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Fluctuations of ergodic averages for amenable group actions

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Abstract. We show that for any countable amenable group action, along certain Følner sequences (those that have for any $c > 1$ a two-sided c-tempered tail), one has a universal estimate for the number of fluctuations in the ergodic averages of L^∞ functions. This estimate gives exponential decay in the number of fluctuations. Any two-sided Følner sequence can be thinned out to satisfy the above property. In particular, any countable amenable group admits such a sequence. This extends results of S. Kalikow and B. Weiss [1] for $\mathbb{Z}^d$ actions and of N. Moriakov [3] for actions of groups with polynomial growth.

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1. Introduction

A real-valued sequence is said to *fluctuate N times across a gap* (α, β), if there are integers $n_1 < n_2 < \cdots < n_{2N}$ such that for odd i, $a_{n_i} \le \alpha$, and for even i, $a_{n_i} \ge \beta$. Let $(X, \mu, \mathcal{B}, (T_g)_{g \in G})$ be a measure-preserving action of a countable amenable group G, and fix some (left) Følner sequence (F_n) in G. For any N, we define the set D_N by

$$D_N = D_{(F_n), f, \alpha, \beta, N} = \{x \colon \mathsf{A}_n f(x) \text{ fluctuates across } (\alpha, \beta) \text{ at least } N \text{ times}\}$$

where $\mathsf{A}_n f = \frac{1}{|F_n|} \sum_{g \in F_n} f \circ T_g$ denotes the sequence of ergodic averages of a function f on X along (F_n). In [1] it was shown that for $G = \mathbb{Z}^d$ and $F_n = [-n, n]^d$, the following holds:

Theorem 1.1. *For any $0 < \alpha < \beta$, there are constants $0 < c_0 < 1$ and $c_1 > 0$, such that for every m.p.s. $(X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ and every measurable $f \ge 0$, one has*

$$\mu(D_N) \le c_1 c_0^N \quad \textit{for all } N.$$

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In [3] this result was extended to measure-preserving actions of groups of polynomial growth, where the fixed Følner sequence is taken to be balls of increasing radii, that is, $F_n = S^n$ where S is a finite symmetric set of generators which contains the unit.

The aim of this paper is to extend these results to general actions of amenable groups. In this context, the notion of temperedness is of importance: a sequence (F_n) is *left c-tempered* if, for all n,

$$\Big|\bigcup_{i<n} F_i^{-1} F_n\Big| \leq c|F_n|,$$

right c-tempered if, for all n,

$$\Big|\bigcup_{i<n} F_n F_i^{-1}\Big| \leq c|F_n|,$$

and *c-bi-tempered* if it is both left and right c-tempered. In this paper, a sequence that for any $c > 1$ has some tail which is c-bi-tempered, will be called *strongly tempered*. Notice that any two-sided Følner sequence can be thinned out to be strongly tempered.

The class of tempered Følner sequences is the most general class of sequences that are known to satisfy the pointwise ergodic theorem [2, 5]. That is, the averages along any (left) tempered Følner sequence of any integrable function converges a.e. Consequently, if the fixed Følner sequence (F_n) is tempered, then for any $\alpha < \beta$ and any integrable function f, the measure of $D_N = D_{(F_n),f,\alpha,\beta,N}$ decreases to zero as $N \to \infty$. Thus, along such sequences, one might hope to have some control over the rate of $\mu(D_N)$, as in Theorem 1.1:

Question. *Does every amenable group have a Følner sequence that satisfies* (*in some sense*) *Theorem* 1.1? *Can one find for any Følner sequence a subsequence with this property*?

Our main result is the following theorem and its corollary, which says that one can successfully bound the rate of decrease of $\mu(D_N)$ in any amenable group, provided that f is bounded and that the averages are taken along strongly tempered Følner sequences.

Theorem 1.2. *For any $\alpha < \beta$ and $S > 0$, there exist $\lambda > 0$ and $0 < c_0 < 1$, such that for any $(1+\lambda)$-bi-tempered Følner sequence (F_n), any m.p.s. $(X, \mu, \mathcal{B}, (T_g)_{g\in G})$ and any $f \in L^\infty_\mu(X)$ with $\|f\|_\infty \leq S$, one has*

$$\mu(D_N) \leq c_1 c_0^N \quad \textit{for all } N$$

for some $c_1 > 0$ which depends only on the sequence (F_n) (and neither on the m.p.s. nor on the function f).

If (F_n) is strongly tempered, then for any gap $(\alpha, \beta) \subset \mathbb{R}$ and any $S > 0$, some tail of the sequence, say $(F_n)_{n>n_0}$, satisfies the hypothesis of Theorem 1.2, while the first n_0 elements of (F_n) attribute at most $O(n_0)$ fluctuations. Thus, enlarging c_1 depending on that n_0, we get:

Corollary 1.3. *Let (F_N) be a strongly tempered Følner sequence. For any $\alpha < \beta$ and $S > 0$, there exist $0 < c_0 < 1$ and $c_1 > 0$, such that for any m.p.s. $(X, \mu, \mathcal{B}, (T_g)_{g\in G})$ and any $f \in L^\infty_\mu(X)$ with $\|f\|_\infty \leq S$, one has*

$$\mu(D_N) \leq c_1 c_0^N \quad \textit{for all } N.$$

As the proof of Theorem 1.2 indicates, the bi-temperedness condition could be slightly relaxed and was chosen for the clarity of presentation. Also, the dependency of c_1 on the sequence (F_n) could be replaced by restricting the theorem to sequences with some certain properties. For example, assuming $e \in F_1$ would be enough for determining c_1, regardless of what (F_n) is.

