Properly discontinuous actions versus uniform embeddings

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Abstract. Whenever a finitely generated group G acts properly discontinuously by isometries on a metric space X, there is an induced uniform embedding (a Lipschitz and uniformly proper map) $\rho\colon G \to X$ given by mapping G to an orbit. We study when there is a difference between a finitely generated group G acting properly on a contractible n-manifold and uniformly embedding into a contractible n-manifold. For example, Kapovich and Kleiner showed that there are torsion-free hyperbolic groups that uniformly embed into a contractible 3-manifold but do not act on a contractible 3-manifold. We show that k-fold products of certain examples do not act on contractible $3k$-manifolds.

1. Introduction

For a finitely generated group G, the *action dimension* of G, denoted actdim(G), is the minimal dimension of contractible manifold M that admits a properly discontinuous G-action. If G is torsion-free, then the quotient M/G is a manifold model for the classifying space BG, so the action dimension is precisely the minimal dimension of such a model. The *geometric dimension* is the minimal dimension of a CW-model for BG.

Given a properly discontinuous action of G on M, and given any choice of basepoint m_0, there is an orbit map $\rho\colon G \to M$ defined by $g \to g.m_0$. After choosing a proper G-invariant metric on M, this map is Lipschitz and uniformly proper; we call such a map a *uniform embedding*. Furthermore, the manifold M is uniformly contractible around the image of G (see Section 2 for precise definitions). The *uniformly proper dimension* of G, denoted updim(G), is the minimal dimension of contractible manifold M, equipped with a proper metric, so that there is a uniform embedding $\rho\colon G \to M$ so that M is uniformly contractible around the image of G. The orbit map of a properly discontinuous action shows

that
$$\mathrm{updim}(G) \le \mathrm{actdim}(G)$$

We now review some of the known relations between $\mathrm{updim}(G)$ and $\mathrm{actdim}(G)$. Somewhat surprisingly, the two dimensions coincide for most of the examples where they have been computed. Such groups include lattices in Lie groups [2], mapping class groups [9], many Artin groups [1] and [6], and torsion-free lattices in Euclidean buildings [21]. These results all come from computing a lower bound to $\mathrm{updim}(G)$, called the *obstructor dimension*, which was defined by Bestvina, Kapovich, and Kleiner [3]. We will come to this later in the introduction.

There are also examples where $\mathrm{updim}(G)$ is strictly less than $\mathrm{actdim}(G)$, but equal to $\mathrm{actdim}(H)$ for H a finite index subgroup of G. These examples are relatively easy to construct when G is allowed to have torsion, for example there are many virtually free groups which do not act properly on the plane (such as the free product of alternating groups $A_5 * A_5$). Torsion-free examples were constructed by Kapovich and Kleiner in [17] and Hruska, Stark, and Tran in [15]. In both cases, the groups constructed were virtually 3-manifold groups but not 3-manifold groups. The constructions have a similar flavor, roughly one glues surfaces together along simple closed curves using degree k covering maps for $k > 1$. In both cases, the obstruction to properly acting on a contractible 3-manifold comes from analyzing the action of the group on collections of codimension-one hypersurfaces in the universal cover EG of BG, and applying the coarse Jordan separation theorem of [17].

There are fewer known examples where
$$\mathrm{updim}(G) < \min_{[G:H]<\infty} \mathrm{actdim}(H).$$

In fact, the only common examples we know are the Baumslag–solitar groups
$$\mathrm{BS}(m,n) = \langle x, y \mid xy^m x^{-1} = y^n \rangle$$

for $m \ne n$. These uniformly embed into a uniformly contractible 3-manifold (which is a thickening of the Cayley 2-complex), but for a variety of reasons are not 3-manifold groups if $m \ne n$ (and this is true for finite index subgroups as well). A theorem of Stallings [22] implies that for groups with a finite BG, $\mathrm{actdim}(G)$ is bounded above by twice the geometric dimension of G. In particular, $\mathrm{actdim} = 4$ for the two previous examples (there are also obvious 4-dimensional manifold models of BG).

Kapovich and Kleiner have higher-dimensional results in this direction; for example they show that the group $\mathrm{BS}(m,n) \times \mathbb{Z}^k$ does not act properly on a uniformly contractible, open $(3+k)$-manifold (or more generally a coarse $\mathrm{PD}(3+k)$-space).

Note that we have the obvious inequalities
$$\mathrm{updim}(\Gamma_1 \times \Gamma_2) \le \mathrm{updim}(\Gamma_1) + \mathrm{updim}(\Gamma_2)$$

and

$$\text{actdim}(\Gamma_1 \times \Gamma_2) \leq \text{actdim}(\Gamma_1) + \text{actdim}(\Gamma_2).$$

It is still open if this last inequality is strict for products of the above examples.

Question 1.1. *Let G be the k-fold direct product of the examples in* [17] *or* [15]. *What is* $\text{actdim}(G)$? *Same question for products of Baumslag–solitar groups with $m \neq n$.*

It follows from [3] that the uniformly proper dimension of these is $= 3k$, hence $3k \leq \text{actdim}(G) \leq 4k$. The difficulty here is that all of the above computations rely on studying the action of G on codimension-one hypersurfaces inside EG, and showing that this action in incompatible with a group acting on hypersurfaces in a contractible 3-manifold. After crossing with $\mathbb{Z}^n$ (or π_1 of any closed aspherical manifold), there are still codimension-one hypersurfaces in EG. However, the k-fold product of these examples now has codimension-k hypersurfaces inside EG, and the same analysis doesn't apply. For such products, we have the following theorem, which handles some cases where the planes have larger codimension.

Theorem 1.2. *Let G be the k-fold direct product of the examples in* [17] *or* [15]. *Then*

$$\text{actdim}(G) \geq 3k + 1$$

if one of the degrees of the covering map is a multiple of 4.

We note that similar results hold for some virtually free groups, see Section 6 for the precise statements. The conditions on the degree of the cover are an unfortunate fault of our method. Of course, these groups have finite index subgroups which have $\text{actdim}(G) = 3k$. Interestingly, Dranishnikov has constructed examples with $\text{gd}(G_1 \times G_2) < \text{gd}(G_1) + \text{gd}(G_2)$, see [7], but has also shown that $\text{gd}(G \times G) = 2\,\text{gd}(G)$, see [8]. We assume that the general formula $\text{actdim}\left(\prod_k G\right) = 3k + 1$ holds for all of these examples, and will work on this in a subsequent paper.

Before describing our methods for the lower bounds, let us recall the *obstructor dimension* $\text{obdim}(G)$ of a group G as defined in [3]. This is based on the $\mathbb{Z}/2$-valued van Kampen obstruction to embedding finite subcomplexes into $\mathbb{R}^n$, which is an n-dimensional class, denoted $\text{vk}^n_{\mathbb{Z}/2}$, in the cohomology of the unordered 2-point configuration space $\mathcal{C}(K)$ with $\mathbb{Z}/2$-coefficients, see Subsection 2.2 for details. A finite complex K is an *n-obstructor* if $\text{vk}^n_{\mathbb{Z}/2}(K) \neq 0$; in particular this implies that K does not embed into $\mathbb{R}^n$.

Roughly speaking, the *obstructor dimension* of a finitely generated group G is the maximal $n + 2$ so that there is an n-obstructor K and a uniformly proper embedding $f \colon K \times \mathbb{R}^+ \to EG$. Bestvina, Kapovich and Kleiner show that

$$\text{obdim}(G) \leq \text{updim}(G).$$

The moral we follow is that if obstructor complexes give lower bounds for updim(G), then simplicial complexes with a group action which do not equivariantly embed into $\mathbb{R}^n$ should give lower bounds for actdim(G). Again, we require a cohomological obstruction to equivariantly embedding the complex. In this case, we use an ambient isotopy invariant, called the *Wu invariant*, of an embedding of K into $\mathbb{R}^{n+1}$. This invariant also lives in the nth degree cohomology of $\mathcal{C}(K)$, though with twisted integral coefficients, and its image in $H^n(\mathrm{Conf}(K); \mathbb{Z}/2)$ upon reducing the coefficients to $\mathbb{Z}/2$ is precisely the $\mathbb{Z}/2$-valued van Kampen obstruction.

