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A complex Euclidean reflection group with a non-positively curved complement complex

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Abstract. The complement of a hyperplane arrangement in $\mathbb{C}^n$ deformation retracts onto an n-dimensional cell complex, but the known procedures only apply to complexifications of real arrangements (Salvetti) or the cell complex produced depends on an initial choice of coordinates (Björner–Ziegler). In this article we consider the unique complex Euclidean reflection group acting cocompactly by isometries on $\mathbb{C}^2$ whose linear part is the finite complex reflection group known as G_4 in the Shephard-Todd classification and we construct a choice-free deformation retraction from its hyperplane complement onto a 2-dimensional complex K where every 2-cell is a Euclidean equilateral triangle and every vertex link is a Möbius–Kantor graph. The hyperplane complement contains non-regular points, the action of the reflection group on K is not free, and the braid group is not torsion-free. Despite all of this, since K is non-positively curved, the corresponding braid group is a CAT(0) group.

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Introduction

The complement of a hyperplane arrangement in $\mathbb{C}^n$ is obtained by removing the union of its hyperplanes. When the arrangement under consideration is the complexification of a real arrangement, there is a classical construction due to Salvetti that provides a deformation retraction onto an n-dimensional cell complex now known as the *Salvetti complex* of the arrangement [19]. Björner and Ziegler extended Salvetti's construction so that it works for an arbitrary complex hyperplane arrangement, but their construction depends on an initial choice of a coordinate system [3]. In this article we deformation retract the complement of a specific infinite affine hyperplane arrangement in $\mathbb{C}^2$ onto a 2-dimensional piecewise Euclidean complex that involves no choices along the way. The arrangement we consider is the set of hyperplanes for the reflections in a complex Euclidean reflection group that we denote $\textsc{Refl}(\widetilde{G}_4)$. This is the unique complex Euclidean reflection group acting cocompactly by isometries on $\mathbb{C}^2$ whose linear part is the finite complex reflection group known as G_4 in the Shephard-Todd classification.

Theorem A. *The hyperplane complement of* $\text{Refl}(\widetilde{G}_4)$ *deformation retracts onto a non-positively curved piecewise Euclidean* 2*-complex.*

Our construction is easy to describe. We use the set of 0-dimensional hyperplane intersections to form Voronoi cells and then construct a deformation retraction from the hyperplane complement onto the portion of the Voronoi cell structure contained in the complement. For the group $\text{Refl}(\widetilde{G}_4)$ all of the Voronoi cells are isometric and their shape is that of the regular 4-dimensional polytope known as the 24-cell. The 0-dimensional intersection at the center of each Voronoi cell means that as a first step one can remove its interior by radially retracting onto its 3-dimensional polytopal boundary. This procedure works for this particular complex Euclidean reflection group but it appears that this is one of the few cases where it can be carried out without significant modifications. See Remark 9.2.

Recall that for any group G acting on a space X a point $x \in X$ is said to be *regular* when its G-stabilizer is trivial (and *irregular* otherwise). For complex spherical reflection groups, one consequence of Steinberg's theorem is that the hyperplane complement is exactly the set of regular points [21, 14, 15]. For complex Euclidean reflection groups the two spaces can be distinct, as discussed in [18], and they are distinct in this case, i.e. there are points in $\mathbb{C}^2$ that are irregular under the action of $\text{Refl}(\widetilde{G}_4)$, even though they are not fixed by any reflection in the group. In fact, these points are *isolated* in the sense that each such point has a neighborhood where every other point in the neighborhood is regular.

Theorem B. *Under the action of* $\text{Refl}(\widetilde{G}_4)$ *on* $\mathbb{C}^2$*, there are isolated irregular points.*

Isolated irregular points, of necessity, are disjoint from the union of the hyperplanes fixed by the reflections, and thus contained in the hyperplane complement. These isolated irregular points are the vertices of the 2-complex K and we use K to study the structure of the braid group of $\text{Refl}(\widetilde{G}_4)$ acting on $\mathbb{C}^2$. Recall that the *space of regular orbits* is the quotient of the subset of regular points by the free G-action and the *braid group of G acting on X* is the fundamental group of the space of regular orbits. The name "braid group" alludes to the fact that when the symmetric group Sym_n acts on $\mathbb{C}^n$ by permuting coordinates, the braid group of this action is Artin's classical braid group Braid_n. Let $\text{Braid}(\widetilde{G}_4)$ denote the braid group of $\text{Refl}(\widetilde{G}_4)$ acting on $\mathbb{C}^2$. The well-behaved geometry of K and the isolated fixed points in the hyperplane complement lead to an unusual mix of properties for a braid group of a reflection group.

Theorem C. *The group* $\text{Braid}(\widetilde{G}_4)$ *is a* CAT(0) *group and it contains elements of order* 2*.*

The group $\text{BRAID}(\widetilde{G}_4)$ is a CAT(0) group because it acts properly discontinuously and cocompactly by isometries on the CAT(0) universal cover of K and it has elements of order 2 that stabilize lifts of the isolated fixed points in the hyperplane complement. Since every finitely generated Coxeter group is a CAT(0) group that contains 2-torsion, this combination is not unusual in the broader world of CAT(0) groups. However, torsion is unusual in the braid group of a reflection group. The braid groups of finite complex reflection groups are torsion-free [2], as are the braid groups of complexified Euclidean Coxeter groups, also known as Euclidean Artin groups or affine Artin groups [16]. In fact, it is conjectured that the braid groups of all complexified Coxeter groups, i.e. all Artin groups, are torsion-free [12]. Thus, this example is a departure from the norm.

The article is structured from general to specific. Sections 1 and 2 contain basic definitions and results about complex spherical and complex Euclidean reflection groups. Section 3 briefly focuses attention on the groups acting on the complex Euclidean line. This section highlights our approach and it contains an explicit computation used to prove the main theorems. The next sections develop tools for visualizing and describing the action of the complex spherical reflection group $\text{REFL}(G_4)$ on $\mathbb{C}^2$. Concretely, Section 4 uses the quaternions to describe the 4-dimensional regular polytope known as the 24-cell, and Section 5 introduces a novel way to visualize the natural action of $\text{REFL}(G_4)$ on this polytope. Section 6 describes how quaternions can be used to give efficient linear-like descriptions of arbitrary isometries of the complex Euclidean plane, Section 7 applies these tools to the complex spherical reflection group $\text{REFL}(G_4)$, and Section 8 extends them to the complex Euclidean reflection group $\text{REFL}(\widetilde{G}_4)$. Finally, Section 9 contains the proofs of our three main results. The authors would like to thank the anonymous referee for their detailed comments on an earlier version of this article.

1. Complex spherical reflection groups

This section reviews the notion of a complex spherical reflection group. Recall that a *geometry* is a proper metric space in which metric balls are compact and that a group acting on a geometry is acting *geometrically* if the action is properly discontinuous and cocompact by isometries. In this language, complex spherical reflection groups act geometrically on the complex unit sphere.

Definition 1.1. We call $V = \mathbb{C}^n$ a *complex spherical geometry* when it comes equipped with a positive definite hermitian inner product that is linear in the second coordinate and conjugate linear in the first. The *unitary* linear transformations that preserve this inner product act on the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ and they form the *unitary group* $U(n)$.

A complex reflection is an elementary unitary isometry.

Definition 1.2. Let V be a complex spherical geometry. A *complex* (*spherical*) *reflection* r is a non-trivial unitary transformation of V that multiples some unit vector v by a unit complex number $z \in \mathbb{C}$ and fixes the orthogonal complement of v pointwise. The formula for the reflection $r = r_{v,z}$ is $r(w) = w - (1-z)\langle v, w\rangle v$. The reflection r has finite order if and only if $z = e^{ai}$ where a is a rational multiple of π. When this occurs, r is called a *proper* complex reflection. All reflections in this article are proper and we drop the adjective. The complex reflection $r_{v,z}$ is *primitive* when the complex number z is of the form $z = e^{\frac{2\pi}{m}i}$ for some positive integer m.

We are interested in groups generated by complex reflections.