In contrast to Theorem 1.2, we show that the temperedness property (with any fixed $c > 1$) alone, isn't enough to bound the rate of decrease of $\mu(D_N)$ for any given gap (α, β). More precisely, we show that in any measure-preserving $\mathbb{Z}$-action $(X, \mu, \mathcal{B}, \{T^n\}_{n\in\mathbb{Z}})$ one has the following:

Theorem 1.4. *Let $(X, \mathcal{B}, \mu, \{T^n\}_{n\in\mathbb{Z}})$ be an m.p.s. and let $\omega(n) \searrow 0$ be any sequence that decreases to 0. For any $\lambda > 0$, there are some $\alpha < \beta$, a bounded function $0 \leq f \leq 1$ and a $(1+\lambda)$-tempered Følner sequence (F_n), for which $\mu(D_{(F_n),f,\alpha,\beta,N}) > \omega(N)$ for all but finitely many N.*

Although this shows that the requirement for (F_n) to have a left $(1+\lambda)$-tempered tail for any λ is essential for Corollary 1.3 to take place, it is not clear whether the other requirements are. More generally, the following question remains open:

Question 1.5. *Does every left Følner sequence in a countable amenable group G have a subsequence which satisfies the conclusion of Corollary 1.3?*

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2. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Towards this end, we need a few definitions and lemmas.

Definition 2.1. Given $0 < \lambda < 1$, we say that a sequence (F_n) is λ-*good* if the following two conditions hold:

(i) for any n, $\left|\bigcup_{i<n} F_i^{-1}F_n \backslash F_n\right| \leq \lambda|F_n|$;

(ii) for any $i < n$ and $f \in F_i$, $|F_n \backslash F_n f| < \lambda|F_n|$.

Proposition 2.2. *Let* $0 < \lambda < \lambda' < 1$. *For any* $(1+\lambda)$*-bi-tempered two-sided Følner sequence* (F_n)*, there is some* n_0 *such that* $(F_n)_{n\geq n_0}$ *is* λ'*-good.*

Proof. Pick some $g_0 \in F_1$. Since the sequence is (left) Følner, there is some n_1 such that for all $n \geq n_1$,

$$|g_0^{-1}F_n \cap F_n| > (1-\lambda'+\lambda)|F_n|$$

By the (left) temperedness property of (F_n), we have

$$\begin{aligned}\left|\bigcup_{n_1\leq i<n} F_i^{-1}F_n \backslash F_n\right| &\leq \left|\bigcup_{1\leq i<n} F_i^{-1}F_n\right| - \left|\bigcup_{1\leq i<n} F_i^{-1}F_n \cap F_n\right| \\ &< (1+\lambda)|F_n| - (1-\lambda'+\lambda)|F_n| \\ &= \lambda'|F_n|\end{aligned}$$

and (i) of Definition 2.1 takes place for the sequence $(F_n)_{n\geq n_1}$. The same proof applies from the right, thus we get some n_2 such that for any $n \geq n_2$

$$\left|\bigcup_{n_2\leq i<n} F_n F_i^{-1} \backslash F_n\right| \leq \lambda'|F_n|$$

but now, for any $i < n$ and $f \in F_i$,

$$|F_n \backslash F_n f| = |F_n f^{-1} \backslash F_n| \leq \left|\bigcup_{n_2\leq i<n} F_n F_i^{-1} \backslash F_n\right| \leq \lambda'|F_n|$$

which is (ii) of Definition 2.1 for the sequence $(F_n)_{n\geq n_2}$. Take $n_0 = \max\{n_1, n_2\}$. □

The following theorem is a version of Theorem 1.2 for λ-good Følner sequences, from which we will deduce Theorem 1.2:

Theorem 2.3. *For any* $\alpha < \beta$ *and* $S > 0$*, there exist* $\lambda > 0$*,* $0 < c_0 < 1$ *and* $c_1 > 0$*, such that for any* λ*-good (left) Følner sequence* (F_n)*, any m.p.s.* $(X, \mu, \mathcal{B}, (T_g)_{g\in G})$ *and any* $f \in L^\infty_\mu(X)$ *with* $\|f\|_\infty \leq S$*, one has*

$$\mu(D_N) \leq c_1 c_0^N \quad \textit{for all } N.$$

We remark that as opposed to Theorem 1.2, here the constant c_0 doesn't depend on (F_n).

Once Theorem 2.3 is valid, the proof of Theorem 1.2 is immediate:

Proof of Theorem 1.2. For $[\alpha, \beta]$ and $S > 0$, let λ' be the value for which any λ'-good Følner sequence satisfies the conclusion of Theorem 2.3 with c_0' and c_1'. Take $0 < \lambda < \lambda'$, then by Proposition 2.2, for any $(1+\lambda)$-bi-tempered two-sided Følner sequence (F_n), there is some n_0, such that $(F_n)_{n \geq n_0}$ is λ'-good, and thus for any m.p.s. $(X, \mu, \mathcal{B}, (T_g)_{g \in G})$, any $f \in L^\infty_\mu(X)$ with $\|f\|_\infty \leq S$ and any N,

$$\mu(D_{(F_n)_{n\geq 1},N}) \leq \mu(D_{(F_n)_{n \geq n_0}, N-n_0}) \leq c_1' c_0'^{N-n_0}$$

thus for $c_0 = c_0'$, $c_1 = c' c_0'^{-n_0}$ the conclusion follows. □

Thus it remains to prove Theorem 2.3, which will be our task for the rest of this section.

Definition 2.4. Given $\epsilon > 0$, a collection $(F_j)_{j=1}^L$ of finite subsets of G is said to be *ϵ-disjoint* if there are pairwise disjoint sets $E_j \subset F_j$ such that $|E_j| \geq (1-\epsilon)|F_j|$ for all $1 \leq j \leq L$.

We record here a version of the ϵ-disjointification lemma [5, Lemma 9.2], which will be used again and again:

Lemma 2.5 (ϵ-disjointification lemma). *Let $F_1, \dots, F_L$ be a sequence of finite subsets of a group G which is 2-tempered, let $C \subset G$ be finite, and suppose that $C_1, \dots, C_L$ are disjoint subsets of C. For any $0 < \epsilon \leq \frac{1}{2}$, there are subsets $D_j \subset C_j$, such that*

(i) the collection $\{F_j d : d \in D_j,\ 1 \leq j \leq L\}$ is ϵ-disjoint;

(ii) $\left|\bigcup_{j=1}^L F_j D_j\right| \geq \frac{\epsilon}{5}|C|$.

The following proposition, which is analogous to the effective Vitali covering argument of Kalikow and Weiss [1], will be used as a key step throughout the proof of Theorem 2.3.