If K is a graph, then the Wu invariant has been often used to study embeddings of K into $\mathbb{R}^3$, see for example [12] and [24]. Flapan in [11] also used the linking number of images of subgraphs to obstruct certain equivariant embeddings of the complete graph K_n into $\mathbb{R}^3$. If a finite group H acts on K, and an embedding $f: K \to \mathbb{R}^n$ is equivariant with respect to some representation $\rho: H \to \mathrm{Homeo}^+(\mathbb{R}^n)$, then the Wu invariant of f is fixed under the H-action on $H^n(\mathrm{Conf}(K))$ (since all orientation-preserving homeomorphisms of $\mathbb{R}^n$ are isotopic to the identity). We roughly define an *equivariant obstructor* to be a finite H-complex K which is an n-obstructor and which does not admit an invariant Wu class.

We then define the *equivariant obstructor dimension*, eqobdim(G), of a finitely generated group G to be the maximal $n + 2$ so that there is an equivariant n-obstructor K and a uniformly proper embedding $f: K \times \mathbb{R}^+ \to EG$ which is coarsely H-equivariant, see Section 4. It will follow from the definitions that obdim(G) $\leq$ eqobdim(G) $\leq$ obdim(G) + 1. We will show that

$$\mathrm{eqobdim}(G) \leq \mathrm{actdim}(G)$$

and

$$\mathrm{eqobdim}(G_1 \times G_2) \geq \mathrm{eqobdim}(G_1) + \mathrm{eqobdim}(G_2) - 1$$

which will give the lower bound of Theorem 1.2.

This paper is structured as follows. If Section 2, we review some necessary background information. In Section 3 we define an equivariant obstructor complex and show that such a complex does not equivariantly embed into $\mathbb{R}^n$. In Sections 4 and 5 we develop the coarse analogue of this. In Section 6, we apply this to give lower bounds to the action dimension of a number of examples.

2. Background

2.1. Uniformly proper dimension and coarse topology. Recall that a metric space is *proper* if closed metric balls are compact, and that a map between two spaces is proper if preimages of compact sets are compact. Two maps $f_0, f_1: X \to Y$ are *properly homotopic* if there is a proper map $F: X \times I \to Y$ so that $F|_{X \times 0} = f_0$ and $F|_{X \times 1} = f_1$.

Let X and Y be two proper metric spaces. A proper map $f: X \to Y$ is *uniformly proper* if there exists a proper function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ so that

$$d_Y(f(x_1), f(x_2)) \geq \phi(d_X(x_1, x_2))$$

for all $x_1, x_2 \in X$, where $\mathbb{R}^+ = [0, \infty)$. If f is also Lipschitz, this is sometimes referred to in the literature as a *coarse embedding*, though we will call these *uniform embeddings*. If ϕ is a linear function, then f is a *quasi-isometric embedding*. If G is a finitely generated group and H is a finitely generated subgroup, the inclusion of H into G is a uniform embedding with respect to the word metrics on G and H (and this map will not be a quasi-isometric embedding if H is distorted in G).

Recall that a space X is *uniformly contractible* if there is a function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ so that for every $x \in X$ the ball $B(x, R)$ contracts inside $B(x, \phi(R))$. Given a subspace $Y \subset X$, we say X is *uniformly contractible around Y* if the above holds for all points $y \in Y$. Of course, if X is contractible and admits a proper, cocompact group action, then it is uniformly contractible.

Definition 2.1. Given a finitely generated group G, the *uniformly proper dimension* of G is the minimal n so that there is a contractible n-manifold M^n, equipped with a proper metric, and a uniform embedding $\rho: G \to M^n$ so that M^n is uniformly contractible around $\rho(G)$.

The uniformly contractibility assumption is essential, as Bestvina, Kapovich, and Kleiner noted that any finitely generated group has a uniform embedding into $\mathbb{R}$ for some proper metric on $\mathbb{R}$. Note also that $\mathrm{updim}(G)$ is a quasi-isometry invariant of G, whereas $\mathrm{actdim}(G)$ is not. We record the following well-known lemma.

Lemma 2.2. *Suppose that a finitely generated group G acts properly discontinuously by isometries on a proper metric space X. Then each orbit map $\rho: G \to X$, which takes g to gx_0 for a choice of $x_0 \in X$, is a uniform embedding. If X is contractible, then X is uniformly contractible around $\rho(G)$. In particular, $\mathrm{updim}(G) \leq \mathrm{actdim}(G)$.*

Proof. Since the action is properly discontinuous, the orbit map is proper. Let $g_1, \ldots g_n$ be a generating set for G, and let N be the maximum of the values

$d_X(x_0, g_i x_0)$ for $i \in \{1, \ldots, n\}$. Then if g is a group element with $d(1, g) = m$, we have that $d_X(x_0, gx_0) \leq mN$. Therefore, since G acts by isometries, we have that $d_X(g_1 x_0, g_2 x_0) = d_X(g_2^{-1} g_1 x_0, x_0) \leq d_G(g_1, g_2)N$, so ρ is N-Lipschitz. To prove uniform properness, let

$$\phi(n) = \min_{\substack{g \in G \\ d_G(1,g)=n}} d(x_0, gx_0).$$

Then ϕ is proper since the action is properly discontinuous, and obviously

$$d_X(f(1), f(g)) \geq \phi(d_G(1, g)).$$

Since G acts by isometries, $d_X(f(g_1), f(g_2)) \geq \phi(d_G(g_1, g_2))$ for all $g_1, g_2 \in G$. The uniformly contractible statement is immediate as G acts cocompactly on $\rho(G)$. □

Given a finitely generated group G, we will denote by EG any contractible complex X that admits a proper and cocompact cellular action by G. Such an EG may not exist, but for the rest of the paper we will only work with groups that act on such spaces. If G is torsion-free, then this is the same as the universal cover of a finite classifying space BG. If G contains torsion, it is usually assumed that the fixed point sets of these torsion elements in EG are contractible; however we do not need this assumption. We will always assume that EG is equipped with a proper G-invariant metric. We will need the following lemma.

Lemma 2.3. *Let G be a finitely generated group, and suppose that G acts properly on a contractible metric space Y by isometries. Then the orbit map $\rho: G \to Y$ extends to a uniformly proper Lipschitz map $\rho: EG \to Y$ so that $d_Y(g\rho(x), \rho(gx)) < C$ for all $x \in EG$ and some constant $C > 0$ (such a ρ is called quasi-equivariant [10]).*

Proof. There is a quasi-isometry $f: G \to EG$. Let $f^{-1}: EG^{(0)} \to G$ denote the quasi-inverse restricted to the 0-skeleton of EG. Then f^{-1} is quasi-equivariant, and the composition with the orbit map $\rho: G \to Y$ gives a quasi-equivariant map from $EG^{(0)} \to Y$. Since Y is uniformly contractible around the image of ρ, we can extend this map to the simplices of EG so that the diameter of the image of each simplex is uniformly bounded. Since the orbit map was a uniform embedding from G to Y, the extension is also a uniform embedding, and since the we started with a quasi-equivariant map on $EG^{(0)}$, we get a quasi-equivariant extension. □

2.2. $\mathbb{Z}/2$-valued van Kampen obstruction. Let $\widetilde{\mathcal{C}}(K)$ denote the simplicial configuration space of ordered pairs of distinct simplices in K, i.e., if Δ denotes the simplicial diagonal $\Delta = \{(\sigma, \tau) | \sigma \cap \tau \neq \emptyset\}$ then

$$\widetilde{\mathcal{C}}(K) = (K \times K) - \Delta.$$

There is an involution ι on $\widetilde{\mathcal{C}}(K)$ which switches the factors, let $\mathcal{C}(K)$ denote the quotient. The induced double cover $\bar{\iota}\colon \widetilde{\mathcal{C}}(K) \to \mathcal{C}(K)$ is classified by a map $c\colon \mathcal{C}(K) \to \mathbb{R}P^\infty$. The $\mathbb{Z}/2$-*valued van Kampen obstruction* in degree m is the cohomology class $\mathrm{vk}_{\mathbb{Z}/2}^m(K) \in H^m(\mathcal{C}(K); \mathbb{Z}/2)$ defined by

$$\mathrm{vk}_{\mathbb{Z}/2}^m(K) = c^*(w_1^m),$$

where w_1 is the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$. If K embeds into $\mathbb{R}^m$, then a classifying map factors through $\mathbb{R}P^{m-1}$, and hence $\mathrm{vk}_{\mathbb{Z}/2}^m(K) = 0$. Therefore, $\mathrm{vk}_{\mathbb{Z}/2}^m(K)$ is an obstruction to embedding K in $\mathbb{R}^m$.

