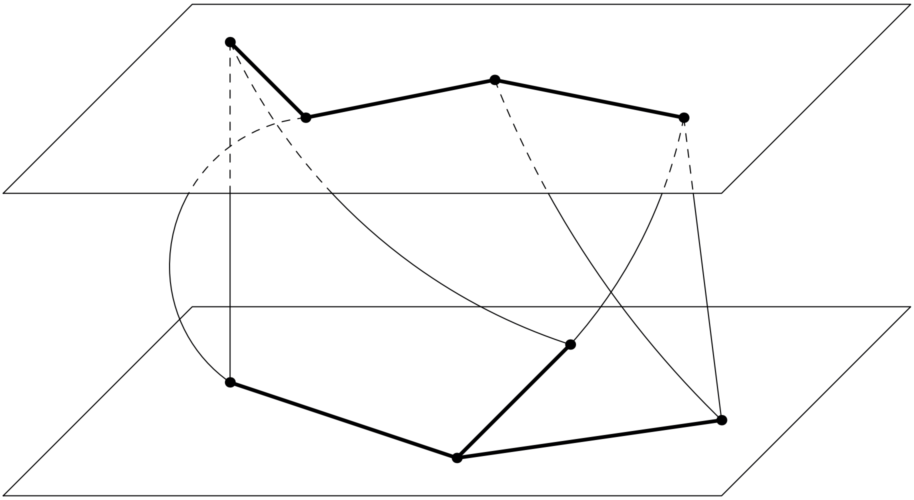
Consider a bi-thorn $\mathfrak{Bt}(R, Q; \theta)$. Let a be a vertex of $\partial(R^\circ)$, a' be a unique adjacent vertex of R°. Let b a vertex of $\partial(Q^\circ)$ and b' the adjacent vertex. We say that a, b are *similar* if θ sends all spikes at a to spikes at b, see Figure 7. In this situation, we can cut the mid-edges of $a'a$ and $b'b$. The thorn splits into two pieces. We remove the piece with two vertices a and b and modify θ saying that it sends the mid-edge of $a'a$ to the mid-edge of $b'b$. In this way we get a new thorn.

We say that a bi-thorn is *minimal* if it has not a pair of similar vertices.

Proposition 3.2. *There is a canonical one-to-one correspondence between the double coset space* $\mathrm{Aut}(\mathcal{T}_n) \setminus \mathrm{Hier}(\mathcal{T}_n) / \mathrm{Aut}(\mathcal{T}_n)$ *and the set of minimal bi-thorns.*



a)



b)

Figure 7. a) A bi-thorn. The left vertices of the upper and lower thorns are similar. b) We cut off the left vertices and get a minimal bi-thorn (an additional "vertical" arc appears instead of two cut vertical arcs).

Let us construct the correspondence. Let $g \in \mathrm{Hier}(\mathcal{T}_n)$. Take a ball $B = \mathrm{Ba}[U]$ and assume that gB is a ball, $gB = \mathrm{Ba}[V]$. We say that g *regards* the ball B if the map $g\colon B \to gB$ is induced by an isomorphism of the branches $U \to V$.

Let g regard a ball B. Then there is a unique maximal ball $C = \widetilde{B} \supset B$ regarded by g. Thus we get a partition

$$\mathrm{Abs}(\mathcal{T}_n) = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_N$$

consisting of maximal balls regarded by g and the corresponding partition

$$\mathrm{Abs}(\mathcal{T}_n) = gC_1 \sqcup gC_2 \sqcup \cdots \sqcup gC_N$$

consisting of balls regarded by g^{-1}. We take thorns R and Q corresponding to this partitions, by construction g determines a bijection between their spikes. □

Corollary 3.3. *Fix* $g \in \mathrm{Hier}(\mathcal{T}_n)$. *Fix an* $\mathrm{Aut}(\mathcal{T}_n)$*-orbit* $\mathcal{O}$ *in* $\mathrm{Clop}^\star(\mathcal{T}_n)$. *Then for all but a finite number of elements* $\Omega \in \mathcal{O}$ *we have* $g\Omega \in \mathcal{O}$.