Proposition 2.6. *For any $\epsilon > 0$, once $\lambda > 0$ is small enough and $q \in \mathbb{N}$ is large enough, the following holds for any λ-good Følner sequence (F_n).*

Let $C \subset G$ be a finite subset, and suppose that for each $c \in C$ there is associated a subsequence of $(F_n c)$ of length q:

$$F_{n_1(c)}c, \dots, F_{n_q(c)}c, \quad n_1(c) < \dots < n_q(c).$$

Then there exists an ϵ-disjoint collection $\{F_{n(d)}d\}_{d \in D}$ where $D \subset C$ and $n(d) \in \{n_1(d), \dots, n_q(d)\}$, which satisfies at least one of the following properties:

(1) *either* $\left|\bigcup_{d\in D} F_{n(d)}d\right| \geq 2|C|$,

(2) *or* $\left|\bigcup_{d\in D} F_{n(d)}d \cap C\right| \geq (1-\epsilon)|C|$.

As it can be seen from the proof below, for (F_n) to satisfy the conclusion, one can assume that (F_n) is a Følner sequence that merely admits property (i) of being λ-good (Definition 2.1).

Proof. Define

$$\mathcal{C} = \{(c, n_i(c)) : c \in C, 1 \leq i \leq q\}$$

let $m = \max\{n : \text{there exists } c \in G \text{ such that } (c,n) \in \mathcal{C}\}$, and consider the m-section of $\mathcal{C}$:

$$C_m = \{c : (c,m) \in \mathcal{C}\}$$

Assuming $\lambda \leq 1$, the ϵ-disjointification lemma guarantees there is a subset $D_m \subset C_m$, such that

(a) the collection $\{F_m d\}_{d\in D_m}$ is ϵ-disjoint, and

(b) $|F_m D_m| \geq \frac{\epsilon}{5}|C_m|$.

Let $1 \leq k \leq m-1$, and suppose we have already defined subsets $(D_{m-i})_{i=0}^{k-1}$ of C. Denote

$$W_{m-k+1} = C \setminus \bigcup_{n=m-k+1}^{m} \bigcup_{i<n} F_i^{-1} F_n D_n \tag{1}$$

$$C_{m-k} = \{c \in W_{m-k+1} : (c, m-k) \in \mathcal{C}\} \tag{2}$$

and use again the ϵ-disjointification lemma to take some $D_{m-k} \subset C_{m-k}$ so that

(a)′ the collection $\{F_{m-k} d\}_{d\in D_{m-k}}$ is ϵ-disjoint, and

(b)′ $|F_{m-k} D_{m-k}| \geq \frac{\epsilon}{5}|C_{m-k}|$.

The restriction $C_{m-k} \subset W_{m-k+1}$ (2) together with (1) guarantees that

$$\bigcup_{n=m-k+1}^{m} F_n D_n \cap F_{m-k} D_{m-k} = \emptyset.$$

We end up (after m steps) with a pairwise disjoint subsets $D_1 \subset C_1, \ldots, D_m \subset C_m$ where $\bigsqcup_{n=1}^{m} C_n \times \{n\} \subset \mathcal{C}$, and such that each $\{F_n d\}_{d\in D_n}$ is ϵ−disjoint, the unions $\bigcup_{d\in D_n} F_n d = F_n D_n$ are disjoint to each other and are of size $|F_n D_n| \geq \frac{\epsilon}{5}|C_n|$. Let $\mathcal{D} = \bigsqcup_{n=1}^{m} D_n \times \{n\}$. We claim that the collection $\{F_n d\}_{(d,n)\in\mathcal{D}}$ satisfies the conclusion of the lemma. We just pointed out that it is indeed an ϵ-disjoint collection. Suppose it doesn't satisfy property (2) of the conclusion, that is,

$$\left|C \setminus \bigcup_{k=1}^{m} F_k D_k\right| \geq \epsilon|C|. \tag{3}$$

We distinguish between two cases:

I. One has

$$2\lambda\Big|\bigcup_{k=1}^{m} F_k D_k\Big| \geq \frac{1}{2}\Big|C\setminus\bigcup_{k=1}^{m} F_k D_k\Big|$$

then, together with (3) one gets

$$\Big|\bigcup_{k=1}^{m} F_k D_k\Big| \geq \frac{1}{4\lambda}\Big|C\setminus\bigcup_{k=1}^{m} F_k D_k\Big| \geq \frac{\epsilon}{4\lambda}|C|$$

and for small enough λ ($\lambda \leq \frac{\epsilon}{8}$), the last inequality gives property (1) in the conclusion, so we're done.

II. For the other case,

$$2\lambda\Big|\bigcup_{k=1}^{m} F_k D_k\Big| < \frac{1}{2}\Big|C\setminus\bigcup_{k=1}^{m} F_k D_k\Big| \tag{4}$$

we bound from below the size of $W_2 = C\setminus\bigcup_{n=2}^{m}\bigcup_{i<n} F_i^{-1}F_n D_n$:

$$\begin{aligned}
|W_2| &\geq \Big|C\setminus\bigcup_{k=1}^{m} F_k D_k\Big| - \Big|\bigcup_{(d,n)\in\mathfrak{D}}\bigcup_{i<n} F_i^{-1}F_n d\setminus F_n d\Big| \\
&\geq \Big|C\setminus\bigcup_{k=1}^{m} F_k D_k\Big| - \lambda\sum_{(d,n)\in\mathfrak{D}}|F_n d| \\
&\geq \Big|C\setminus\bigcup_{k=1}^{m} F_k D_k\Big| - \frac{\lambda}{1-\epsilon}\Big|\bigcup_{k=1}^{m} F_k D_k\Big| \\
&\geq \frac{1}{2}\Big|C\setminus\bigcup_{k=1}^{m} F_k D_k\Big|
\end{aligned}$$