Note that $\mathrm{vk}_{\mathbb{Z}/2}^m(K) \neq 0$ if and only if there is a cycle $\Phi \in H_m(\mathcal{C}(K); \mathbb{Z}/2)$ so that the evaluation $\langle \mathrm{vk}_{\mathbb{Z}/2}^m(K), \Phi \rangle \neq 0$. Bestvina, Kapovich and Kleiner define a m-obstructor as a slight strengthening of this.

Definition 2.4. A finite complex K is an *m-obstructor* if there is a cycle $\Phi \in H_m(\mathcal{C}(K); \mathbb{Z}/2)$ satisfying

- $\langle \mathrm{vk}_{\mathbb{Z}/2}^m(K), \Phi \rangle \neq 0$;
- if v is a vertex, then the collection $\{\sigma, v\} \in \Phi$ has even cardinality.

Example. The following are examples of obstructor complexes [3].

- The disjoint union of an m-sphere (say triangulated as $*_m S^0$) and a point is an m-obstructor.
- The cone on an m-obstructor is an $(m+1)$-obstructor.
- The join of an m_1-obstructor and an m_2-obstructor is an $(m_1 + m_2 + 2)$-obstructor.

Definition 2.5. Let $\mathrm{Cone}_\infty(K) = K \times \mathbb{R}^+ / K \times 0$. A proper map of $\mathrm{Cone}_\infty(K)$ into a metric space is *expanding* if for any pair of disjoint simplices σ, τ in K, the distance between $\sigma \times [t, \infty)$ and $\tau \times [t, \infty)$ goes to infinity as $t \to \infty$.

Definition 2.6. The *obstructor dimension* of G, denoted $\mathrm{obdim}(G)$, is the maximal $n+2$ so that there is a proper expanding map $f\colon \mathrm{Cone}_\infty(K) \to EG$ where K is an n-obstructor.

Remark 2.7. Bestvina, Kapovich and Kleiner give a more general definition of obstructor dimension which does not involve the space EG (and in particular works for all finitely generated groups). For the groups we are interested in, the two notions coincide.

The main theorem of [3] is that $\mathrm{obdim}(G) \leq \mathrm{updim}(G)$. It also follows from a Join Lemma for the van Kampen obstruction that

$$\mathrm{obdim}(G_1 \times G_2) = \mathrm{obdim}(G_1) + \mathrm{obdim}(G_2).$$

2.3. Integral van Kampen and Wu Invariants. The trivial and nontrivial $\mathbb{Z}/2$-module structures on $\mathbb{Z}$ will be denoted by $\mathbb{Z}$ and $\mathbb{Z}^-$, respectively. Recall that if $\widetilde{X}$ is a complex with free $\mathbb{Z}/2$-action and X is the quotient, then for any $\mathbb{Z}[\mathbb{Z}/2]$-module M, the groups $H_*(X, M)$ are the homology groups of the chain complex $C_*(\widetilde{X}) \otimes_{\mathbb{Z}/2} M$. Similarly, the groups $H^*(X, M)$ are the cohomology groups of the complex $\mathrm{Hom}_{\mathbb{Z}/2}(C_*(\widetilde{X}), M)$. If $M = \mathbb{Z}$, then these complexes can be identified with $C_*(X, \mathbb{Z})$ and $C^*(X, \mathbb{Z})$ respectively. If $M = \mathbb{Z}^-$, then $C_*(X, \mathbb{Z}^-)$ can be identified with the quotient complex

$$C_*(\widetilde{X}, \mathbb{Z})/(c \sim -\iota_* c)$$

and $C^*(X, \mathbb{Z}^-)$ with the subcomplex

$$\{ f \in C^*(\widetilde{X}, \mathbb{Z}) \mid f(c) \sim -f(\iota_* c) \}.$$

In our setting, note that the $\mathbb{Z}/2$-action ι_* on $C_*(\widetilde{\mathcal{C}}(K), \mathbb{Z})$ sends the chain (σ, τ) to the chain $(-1)^{\dim \sigma \dim \tau}(\tau, \sigma)$.

We have

$$H^i(\mathbb{R}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0; \\ \mathbb{Z}/2 & \text{if } i > 0 \text{ and is even}; \\ 0 & \text{if } i \text{ is odd}. \end{cases}$$

The sequence $0 \to \mathbb{Z} \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}^- \to 0$ induces a long exact sequence in cohomology. Since $H^*(\mathbb{R}P^\infty; \mathbb{Z}[\mathbb{Z}/2]) = H^*(S^\infty, \mathbb{Z}) = 0$ for $* > 0$, we have

$$H^i(\mathbb{R}P^\infty; \mathbb{Z}^-) = \begin{cases} \mathbb{Z}/2 & \text{if } i \text{ is odd}; \\ 0 & \text{if } i \text{ is even}. \end{cases}$$

Let e_1 denote the nontrivial element of $H^1(\mathbb{R}P^\infty; \mathbb{Z}^-) \cong \mathbb{Z}/2$. For the rest of the paper we will let ε denote the sign of $(-1)^n$ (if n is obvious from context), so that e_1^n is the nontrivial element of $H^n(\mathbb{R}P^\infty; \mathbb{Z}^\varepsilon) \cong \mathbb{Z}/2$. For K a finite complex, the *integral degree n van Kampen obstruction*, denoted $\mathrm{vk}^n(K)$, is given by

$$\mathrm{vk}^n(K) = c^*(e_1^n) \in H^n(\mathcal{C}(K); \mathbb{Z}^\varepsilon).$$

Now, if f is an embedding of K into $\mathbb{R}^{n+1}$, then f determines a $\mathbb{Z}/2$-equivariant *Gauss map* $\widetilde{F}$ from $\widetilde{\mathcal{C}}(K)$ to $\mathbb{S}^n$;

$$\widetilde{F}((x, y)) = \frac{f(x) - f(y)}{\| f(x) - f(y) \|}.$$

There is an induced map $F: \mathcal{C}(K) \to \mathbb{R}P^n$. Again by the coefficient long exact sequence,

$$H^n(\mathbb{R}P^n; \mathbb{Z}^{-\varepsilon}) \cong \mathbb{Z}.$$

Let η denote the generator of $H^n(\mathbb{R}P^n; \mathbb{Z}^{-\varepsilon})$. The *Wu invariant* of f, denoted $\mathrm{Wu}_f^n(K)$, is the pullback

$$\mathrm{Wu}_f^n(K) = F^*(\eta) \in H^n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon}).$$

If f and g are two embeddings with $\mathrm{Wu}_f^n(K) \neq \mathrm{Wu}_g^n(K)$, then f and g are not ambient isotopic (since such an isotopy would induce a $\mathbb{Z}/2$-equivariant homotopy between the Gauss maps $\widetilde{F}$ and $\widetilde{G}$). More interestingly, Wu is a complete ambient isotopy invariant for embedding of n-complexes into $\mathbb{R}^{2n+1}$ for $n > 1$, see [25]. It is easy to construct examples of embeddings with $\mathrm{Wu}_f \neq \mathrm{Wu}_g$. For example, if K is the disjoint union of two circles, then the degree two Wu invariant evaluated on the fundamental class of $\mathcal{C}(K)$ is twice the linking number of an embedding of K into $\mathbb{R}^3$.

On the other hand, the class vk does not depend on the embedding. See Figure 1 for an explanation of how the Wu invariant evaluates differently on cells than the van Kampen obstruction. Roughly, both invariants admit geometric representatives obtained by taking an embedding into $\mathbb{R}^{n+1}$, generically projecting to $\mathbb{R}^n$, and then counting signed intersections between disjoint simplices with $\dim \sigma + \dim \tau = n$. The Wu invariant is more refined as it remembers which simplices are "higher" from the point of view of the projection.