Proof. According the previous proof, g canonically determines a pair of sub-thorns R and Q of the Bruhat–Tits tree. The orbit $\mathcal{O}$ corresponds to a certain reduced thorn T. Elements Ω of the orbit correspond to sub-thorns S in $\mathcal{T}_n$ isomorphic to T. Clearly, if $S \cap R = \varnothing$, then $g\Omega \in \mathcal{O}$. □

4. A family of spherical representations

4.1. The infinite symmetric group with Young subgroup. Fix k. Consider k countable sets $\Pi_1, \dots, \Pi_k$ and their disjoint union

$$\mathbf{\Pi} := \Pi_1 \sqcup \cdots \sqcup \Pi_k.$$

First, consider the group G of all finitely supported permutations of $\mathbf{\Pi}$ and its (Young) subgroup K preserving each Π_j. Then $G \supset K$ is a spherical pair and according Nessonov [45] all K-spherical functions on G have the following form Φ_S (a detailed exposition of a proof of this theorem is contained also in [42], Section 8). Consider a positive (semi)definite matrix S of size $k \times k$ with $s_{jj} = 1$. Then

$$\Phi_S(\sigma) = \prod_{p,q=1}^{k} s_{pq}^{\theta_{p,q}(\sigma)}, \quad \sigma \in G, \tag{4.1}$$

where $\theta_{p,q}(\sigma)$ is the number of elements $\alpha \in \Pi_p$ such that $\sigma\alpha \in \Pi_q$.

To construct the corresponding unitary representations of G we consider a Euclidean space V and a collection of unit vectors $e_1, \dots, e_k$ such that one has $\langle e_p, e_q\rangle_V = s_{p,q}$ (we can assume that V is spanned by these vectors). Consider the tensor product[4]

$$\bigotimes_{p=1}^{k}\Bigl(\bigotimes_{\alpha\in\Pi_p}(V,e_p)\Bigr),$$

we see that factors are enumerated by elements of the set $\mathbf{\Pi}$. The group G acts by permutations of the factors. A unique K-fixed vector is

$$\mathcal{E} := \otimes_{p=1}^{k} e_p^{\otimes\infty}.$$

The G-cyclic span of the vector $\mathcal{E}$ is an irreducible spherical representation of G and the spherical function is given by the formula (4.1).

Second, we notice that our representation can be extended by the continuity to a larger group $\mathbf{G}$. It consists of all permutations σ of the set Π such that for all p for all but a finite number of $\alpha \in \Pi_p$, we have $\sigma\alpha \in \Pi_p$ (permutations of factors in the tensor product are well defined for such σ). The spherical subgroup $\mathbf{K}$ consists of all permutations preserving each subset Π_p.

4.2. Embeddings of Hier($\mathcal{T}_n$) to the group G. Consider a collection of reduced thorns $T_1, \dots, T_N$, let they correspond to the same ι in the decomposition (3.1). Consider the corresponding $\mathrm{Aut}(\mathcal{T}_n)$-orbits $\mathcal{O}_1, \dots, \mathcal{O}_N$ in $\mathrm{Clop}_\iota^\star(\mathcal{T}_n)$ and the complement $\mathcal{P}$ to the union of these orbits. Thus we get a partition

$$\mathrm{Clop}_\iota^\star(\mathcal{T}_n) = \mathcal{P}\sqcup\mathcal{O}_1\sqcup\cdots\sqcup\mathcal{O}_N.$$

Consider the group $\mathbf{G}$ corresponding to this partition. By Corollary 3.3, the group $\mathrm{Hier}(\mathcal{T}_n)$ is contained in $\mathbf{G}$. Obviously, $\mathrm{Aut}(\mathcal{T}_n) \subset \mathbf{K}$. So we can apply the Nessonov construction.