(the second inequality follows from property (i) of Definition 2.1, the third by the ϵ-disjointness of the collection, and the last one by (4), together with the assumption $\epsilon \leq \frac{1}{2}$). Any element in W_2 appears as the left coordinate of q different elements in $\bigcup_{k=1}^{m} C_k \times \{k\}$, thus,

$$\begin{aligned}\Big|\bigcup_{(d,n)\in\mathfrak{D}} F_n d\Big| &= \sum_{k=1}^{m} |F_k D_k| \\ &\ge \frac{\epsilon}{5}\sum_{k=1}^{m} |C_k| \\ &= \frac{\epsilon}{5}\Big|\bigcup_{k=1}^{m} C_k \times \{k\}\Big| \\ &\ge \frac{\epsilon}{5} q|W_2| \\ &\ge \frac{\epsilon}{10} q\Big|C \setminus \bigcup_{k=1}^{m} F_k D_k\Big| \\ &\ge \frac{\epsilon^2}{10} q|C|\end{aligned}$$

assuming $q \ge \frac{20}{\epsilon^2}$, the lemma is proved. □

Proof of Theorem 2.3. For any $x \in X$, the number of fluctuations of $\mathsf{A}_n f(x)$ across (α, β) is equal to the number of fluctuations of $\mathsf{A}_n[f + \|f\|_\infty](x)$ across $(\alpha + \|f\|_\infty, \beta + \|f\|_\infty)$. Consequently, for any N,

$$D_{(F_n),f,\alpha,\beta,N} = D_{(F_n),f+\|f\|_\infty,\alpha+\|f\|_\infty,\beta+\|f\|_\infty,N}.$$

Notice that $\|f + \|f\|_\infty\|_\infty \le 2\|f\|_\infty$, and besides trivial cases, one has $0 < \alpha + \|f\|_\infty$. Hence, for any $S > 0$ and $\alpha < \beta$, any estimate of $\mu(D_N)$, where D_N is defined with respect to any nonnegative function $0 \le f \le 2S$ and the gap $[\alpha + S, \beta + S] \subset (0, \infty)$, is an estimate of $\mu(D_N)$, where D_N is defined with respect to any function $\|f\|_\infty \le S$ and the gap $[\alpha, \beta] \subset \mathbb{R}$. Thus from now on, we shall assume $0 \le f \le S$ and $0 < \alpha < \beta$.

Write $D_{N,M} = \{x\colon (\mathsf{A}_n f(x))_{n=1}^{M}$ fluctuates across (α, β) at least N times$\}$. Fix $x \in X$, $M \in \mathbb{N}$, and let $\Omega \subset G$ be a set which is sufficiently invariant with respect to $\bigcup_{n=1}^{M} F_n$, so that the set

$$B = \Big\{g \in \Omega\colon \bigcup_{n=1}^{M} F_n g \subset \Omega\Big\}$$

has size close to $|\Omega|$. We will give an upper bound to the relative density $\frac{|C|}{|\Omega|}$, where

$$C = C_{x,M} = \{c \in B\colon cx \in D_{N,M}\}.$$

This upper bound won't depend on x or M, and thus by the transference principle, it will give an upper bound for $\mu(D_N)$, as it is shown at the end of the proof.

Take

$$\delta = \min\left\{\frac{1}{2}\left(\frac{\beta}{\alpha} - 1\right), \frac{1}{2}\right\},$$

and choose $\frac{1}{4} > \epsilon > 0$ small enough so that the following three inequalities hold:

$$\frac{(\beta - 4\epsilon S)(1-\epsilon)}{\alpha} \geq 1 + \delta > 1, \tag{5}$$

$$(1-\epsilon)(1+\delta) \geq (1 + \delta/2), \tag{6}$$

$$(1-\epsilon) \geq (1+\delta/2)^{-1}. \tag{7}$$

Take $q \in \mathbb{N}$ and $0 < \lambda \leq \epsilon/2$ so that the conclusion of Lemma 2.6 will take place with $\epsilon/2$.

The first step is to replace C with a union of ϵ-disjoint collections of size not much less than $|C|$, where for each set in the collection, the average of f at x on it is above β. For that, use the first group of q fluctuations to find for each $c \in C$ an increasing sequence $n_1(c) < \cdots < n_q(c)$ such that $\mathsf{A}_{n_i(c)} f(cx) \geq \beta$ for each $1 \leq i \leq q$. Then, by applying Proposition 2.6, one takes an ϵ-disjoint collection $(F_n c)_{(c,n)\in\mathcal{B}_1}$, where its union $C_1 = \bigcup_{(c,n)\in\mathcal{B}_1} F_n c$ is in Ω and of size $|C_1| \geq (1-\epsilon)|C|$. The next step will be done recursively $\left(\left\lfloor \frac{N}{2q} \right\rfloor - 1\right)$ times, thus we introduce it in a more general form:

Lemma 2.7. *Let $f, (F_n), C, \alpha, \beta, \delta, \epsilon, N$ and q be as above. Let $N_k \leq N - 2q$, and suppose that $\mathcal{B}_k \subset C \times \mathbb{N}$ is a collection of tuples such that*

(i) *for each $(c,n) \in \mathcal{B}_k$ the average $\mathsf{A}_n f(cx)$ is one of cx's first N_k upcrossings to above β;*

(ii) *the collection $(F_n c)_{(c,n)\in\mathcal{B}_k}$ is ϵ-disjoint.*

Then there exists a collection $\mathcal{B}_{k+1} \subset C \times \mathbb{N}$ of tuples such that

(i) *for each $(c,n) \in \mathcal{B}_{k+1}$, the average $\mathsf{A}_n f(cx)$ is one of cx's first $N_k + 2q$ upcrossings to above β;*

(ii) *the collection $(F_n c)_{(c,n)\in\mathcal{B}_{k+1}}$ is ϵ-disjoint;*

(iii) $\left|\bigcup_{(c,n)\in\mathcal{B}_{k+1}} F_n c\right| \geq (1+\delta/2)\left|\bigcup_{(c,n)\in\mathcal{B}_k} F_n c\right|$.