Definition 1.3. A group G is called a *complex spherical reflection group* if it is generated by complex reflections acting on a complex spherical geometry V so that the action restricted to the unit sphere is geometric. They are also known as *finite complex reflection groups* since the action is geometric if and only if the group is finite. If there is an orthogonal decomposition $V = V_1 \oplus V_2$ preserved by all of the elements of G, then G is *reducible* and it is *irreducible* when such a decomposition does not exist.

Shephard and Todd completely classified the irreducible complex spherical reflection groups in 1954. There is a single triply-indexed infinite family and 34 exceptional cases denoted G_4 through G_{37} [20, 7]. We write $\text{Refl}(G_k)$ to denote the complex spherical reflection group of type G_k. This is done to distinguish the reflection group from its corresponding braid group. Our focus is on the braid group of the unique Euclidean extension of $\text{Refl}(G_4)$, the smallest exceptional complex spherical reflection group.

2. Complex Euclidean reflection groups

In this section, we transition to complex Euclidean reflection groups, by replacing the underlying vector space with an affine space, where all points are on an equal footing.

Definition 2.1. For any vector space V, the corresponding *affine space* is a set E together with a simply transitive V-action on E; the image of $x \in E$ under $v \in V$ is written $x + v$. For each linear subspace $U \subset V$ and point $x \in E$ there is an *affine subspace* $x + U \subset E$ and the functions $f : E \to E$ that send affine subspaces to affine subspaces are *affine maps*. We write $\text{Aff}(E)$ for the group of all affine maps under composition. For each vector $v \in V$ there is a *translation map* t_v that sends each point x to $x + v$. The translations form a normal abelian subgroup.

When the vector space V is a complex spherical geometry, one can restrict attention to those affine maps that preserve the inner product.

Definition 2.2. If E is an affine space for a complex vector space V and V is a complex spherical geometry, then E is a *complex Euclidean geometry*. Since an ordered pair (x, x') of points in E determines a vector $v_{x,x'} \in V$ that sends x to x', an ordered quadruple (x, x', y, y') of points in E determines an ordered pair of vectors $(v_{x,x'}, v_{y,y'})$ in V. An affine map $f : E \to E$ is called a *complex Euclidean isometry* when f preserves the hermitian inner product of the ordered pair of vectors derived from an ordered quadruple of points in E. We write $\text{Isom}(E)$ for the group of all complex Euclidean isometries.

Translations are complex Euclidean isometries and an affine map fixing a point x is a complex Euclidean isometry if and only if the corresponding linear transformation of V is unitary. The notion of a reflection extends to this new context.

Definition 2.3. An isometry of a complex Euclidean space E is called a *complex* (*Euclidean*) *reflection* if it can be viewed as a complex spherical reflection (Definition 1.2) after choosing an appropriate origin. A *crystallographic complex Euclidean reflection group* is a group G generated by complex reflections that acts geometrically on a complex Euclidean space [17]. The image of G under the projection map from $\text{Isom}(E) \to U(V)$ is called its *linear part* and the kernel is its *translation part*.

The *reducibility* of a complex Euclidean reflection group is determined by its linear part and two such groups G and G' acting on complex Euclidean spaces E and E' are called *equivalent* when there is an invertible affine map from E to E' so that the action of G on E corresponds to the action of G' on E'.

Remark 2.4. The irreducible crystallographic complex Euclidean reflection groups were classified by Popov in [17], although Goryunov and Man found one additional 2-dimensional example [13]. There are roughly 30 infinite families and 20 isolated examples.

There is a unique complex Euclidean reflection group whose linear part is the spherical group $\text{Refl}(G_4)$. We write $\text{Refl}(\widetilde{G}_4)$ to denote this group. Popov denotes it $[K_4]$.

3. Isometries of the complex Euclidean line

This section focuses on complex Euclidean reflection groups acting geometrically on the complex Euclidean line. It includes an explicit computation needed in the proof of Theorem B and it previews the Voronoi cell argument in this easy-to-visualize context. Isometries of the complex Euclidean line are functions of the form $f(x) = e^{ai}x + z$ where a is real and z is an arbitary complex number. A reflection must fix a point z_0 and it can be written as $e^{ai}(x - z_0) + z_0$. There are very few 1-dimensional complex Euclidean reflection groups.

Theorem 3.1. *If G is a complex Euclidean reflection group that acts geometrically on the complex Euclidean line, then every reflection in G has order* 2, 3, 4 *or* 6 *and its reflections of maximal order generate G. When the maximal order is* 2 *there is a* 1*-parameter family of such groups, but when it is* 3, 4 *or* 6 *there is a unique such group up to affine equivalence.*

Complex reflections in this context are real rotations.

Example 3.2. There are three real irreducible 2-dimensional Euclidean Coxeter groups (of type $\widetilde{A}_2$, $\widetilde{B}_2$ and $\widetilde{G}_2$) and each is the symmetry group of a triangular tiling of $\mathbb{R}^2$. Each rigid case mentioned in Theorem 3.1 is the index 2 subgroup of the orientation-preserving isometries in one of these Coxeter groups. These subgroups are denoted $[K_3(m)]$ in Popov's notation and $\textsc{Refl}(\widetilde{G}_3(m))$ in ours, where $\frac{\pi}{m}$ is the smallest angle in the triangular tiling mentioned above. The complex reflections which generate the group $\textsc{Refl}(\widetilde{G}_3(m))$ are the real rotations that fix a point where $2m$ triangles meet, rotating through an angle of $\frac{2\pi}{m}$.

Fixed points of the reflections are related to the translation subgroup.

Remark 3.3. Consider a complex Euclidean reflection group acting on $\mathbb{C}$ where the origin is fixed by a primitive reflection r of maximal order. If T_0 denotes the images of the origin under the action of the translation subgroup and FP_m denotes the fixed points of the primitive reflections of order m, then the computation $t_v \circ r \circ t_v^{-1}(x) = z(x - v) + v = zx + (1 - z)v = t_{(1-z)v} \circ r(x)$ with $z = e^{\frac{2\pi}{m}i}$ shows that $(1 - z) \cdot \mathrm{FP}_m = T_0$. In the group $\textsc{Refl}(\widetilde{G}_3(6))$, for example, $2 \cdot \mathrm{FP}_2 = (1 - \omega) \cdot \mathrm{FP}_3 = \mathrm{FP}_6 = T_0$, where $\omega = e^{2\pi i/3}$ is a primitive cube-root of unity.

The corresponding braid group can be computed using Voronoi cells.

Example 3.4. Let G be $\textsc{Refl}(\widetilde{G}_3(m))$ with $m \in \{3, 4, 6\}$ and let S be the set of fixed points for the reflections in G. The vertices of the Voronoi cells for this set S are the incenters of the triangles in the corresponding triangular tiling. The case $m = 3$ is illustrated in Figure 1. The Voronoi cells can be used to understand the corresponding braid group. Once the fixed points of the reflections are removed, the remainder deformation retracts to the 1-skeleton of the Voronoi cell structure. The braid group acts freely on the 1-skeleton but it does not act transitively on the vertices. The quotient graph has 2 vertices with 3 edges connecting them, a graph whose fundamental group is the free group of rank 2.

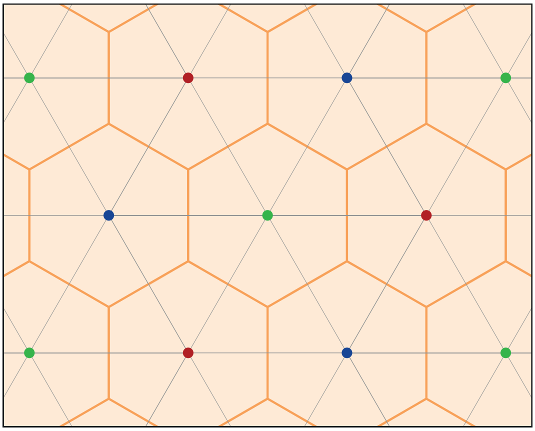

Figure 1. The Voronoi cell structure for the complex Euclidean reflection group $\text{REFL}(\widetilde{G}_3(3))$ is a hexagonal tiling of $\mathbb{C}$ and the hyperplane complement deformation retracts to its 1-skeleton.