There are obvious homomorphisms $\mathbb{Z}^{+/-} \to \mathbb{Z}/2$. Under these change of coefficients, $e_1 \in H^1(\mathbb{R}P^\infty, \mathbb{Z}^-)$ maps to ω_1, and $\eta \in H^n(\mathbb{R}P^n, \mathbb{Z}^{-\varepsilon})$ maps to ω_1^n. Therefore, both the integral van Kampen obstruction and the Wu invariant reduce to $\mathrm{vk}_{\mathbb{Z}/2}^n$ via change of coefficients.

There is also a natural evaluation map

$$\langle\, , \,\rangle \colon H^n(\mathcal{C}(K), \mathbb{Z}^{-\varepsilon}) \times H_n(\mathcal{C}(K), \mathbb{Z}^{-\varepsilon}) \longrightarrow \mathbb{Z}^{-\varepsilon}$$

which comes from the identifications

$$\begin{aligned}
C^n(\mathcal{C}(K), \mathbb{Z}^{-\varepsilon}) &= \mathrm{Hom}_{\mathbb{Z}/2}(C_n(\mathcal{C}(K), \mathbb{Z}), \mathbb{Z}^{-\varepsilon}) \\
&\cong \mathrm{Hom}_{\mathbb{Z}/2}(C_n(\mathcal{C}(K), \mathbb{Z}) \otimes_{\mathbb{Z}/2} \mathbb{Z}^{-\varepsilon}, \mathbb{Z}^{-\varepsilon}).
\end{aligned}$$

Therefore, a representative for a cohomology class gives a $\mathbb{Z}/2$-homomorphism from $C_n(K, \mathbb{Z}^{-\varepsilon})$ to $\mathbb{Z}^{-\varepsilon}$. This passes to a well-defined homomorphism on cohomology and homology. We record the following lemma.

Lemma 2.8. *Let $\phi \in H^n(\mathcal{C}(K), \mathbb{Z}^{-\varepsilon})$ and $\psi \in H_n(\mathcal{C}(K), \mathbb{Z}^{-\varepsilon})$, let $p^{-\varepsilon}$ be the nontrivial homomorphism $\mathbb{Z}^{-\varepsilon} \to \mathbb{Z}_2$, and let $\phi_{\mathbb{Z}/2}$ and $\psi_{\mathbb{Z}/2}$ denote the images in $H_n(\mathcal{C}(K), \mathbb{Z}/2)$ and $H_n(\mathcal{C}(K), \mathbb{Z}/2)$ of ϕ and ψ respectively. Then*

$$p^{-\varepsilon}(\langle \phi, \psi \rangle) = \langle \phi_{\mathbb{Z}/2}, \psi_{\mathbb{Z}/2} \rangle.$$
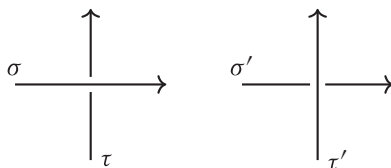
$$\mathrm{vk}(\sigma, \tau) = \mathrm{vk}(\sigma', \tau') = -\mathrm{vk}(\tau, \sigma),$$
$$\mathrm{Wu}(\sigma, \tau) = -\mathrm{Wu}(\sigma', \tau') = \mathrm{Wu}(\tau, \sigma)$$

Figure 1. An example of how the van Kampen obstruction and Wu invariant evaluate on a pair of cells. In this case, these edges are part of an embedded graph in $\mathbb{R}^3$. In both cases, vk and Wu switch sign upon changing the orientation of σ or τ. In this case, $\iota(\sigma, \tau) = -(\tau, \sigma)$, so $\mathrm{Wu}(\sigma, \tau) = \mathrm{Wu}(\tau, \sigma)$ and $\mathrm{vk}(\sigma, \tau) = -\mathrm{vk}(\tau, \sigma)$.

3. Equivariant obstructors

We now use the Wu invariant to obstruct certain equivariant embeddings of finite complexes into Euclidean space. In the next section, we will develop a coarsened version which obstructs properly discontinuous actions.

Suppose that H is a finite group and K is an H-complex. Suppose that $\rho: H \to \mathrm{Homeo}(\mathbb{R}^{n+1})$ is a homomorphism. An H-equivariant embedding of K is an embedding $f: K \to \mathbb{R}^{n+1}$ satisfying

$$f(hk) = \rho(h)(f(k)) \quad \text{for all } k \in K, h \in H.$$

If $\rho(h)$ is orientation-preserving, it is isotopic to the identity by the solution to the Stable Homeomorphism Conjecture by Kirby and Quinn [18] and [20]. Therefore, if f is H-equivariant than $\mathrm{Wu}_f = \mathrm{Wu}_{\rho(h) \circ f} = \mathrm{Wu}_{f \circ h}$ for all $h \in H$. We will want to assume $\rho(h)$ preserves orientation, so we consider only the elements of H that are squares. For H a finite group, let $S(H)$ be the subgroup generated by the squares of elements of H. For a cycle

$$\Phi = \sum_{(\sigma, \tau) \in \widetilde{\mathcal{C}}(K)} (\sigma, \tau) \otimes a_{(\sigma, \tau)} \in H_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon}),$$

let $\Phi_{\mathbb{Z}/2}$ denote its image in $H_n(\mathcal{C}(K); \mathbb{Z}/2)$ under the change of coefficients homomorphism.

Definition 3.1. Let K be a finite complex. A cycle Φ in $H_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$ is an *evaluation n-cycle* if

- $\langle \mathrm{vk}^n_{\mathbb{Z}/2}(K), \Phi_{\mathbb{Z}/2} \rangle \neq 0$;

- if σ is an n-cell, then

$$\sum_{\substack{v \in K^{(0)} \\ (\sigma,v) \in \tilde{C}(K)}} a_{(\sigma,v)} = 0.$$

As in the definition of obstructor, the first condition is the more important one, and the second condition guarantees that certain join formulas will hold. Since Wu_f^n maps to $\mathrm{vk}_{\mathbb{Z}/2}^n$ under change of coefficients, we have the following lemma.

Lemma 3.2. *If $f \colon K \to \mathbb{R}^{n+1}$ is an embedding and $\Phi \in H_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$ is an evaluation n-cycle, then $\mathrm{Wu}_f^n(K)$ evaluates nontrivially on Φ.*

It will be convenient later to have the following refinement of $\mathrm{obdim}(G)$.

Definition 3.3. The $\mathbb{Z}$-*valued obstructor dimension* of G, denoted $\mathrm{obdim}_{\mathbb{Z}}(G)$, is the maximal $n + 2$ so that there is a proper expanding map $f \colon \mathrm{Cone}_\infty(K) \to EG$ where K is an evaluation n-cycle.

Remark 3.4. It is obvious that $\mathrm{obdim}_{\mathbb{Z}}(G) \le \mathrm{obdim}(G)$. On the other hand, many of the complexes used to compute obstructor dimension contain evaluation cycles that reduce to the nontrivial $\mathbb{Z}/2$-valued obstructor cycles. For example, if K is the n-fold join of 3 points, then $H_{2n-2}(\mathcal{C}(K), \mathbb{Z}^{-\varepsilon})$ surjects onto $H_{2n-2}(\mathcal{C}(K), \mathbb{Z}/2)$ (this follows for example from the Join Lemma below). Note that in this case $H_{2n-2}(\mathcal{C}(K), \mathbb{Z}^\varepsilon) = 0$.

If H acts on K cellularly, then there is an induced action on $\mathcal{C}(K)$, and hence an action on $H_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$. We always assume that H acts trivially on $\mathbb{Z}^{-\varepsilon}$.