Proof of Lemma 2.7. Denote $C_k = \bigcup_{(c,n)\in\mathcal{B}_k} F_n c$. To each $g \in C_k$ we will associate a subsequence of $(F_n g)$ of length q, in order to apply Lemma 2.6 to the set C_k. For any $g \in C_k$, choose some $c = c(g)$ so that $(c,n) \in \mathcal{B}_k$ for some n and $g \in F_n c$. Associate to g the indices of the next q downcrossings to below α of c, $n < n_1(c) < \cdots < n_q(c)$. By Proposition 2.6, there is an $\epsilon/2$-disjoint collection $(F_n g)_{(g,n)\in\mathcal{B}'_k}$, with union $C'_k = \bigcup_{(g,n)\in\mathcal{B}'_k} F_n g \subset \Omega$ that satisfies one of the two options in the conclusion of Proposition 2.6. Next, we define another index set $\mathcal{B}''_k$ to be

$$\mathcal{B}''_k = \{(c,n)\colon \text{there exists } (g,n) \in \mathcal{B}'_k \text{ such that } c(g) = c\}$$

and the union of its associated collection

$$C_k'' = \bigcup_{(c,n)\in\mathcal{B}_k''} F_n c.$$

For any $(c,n) \in \mathcal{B}_k''$, let $(g,n) \in \mathcal{B}_k'$ be such that $c(g) = c$. Then, (F_n) being λ-good, by (ii) of Definition 2.1,

$$|F_n g \triangle F_n c(g)| < \lambda |F_n| \leq \epsilon/2 |F_n|.$$

That, together with $(F_n g)_{(g,n)\in\mathcal{B}_k'}$ being $\epsilon/2$-disjoint, implies that

$$(F_n c)_{(c,n)\in\mathcal{B}_k''} \text{ is } \epsilon\text{-disjoint} \tag{8}$$

and that

$$\begin{aligned} |C_k'' \cap C_k'| &\geq \sum_{(g,n)\in\mathcal{B}_k'} ((1-\epsilon/2)|F_n g| - |F_n g \setminus F_n c(g)|) \\ &\geq (1-\epsilon) \sum_{(g,n)\in\mathcal{B}_k'} |F_n g| \\ &\geq (1-\epsilon)|C_k'|. \end{aligned} \tag{9}$$

This relation together with C_k' being as in the conclusion of Proposition 2.6, gives one of the following two options:

1. either $|C_k'| \geq 2|C_k|$, in which case (9) implies that

$$|C_k''| \geq 2(1-\epsilon)|C_k|, \tag{10}$$

2. or $|C_k'| < 2|C_k|$, but $|C_k' \cap C_k| \geq (1-\epsilon/2)|C_k|$, which implies

$$\begin{aligned} |C_k'' \cap C_k| &\geq |C_k' \cap C_k| - |C_k' \setminus C_k''| \\ &\geq (1-\epsilon/2)|C_k| - \epsilon|C_k'| \\ &\geq (1-\epsilon/2)|C_k| - 2\epsilon|C_k| \\ &> (1-3\epsilon)|C_k|. \end{aligned} \tag{11}$$

In both cases one can conclude that $|C_k''| \geq (1+\delta)|C_k|$. For the first case (10), $\epsilon < \frac{1}{4}$ and $\delta \leq \frac{1}{2}$ gives

$$2(1-\epsilon) \geq 1.5 \geq 1+\delta.$$

For the second case (11), this can be observed by the next calculation.

By (8), there are pairwise disjoint sets $E''_{(n,c)} \subset F_n c$ (for each $(n,c) \in \mathcal{B}''_k$), with $|E''_{(n,c)}| \geq (1-\epsilon)|F_n c|$. Thus

$$\begin{aligned}\sum_{g\in C''_k} f(gx) &\leq \sum_{(c,n)\in\mathcal{B}''_k} \sum_{g\in F_n c} f(gx)\\ &\leq \Big(\sum_{\mathcal{B}''_k} |F_n c|\Big)\alpha\\ &\leq \frac{1}{1-\epsilon}\Big(\sum_{\mathcal{B}''_k} |E''_{(n,c)}|\Big)\alpha\\ &\leq \frac{1}{1-\epsilon}|C''_k|\alpha.\end{aligned}$$

On the other hand, the collection $(F_n c)_{(c,n)\in\mathcal{B}_k}$ is $\epsilon/2$-disjoint, and so, there are pairwise disjoint sets $E_{(n,c)} \subset F_n c$ (for each $(n,c) \in \mathcal{B}_k$), with $|E_{(n,c)}| \geq (1-\epsilon/2)|F_n c|$. Thus

$$\begin{aligned}\sum_{g\in C_k} f(gx) &\geq \sum_{(c,n)\in\mathcal{B}_k} \sum_{g\in E_{(n,c)}} f(gx)\\ &= \sum_{\mathcal{B}_k}\Big(\sum_{F_n c} f(gx) - \sum_{F_n c\setminus E_{(n,c)}} f(gx)\Big)\\ &\geq \sum_{\mathcal{B}_k} |F_n c|\Big(\beta - \frac{\epsilon}{2}S\Big)\\ &\geq |C_k|\Big(\beta - \frac{\epsilon}{2}S\Big)\end{aligned}$$

if $|C''_k \cap C_k| \geq (1-3\epsilon)|C_k|$ as in (11), then

$$\begin{aligned}|C_k|\Big(\beta - \frac{\epsilon}{2}S\Big) &\leq \sum_{g\in C_k} f(gx)\\ &\leq |C_k|3\epsilon S + \sum_{g\in C''_k} f(gx)\\ &\leq |C_k|3\epsilon S + \frac{1}{1-\epsilon}|C''_k|\alpha.\end{aligned}$$

Thus, with our choice of ϵ with respect to δ (5), we get that

$$|C''_k| \geq (1+\delta)|C_k|.$$

In the same manner we constructed $\mathcal{B}''_k$, we use the next q upcrossings to above β to construct a collection $\mathcal{B}_{k+1}$ such that $(F_n c)_{\mathcal{B}_{k+1}}$ is an ϵ-disjoint collection of upcrossings, with union $C_{k+1} = \bigcup_{\mathcal{B}_{k+1}} F_n g$ in Ω that satisfies one of the two options in the conclusion of Proposition 2.6. In particular, we have

$$\begin{aligned} |C_{k+1}| &\geq (1-\epsilon)|C_k''| \\ &\geq (1-\epsilon)(1+\delta)|C_k| \\ &\geq (1+\delta/2)|C_k| \end{aligned}$$