4. Quaternions and the 24-cell

This section describes the 24-cell using the quaternions.

Definition 4.1. Let $\mathbb{H}$ denote the *quaterions* viewed as a 4-dimensional Euclidean space with orthonormal basis $\{1, i, j, k\}$ and identify the reals $\mathbb{R}$ with the $\mathbb{R}$-span of 1. The *unit quaternions* of norm 1 correspond to the unit 3-sphere. The convex hull of the set

$$\Phi = \{\pm 1, \pm i, \pm j, \pm k\} \cup \left\{\frac{\pm 1 \pm i \pm j \pm k}{2}\right\}$$

is a 4-dimensional regular polytope known as the 24-*cell* [8]. The centers of its 24 regular octahedral facets are at the points $\frac{i-j}{2} \cdot \Phi$, a scaled and rotated version of Φ obtained by left multiplying every element of Φ by $\frac{i-j}{2}$. In particular, $\frac{i-j}{2} \cdot \Phi$ consists of the 24 quaternions of the form $\frac{\pm u \pm v}{2}$ for $u, v \in \{1, i, j, k\}$ with $u \neq v$. We use Φ for this set because it is the conventional letter used for root systems and the type D_4 root system is the set $\Phi_{D_4} = (i - j) \cdot \Phi$.

The unit quaternions are a compact Lie group (and those in Φ are a finite subgroup). The unique real subalgebra of $\mathbb{H}$ identifies $\mathbb{S}^3$ with $\text{SPIN}(3)$, the double cover of $SO(3)$, but $\mathbb{H}$ contains a continuum of subalgebras isomorphic to $\mathbb{C}$, and each one produces a complex spherical structure and an identification of $\mathbb{S}^3$ with $\text{SU}(2)$, see [9].

Definition 4.2. For each purely imaginary unit quaternion u, $u^2 = -1$ and the $\mathbb{R}$-span of 1 and u is a subalgebra of $\mathbb{H}$ isomorphic to the complex numbers with u playing the role of $\sqrt{-1}$. More generally, note that every nonreal quaternion q_0 determines a complex subalgebra of $\mathbb{H}$ in which q_0 has positive imaginary part. Concretely, the $\mathbb{R}$-span of 1 and q_0 is a complex subalgebra and the isomorphism with $\mathbb{C}$ identifies $\sqrt{-1}$ with the normalized imaginary part of q_0. We call this the *complex subalgebra determined by* q_0. Let $\mathbb{H}_{q_0}$ denote the *complexified* quaternions with a fixed complex subalgebra $\mathbb{C}$ determined by the nonreal quaternion q_0. The right cosets $q\mathbb{C}$ of $\mathbb{C}$ inside $\mathbb{H}_{q_0}$ partition the nonzero quaternions into right *complex lines*. Vector addition and this type of right scalar multiplication turn $\mathbb{H}_{q_0}$ into a 2-dimensional right vector space over this subalgebra $\mathbb{C}$. In addition, there is a unique positive definite hermitian inner product on this 2-dimensional $\mathbb{C}$-vector space so that the unit quaternions have length 1 with respect to this inner product.

We use elements in Φ to define a complex structure on $\mathbb{H}$.

Definition 4.3. Let $\omega = \frac{-1+i+j+k}{2}$ and let $\zeta = \frac{1+i+j+k}{2}$, and note that ω is a cube-root of unity, ζ is a sixth-root of unity and $\zeta^2 = \omega$. For the remainder of the article we identify $\mathbb{C}$ with the subalgebra of $\mathbb{H}$ that contains both ω and ζ. Since Φ is a group of order 24 and ζ is an element in Φ of order 6, we can partition Φ into the four cosets $q\langle\zeta\rangle$ with $q \in \{1, i, j, k\}$. Thus every element in Φ is of the form $q\zeta^\ell$ with $q \in \{1, i, j, k\}$ and ℓ an integer mod 6 and Φ is contained in the union of the four complex lines $1\mathbb{C}$, $i\mathbb{C}$, $j\mathbb{C}$ and $k\mathbb{C}$.

Those who prefer computations over $\mathbb{C}$ can select an ordered basis and work with coordinates. Note that we use the letter z rather than q when we wish to emphasize that a particular quaternion lives in the distinguished copy of $\mathbb{C}$.

Definition 4.4. Let $\mathbb{H}_{q_0}$ be the complexified quaternions. Every ordered pair of nonzero quaternions q_1 and q_2 that belong to distinct complex lines form an *ordered basis* of $\mathbb{H}_{q_0}$ viewed as a 2-dimensional right complex vector space. In particular, their right $\mathbb{C}$-linear combinations $q_1\mathbb{C} + q_2\mathbb{C}$ span all of $\mathbb{H}_{q_0}$ and for every $q \in \mathbb{H}_{q_0}$ there are unique *coordinates* $z_1, z_2 \in \mathbb{C}$ such that $q = q_1z_1 + q_2z_2$. When the basis $\mathcal{B} = \{q_1, q_2\}$ is ordered we view the coordinates of q as a column vector.

In $\mathbb{H}_j$ with ordered basis $\mathcal{B} = \{1, i\}$, for example, the quaternion $q = a + bi + cj + dk$ has coordinates $z_1 = a + cj$ and $z_2 = b + dj$ because $q = 1(a + cj) + i(b + dj)$. In other words, in $\mathbb{H}_j$

$$q = a + bi + cj + dk = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} a + cj \\ b + dj \end{bmatrix}_{\mathcal{B}}.$$

Since our chosen $\mathbb{C}$ in $\mathbb{H}$ contains ω and ζ, we are working with $\mathbb{H}_\omega = \mathbb{H}_\zeta$ and the computations there are slightly more complicated. To compensate, we at least simplify the notation for unit complex numbers. The copy of $\mathbb{C}$ in $\mathbb{H}_\omega$ does not contain the quaternion i and, in fact, the element that plays the role of $\sqrt{-1}$ is $\frac{i+j+k}{\sqrt{3}}$. Nevertheless, we still write $z = e^{ai}$ for the numbers on the unit circle in this copy of $\mathbb{C}$. This misuse of the letter i only occurs as an exponent and only in this particular formulation.

5. Visualizing the 24-cell

In 2007 John Meier and the second author developed a technique for visualizing the regular 4-dimensional polytopes as a union of spherical lenses that has been very useful for understanding the various groups that act on these polytopes. To our knowledge this is the first time that this technique has appeared in print.

Definition 5.1. A *lune* is a portion of a 2-sphere bounded by two semicircular arcs with a common 0-sphere boundary and its shape is completely determined by the angle at which these semicircles meet. A *lens* is a portion of the 3-sphere determined by two hemispheres sharing a common great circle boundary and the shape of a lens is completely determined by the dihedral angle between these hemispheres along the great circle where they meet. This is the 3-dimensional analog of a lune.

In the same way that lunes can be used to display the map of a 2-sphere such as the earth in $\mathbb{R}^2$ with very little distortion, lenses can be used to display a map of the 3-sphere in $\mathbb{R}^3$ with very little distortion.

Definition 5.2. To visualize the structure of the 24-cell it is useful to use the 6 lenses displayed in Figure 2. Each of the six figures represents one-sixth of the 3-sphere. The outside circle is a great circle in $\mathbb{S}^3$, the solid lines live in the hemisphere that bounds the front of the lens, the dashed lines live in the hemisphere that bounds the back of the lens and the dotted lines live in the interior of the lens. The dihedral angle between the front and back hemispheres, along the outside boundary circle is $\frac{\pi}{3}$ and all the edges are length $\frac{\pi}{3}$. The six lenses are arranged so that the front hemisphere of each lens is identified with the back hemisphere of the next one in counter-clockwise order. Each lens contains one complete octahedral face at its center and six half octahedra, three bottoms halves corresponding to the squares in the front hemisphere and three top halves corresponding to the squares in the back hemisphere. The label at the center of each lens is the coordinate of the center of the Euclidean octahedron spanned by the six nearby vertices. The arrows in Figure 2 indicate how the 24 vertices move under the map which right multiplies by ζ. The arrows glue together to form 4 (oriented) *moved hexagons* with vertices $q\langle\zeta\rangle$ that live in the 4 complex lines $q\mathbb{C}$ where q is $1, i, j$ or k.