Remark. Fix $\iota = 0, 1, \dots, n-2$. Consider a Hilbert space V and a countable set of unit vectors e_T enumerated by reduced thorns whose number of spikes is ι modulo $n-1$. Let this set have a unique limit point e (and hence a sequence composed of e_S in any order converges to e. For a clopen subset Ω denote by $T(\Omega)$ the corresponding reduced thorn. Consider the following tensor product

$$\mathcal{H} := \bigotimes_{\Omega\in\mathrm{Clop}_\iota^\star}(V, e_{T(\Omega)}).$$

The action of the group $\mathrm{Hier}(\mathcal{T}_n)$ in $\mathcal{H}$ by permutations of factors is well defined if and only if the following product absolutely converges for all hierarchomorphisms g:

$$\Phi(g) = \prod_{\Omega\in\mathrm{Clop}_\iota^\star}\langle e_{T(g\Omega)}, e_{T(\Omega)}\rangle_V.$$

[4] Recall that a definition of a tensor product of an infinite family H_j of Hilbert spaces requires a fixing of a distinguished unit vector $\xi_j \in H_j$ in each factor, a tensor product depends on a choice of ξ_j. For details, see, e.g., [19], Appendix A.

Clearly, if the sequence e_T converges fast enough, then this is the case. In this situation, we get a spherical representation of $\mathrm{Hier}(\mathcal{T}_n)$ in $\mathcal{H}$ with the spherical vector $\otimes_{\Omega\in\mathrm{Clop}_\iota^\star} e_{T(\Omega)}$ and the spherical function $\Phi(g)$.

It can be interesting to find precise conditions for a family e_T providing well-definiteness of this construction.

Addendum. Several comments on the sphericity phenomenon

A.1. General remarks on sphericity. Thus $\mathrm{Aut}(\mathcal{T}_n)$ is a noncompact spherical subgroup in a locally compact group $\mathrm{Hier}(\mathcal{T}_n)$. According to [43], the subgroup $\mathrm{PSL}(2,\mathbb{R})$ is spherical in the group $\mathrm{Diff}^3(S^1)$ of C^3-diffeomorphisms of the circle S^1. We explain why this seems distinguished.

Phenomenon of sphericity was discovered by Gelfand in 1950, [15]. He showed that maximal compact subgroups K in semisimple Lie groups $G \supset K$ are spherical (as $\mathrm{GL}(n,\mathbb{R}) \supset \mathrm{O}(n)$ or $\mathrm{Sp}(2n,\mathbb{R}) \supset \mathrm{U}(n)$). Also symmetric subgroups in semisimple compact Lie groups are spherical (as $\mathrm{U}(n) \supset \mathrm{O}(n)$ or $\mathrm{O}(2n) \supset \mathrm{U}(n)$). The third family of spherical pairs is Cartan motion groups (as the semidirect product of $\mathrm{O}(n)$ and the additive group of real symmetric matrices of order n, in this case the subgroup $\mathrm{O}(n)$ is spherical). This case is degenerate in a certain sense.

Related spherical representations of semisimple groups played a distinguished role in theory of unitary representations, and spherical functions were an important standpoint for development of modern theory of multi-dimensional special functions.

In 1979 Krämer [28] observed that simple compact Lie groups can have non-symmetric spherical subgroups as $\mathrm{O}(2n+1) \supset \mathrm{U}(n)$ or $\mathrm{Sp}(2n+2) \supset \mathrm{Sp}(2n) \times \mathrm{SO}(2)$, in the most of cases such pairs can be obtained from Gelfand pairs $G \supset K$ by a minor enlargement of G or minor reduction of K. Mikityuk and Brion extended the Krämer classification to semisimple compact groups. There is also a story with finite spherical pairs $G \supset K$, see, e.g., [9].

On the other hand infinite-dimensional limits of Gelfand pairs (as $\mathrm{GL}(\infty,\mathbb{R}) \supset \mathrm{O}(\infty)$) are spherical. G. Olshanski [48] and [49] understood that such pairs have a substantial representation theory, later there appeared related harmonic analysis. For infinite-dimensional (large) groups the phenomenon of sphericity is more usual than for Lie group, and at least representation theory can be developed in quite wide generality (see, e.g., [44], [45], [42], and [41]), in Subsection 4.1 we used a construction of this kind. In a known zoo, spherical subgroups are "heavy groups" in the sense of [40] (as the complete unitary group, the complete symmetric group, the group of all measure preserving transformations).

Two examples mentioned in the beginning of the present subsection are outside these two families. In one case a noncompact Lie group $\mathrm{SL}(2,\mathbb{R})$ is a spherical subgroup in an infinite-dimensional group $\mathrm{Diff}^3(S^1)$, in another case a noncompact subgroup $\mathrm{Aut}(\mathcal{T}_n)$ is spherical in a locally compact group $\mathrm{Hier}(\mathcal{T}_n)$.