(the last inequality follows from the assumption $(1-\epsilon)(1+\delta) \geq (1+\delta/2)$), and Lemma 2.7 is proved. △

Back to the proof of Theorem 2.3, from Lemma 2.7 it follows that there exist finite subsets of Ω, $C_1, \ldots, C_{\lfloor \frac{N}{2q} \rfloor}$ such that

$$\begin{aligned} |\Omega| \geq |C_{\lfloor \frac{N}{2q} \rfloor}| &\geq (1+\delta/2)^{\lfloor \frac{N}{2q} \rfloor - 1}|C_1| \\ &\geq (1+\delta/2)^{\lfloor \frac{N}{2q} \rfloor - 1}(1-\epsilon)|C| \\ &\geq (1+\delta/2)^{\frac{N}{2q} - 3}|C| \end{aligned}$$

(the last inequality follows partially from the assumption $(1-\epsilon) \geq (1+\delta/2)^{-1}$). Since

$$\mu(D_{N,M}) = \frac{1}{|\Omega|} \int \sum_{g \in \Omega} \mathbf{1}_{D_{N,M}}(gx) d\mu(x) \leq \int \frac{|C_{x,M}|}{|\Omega|} d\mu(x) + \Big(1 - \frac{|B|}{|\Omega|}\Big),$$

where $\big(1 - \frac{|B|}{|\Omega|}\big)$ can be made arbitrarily small (by taking Ω to be arbitrarily invariant), one have

$$\mu(D_{N,M}) \leq \int \frac{|C_{x,M}|}{|\Omega|} d\mu(x) \leq (1+\delta/2)^{-(\frac{N}{2q} - 3)}$$

Thus the claim of the theorem takes place with $c_0 = (1+\delta/2)^{-\frac{1}{2q}}, c_1 = (1+\delta/2)^3$. □

3. Proof of Theorem 1.4

Proof of Theorem 1.4. Let $\lambda > 0$. We first construct finite sequences of subsets of $\mathbb{Z}$, which have good fluctuation and invariance properties, and then concatenate such sequences to get the whole sequence (F_n) in question. Fix $l, N \in \mathbb{N}$, and let $\phi_l : \mathbb{N}_0 \to \{0, 1\}$ be the indicator function

$$\phi_l = \mathbf{1}_{(2l\mathbb{N}_0 + [0, l-1])}$$

(here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.) We define a sequence of subsets $(A_n)_{n=1}^{2N} = (A_n^{l,N})_{n=1}^{2N}$, recursively,

$$A_{n+1} = \begin{cases} [0, \frac{2}{\lambda} M_n] \cup \Big((2l\mathbb{N}_0 + [0, l-1]) \cap \Big[0, \frac{2+\lambda}{\lambda} M_n\Big]\Big) & \text{if } n+1 \text{ is odd,} \\ \Big[0, \frac{2}{\lambda} M_n\Big] \cup \Big((2l\mathbb{N}_0 + [l, 2l-1]) \cap \Big[0, \frac{2+\lambda}{\lambda} M_n - 2l + 1\Big]\Big) & n+1 \text{ is even,} \end{cases}$$

where $M_0 = l^2$, and $M_n = \max(A_n)$ for $n > 0$. This sequence has the following properties.

(a) $(A_n)_{n=1}^{2N}$ is $(1+\lambda)$-tempered: for any n,

$$A_n - \bigcup_{i=0}^{n-1} A_i \subset A_n - [0, M_{n-1}] \subset [-M_{n-1}, M_n]$$

thus

$$\Big| A_n - \bigcup_{i=0}^{n-1} A_i \Big| \le M_n + M_{n-1} \le \frac{2+\lambda}{\lambda} M_{n-1} + M_{n-1} \le (1+\lambda)|A_n|.$$

(b) A_n is $([-\sqrt{l}, \sqrt{l}], 2/\sqrt{l})$-invariant for all n; that is, for any $b \in [-\sqrt{l}, \sqrt{l}]$, one has $\frac{|(b+A_n)\triangle A_n|}{|A_n|} \le 2/\sqrt{l}$. This follows immediately from the fact that A_n is a union of segments, the first one of size at least l^2, and all but the last one of size at least l.

(c) Assuming l large enough, there are some $0 < \alpha < \beta$ such that for any $0 \le i \le l/4$, and any k, averaging $\phi_l(z + 2lk + i)$ as a function of z along A_n, the sequence of averages fluctuates across the gap (α, β) N times. Averaging along A_n of odd n gives

$$\begin{aligned} \frac{1}{|A_n|} \sum_{z \in A_n} \phi_l(z + 2lk + i) &= \frac{|(2l\mathbb{N}_0 + [0, l-1] - i) \cap A_n|}{|A_n|} \\ &\ge \frac{1}{|A_n|} \cdot \Big[l \cdot \Big(\frac{\frac{2}{\lambda} M_{n-1}}{2l} - 1 \Big) + \frac{3}{4} l \cdot \Big(\frac{M_{n-1}}{2l} - 1 \Big) \Big] \\ &\ge \frac{\big(\frac{1}{\lambda} + \frac{3}{8}\big) M_{n-1} - 2l}{\big(\frac{2}{\lambda} + \frac{1}{2}\big) M_{n-1} + l} \\ &\ge \frac{1}{2} + \frac{\lambda}{4(4+\lambda)} - \frac{4}{l} \end{aligned}$$

while for even n,

$$\frac{1}{|A_n|}\sum_{z\in A_n}\phi_l(z+2lk+i) \le \frac{1}{|A_n|}\cdot\Big[l\cdot\Big(\frac{\frac{2}{\lambda}M_{n-1}}{2l}+1\Big)+\frac{1}{4}l\cdot\Big(\frac{M_{n-1}}{2l}+1\Big)\Big]$$
$$\le \frac{\big(\frac{1}{\lambda}+\frac{1}{8}\big)M_{n-1}+2l}{\big(\frac{2}{\lambda}+\frac{1}{2}\big)M_{n-1}-l}$$
$$\le \frac{1}{2}-\frac{\lambda}{4(4+\lambda)}+\frac{4}{l}$$

(for the error summand $\frac{4}{l}$, we used $M_{n-1}\ge M_0=l^2$ and assumed $l\ge 4$.) Taking l large enough, one gets that the claim above takes place with $\alpha=\frac{1}{2}-\frac{\lambda}{5(4+\lambda)}$ and $\beta=\frac{1}{2}+\frac{\lambda}{5(4+\lambda)}$.