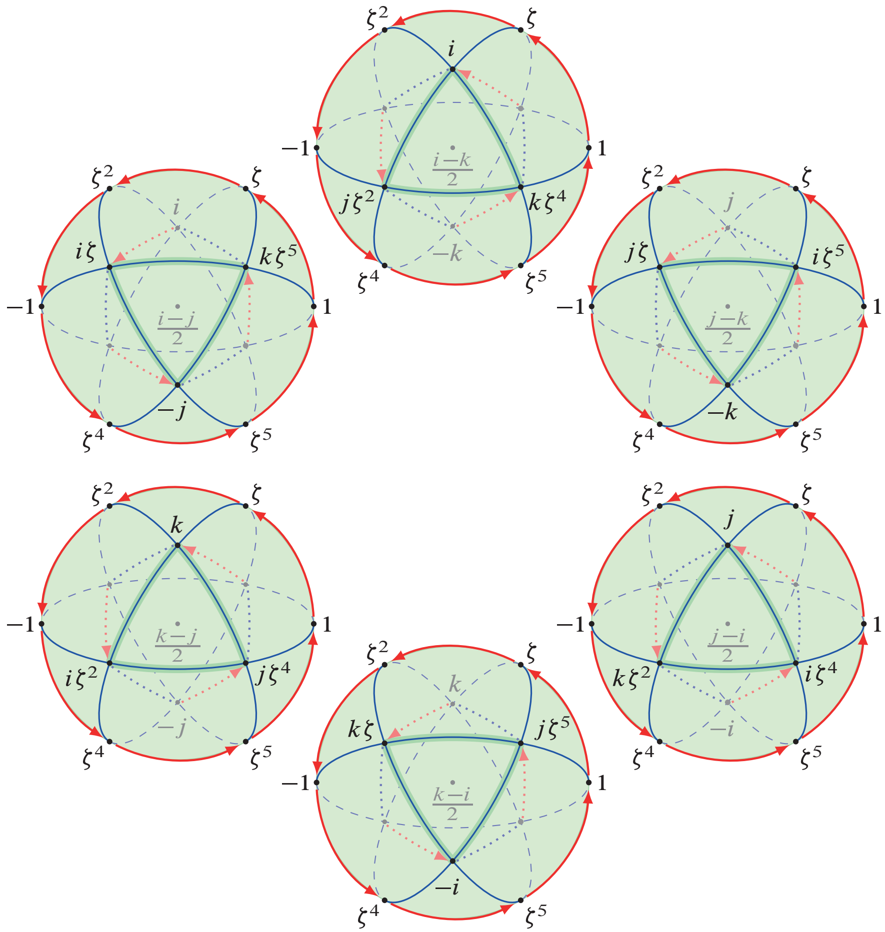

Figure 2. Six lenses that together display the structure of the 24-cell. Each figure represents a one-sixth lens in the 3-sphere with dihedral angle $\frac{\pi}{3}$ between its front and back hemispheres. They are arranged so that every front hemisphere is identified with the back hemisphere of the next one when ordered in a counter-clockwise way.

6. Complex spherical isometries of the quaternions

Every spherical isometry of $\mathbb{C}^2$ can be described using quaternions. This material is elementary and similar treatments can be found in [9, 15, 11]. Even these references, however, do not treat the case where a non-standard complex structure is being preserved. We begin with general maps given by left and right multiplication.

Definition 6.1. For each quaternion q there are left and right multiplication maps $L_q(x) = qx$ and $R_q(x) = xq$ from $\mathbb{H}$ to itself. When q is a unit quaternion, both L_q and R_q are orientation preserving Euclidean isometries that fix the origin, send the unit 3-sphere to itself and move every point in $\mathbb{S}^3$ the same distance. For each pair of unit quaternions q and q', there is a function $L_q(R_{q'}(x)) = R_{q'}(L_q(x)) = qxq'$ that we call a *spherical map*. Every orientation preserving isometry of $\mathbb{S}^3$ can be uniquely represented as a spherical map, up to negating both q and q'. This correspondence essentially identifies the topological space $\mathbb{S}^3 \times \mathbb{S}^3$ of pairs of unit quaternions with $\mathrm{Spin}(4)$, the double cover of $\mathrm{SO}(4)$ [9].

The spherical maps that preserve a complex structure are special.

Definition 6.2. When the quaternions have a fixed complex structure, only some spherical maps preserve it, and we call these *complex spherical maps*. For every unit quaternion q, L_q is a complex spherical map. However, right multiplication maps are complex spherical maps only if of the form R_z where $z = e^{ai}$ is a unit complex number. In this case, R_z stabilizes each individual complex line $q\mathbb{C}$ setwise and rotates it by through an angle of a radians. As z varies through the unit complex numbers, this motion is called the *Hopf flow*.

The next proposition records the fact that left multiplication and the Hopf flow are sufficient to generate all complex spherical isometries.

Proposition 6.3. *The spherical maps that preserve the complex structure of* $\mathbb{H}_{q_0}$ *are precisely those of the form* $x \mapsto qxz$ *where* q *is a unit quaternion and* z *is a unit complex number in the chosen complex subalgebra.*

As with general spherical maps, each complex spherical map can be represented in two ways because of the equality $qxz = (-q)x(-z)$. This gives a map from $\mathbb{S}^3 \times \mathbb{S}^1 \twoheadrightarrow \mathrm{U}(2)$ with kernel $\{\pm 1\}$, which corresponds to the short exact sequence $\mathrm{O}(1) \hookrightarrow \mathrm{Sp}(1) \times \mathrm{U}(1) \twoheadrightarrow \mathrm{U}(2)$.

Definition 6.4. Let $\mathbb{H}_{q_0}$ be the quaternions with a complex structure, let q_1 be any unit quaternion orthogonal to both 1 and q_0, and let $z = e^{ai}$ with a real be a unit complex number. The complex spherical map $L_z \circ R_z(x) = zxz$ is a complex reflection because it fixes the complex line $q_1\mathbb{C}$ pointwise and rotates the complex line $\mathbb{C} = 1\mathbb{C}$ through an angle of $2a$ radians. In the notation of Definition 1.2 this map is r_{1,z^2}. To create an arbitrary complex spherical reflection r_{q,z^2} with q a unit quaternion, it suffices to conjugate r_{1,z^2} by L_q since the composition $L_q \circ r_{1,z^2} \circ L_{q^{-1}}$ defined by the equation $x \mapsto (qzq^{-1})xz$ rotates the complex line $q\mathbb{C}$ through an angle of $2a$ and fixes the unique complex line orthogonal to $q\mathbb{C}$ pointwise.

This explicit description makes complex reflections easy to detect. Both directions are easy quaternionic exercises.

Proposition 6.5. *Let $\mathbb{H}_{q_0}$ be the quaternions with a complex structure. A complex spherical map $f(x) = qxz$ with q a unit quaternion and z a unit complex number is a complex reflection if and only if* $\textsc{Real}(q) = \textsc{Real}(z)$.

7. The group $\textsc{Refl}(G_4)$

The complex spherical reflection group $\textsc{Refl}(G_4)$ acts on $\mathbb{C}^2$ and its hyperplane complement deformation retracts to a non-positively curved 2-complex in the 2-skeleton of the 24-cell. This model for the $\textsc{Refl}(G_4)$ hyperplane complement is being introduced here.