Definition 3.5. An H-complex K is an *equivariant $(n + 1)$-obstructor* if there exists an evaluation cycle $\Phi \in H_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$ and a nonempty subset $A \subset S(H)$ so that

$$\sum_{h \in A} h_* \Phi = 0.$$

Lemma 3.6. *Suppose that K is a finite H-complex and $f \colon K \to \mathbb{R}^{n+1}$ is an embedding. Then $\mathrm{Wu}_{f \circ h} = h^*(\mathrm{Wu}_f)$. In particular, for all $\Phi \in H_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$,*

$$\langle \mathrm{Wu}_f^n(K), h_* \Phi \rangle = \langle \mathrm{Wu}_{f \circ h}^n(K), \Phi \rangle.$$

Proof. This follows immediately from the equalities

$$h^* \mathrm{Wu}_f^n((\sigma, \tau)) = \mathrm{Wu}_f^n((h\sigma, h\tau)) = \mathrm{Wu}_{f \circ h}^n(\sigma, \tau). \qquad \square$$

Lemma 3.7. *If K is an equivariant $(n + 1)$-obstructor, then K does not equivariantly embed into $\mathbb{R}^{n+1}$.*

Proof. Suppose there was a representation $\rho\colon H \to \mathrm{Homeo}(\mathbb{R}^{n+1})$ and an equivariant embedding $f\colon K \to \mathbb{R}^{n+1}$. Every $h \in S(H)$ acts by orientation-preserving homeomorphisms on $\mathbb{R}^{n+1}$, so in particular f is isotopic to $f \circ h = \rho(h) \circ f$. Let $A \subset S(H)$ with $\sum_{h \in A} h_* \Phi = 0$, where Φ is an evaluation cycle. We then have that

$$0 = \left\langle \mathrm{Wu}_f^n(K), \left(\sum_{h \in A} h_* \Phi \right) \right\rangle = \sum_{h \in A} \langle \mathrm{Wu}_f^n(K), h_* \Phi \rangle$$

$$= \sum_{h \in A} \langle \mathrm{Wu}_{f \circ h}^n(K), \Phi \rangle = |A| \langle \mathrm{Wu}_f^n(K), \Phi \rangle \neq 0,$$

where the last inequality is by Lemma 3.2. This is a contradiction. $\qquad\square$

Note that if K is an equivariant $(n + 1)$-obstructor then K does not embed into $\mathbb{R}^n$.

3.1. Examples of equivariant obstructors.

We now give our main examples of finite complexes which are equivariant $(n + 1)$-obstructors. All our examples are iterated joins or cones of the following.

Lemma 3.8. *Suppose that $\mathbb{Z}/4 = \langle h \rangle$ acts on $K = 5$ points by cyclically permuting 4 of the points and fixing one. Then K is an equivariant 1-obstructor.*

Proof. Let A denote the fixed point and $\{1, 2, 3, 4\}$ the other points. Let $\Phi \in H_0(\mathcal{C}(K); \mathbb{Z}^-)$ be given by

$$\Phi = \{(A, 1) \otimes 1, (1, 3) \otimes 1, (3, A) \otimes 1\}.$$

Then $\langle vk_{\mathbb{Z}/2}(K), \Phi_{\mathbb{Z}/2} \rangle \neq 0$, $\Phi + h_*^2 \Phi = 0 \in H_0(\mathcal{C}(K); \mathbb{Z}^-)$, and Φ satisfies the second condition of equivariant obstructors. $\qquad\square$

In the next two lemmas, we are explicitly identifying $H_n(\mathcal{C}(K), \mathbb{Z}^{-\varepsilon})$ with the homology of the quotient complex $C_n(\widetilde{\mathcal{C}}(K), \mathbb{Z})/(c \sim (-1)^{-\varepsilon} \iota_* c)$. The next two lemmas mirror the Cone Lemma and the Join Lemma from [3], but unfortunately we have to keep track of signs. We will let Cone K denote the finite cone $K \times I / K \times 0$, in order to distinguish it from $\mathrm{Cone}_\infty(K)$.

Cone Lemma. *Suppose that K is an H-complex. Let* Cone K *be the cone of K, and extend the action of H by fixing the cone point. If K is an equivariant n-obstructor, then* Cone K *is an equivariant $(n + 1)$-obstructor.*

Proof. Let $\Phi \in C_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$ be an evaluation cycle. Define

$$\Phi' \in C_{n+1}(\mathcal{C}(\mathrm{Cone}\, K); \mathbb{Z}^{-\varepsilon})$$

by
$$\Phi' = \{(-1)^{\dim \sigma}(\sigma, \mathrm{Cone}(\tau)) - (\mathrm{Cone}(\sigma), \tau))|(\sigma, \tau) \in \mathrm{Supp}\,\Phi\}$$
and then extending linearly.

We first claim that this procedure is well-defined. If $(\sigma, \tau) \in \Phi$, then
$$(\sigma, \tau) \sim (-1)^{\dim \sigma\, \dim \tau + n - 1}(\tau, \sigma),$$
which produces
$$(-1)^{\dim \sigma\, \dim \tau + \dim \sigma - 1}(\tau, \mathrm{Cone}(\sigma)) + (-1)^{\dim \sigma\, \dim \tau + n}(\mathrm{Cone}(\tau), \sigma).$$

So, using the fact that $n = \dim \sigma + \dim \tau$, we are done by the equalities:
$$\dim \sigma + \dim \sigma(\dim \tau + 1) + n = \dim \sigma\, \dim \tau + n \quad \mathrm{mod}\ 2,$$
$$(\dim \sigma + 1)\dim \tau + n - 1 = \dim \sigma\, \dim \tau + \dim \sigma - 1 \quad \mathrm{mod}\ 2.$$

We now check that Φ' is a cycle. Since $\partial(\sigma, \tau) = (\partial\sigma, \tau) + (-1)^{\dim \sigma}(\sigma, \partial\tau)$, we have exactly set it up so that $\partial\Phi'(\sigma, \tau) = 0$. Since Φ is an evaluation cycle, for a cell $(\sigma, \mathrm{Cone}(\alpha))$, we have
$$\sum_{\alpha \subset \tau} a_{\sigma, \tau} = 0.$$
The cells $(\sigma, \mathrm{Cone}(\tau))$ are the cells containing $(\sigma, \mathrm{Cone}(\alpha))$, and since these are multiplied by the same constant, we have $\partial\Phi'((\sigma, \mathrm{Cone}(\alpha)) = 0$. For the non-triviality condition, $\Phi'_{\mathbb{Z}/2}$ is precisely the chain constructed in the Cone Lemma of [3], which was shown to have $\mathrm{vk}^n_{\mathbb{Z}/2}(\Phi'_{\mathbb{Z}/2}) \neq 0$ as long as $\mathrm{vk}^n_{\mathbb{Z}/2}(\Phi_{\mathbb{Z}/2}) \neq 0$. If $\sum_{h \in A} h\Phi = 0$ then obviously $\sum_{h \in A} h\Phi' = 0$. Finally, it is straightforward to check that Φ satisfies the second condition of equivariant obstructors if Φ does. $\qquad\square$

Join Lemma. *Let K be an H-complex which is an $(n+1)$-equivariant obstructor, and suppose that J is a complex with an evaluation m-cycle. Let H act on $K * J$ by permuting the K-factor and fixing J. Then $K * J$ is an equivariant $(n+m+3)$-obstructor.*

Proof. Let Φ_K be an evaluation cycle in $H_n(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$ with
$$\sum_{h \in A} h_*(\Phi_K) = 0$$
and let Φ_J be the evaluation m-cycle in $H_n(\mathcal{C}(J); \mathbb{Z}^{-\varepsilon})$. Form a new cycle $\Phi \in H_{n+m+2}(\mathcal{C}(K * J); \mathbb{Z}^{-\varepsilon})$ by putting in for every $(\sigma, \tau) \in \Phi_K$ and $(\sigma', \tau') \in \Phi_J$ the chains
$$(-1)^{\dim \sigma(\dim \tau' + 1) + \dim \sigma'}(\sigma * \sigma', \tau * \tau')$$
$$+ (-1)^{\dim(\sigma+1)(\dim \sigma' + 1) + \dim \sigma'\, \dim \tau'}(\sigma * \tau', \tau * \sigma')$$
and extending linearly.

We need to show that this procedure is well-defined after passing to the quotient complexes $C_*(\widetilde{\mathcal{C}}(K))/\iota$ and $C_*(\widetilde{\mathcal{C}}(J))/\iota$. In particular, we need to show that the involuted (σ, τ) and (σ', τ') produce equivalent chains. For these calculations, we will shorten $\dim \sigma$ in the exponents to σ.