Now construct the whole sequence as follows. Take $l_0>100$ and also large enough so that property (c) takes place, and then define $(l_m)_{m=1}^{\infty}$ recursively by the rule

$$l_{k+1}=\max(A_{2l_m}^{l_m,2l_m}).$$

We define (F_n) to be the concatenation of the sequences $\{(A_n^{l_m,2l_m})_{n=1}^{2l_m}\}_{m=0}^{\infty}$. Using properties (a) and (b) above together with the definition of $(l_m)_{m=1}^{\infty}$, one can observe that this sequence is a $(1+\lambda)$-tempered Følner sequence.

Recall that for a given function f on a m.p.s. $(X,\mathcal{B},\mu,\{T_g\}_{g\in G})$, a gap $(\alpha,\beta)\subset\mathbb{R}$, a sequence (F_n) of subsets of G, and $N,M\in\mathbb{N}$, we write

$$D_N=\{x\colon\{\mathsf{A}_n f(x)\}_{n=1}^{\infty}\text{ fluctuates across }(\alpha,\beta)\text{ at least }N\text{ times}\},$$
$$D_{N,M}=\{x\colon\{\mathsf{A}_n f(x)\}_{n=1}^{M}\text{ fluctuates across }(\alpha,\beta)\text{ at least }N\text{ times}\},$$

where the sequence (F_n), the function f and the gap (α,β) are understood from the context. At some places we shall write D_N^f and $D_{N,M}^f$ to specify the function f for which the sets refer to.

We will construct the function in question by applying iteratively infinitely many times the following lemma:

Lemma 3.1. *Let* $f\colon X\to[0,1]$. *For any* $\epsilon>0$, $1>\delta>0$, *and* $n',N',N''\in\mathbb{N}$, *there exists a measurable function* $\hat{f}\colon X\to[0,1]$ *such that the following holds*:

(i) $\mu(D_{N''}^{\hat{f}})>\frac{1}{10}\delta$;

(ii) $\mu((f(T^ix))_{i=0}^{L-1}\ne(\hat{f}(T^ix))_{i=0}^{L-1})\le\delta$, *where* $L:=\max\big(\bigcup_{n=1}^{n'}F_n\big)$;

(iii) *for all* $N\le N'$, $\mu(D_N^{\hat{f}})\ge\min\big\{\mu(D_{N,n'}^f)-\epsilon,\frac{1}{10}\big\}$.

Proof. We will assume without loss of generality that ϵ is small enough so that $\epsilon<\min\big\{\frac{\delta}{100},1-\delta\big\}$. Take an $m\in\mathbb{N}$ that satisfies $l_m\ge N''$. Let $B\subset X$ be a base for a Rokhlin tower of height h and total measure $> 1-\epsilon/4$, where h is large enough to satisfy

$$h > \frac{(L + \max A^{l_m,2l_m}_{2l_m})}{\epsilon/4}. \tag{12}$$

and also large enough to guarantee that

$$\mu\Big(x \in B\colon \frac{1}{h}\sum_{i=0}^{h-1} \mathbf{1}_{D^f_{N,n'}}(T^i x) > \mu(D^f_{N,n'}) - \epsilon/4 \text{ for all } N \le N'\Big) > (1-\epsilon/4)\mu(B) \tag{13}$$

in words, for all $N \le N'$, for at least $1-\epsilon/4$ of the x's in B, their orbit along the tower spends more than $\mu(D^f_{N,n'}) - \epsilon/4$ of the time in the set $D^f_{N,n'}$ (For the validity of such a requirement, see for example [4, Theorem 7.13]).

Take $B' \subset B$ of measure $\mu(B') = 0.99\delta/h$ (this can be achieved because $1-\epsilon > \delta$), and define $\hat{f}$ to be

$$\hat{f}(x) = \begin{cases} \phi_{l_m}(i) & x \in T^i B',\ 0 \le i \le h-1, \\ f(x) & x \in X \setminus \bigcup_{i=0}^{h-1} T^i B'. \end{cases}$$

The validity of property (c) above for $(A^{l_m,2l_m}_n)_{n=1}^{2l_m}$ and thus for (F_n), together with the definition of $\hat{f}$ as ϕ_{l_m} on the tower above B', implies that for any $0 \le i \le l_m/4$ and any $k \ge 0$ such that $2kl_m + i < h - \max A^{l_m,2l_m}_{l_m}$, one has

$$T^{2kl_m+i} B' \subset D^{\hat{f}}_{l_m} \subset D^{\hat{f}}_{N''}.$$

The density of these levels in the tower is at least

$$\Big(\frac{l_m}{4} - 1\Big)\cdot\Big(\frac{h}{2l_m} - 1\Big)/h > \frac{1}{8} - \frac{l_m}{h} - \frac{1}{l_m}$$

and since $l_m \ge l_0 > 100$ and $\frac{l_m}{h} \le \frac{\max A^{l_m,2l_m}_{l_m}}{h} \le \epsilon/4 < \frac{1}{100}$, the last expression is at least $\frac{1}{9}$. Thus

$$\mu(D^{\hat{f}}_{N''}) \ge \frac{1}{9}\mu\Big(\bigcup_{n=0}^{h-L} T^n B'\Big) = \frac{1}{9}(0.99\delta) > \frac{1}{10}\delta \tag{14}$$

which gives property (i) of the conclusion.