Definition 7.1. We define the group $\textsc{Refl}(G_4)$ as the complex spherical reflection group generated by order 3 reflections $r_{1,\omega}(x) = \zeta x \zeta$ and $r_{i,\omega}(x) = \zeta^i x \zeta$, where $\zeta^i = (-i)\zeta i = i\zeta(-i) = \zeta^{-i}$. For simplicity we abbreviate these as $r_1 = r_{1,\omega}$ and $r_i = r_{i,\omega}$. The resulting group also includes the order 3 reflections $r_j = r_{j,\omega}(x) = \zeta^j x \zeta$ and $r_k = r_{k,\omega}(x) = \zeta^k x \zeta$ as well as the reflections $r_q^2 = r_{q,\omega^2}$ for $q \in \{1, i, j, k\}$. This group includes the map that multiplies by -1, so it also includes the negatives of these eight reflections, which are no longer reflections. Finally, $\textsc{Refl}(G_4)$ contains elements which left multiply by $\pm q$ with $q \in \{1, i, j, k\}$. Thus the full list of all 24 elements in $\textsc{Refl}(G_4)$ is $\{\pm L_q\} \cup \{\pm r_q\} \cup \{\pm r_q^2\}$ with $q \in \{1, i, j, k\}$.

This presentation of $\textsc{Refl}(G_4)$ is new, but closely related to that given in [15, Chapter 6]. These authors, however, write matrices using the complex structure $\mathbb{H}_j$. The computations are simpler, but the underlying symmetry is obscured. We use the lens diagram in Figure 2 to understand how the reflections in $\textsc{Refl}(G_4)$ act on the 24-cell. Since radial projection identifies points in the boundary of the 24-cell with points in the 3-sphere, we can use the piecewise spherical structure in the lens picture as a proxy for the piecewise Euclidean structure of the 24-cell boundary.

Definition 7.2. Each reflection r_q with $q \in \{1, i, j, k\}$ rotates the complex line $q\mathbb{C}$ and fixes the orthogonal complex line pointwise. Concretely, the fixed hyperplanes for the reflections r_1, r_i, r_j and r_k are $(i - j)\mathbb{C}$, $(1 + k)\mathbb{C}$, $(1 - k)\mathbb{C}$ and $(i + j)\mathbb{C}$, respectively. The moved lines intersect the boundary of the 24-cell in the four moved hexagons (Definition 5.2). Each fixed line also intersects the boundary of the 24-cell in a hexagon, but these *fixed hexagons* are disjoint from the 1-skeleton of the 24-cell. The smallest subcomplex of the 24-cell boundary containing a fixed hexagon is a union of 6 octahedra, overlapping on triangles, that we call a *necklace*. Thus each reflection r_q has a moved hexagon in the 1-skeleton and a fixed hexagon in the interior of its antipodal necklace.

To see this, consider the action of r_1 on the 6 lens picture (Figure 2).

Remark 7.3. In Figure 2, each of the six lenses is stabilized and rotated by r_1 and the common boundary circle of the lenses represents the moved hexagon of r_1. Recall from Definition 5.2, that the 4 moved hexagons correspond to the 4 complex lines $q\mathbb{C}$, with $q \in \{1, i, j, k\}$. These are permuted by the action of r_1. The moved hexagon containing 1 is rotated by $\frac{2\pi}{3}$, the three moved hexagons containing i, j and k are cyclically permuted. In the top lens, for example, the three points $\{1, \zeta^2, \zeta^4\} \in 1\mathbb{C}$ are cyclically permuted and three points $\{i, j\zeta^2, k\zeta^4\}$, in $i\mathbb{C}$, $j\mathbb{C}$ and $k\mathbb{C}$, respectively, are cyclically permuted. The fixed hexagon for r_1 is formed from the line segment in each lens connecting the center of the back hemisphere to the center of the front hemisphere through the center of its central octahedron. These six octahedra form the r_1 necklace.

We now extend these observations to all of the reflections.

Remark 7.4. The reflections r_q with $q \in \{1, i, j, k\}$ are conjugate in $\textsc{Refl}(G_4)$, and therefore geometrically similar. The reflection r_q rotates the complex line $q\mathbb{C}$ and it cyclically permutes the other three complex lines. Thus, it rotates one moved hexagon and permutes the other three moved hexagons. The necklaces containing the four fixed hexagons have disjoint interiors and they partition the 24 octahedra into 4 necklaces with 6 octahedra each (see Figure 3). The boundary of each necklace contains three moved hexagons and each moved hexagon is in the boundary of three of the necklaces. The action of r_q fixes its fixed hexagon and rotates the surrounding necklace around this core curve. The other three necklaces, which contain the moved hexagon, are cyclically permuted as the moved hexagon rotates.

The portion of the 24-cell that avoids the fixed hyperplanes of the reflections in $\textsc{Refl}(G_4)$ is of particular interest.

Definition 7.5. Let K_0 be the largest subcomplex of 24-cell P that avoids any point fixed by a reflection in $\textsc{Refl}(G_4)$. The interior of P is removed because the origin is fixed by all reflections. In addition, all 24 octahedral facets of P, and some of the equilateral triangles in its 2-skeleton are removed because they intersect the fixed hexagons of the reflections. What remains is the entire 1-skeleton of P and some of its triangles. From the description of the fixed points given in Remark 7.4 we see that a triangular face of P is excluded precisely when all three of its vertices belong to distinct complex lines and included when two of the vertices belong to the same complex line. In Figure 2 the included triangles can be characterized as those which contain an arrow (representing right multiplication by ζ) as one of its edges.

One nice property of complex K_0 is that it is non-positively curved. We only need a few basic facts. See [4] for additional details.

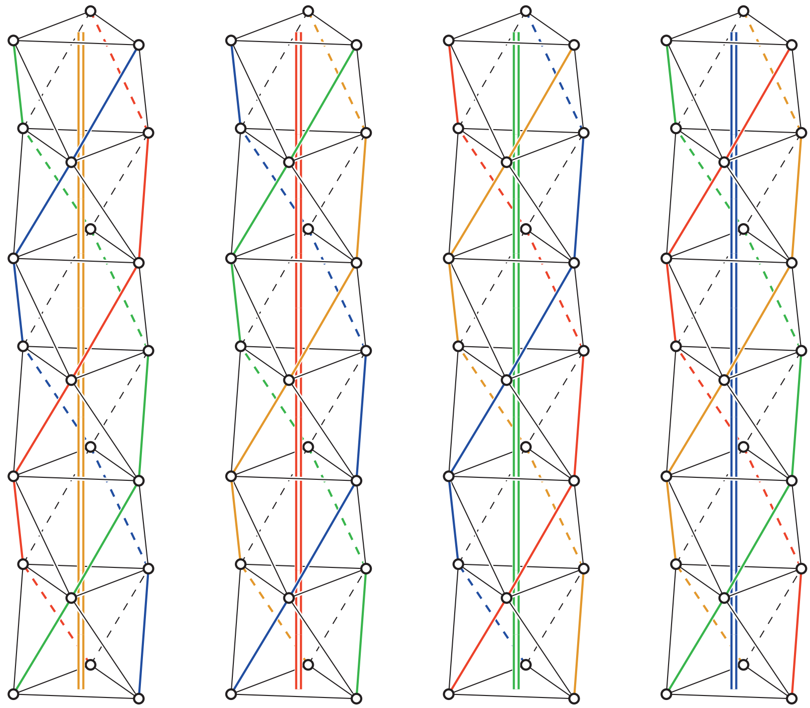

Figure 3. The 4 octahedral necklaces centered around the fixed orthogonal circles/hexagons are created by identifying the top and bottom triangle in each pillar. The triangles in the boundaries of the necklaces can be pairwise identified to form the boundary of the 24-cell homemorphic to a 3-sphere.

Remark 7.6. A piecewise Euclidean 2-complex is *non-positively curved* when every simple loop in every vertex link has length at least 2π. The universal cover of a non-positively curved 2-complex is a complete, contractible CAT(0) space, and a group that acts geometrically on a such a space is a CAT(0) *group*.