- (σ, τ) and (τ', σ'). In this case, the procedure gives

$$(-1)^{\sigma(\sigma'+1)+\tau'}(\sigma * \tau', \tau * \sigma') + (-1)^{(\sigma+1)(\tau'+1)+\tau'\sigma'}(\sigma * \sigma', \tau * \tau').$$

Since $(\tau', \sigma') \sim (-1)^{\sigma'\tau'+\tau'+\sigma'+1}(\sigma', \tau')$ we need to check that

$$\sigma'\tau' + \tau' + \sigma' + 1 + (\sigma+1)(\tau'+1) + \tau'\sigma' = \sigma(\tau'+1) + \sigma' \mod 2,$$
$$(\sigma+1)(\sigma'+1) + \sigma'\tau' = \sigma(\sigma'+1) + \tau' + \sigma'\tau' + \sigma' + \tau' + 1 \mod 2.$$

which is easily verifed.

- (τ, σ) and (σ', τ') In this case, the procedure gives

$$(-1)^{\tau(\tau'+1)+\sigma'}(\tau * \sigma', \sigma * \tau') + (-1)^{(\tau+1)(\sigma'+1)+\sigma'\tau'}(\tau * \tau', \sigma * \sigma').$$

Since $(\sigma * \sigma', \tau * \tau') \sim (-1)^{(\sigma+\sigma')(\tau+\tau')}(\tau * \tau', \sigma * \sigma')$, we need to show that

$$\sigma(\tau'+1) + \sigma' + (\sigma+\sigma')(\tau+\tau')$$
$$= \sigma\tau + \sigma + \tau + 1 + (\tau+1)(\sigma'+1) + \sigma'\tau' \mod 2,$$
$$(\sigma+1)(\sigma'+1) + \sigma'\tau' + (\sigma+\tau')(\tau+\sigma')$$
$$= \sigma\tau + \sigma + \tau + 1 + \tau(\tau'+1) + \sigma' \mod 2,$$

which again is easily verified.

Again, $\Phi_{\mathbb{Z}/2}$ is the cycle constructed in the Join Lemma of [3]. To see that Φ is a cycle, assume that we have a $(n + m + 1)$-cell in $\mathcal{C}(K * J)$. We can assume without loss of generality that this cell is of the form $(\sigma_1 * \alpha_2, \tau_1 * \alpha_2')$, where $\dim(\sigma_1) + \dim(\tau_1) = n$ and $\dim(\alpha_2) + \dim(\alpha_2') = m - 1$. Since Φ_J is a cycle, we have that the sum

$$\sum_{\substack{(\sigma_2,\alpha_2')\in\Phi_J \\ \alpha_2 \subset \sigma_2}} a_{(\sigma_2,\alpha_2')} + \sum_{\substack{(\alpha_2,\tau_2)\in\Phi_J \\ \alpha_2' \subset \tau_2}} a_{(\alpha_2,\tau_2)} = 0.$$

Since for each of these cells,

$$(\sigma_1 * \sigma_2, \tau_1 * \alpha_2') \otimes a_{(\sigma_1,\tau_1)} a_{(\sigma_2,\alpha')} \quad \text{and} \quad (\sigma_1 * \alpha_2, \tau_1 * \tau_2) \otimes a_{(\sigma_1\tau_1)} a_{(\alpha_2,\tau_2)}$$

are precisely the cells in Φ containing $(\sigma_1 * \alpha_2, \tau_1 * \alpha_2')$ and have the same sign, it follows that Φ is a cycle (if $\alpha_2 = \emptyset$ then we require the second item in the definition of evaluation cycle to prove this).

Now, suppose that $\sum_{h\in A} h_* \Phi_K = 0$. Then for any $(\sigma, \tau) \in \mathcal{C}(K)$, the sum $\sum_{h\in A} a_{h\sigma,h\tau} = 0$. This immediately implies that for any $(\sigma * \sigma', \tau * \tau')$, the sum $\sum_{h\in A} a_{h(\sigma*\sigma'),h(\tau*\tau')} = 0$, so $\Phi = 0$. The second condition of an equivariant obstructor is trivially satisfied since no simplices in Φ are paired with vertices. $\quad\square$

Remark 3.9. The usual homological tool to analyze group actions on $\mathbb{S}^n$ or $\mathbb{R}^n$ is *Smith theory*, and this handles far more examples than our method does. For example, if p is a prime, then the fixed set of a orientation preserving $\mathbb{Z}/p$-action on S^n is a homology r-sphere with $r < n-1$. This immediately obstructs $(p+1)$-points with $\mathbb{Z}_p$-action as above equivariantly embedding into S^1, as well as all joins of this complex with the product action embedding into S^{2n+1}.

The reason that we do not use Smith theory is that we do not have an adequate version of coarse Smith theory that could handle the examples of groups that we were interested. A coarse version of Smith theory has been developed by Hambleton and Savin [14], but it does not seem to be applicable to our examples. In particular, they relate the coarse topology of an ambient G-space X to the coarse topology of a "bounded fixed point set." This consists of points in X which are fixed up to bounded distance by every element of G, which in our main examples (when G is torsion-free) is always empty. In Section 6 we will consider some examples with torsion, and the methods of [14] do probably obstruct actions on uniformly contractible manifolds.

Remark 3.10. This use of the Wu invariant is our attempt to build an "equivariant van Kampen obstruction." A natural place for such an invariant to live is in the equivariant cohomology group $H_H^*(\mathcal{C}(K), \mathbb{Z}^{\pm\varepsilon})$, but we couldn't make this work. One difficulty is that if the H-action on $\mathbb{R}^{n+1}$ is not affine, then there is not an induced H-action on $\mathbb{R}P^n$. The larger problem is that the usual applications of equivariant obstruction theory require knowing both the H-action on the domain and range, whereas we are only given the H-action on the domain.

4. Coarse Wu Invariant

Let K be a finite complex. Equip $\mathbb{R}^{n+1}$ with a proper metric, and suppose that $f : \mathrm{Cone}_\infty K \to \mathbb{R}^{n+1}$ is a proper, expanding map. Then there are induced maps $f_t : \mathrm{Cone}\, K \to \mathbb{R}^{n+1}$ defined by

$$f_t(x, s) = f(x, st) \quad \text{for } s \in I, t \in [0, \infty)$$

Again, Cone K here denotes the finite cone $K \times I / K \times 0$. The basic idea behind defining the coarse Wu invariant is that if f is a proper, expanding map, the f_t will eventually be *almost embeddings*, where an almost embedding maps disjoint simplices of the cone disjointly. An almost embedding suffices to define the Wu invariant, and for large enough t this will stabilize to give a well-defined class in $H^n(\mathcal{C}(\mathrm{Cone}\, K), \mathbb{Z}^{-\varepsilon})$. We also want the Wu invariant to not change if we postcompose f with a homeomorphism of $\mathbb{R}^n$ which is isotopic to the identity. This composition may no longer be expanding, but will be an almost embedding, and furthermore will be isotopic to f via almost embeddings. Therefore, we will

eventually define the coarse Wu invariant for all maps $f \colon \mathrm{Cone}_\infty(K) \to \mathbb{R}^n$ which are isotopic to proper, expanding maps.

Lemma 4.1. *Let $f \colon \mathrm{Cone}_\infty K \to \mathbb{R}^{n+1}$ be a proper, expanding map. Then there exists a $T_f > 0$ so that for all $t > T_f$, f_t is an almost embedding.*

Proof. By the definition of expanding, there exists $T_f' > 0$ so that $\sigma \times [T_f', \infty)$ and $\tau \times [T_f', \infty)$ are disjoint for each pair of disjoint simplices (σ, τ). Since f is proper, there exists $T_f \geq T_f'$ so that $\sigma \times [0, T_f'] \cap \tau \times [T_f, \infty] = \emptyset$ for each such pair. Therefore, f_t for all $t > T_f$ is an almost embedding. $\square$

Therefore, for f proper and expanding, there exists $T_f > 0$ so that for all $t > T_f$, there is a well-defined Gauss map $\widetilde{F}_t \colon \widetilde{\mathcal{C}}(\mathrm{Cone}\, K) \to \mathbb{S}^n$:

$$\widetilde{F}_t((x, s), (y, s')) = \frac{f_t(x, s) - f_t(y, s')}{\| f_t(x, s) - f_t(y, s') \|}$$

where either s or $s' = 1$ since $\widetilde{\mathcal{C}}(\mathrm{Cone}\, K)$ is the simplicial configuration space.