To see why property (ii) of the conclusion holds, notice that

$$\bigcup_{n=0}^{h-L} T^n(B\setminus B') \subset \{x\colon (f(T^i x))_{i=0}^{L-1} = (\hat{f}(T^i x))_{i=0}^{L-1}\}, \tag{15}$$

thus

$$\begin{aligned}\mu((f(T^i x))_{i=0}^{L-1} = (\hat{f}(T^i x))_{i=0}^{L-1}) &\geq 1 - \epsilon/4 - L\mu(B) - h\mu(B') \\ &> 1 - \epsilon/4 - \epsilon/4 - 0.99\delta \\ &> 1 - \delta.\end{aligned}$$

Finally, by (15) we have for all N

$$\bigcup_{n=0}^{h-L} T^n(B \setminus B') \cap D^f_{N,n'} \subset D^{\hat{f}}_{N,n'}$$

and by (13) and (12), we have for all $N \leq N'$,

$$\mu\Big(\bigcup_{n=0}^{h-L} T^n(B\setminus B') \cap D^f_{N,n'}\Big) \geq \mu(D^f_{N,n'})\mu\Big(\bigcup_{n=0}^{h-L} T^n(B\setminus B')\Big) - \frac{3}{4}\epsilon.$$

This, together with the first inequality in (14) gives for all $N \leq N'$

$$\begin{aligned}\mu(D^{\hat{f}}_N) &\geq \mu\Big(\bigcup_{n=0}^{h-L} T^n(B\setminus B') \cap D^{\hat{f}}_N\Big) + \mu\Big(\bigcup_{n=0}^{h-L} T^n B' \cap D^{\hat{f}}_N\Big) \\ &\geq \mu(D^f_{N,n'})\mu\Big(\bigcup_{n=0}^{h-L} T^n(B\setminus B')\Big) - \frac{3}{4}\epsilon + \frac{1}{9}\mu\Big(\bigcup_{n=0}^{h-L} T^n B'\Big) \\ &\geq \min\Big\{\mu(D^f_{N,n'}), \frac{1}{9}\Big\}\mu\Big(\bigcup_{n=0}^{h-L} T^n B\Big) - \frac{3}{4}\epsilon \\ &\geq \min\Big\{\mu(D^f_{N,n'}), \frac{1}{9}\Big\} - \epsilon \\ &\geq \min\Big\{\mu(D^f_{N,n'}) - \epsilon, \frac{1}{10}\Big\}\end{aligned}$$

which gives property (iii) of the conclusion. △

Let $\omega(n) \searrow 0$ be any sequence which decreases to 0. Define $(N_k)_{k=1}^{\infty}$ by

$$N_k = \min\Big\{N : \omega(N) < \frac{1}{10}2^{-k-1}\Big\}.$$

We will construct a function f which satisfies for all k

$$\mu(D^f_{N_k}) \geq \frac{1}{10}2^{-k}$$

and by monotonicity of $\mu(D^f_N)$ and $\omega(N)$, for any $N_k \leq N < N_{k+1}$, $k \geq 1$,

$$\mu(D^f_N) \geq \mu(D^f_{N_{k+1}}) \geq \frac{1}{10}2^{-k-1} > \omega(N_k) \geq \omega(N)$$

and the conclusion of Theorem 1.4 follows.

Take $f_0 \equiv 0$, and define inductively $(f_k)_{k=0}^{\infty}$. Given f_{k-1}, assume that there exists n_{k-1} such that

$$\mu(D^{f_{k-1}}_{N_i,n_{k-1}}) > \frac{1}{10}2^{-i} \quad \text{for all } 1 \le i \le k-1. \tag{16}$$

Take $\epsilon > 0$ small enough so that for all $i \le k-1$,

$$\mu(D^{f_{k-1}}_{N_i,n_{k-1}}) - \epsilon > \frac{1}{10}2^{-i}$$

and apply Lemma 3.1 with $f := f_{k-1}, N' = N_{k-1}$, $N'' = N_k$, $n' = n_{k-1}$, $\delta = 2^{-k}$ while letting f_k be the resulting function $\hat{f}$. This f_k satisfies the hypothesis (16) in the inductive step: by property (iii) of the lemma, for all $i \le k-1$,

$$\mu(D^{f_k}_{N_i}) \ge \min\left\{\mu(D^{f_{k-1}}_{N_i,n_{k-1}}) - \epsilon, \frac{1}{10}\right\} > \frac{1}{10}2^{-i} \tag{17}$$

and by property (i) of the lemma,

$$\mu(D^{f_k}_{N_k}) > \frac{1}{10}2^{-k}. \tag{18}$$

Since $\mu(D^{f_k}_{N,n}) \xrightarrow{n\to\infty} \mu(D^{f_k}_{N})$, there exists large enough n_k such that (17) and (18) will be satisfied with $D^{f_k}_{N,n_k}$ in place of $D^{f_k}_{N}$. Thus the hypothesis (16) of the induction step is indeed satisfied with k in place of $k-1$.

We end up with a sequence $(f_k)_{k=0}^{\infty}$ together with a sequence (n_k) which we can assume to be increasing. By property (ii) of the lemma, (f_k) converges a.e. to some limit, call it f. For each k, let

$$L_k := \max\left(\bigcup_{n=1}^{n_k} F_n\right)$$

then again by property (ii) of the lemma, f satisfies

$$\begin{aligned}
&\mu((f_k(T^n x))_{n=0}^{L_k-1} \ne (f(T^n x))_{n=0}^{L_k-1}) \\
&\quad \le \sum_{i\ge k} \mu((f_i(T^n x))_{n=0}^{L_k-1} \ne (f_{i+1}(T^n x))_{n=0}^{L_k-1}) \\
&\quad \le \sum_{i\ge k} \mu((f_i(T^n x))_{n=0}^{L_i-1} \ne (f_{i+1}(T^n x))_{n=0}^{L_i-1}) \\
&\quad \le \sum_{i\ge k} 2^{-i} \\
&\quad = 2^{-k+1}
\end{aligned}$$

(in the second inequality we used the assumption that $n_i \geq n_{i-1}$ for all i). Thus for any i,

$$\begin{aligned}\mu(D^f_{N_i}) &\geq \mu(D^f_{N_i,n_k}) \\ &\geq \mu(D^{f_k}_{N_i,n_k}) - 2^{-k+1} \\ &> \frac{1}{10}2^{-i} - 2^{-k+1}\end{aligned}$$

taking $k \to \infty$ gives

$$\mu(D^f_{N_i}) \geq \frac{1}{10}2^{-i}$$

and the proof of Theorem 1.4 is complete. □

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