Theorem 7.7. *The hyperplane complement of* $\text{Refl}(G_4)$ *deformation retracts onto a non-positively curved piecewise Euclidean* 2*-complex* K_0 *contained in the boundary of the* 24*-cell. Every* 2*-cell in* K_0 *is an equilateral triangle and every vertex link is a subdivided theta graph.*

Proof. The deformation retraction from the hyperplane complement to K_0 has two stages. First, we can retract the hyperplane complement onto the boundary of the 24-cell, radially pushing away from the origin and pulling away from ∞. This is possible because the origin belongs to all 4 fixed hyperplanes. The end result of this first deformation is the boundary of the 24-cell minus the four fixed hexagons. The second step focuses on the four necklaces that contain the four fixed hexagons in their interiors. Topologically, each necklace is a solid torus with a missing core curve and we can retract away from the missing fixed hexagon onto the boundary torus of the necklace. Geometrically, each punctured triangle retracts onto its boundary and each pierced octahedron retracts onto the annulus formed by the six triangles that are not punctured. See Figure 3. This completes the deformation retraction to K_0. Each vertex of K_0 belongs to 9 triangles and each vertex link has 9 edges of length $\frac{\pi}{3}$. These edges connect to form three paths of length π sharing endpoints. Since every simple loop in this "theta graph" has length at least 2π, K_0 is non-positively curved. ☐

A presentation for the corresponding braid group $\textsc{Braid}(G_4)$ can be derived from its action on K_0. We include this without proof.

Corollary 7.8. *The group* $\textsc{Braid}(G_4)$ *is a* CAT(0) *group isomorphic to the three-strand braid group and it is defined by the presentation* $\langle a, b, c, d \mid abd, bcd, cad \rangle$.

The fact that the braid group of $\textsc{Refl}(G_4)$ is isomorphic to the 3-strand braid group is well-known [1, 5, 6]. The novelty of our presentation is that we use an explicit piecewise Euclidean 2-complex in the 2-skeleton in the boundary of the 24-cell to establishes this connection.

8. The group $\textsc{Refl}(\widetilde{G}_4)$

This section extends results about the spherical reflection group $\textsc{Refl}(G_4)$ to the Euclidean reflection group $\textsc{Refl}(\widetilde{G}_4)$, which is the focus of our main theorems. It establishes key facts about translations, reflections, fixed hyperplanes and Voronoi cells.

Definition 8.1. For every quaternion q the *translation* map $t_q(x) = x + q$ is an orientation preserving isometry of the canonical Euclidean structure of $\mathbb{H}$. When a spherical map is combined with translation by an arbitrary quaternion q'' we call the resulting function $f(x) = qxq' + q''$ a *Euclidean map*. As was the case with spherical maps (Definition 6.1), every Euclidean map is an orientation preserving Euclidean isometry and every orientation preserving Euclidean isometry can be represented as a Euclidean map in precisely two ways (with the second representation obtained by negating q and q').

Once translations are allowed, and the location of the origin is forgotten, $\mathbb{H}_{q_0}$ can be viewed as the complex Euclidean plane. The images of the complex lines $q\mathbb{C}$ under translation are called *affine complex lines* and they are sets of the form $q\mathbb{C} + v$. Every translation preserves this complex Euclidean structure and Propositions 6.3 and 6.5 extend.

Proposition 8.2. *The Euclidean maps that preserve the complex Euclidean structure of $\mathbb{H}_{q_0}$ are precisely those of the form $x \mapsto qxz + v$ where q is a unit quaternion, z is a unit complex number in the chosen complex subalgebra and v is arbitrary.*

Proposition 8.3. *Let $\mathbb{H}_{q_0}$ denote the quaternions with a complex structure. A map $f(x) = qxz + v$ with q a unit quaternion, z a unit complex number and v arbitrary is a complex Euclidean reflection if and only if f has a fixed point and* $\textsc{Real}(q) = \textsc{Real}(z)$.

When a complex reflection is conjugated by a translation, the result is a *parallel* reflection. The group $\textsc{Refl}(\widetilde{G}_4)$ is generated from $\textsc{Refl}(G_4)$ by adding an affine reflection parallel to an existing reflection. See [17].

Definition 8.4. Let $\textsc{Refl}(\widetilde{G}_4)$ denote the group generated by the reflections r_1, r_i and $r_1' = t_{1+k} \circ r_1 \circ t_{1+k}^{-1} = t_2 \circ r_1$. The first two generate $\textsc{Refl}(G_4)$ as before and the third, $r_1'(x) = \zeta x \zeta + 2$, is a complex Euclidean reflection whose action on $\mathbb{H}_\omega$ is a translated version of r_1. The first equation shows that r_1' is a complex Euclidean reflection fixing $1 + k$ and the second equation is useful for computations.

One can also write $r_1' = t_{1+i} \circ r_1 \circ t_{1+i}^{-1}$. Our choice of t_{1+k} as the conjugating translation is motivated by the following computation.

Example 8.5. The sets $\textsc{Fix}(r_1) = (i - j)\mathbb{C}$, $\textsc{Fix}(r_i) = (1 + k)\mathbb{C}$ and $\textsc{Fix}(r_1') = (1 + k) + (i - j)\mathbb{C}$ can be described as

$$\textsc{Fix}(r_1) = \{a + bi + cj + dk \mid a = 0, b + c + d = 0\},$$

$$\textsc{Fix}(r_i) = \{a + bi + cj + dk \mid b = 0, a + c - d = 0\},$$

$$\textsc{Fix}(r_1') = \{a + bi + cj + dk \mid a = 1, b + c + d = 1\}.$$

Solving these equations, one finds that $1 + k \in \Phi_{D_4}$ is the unique point in the intersection $\textsc{Fix}(r_1') \cap \textsc{Fix}(r_i)$. Thus r_1' and r_i generate a copy of $\textsc{Refl}(G_4)$ that uses $1 + k$ as its origin.

The complex spherical reflection group $\textsc{Refl}(G_4)$ acts on the root system Φ and the complex Euclidean reflection group $\textsc{Refl}(\widetilde{G}_4)$ acts on the Hurwitzian integers.

Definition 8.6. The $\mathbb{Z}$-span of Φ inside $\mathbb{H}$ is the set Λ of *Hurwitzian integers*. It consists of all quaternions of the form $\frac{a+bi+cj+dk}{2}$ where a, b, c and d are all even integers or all odd integers. Our notation is derived from the theory of Coxeter groups. The $\mathbb{Z}$-span of a root system Φ is its *root lattice* Λ and, as with Φ, we write $q \cdot \Lambda$ for the $\mathbb{Z}$-span of $q \cdot \Phi$ and $\Lambda_{D_4} = \{(a, b, c, d) \in \mathbb{Z}^4 \mid a + b + c + d \in 2\mathbb{Z}\}$ for the $\mathbb{Z}$-span of Φ_{D_4}. We note that $2 \cdot \Lambda \subset \Lambda_{D_4} \subset \Lambda$ and that each is an index 4 subset of the next.

The Hurwitzian integers have many nice properties. They form a subring of the quaternions with Φ as its group of units. Every element has an integral norm and Λ satisfies a noncommutative version of the Euclidean algorithm. See [9, Chapter 5] for details. The remainder of the section records basic facts about the action of $\mathrm{Refl}(\widetilde{G}_4)$ on $\mathbb{H}_\omega$.

Lemma 8.7. *The elements of the translation subgroup of* $\mathrm{Refl}(\widetilde{G}_4)$ *are precisely those of the form* t_q *with* $q \in 2 \cdot \Lambda$.

Proof. The element $t_2 = r_1' \circ r_1^{-1}$ is a translation in $\mathrm{Refl}(\widetilde{G}_4)$ and conjugating t_2 by elements of $\mathrm{Refl}(G_4)$ shows that all the translations t_q for all $q \in 2 \cdot \Phi$ are also in $\mathrm{Refl}(\widetilde{G}_4)$. Thus $\mathrm{Refl}(\widetilde{G}_4)$ contains the abelian subgroup T that they generate and this consists of all translations of the form $2 \cdot \Lambda$. The subgroup T is normal in $\mathrm{Refl}(\widetilde{G}_4)$ since it is stabilized by the generating set and, because the quotient of $\mathrm{Refl}(\widetilde{G}_4)$ by T is $\mathrm{Refl}(G_4)$, the elements in T are the only translations in $\mathrm{Refl}(\widetilde{G}_4)$. □

Lemma 8.8. *For each* v *in the lattice* Λ_{D_4}*, there is a copy of* $\mathrm{Refl}(G_4)$ *inside* $\mathrm{Refl}(\widetilde{G}_4)$ *fixing* v*. In particular, every point in* Λ_{D_4} *is an intersection of fixed hyperplanes of complex reflections in* $\mathrm{Refl}(\widetilde{G}_4)$.