Lemma 4.2. *For all $t, t' > T_f$, $\widetilde{F}_t$ and $\widetilde{F}_{t'}$ are $\mathbb{Z}/2$-equivariantly homotopic.*

Proof. There is an obvious homotopy of f_t to $f_{t'}$ by the $\{f_s\}_{s \in [t, t']}$. Since each f_s induces a well-defined Gauss map F_s, these give a $\mathbb{Z}/2$-equivariant homotopy between $\widetilde{F}_t$ and $\widetilde{F}_{t'}$. $\square$

We will say that $f, f' \colon \mathrm{Cone}_\infty(K) \to \mathbb{R}^n$ are *isotopic* if there is an ambient isotopy $\{j^s\}_{s \in I}$ of $\mathbb{R}^n$ with j_0 the identity and $j_1 \circ f = f'$. In particular, we are not assuming f and f' are embeddings.

Lemma 4.3. *Suppose that $f \colon \mathrm{Cone}_\infty(K) \to \mathbb{R}^{n+1}$ and $f' \colon \mathrm{Cone}_\infty(K) \to \mathbb{R}^{n+1}$ are isotopic maps and that f is proper and expanding. Then for T_f as above and $t > T_f$, we have that the Gauss maps $\widetilde{F}_t$ and $\widetilde{F}_t'$ are $\mathbb{Z}/2$-equivariantly homotopic.*

Proof. Let $\{j^s\}_{s \in I}$ be an ambient isotopy between f and $\bar{f}$. The homeomorphisms j^s preserve disjointness of simplices, so for each $t > T_f$, the map $j_t^s := j^s \circ f_t$ induces a well-defined Gauss map J_t^s. The $\{J_t^s\}_{s \in I}$ give a $\mathbb{Z}/2$-equivariant homotopy between $\widetilde{F}_t$ and $\widetilde{F}_t'$. $\square$

Definition 4.4. Suppose that $\bar{f} \colon \mathrm{Cone}_\infty K \to \mathbb{R}^{n+1}$ is isotopic to a proper expanding map $f \colon \mathrm{Cone}_\infty K \to \mathbb{R}^{n+1}$, and $T_f > 0$ is defined as above. Suppose that $\widetilde{F}_t \colon \widetilde{\mathcal{C}}(\mathrm{Cone}\, K) \to \mathbb{S}^n$ is the induced Gauss map for f_t as above, and let F_t denote the induced map $\mathcal{C}(\mathrm{Cone}\, K) \to \mathbb{R}P^n$. Let η be the generator of $H^n(\mathbb{R}P^n; \mathbb{Z}^{-\varepsilon})$. The *coarse Wu invariant* of $\bar{f}$, denoted $\mathrm{CWu}_{\bar{f}}^n(K)$, is defined to be

$$F_t^*(\eta) \in H^n(\mathcal{C}(\mathrm{Cone}\, K); \mathbb{Z}^{-\varepsilon})$$

for $t > T_f$.

The coarse Wu invariant is well-defined (i.e. does not depend on t or f) by Lemmas 4.2 and 4.3.

Lemma 4.5. *Suppose that* $f: \text{Cone}_\infty K \to \mathbb{R}^{n+1}$ *is a proper, expanding map, and* g *is an orientation-preserving homeomorphism of* $\mathbb{R}^{n+1}$. *Then* $\text{CWu}_f^n(K) = \text{CWu}_{g \circ f}^n(K)$.

Proof. Since g is orientation-preserving, it is isotopic to the identity. Therefore, f and $g \circ f$ are isotopic, so the coarse Wu invariant is defined for $g \circ f$ and equal to $\text{CWu}_f^n(K)$. ∎

Theorem 4.6. *Suppose that* $f: \text{Cone}_\infty K \to \mathbb{R}^{n+1}$ *be a proper, expanding map. Suppose that* $f': \text{Cone}_\infty K \to \mathbb{R}^{n+1}$ *is proper, expanding, and homotopic to* f *through proper expanding maps* $\{h^s\}_{s \in I}$. *Then* $\text{CWu}_f^n(K) = \text{CWu}_{f'}^n(K)$.

Proof. Let $s \in I$, and let $T_{h^s} \in \mathbb{R}$ so that if $t > T_{h^s}$, then h_t^s induces a well-defined Gauss map. Then there exists $\varepsilon > 0$ so that for any s' with $|s - s'| < \varepsilon$, $h_t^{s'}$ induces a well-defined Gauss map, and hence $\text{CWu}^n(K)$ is locally constant. For each $s \in I$, there is a $T_{h^s} \in \mathbb{R}$ as above. Since I is compact we can take a finite subcover $(s_i - \varepsilon, s_i + \varepsilon)$. Choosing $t > \max(T_{h^{s_i}})$ gives a homotopy between F_t and F_t'. ∎

Note that if K is a complex with an evaluation cycle $\Phi \in H_{n-1}(\mathcal{C}(K); \mathbb{Z}^{-\varepsilon})$, and $f: \text{Cone}_\infty(K) \to \mathbb{R}^{n+1}$ is a proper expanding map, then by the Cone Lemma and Lemma 3.2, there is an evaluation cycle $\Phi' \in H_n(\mathcal{C}(\text{Cone } K; \mathbb{Z}^{-\varepsilon})$ that CWu_f^n evaluates nontrivially on.

Lemma 4.7. *Suppose that* $f: \text{Cone}_\infty(K) \to \mathbb{R}^{n+1}$ *is a proper expanding map, that* $\mathbb{R}^{n+1}$ *is uniformly contractible around the image of* f, *and*

$$f': \text{Cone}_\infty(K) \longrightarrow \mathbb{R}^{n+1}$$

is uniformly bounded distance from f, *i.e. there exists* $C > 0$ *so that*

$$d(f(x,s), f'(x,s)) < C \quad \text{for all } (x,s) \in \text{Cone}(K).$$

Then $\text{CWu}_f^n(K) = \text{CWu}_{f'}^n(K))$.

Proof. Since $\mathbb{R}^n$ is uniformly contractible around the image of f, we can homotope f to f' so that points move a uniformly bounded distance during the homotopy (say $< R$). As before, choose $T > 0$ so that $\sigma \times [T, \infty)$ and $\tau \times [0, \infty)$ have distance $> 2R$ for each pair of disjoint simplices σ and τ in K. This guarantees that a homotopy exists between the Gauss maps for f_t and f_t' for large enough t, and hence the coarse Wu invariants are the same. ∎

5. Equivariant obstructor dimension

We now show that the coarse Wu invariant obstructs proper, expanding maps $\mathrm{Cone}_\infty(K) \to \mathbb{R}^n$ that are "coarsely equivariant," and hence obstructs properly discontinuous actions on $\mathbb{R}^n$. Our notion is different from quasi-equivariance as defined in Section 2. For example, we want different groups acting on the domain and range (a finite group for the domain and usually a torsion-free group for the range), and we also want to allow proper homotopies that preserve disjointness of far away simplices.

Definition 5.1. Let H be a finite group, K a finite H-complex, and extend the H-action to $\mathrm{Cone}_\infty(K)$ by acting trivially on $[0, \infty)$. Let G be a group and EG a contractible, proper, cocompact, G-complex. A proper expanding map $f \colon \mathrm{Cone}_\infty K \to EG$ is H-*preserving* if for each $h \in H$, there is g_h in G so that $f \circ h$ is homotopic to $g_h \circ f$ via a homotopy of proper expanding maps $\{j^s\}$.

Of course, the element g_h may not be unique. We can and will assume that for all $h \in S(H)$, the elements g_h are in $S(G)$.

Definition 5.2. The *equivariant obstructor dimension* of G is the maximal $n + 3$ so that there is an H-equivariant $(n+1)$-obstructor K and an H-preserving proper expanding map $f \colon \mathrm{Cone}_\infty K \to EG$.

From the definitions, we have the following lemma:

Lemma 5.3. $\mathrm{obdim}(G) \leq \mathrm{eqobdim}(G) \leq \mathrm{obdim}(G) + 1$.

The following is our main theorem.

Theorem 5.4. $\mathrm{eqobdim}(G) \leq \mathrm{actdim}(G)$.