A.2. On compactness of stabilizers of vectors in unitary representations. In examples of spherical pairs $G \supset K$ of Lie groups discussed above subgroups K are compact. The general statement "a spherical subgroup in a Lie group is compact" formally is incorrect, but informally this is close to a truth.

There is a theorem of Moore [35] about possible stabilizers of vectors in unitary representation, whose precise formulation is slightly sophisticated. We formulate a simpler statement.

Let G be a connected Lie group, Z the center; denote by $\mathfrak{g} \supset \mathfrak{z}$ their Lie algebras. Denote by $\mathrm{Ad}_{\mathfrak{g}}(\cdot)$ the adjoint representation of G in $\mathfrak{g}$, in fact we have a representation of the quotient group G/Z in the group $\mathrm{GL}[\mathfrak{g}]$ of all linear operators of the space $\mathfrak{g}$.

Let ρ be a faithful irreducible unitary representation of G in a Hilbert space V. An irreducible faithful representation determines an injective homomorphism from Z to the unit circle $\mathbb{T}$ on the complex plane. For this reason $\dim \mathfrak{z} \leqslant 1$, and we have three possibilities: $Z = \mathbb{T}$, Z is finite, Z is a dense subgroup in $\mathbb{T}$.

Proposition A.1. *Let G, ρ, V be as above.*

i. *Let the image of G/Z in the group* $\mathrm{GL}[\mathfrak{g}]$ *be closed.*

ii. *Let the center Z be compact.*

Then

a. *the stabilizer K_v of a nonzero vector v is compact*;

b. *the stabilizer L_v of the line $\mathbb{C}v$ is compact.*

Proof. It is sufficient to prove the statement for the group L_v. By definition L_v contains Z. Since Z is compact, the image of L_v in G/Z is closed. Since the Ad-image of G in $\mathrm{GL}[\mathfrak{g}]$ is closed, the Ad-image of L_v in $\mathrm{GL}[\mathfrak{g}]$ also is closed.

We use Theorem 1.2 of Wang [60] (which is a strong version of the result of Moore [35]). We say that an element $g \in \mathrm{GL}(\mathfrak{g})$ is *pre-periodic* if it is semisimple (i.e., it is diagonalizable after the complexification) and its eigenvalues θ_j satisfy $|\theta_j| = 1$. Equivalently, the closure of the set $\{g^m\}$, where $g \in \mathbb{Z}$, is compact. By [60], for any $g \in L_v$ there is a subgroup M_g such that

1. $M_g \subset K_v$;
2. denote by $\mathfrak{m}_g$ the Lie algebra of M_g, then the image of $\mathrm{Ad}(g)$ in $\mathfrak{g}/\mathfrak{m}$ is pre-periodic.

However, *if a normal subgroup fixes a vector v, then it acts trivially on the whole space.* Indeed, let $r \in G$ and $m \in M_g$. Then

$$\rho(m)\,\rho(g)\,v = \rho(g)\,\rho(g^{-1}mg)\,v = \rho(g)v.$$

Our representation is faithful and therefore the subgroup M_g is trivial. Thus the image L_v/Z of L_v in GL[$\mathfrak{g}$] is closed and consists of pre-periodic elements. It is more or less clear that L_v/Z is compact (to avoid a proof, we can refer to Lemma 1.3 from [61] about a group with a *dense* set of pre-periodic elements). Since Z is compact, L_v also is compact. □

Remark 1. There are several reasons, for which we can not simply say: for unitary representations of Lie groups stabilizers of vectors (lines) are compact.

a. Obviously we must consider faithful representations, since any closed normal subgroup $H \subset G$ can be a kernel of a representation.

b. More serious sources of problems are twinings. Consider the group Isom(2) of orientation preserving isometries of the Euclidean plane. Denote $Q := \mathrm{Isom}(2) \times \mathrm{Isom}(2)$, we can regard an element of this group as a pair of matrices of the form

$$\begin{pmatrix} e^{it} & z \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e^{is} & w \\ 0 & 1 \end{pmatrix}, \qquad \text{where } t, s \in \mathbb{R},\, z, w \in \mathbb{C}. \tag{A.1}$$