Proof. By Example 8.5 this holds for $v = 1 + k$ and if we conjugate the copy of $\mathrm{Refl}(G_4)$ fixing $1 + k$ by an element of the copy fixing the origin we find copies fixing v for all $v \in \Phi_{D_4}$. Next, conjugating the copy at the origin by elements in the copies fixing the points in Φ_{D_4} shows that there is a copy fixing every point that is a sum of two elements in Φ_{D_4}. Continuing in this way shows that there is a copy fixing any point that is a finite sum of elements in Φ_{D_4}, a set equal to Λ_{D_4}. □

Lemma 8.9. *The primitive complex reflections in* $\mathrm{Refl}(\widetilde{G}_4)$ *are of the form* $t_v \circ r_q \circ t_v^{-1}$ *where* $v \in \Lambda_{D_4}$ *and* $q \in \{1, i, j, k\}$.

Proof. All of these primitive complex reflections are in $\mathrm{Refl}(\widetilde{G}_4)$ by Lemma 8.8. On the other hand, every primitive complex reflection r' in $\mathrm{Refl}(\widetilde{G}_4)$ must be parallel to one of the primitive reflections r_q with $q \in \{1, i, j, k\}$ in $\mathrm{Refl}(G_4)$, since quotienting by the normal translation subgroup sends primitive reflections in $\mathrm{Refl}(\widetilde{G}_4)$ to primitive reflections in $\mathrm{Refl}(G_4)$. Finally, if $r' = t_v \circ r_q \circ t_v^{-1}$ and $v \notin \Lambda_{D_4}$, then $r' \circ r_q^{-1} = t_v \circ r_q \circ t_v^{-1} \circ r_q^{-1} = t_u$ with $u \notin 2 \cdot \Lambda$, violating Lemma 8.7. □

Lemma 8.10. *The fixed hyperplanes of the complex reflections in* $\mathrm{Refl}(\widetilde{G}_4)$ *of the form* $t_v \circ r_q \circ t_v^{-1}$ *with* $v \in \Lambda_{D_4}$ *and* $q \in \{1, i, j, k\}$ *can be described as follows*:

$$\mathrm{Fix}(t_v \circ r_1 \circ t_v^{-1}) = \{a + bi + cj + dk \mid a = \ell,\ b + c + d = m\},$$

$$\text{Fix}(t_v \circ r_i \circ t_v^{-1}) = \{a + bi + cj + dk \mid b = \ell,\ a + c - d = m\},$$

$$\text{Fix}(t_v \circ r_j \circ t_v^{-1}) = \{a + bi + cj + dk \mid c = \ell,\ a + d - b = m\},$$

$$\text{Fix}(t_v \circ r_k \circ t_v^{-1}) = \{a + bi + cj + dk \mid d = \ell,\ b + c - a = m\},$$

where ℓ and m are the unique integers so that v satisfies the equations.

Proof. Direct computation. □

Now that the reflections and their fixed hyperplanes have been computed, one sees that the isolated fixed points listed in Lemma 8.8 are the only points that arise as intersections of fixed hyperplanes and this set then determines the structure of the Voronoi cells.

Lemma 8.11. *In the Voronoi cell structure around the set of isolated hyperplane intersections for the group* $\text{Refl}(\widetilde{G}_4)$ *the Voronoi cell around the origin is the standard* 24*-cell with vertices* Φ*, the other Voronoi cells are translates of the* 24*-cell by vectors in* Λ_{D_4} *and the link of each vertex in the Voronoi cell structure is a* 4*-dimensional cube.*

Proof. This is a standard computation. See [8, Section 7.2]. □

Lemma 8.12. *The vertices of the Voronoi cell structure are located at the points in* $\Lambda \setminus \Lambda_{D_4}$ *and the group* $\text{Refl}(\widetilde{G}_4)$ *acts transitively on this set.*

Proof. Since the translates of the 24-cell are centered at the elements of Λ_{D_4}, every vertex of the Voronoi cell structure can be described as $u + v$ with $u \in \Phi$ and $v \in \Lambda_{D_4}$. The set Λ contains Λ_{D_4} as an index 4 sublattice, and the 3 other cosets of Λ_{D_4} are of the form $u + \Lambda_{D_4}$ with $u \in \Phi$. In particular, every element of Λ that is not in Λ_{D_4} differs from an element of Λ_{D_4} by an element in Φ. Transitivity follows from the connectivity of the 1-skeleton of the Voronoi cell structure. Given two vertices and a path between them, every edge in this path is in the boundary of some 24-cell. The local copy of $\text{Refl}(G_4)$ fixing this 24-cell acts transitively on its vertices, making it possible to move the initial vertex of this edge to its terminal vertex. Combining these local transitivities gives the global transitivity. □

The following fact is crucial to the proofs of our main results.

Lemma 8.13. *The intersection of a fixed hyperplane H of a complex reflection in the group* $\text{Refl}(\widetilde{G}_4)$ *with a closed Voronoi cell V is either empty, or it is a closed hexagon passing through the point at the center of V. The boundary of this closed hexagon is the fixed hexagon from the interior of a necklace in the boundary of V.*

Proof. This assertion follows from an explicit description of the minimal cellular neighborhood containing a fixed hyperplane. From the equations listed in Lemma 8.8, one sees that every fixed hyperplane contains at least one point from the set Λ_{D_4}, i.e. the center of some Voronoi cell, and the fixed hyperplane intersects the boundary of this Voronoi cell as described above (Remark 7.4). In particular, it then passes into the interiors of the Voronoi cells that also contain the octahedra and triangles pierced by the fixed hexagon. There is only one other Voronoi cell containing any particular octahedron and only two other Voronoi cells containing any triangle. Given that the hyperplane orthogonally intersects the octahedra and the triangles, the extension of the hyperplane also passes through the centers of the neighboring Voronoi cells. Overall, the intersection of the 3-skeleton of the Voronoi cell structure with a fixed hyperplane gives this affine complex plane the cell structure of a regular hexagonal tiling. The interior of each hexagon lies in the interior of a Voronoi cell, the interior of each edge lies in the interior of an octahedron, and each vertex lies at the center of a triangle. Since the union of the closed Voronoi cells that non-trivially intersect H contain an ϵ-neighborhood of H, every other Voronoi cell intersects H trivially. □

9. Proofs of Main Theorems

In this section we prove all three of our main results. We begin by defining the complement complex K. Throughout this section we use G to denote the group $\text{Refl}(\widetilde{G}_4)$ and B to denote the group $\text{Braid}(\widetilde{G}_4)$.

Definition 9.1. The *complement complex* K is the portion of the Voronoi cell structure of the group G that is disjoint from the union of the fixed hyperplanes of its complex reflections.

By Lemma 8.13, around each fixed hyperplane intersection point the portion of K in the boundary of this particular 24-cell is a copy of the 2-complex K_0 defined in Definition 7.5. Thus K can be viewed as a union of local copies of K_0. The proof of our first main theorem reduces to an observation about the structure of the vertex links of K, which turn out to be isomorphic to a well-known graph.

Proof of Theorem A. The proof combines the local deformations described in Theorem 7.7 into a global deformation. The first step is to radially deformation retract from the removed isolated fixed point at the center of each Voronoi cell to its boundary, which can be carried out because of Lemma 8.13. Next, the secondary deformations applied to the punctured equilateral triangles and the skewered octahedra are compatible regardless of which Voronoi cell one views them as belonging to. The link of a vertex in the complement complex K is the portion of the 1-skeleton of the 4-cube shown in Figure 4. This is a 16 vertex 3-regular graph known as the *Möbius–Kantor graph* [10]. The 8 removed edges correspond to the equilateral triangles whose center lies in one of the fixed hyperplanes. The portion of this graph that lives in one of the eight 3-cubes in the 4-cube is the subdivided theta graph that is the link of this vertex inside the corresponding copy of K_0 inside a particular 24-cell. Finally, every edge in every vertex link has length $\frac{\pi}{3}$ and since Möbius–Kantor graphs have no simple cycles of combinatorial length less than 6, there are no simple loops of length less than 2π, the vertex links are CAT(1) and K is non-positively curved. □

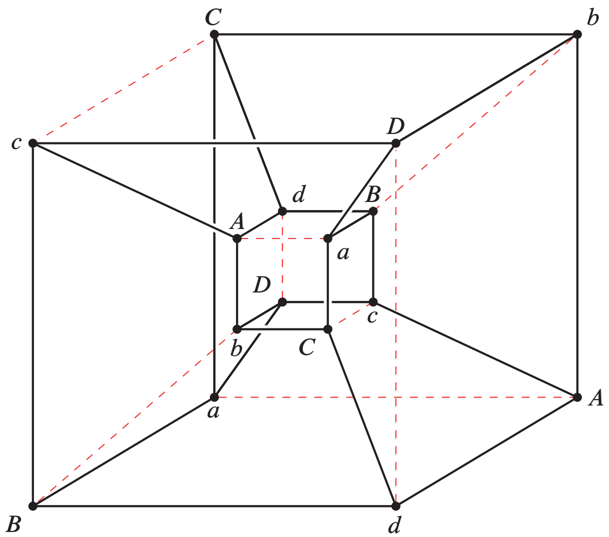

Figure 4. The Möbius–Kantor graph as a subgraph of the 1-skeleton of a 4-cube with 8 edges removed.

Remark 9.2. Although we have attempted to extend our main theorems to other complex Euclidean reflections groups acting on $\mathbb{C}^2$, it is the analog of Lemma 8.13 where those attempts have failed. If a fixed hyperplane intersects the boundary of a Voronoi cell without passing through its center, then the initial deformation retraction onto a portion of the 3-skeleton of the Voronoi cell structure might not be possible due to the fact that points from the interior of the Voronoi cell are being pushed to points in the boundary that are missing.

Our second main theorem asserts that there are isolated irregular points inside the hyperplane complement. We prove this in two steps. We first prove that irregular points exist in the hyperplane complement, and then we classify the full set of irregular points, in order to see that these other irregular points are isolated.

Theorem 9.3. *Under the action of G, the vertices of the complement complex K are irregular points contained in the complement of the fixed hyperplanes.*

Proof. Let T be the set of translations in G, let T_0 the images of the origin under the translations in T and let FP_A be the set of points fixed by some element in G whose linear part is the antipodal map. As in Remark 3.3, the simplification $-(x-v)+v = -x+2v$ shows that $2\cdot\mathrm{FP}_A = T_0$. Since $T_0 = 2\cdot\Lambda$, this means that $\mathrm{FP}_A = \Lambda$. The vertices of the complement complex K are contained in Λ (Lemma 8.12), so there is an element of order 2 fixing each of them. By Lemma 8.13, the fixed hyperplanes of the reflections avoid the vertices of K. □

Theorem 9.4. *Under the action of G, the only irregular points in the hyperplane complement are the vertices of the complement complex K, and they are isolated irregular points.*

Proof. Suppose a point x is stabilized by a non-trivial element s. If x does not lie on a fixed hyperplane, the linear part of s must be something other than a complex reflection, and every such s has a power that is the antipodal map. Concretely, the second power of $\pm L_q$ and the third powers of $-r_q$ and of $-r_q^2$ are antipodal maps, and these are the only possibilities for the linear part of s. In particular, the point x must lie in $\mathrm{FP}_A = \Lambda$. By Lemma 8.8, all of the points in Λ are either vertices of K or contained in the fixed hyperplanes. Thus x is a vertex of K. The second assertion follows from the complete classification of irregular points. □

Taken together, Theorems 9.3 and 9.4 prove Theorem B. For later use we record the following properties of the quotient K/G.

Proposition 9.5. *The quotient K/G is a compact 2-complex.*

Proof. The action of G on K is not free, but it is proper and cellular with trivial stabilizers for every cell of positive dimension (Theorem 9.4). As a consequence, the quotient remains a 2-complex. In addition, the translations in G show that every cell orbit contains a representative in the boundary of the 24-cell containing the origin. Thus there are finitely many orbits of cells and the quotient is compact. □

Remark 9.6. The quotient K/G is the presentation complex for the presentation $\langle a,b,c,d \mid abd, bcd, cad, cba\rangle$. It has one vertex, four edges and four triangles. The finite group defined by this presentation is called the binary tetrahedral group and the universal cover of this complex is the 2-skeleton of the 24-cell. When this same one-vertex presentation 2-complex K/G is viewed as the *orbifold* quotient of G acting on K, the infinite complex Euclidean reflection group G is its orbifold fundamental group.

We now use the complete classification of irregular points to compute the corresponding braid group. Since the braid group of a group action is defined as the fundamental group of the space of regular orbits, and the vertices of K are not regular points, the complement complex K needs to be modified.

Definition 9.7. The modified complex K' is constructed in three steps through two intermediate spaces. Let K_1 be the union of the complement complex K and the set of small closed balls of radius $\epsilon > 0$ in $\mathbb{C}^2$ centered at each of the vertices of K. Next, let K_2 be the metric space obtained by removing from K_1 the points corresponding to the vertices of K. Finally, let K' be the space obtained by removing from K_1 the open balls of radius ϵ centered at each of the vertices of K. We call K' the *modified complement complex*.

The modified complex K' is related to the space of regular points.

Proposition 9.8. *The space K' is homotopy equivalent to the space of regular points.*

Proof. We construct a deformation retraction from the space of regular points to K' in two steps using the intermediate complexes K_1 and K_2 (Definition 9.7). First, we modify the deformation retraction of Theorem A, from the hyperplane complement to K, by simply stopping the retraction whenever a point is distance ϵ from a vertex. This shows that the hyperplane complement deformation retracts to the space K_1. Moreover, removing the stationary vertices of K from this deformation retraction shows that the space of regular points is homotopy equivalent to K_2. Finally, the space K_2 is homotopy equivalent to K' since around each deleted vertex, the punctured 3-ball radially deformation retracts onto its boundary 2-sphere. □

The action of G on K' is free and the fundamental group of the quotient K'/G is the group B, by definition. The quotient K'/G is closely related to the quotient complex K/G.

Proposition 9.9. *K'/G is a copy of the one vertex 2-complex K/G with a neighborhood of the unique vertex removed and a real projective plane attached in its place.*

Proof. The proof uses the intermediate spaces K_1 and K_2 (Definition 9.7). The relationship between K_1/G and K_2/G is as follows. The former is a modification of K/G where the neighborhood of the unique vertex becomes a cone on an $\mathbb{R}P^2$ with the vertex as its cone point, and the latter is this space with the cone point removed. Thus K'/G is a copy of K/G with a neighborhood of the vertex removed and a real projective plane attached instead. □

The group B acts freely by deck transformations on the universal cover of the quotient K'/G, which is the same as $\widetilde{K}'$, the universal cover of K'. And note that the quotient $\widetilde{K}'/B$ is the same as K'/G. From this we deduce an action on the universal cover of K.

Theorem 9.10. *The group B acts geometrically on $\widetilde{K}$, the* CAT(0) *universal cover of the complement complex K. In particular, the action is free away from the vertex set and every vertex stabilizer has order* 2*.*

Proof. Because 2-spheres are simply connected, the space $\widetilde{K}'$ is a modified version of $\widetilde{K}$, the CAT(0) universal cover of K. The local modifications around each vertex are identical to the ones described in Definition 9.7. In short, $\widetilde{K}$ is a modified version of $\widetilde{K}'$, K is a modified version of K' and $\widetilde{K}/B = K/G$ is a modified version of $\widetilde{K}'/B = K'/G$. This shows that the action of B on $\widetilde{K}$ is cocompact, because $\widetilde{K}/B = K/G$ is a compact 2-complex by Proposition 9.5. Finally, the free and isometric action of B on $\widetilde{K}'$ induces a proper isometric action of B on $\widetilde{K}$. The only non-trivial stabilizers are order 2 and they only occur at the vertices of $\widetilde{K}$, which arise from the collapsing of each tiny 2-sphere in $\widetilde{K}'$ to a point. □

And this proves our third main result, Theorem C.

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