Denote by $S \subset Q$ the subgroup consisting of pairs of matrices with $z = w = 0$, i.e., $S = \mathrm{SO}(2) \times \mathrm{SO}(2)$. It is more-or-less obvious that $Q \supset S$ is a spherical pair (the Wigner–Mackey trick, see, e.g., [27], 13.3, Theorem 1, immediately gives a classification of irreducible unitary representations of Q). Next, choose an irrational real θ and take the subgroup $G \subset Q$ consisting of pairs of matrices (A.1) satisfying the condition $s = \theta t$, consider the corresponding subgroup $K = S \cap G$. The group G is the *Mautner group* (see, e.g., [1]). Clearly, restricting an S-spherical representation of Q to G we get a K-spherical representation of G. However, $K \simeq \mathbb{R}$ is not compact.

c. Consider the universal covering $G^{\sim}$ of the group $G = \mathrm{SL}(2, \mathbb{R})$ and the universal covering $R^{\sim}$ of the subgroup of rotations, $R^{\sim} \simeq \mathbb{R}$. Let ρ be a faithful irreducible representation of $G^{\sim}$ (see [51]). Then $R^{\sim}$ has a discrete spectrum. For an eigenvector v we have $L_v = R^{\sim}$ and $K_v \simeq \mathbb{Z}$. Both subgroups are non-compact. However, this non-compactness again is artificial, in our case L_v/Z is compact in $G^{\sim}/Z$.

Remark 2. If G can be covered by a real algebraic group, then conditions (i) and-(ii) are fulfilled automatically.

Notice that for p-adic groups stabilizers of vectors in unitary representations in interesting cases are compact (such stabilizers were topic of works of Wang [60] and [61]).

A.3. The Mautner phenomenon for the groups for Hier($\mathfrak{T}_n$). Let ρ be a unitary representation of a group G. Assume that a subgroup K fixes some vector v. Then quite often v is automatically fixed by certain larger group $\widetilde{K}$. For $G = \mathrm{Hier}(\mathfrak{T}_n)$ we have the following statement:

Proposition A.2. *Let ρ be a unitary representation of* $\mathrm{Hier}(\mathfrak{T}_n)$*, let v be a vector in the space of the representation.*

a. *Let $h \in \mathrm{Aut}(\mathfrak{T}_n)$ be a hyperbolic element and $\rho(h)v = v$. Then v is fixed by the whole subgroup* $\mathrm{Aut}_+(\mathfrak{T}_n)$.

b. *Let v be fixed by a parabolic subgroup $P_c \subset \mathrm{Aut}(\mathfrak{T}_n)$. Then v is fixed by the whole subgroup* $\mathrm{Aut}(\mathfrak{T}_n)$.

This is obvious: nontrivial irreducible representations of $\mathrm{Aut}(\mathfrak{T}_n)$ have no fixed vectors with respect to these subgroup (of course, this argument requires to look at Olshanski's list [47]). □

A trivial spherical representation of Hier($\mathfrak{T}_n$). Recall that the homogeneous space $\mathrm{Hier}(\mathfrak{T}_n)/\mathrm{Aut}(\mathfrak{T}_n)$ is countable and is equipped with the discrete topology. Therefore we have a quasi-regular representation of $\mathrm{Hier}(\mathfrak{T}_n)$ in ℓ^2 on this space. The natural orthogonal basis δ_z in ℓ_2 is indexed by points $z \in \mathrm{Hier}(\mathfrak{T}_n)/\mathrm{Aut}(\mathfrak{T}_n)$, where δ_z is the delta-function supported by z.

Proposition A.3. a. *The representation of* $\mathrm{Hier}(\mathfrak{T}_n)$ *in* $\ell^2\big(\mathrm{Hier}(\mathfrak{T}_n)/\mathrm{Aut}(\mathfrak{T}_n)\big)$ *is irreducible and spherical, the spherical vector is δ_{z_0}, where the* initial point z_0 *of the coset space corresponds to the unit of the group. The spherical function is* 1 *on* $\mathrm{Aut}(\mathfrak{T}_n)$ *and* 0 *outside this subgroup.*

b. *Let G be a topological group, L a closed subgroup, let the homogeneous space G/L be countable and discrete. Let all orbits of L on G/L except $\{z_0\}$ be infinite. Then the representation of G in $\ell^2(G/L)$ is irreducible and spherical. The spherical vector is δ_{z_0} and the spherical function is zero outside L.*

Proof. (b) An L-invariant function on G/L must be constant on orbits of L. Since a vector in ℓ^2 can not have infinite many nonzero equal coordinates, we get that δ_{z_0} is the unique L-invariant vector. By the same argument as in the proof of Proposition 2.6, the G-cyclic span of δ_{z_0} is an irreducible subrepresentation in ℓ^2. However, this cyclic span contains all basis vectors δ_z.

(a) Keeping in mind Proposition 3.2, for any element of $\mathrm{Hier}(\mathfrak{T}_n)/\mathrm{Aut}(\mathfrak{T}_n)$ we can assign a bi-thorn $\mathfrak{Bt}(R, Q; \theta)$ and an embedding of the thorn Q to $\mathfrak{T}_n$. The group $\mathrm{Aut}(\mathfrak{T}_n)$ acts preserving the bi-thorn and changing embeddings. Clearly, if the bi-thorn $\mathfrak{Bt}(R, Q; \theta)$ is non-empty, then orbits are infinite. So, we can apply the statement b). □

A.4. A question about unitary representations of discrete groups. It is well known that questions about unitary representations of discrete groups quite often are tricky. By the Thoma theorem [57], discrete groups are not type I unless they have Abelian normal subgroups of finite index. Absence of type I property implies numerous unpleasant phenomena (see, at least, the Glimm theorem [18] about a bad Borel structure on the dual space). However, we formulate the following informal question.

Question A.4. *Consider a pair of countable groups $\Gamma \supset \Delta$ and let all orbits of Δ on Γ/Δ be infinite (except the initial point). Find such pairs with "interesting" Δ-spherical representations of Γ.*

Apparently, interesting situations are rare. However, there is a famous example of such a pair, which was basically discovered in 1964 by Thoma [56] (see [49]). We take the group $\mathrm{S}(\infty)$ of finitely supported permutations of $\mathbb{N}$, let Γ be $\mathrm{S}(\infty) \times \mathrm{S}(\infty)$ and $\Delta \simeq \mathrm{S}(\infty)$ be the diagonal subgroup. This was the starting point of a large program in representation theory of infinite symmetric groups. We only mention that in this case spherical representations can be extended by continuity to a larger (continuous) group (see [49] and [42]).

The pair of discrete groups $G \supset K$ from Subsection 4.1 is spherical (and again we have a continuous extension to a larger group $\mathbf{G}$). A big zoo of examples of spherical representations in [42] has a similar nature.

Next, consider the Thompson group Th realized as the group of all continuous piece-wise $\mathrm{PSL}(2,\mathbb{Z})$-transformations of the real projective line $\mathbb{RP}^1$, see [50] and [21], by this construction Th is embedded to $\mathrm{Hier}(\mathcal{T}_2)$ and $\mathrm{PSL}(2,\mathbb{Z})$ is contained in $\mathrm{Aut}(\mathcal{T}_2)$.

Proposition A.5. *Consider a unitary $\mathrm{Aut}(\mathcal{T}_2)$-spherical representation ρ of the group $\mathrm{Hier}(\mathcal{T}_2)$ with spherical vector v. Then the* Th*-cyclic span of v is a $\mathrm{PSL}(2,\mathbb{Z})$-spherical representation of* Th.

Proof. It sufficient to show that the restriction of ρ to $\mathrm{PSL}(2,\mathbb{Z})$ does not contain additional $\mathrm{PSL}(2,\mathbb{Z})$-fixed vectors. We take an hyperbolic element h of $\mathrm{PSL}(2,\mathbb{Z})$, say, $h = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. It is hyperbolic in $\mathrm{Aut}(\mathcal{T}_2)$. By Proposition A.2(a), vectors fixed by h are fixed by the whole group $\mathrm{Aut}_+(\mathcal{T}_2)$, and a vector fixed by this subgroup is unique. □

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