Proof. Suppose that $\mathrm{eqobdim}(G) = n + 1$ and G acts properly on a contractible n-manifold M^n. We first assume M^n is homeomorphic to $\mathbb{R}^n$. We equip $\mathbb{R}^n$ with a proper G-invariant metric. By assumption, we get a quasi-equivariant map $\rho \colon EG \to \mathbb{R}^n$, which is uniformly proper and Lipschitz, see Lemma 2.3. By precomposing with the H-preserving map $f \colon \mathrm{Cone}_\infty K \to EG$, we get a proper expanding map $\rho \circ f \colon \mathrm{Cone}_\infty K \to \mathbb{R}^n$, where K is an equivariant $(n-1)$-obstructor.

We now show that $\mathrm{CWu}^{n-1}_{\rho \circ f}(K) = \mathrm{CWu}^{n-1}_{\rho \circ f \circ h}(K)$ for all $h \in S(H)$. We have by Lemma 4.5 that for all $h \in S(H)$,

$$\mathrm{CWu}^{n-1}_{\rho \circ f}(K) = \mathrm{CWu}^{n-1}_{g_h \circ \rho \circ f}(K)$$

since $g_h \in S(G)$ and hence $g_h \colon \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving for all $h \in S(H)$, and hence isotopic to the identity.

We have that $\rho \circ f \circ h$ is properly homotopic to $\rho \circ g_h \circ f$ through the maps $\rho \circ j^s$. Since ρ is uniformly proper, $\rho \circ j_s$ is proper and expanding. Therefore, by Theorem 4.6, $\mathrm{CWu}^{n-1}_{\rho \circ f \circ h} = \mathrm{CWu}^{n-1}_{\rho \circ g_h \circ f}$.

Since ρ is quasi-equivariant, we have that $\rho \circ g_h \circ f$ is uniformly bounded distance from $g_h \circ \rho \circ f$. Since $\mathbb{R}^n$ is uniformly contractible around the image of $\rho(G)$, Lemma 4.7 implies that $\mathrm{CWu}^{n-1}_{\rho \circ g_h \circ f} = \mathrm{CWu}^{n-1}_{g_h \circ \rho \circ f}$.

So, therefore

$$\mathrm{CWu}^{n-1}_{\rho \circ f \circ h}(K) = \mathrm{CWu}^{n-1}_{g_h \circ \rho \circ f}(K) = \mathrm{CWu}^{n-1}_{\rho \circ f}(K)$$

for all $h \in S(H)$.

By the Cone Lemma, we have an evaluation cycle $\Phi \in H_{n-1}(\mathrm{Cone}\, K; \mathbb{Z}^{-\varepsilon})$ which $\mathrm{CWu}^{n-1}_{\rho \circ f}(K)$ evaluates nontrivially on.

As in the non-coarse case, this implies that

$$
\begin{aligned}
0 &= \left\langle \mathrm{CWu}^{n-1}_{\rho \circ f}(K), \left(\sum_{h \in A} h_* \Phi \right) \right\rangle = \sum_{h \in A} \langle \mathrm{CWu}^{n-1}_{\rho \circ f}(K), h_* \Phi \rangle \\
&= \sum_{h \in A} \langle \mathrm{CWu}^{n-1}_{\rho \circ f \circ h}(K), \Phi \rangle = |A| \langle \mathrm{CWu}^{n-1}_{\rho \circ f}(K), \Phi \rangle \neq 0
\end{aligned}
$$

which is a contradiction.

For contractible manifolds not homeomorphic to $\mathbb{R}^n$, we can do a stabilization trick. We assume without loss of generality that M^n is open. If G acts properly on M^n, then $\mathbb{Z} \times G$ acts properly on $M^n \times \mathbb{R}$, which is homeomorphic to $\mathbb{R}^{n+1}$, see [23]. Since $\mathrm{eqobdim}(G \times \mathbb{Z}) = \mathrm{eqobdim}(G) + 1$ by the Cone Lemma, this is a contradiction by the above. $\qquad\square$

Lemma 5.5. *If K_1 and K_2 are H_i-complexes and $f_i \colon \mathrm{Cone}_\infty K_i \to EG_i$ are H_i-preserving, then the product map*

$$f_1 \times f_2 \colon \mathrm{Cone}_\infty(K_1 * K_2) = \mathrm{Cone}_\infty K_1 \times \mathrm{Cone}_\infty K_2 \longrightarrow EG_1 \times EG_2$$

is $H_1 \times H_2$-preserving.

Proof. Let $(h_1, h_2) \in H$. By assumption, there are elements g_{h_1} and g_{h_2} and proper homotopies connecting $f_i \circ h_i$ to $g_{h_1} \circ f_i$. The product of these homotopies gives a homotopy between $f_1 \times f_2 \circ (h_1, h_2)$ and $(g_{h_1}, g_{h_2}) \circ f_1 \times f_2$. Since each homotopy is proper and expanding, the product is as well. $\qquad\square$

Therefore, the join lemma for equivariant obstructors immediately gives the following product formula for eqobdim.

Theorem 5.6. $\mathrm{eqobdim}(G_1 \times G_2) \geq \mathrm{eqobdim}(G_1) + \mathrm{eqobdim}(G_2) - 1$.

Since the Join Lemma only requires one of the complexes to have a group action, we can also say something about eqobdim$(G_1 \times G_2)$ when we know eqobdim(G_1) and obdim$_{\mathbb{Z}}(G_2)$.

Lemma 5.7. eqobdim$(G_1 \times G_2) \geq$ eqobdim$(G_1) +$ obdim$_{\mathbb{Z}}(G_2)$.

If G acts properly and cocompactly on a CAT(0) space X, then almost embedded obstructor complexes K into the boundary $\partial_\infty X$ give proper expanding maps of Cone$_\infty(K)$ into X, and hence give lower bounds for obstructor dimension (recall that an almost embedding maps disjoint simplices disjointly). Similarly, if K is an obstructor complex in $\partial_\infty X$ which is invariant setwise under the G-action on $\partial_\infty(X)$, then this should give lower bounds for equivariant obstructor dimension.

Lemma 5.8. *Suppose G acts properly and cocompactly on a CAT(0) space EG, and let $\partial_\infty EG$ be the visual boundary for EG. Suppose K is an H-equivariant n-obstructor, and $i\colon K \to \partial_\infty(G)$ is an almost embedding. Suppose that for all $h \in H$, there is $g_h \in G$ so that $i \circ h(K)$ is homotopic (inside $\partial_\infty(G)$) to $g_h \circ i(K)$ via almost embeddings. Then eqobdim$(G) \geq n + 3$.*

Proof. Choose a basepoint $x_0 \in EG$ and define $f\colon$ Cone$_\infty(K) \to EG$ by coning $i(K)$ to x_0. We claim f is H-preserving. By assumption, the maps $f \circ h($Cone$_\infty(hK))$ and $g_h \circ f($Cone$_\infty K))$ have homotopic boundary values. Use this homotopy to homotope f to $f'\colon$ Cone$_\infty(hK) \to \mathbb{R}^n$ which has the same value on the boundary as $g_h \circ F$. This homotopy is proper and by assumption each map is proper and expanding.

Therefore, $g_h \circ f$ sends (x, t) to a geodesic based at $g_h(x_0)$, and f' sends (x, t) to the asymptotic geodesic based at x_0. Since EG is CAT(0), the distance between $g_h \circ F(x, t)$ and $F'(x, t)$ is uniformly bounded by the distance between $g_h(x_0)$ and x_0. Since EG is uniformly contractible, we can homotope $g_h \circ f$ to f' and move points a uniformly bounded distance. $\square$

6. Examples of Groups

6.1. Virtual RAAG's.

The simplest examples of groups with updim$(G) <$ actdim(G) are virtually free groups that do not act on the plane. We will compute the equivariant obstructor dimension of a more general class of groups which are finite extensions of right-angled Artin groups. We recall the definition.

Definition 6.1. Suppose L^1 is a simplicial graph with vertex set V. The *flag complex determined by L^1* is the simplicial complex L whose simplices are the (vertex sets of) complete subgraphs of L^1. Associated to L^1 there is a RAAG, A_L. A set of generators for A_L is $\{g_v\}_{v \in V}$; there are relations $[g_v, g_{v'}] = 1$ whenever $\{v, v'\} \in$ Edge L^1.

Let T^V denote the product $(S^1)^V$. For each simplex $\sigma \in L$, let $T(\sigma)$ denote the subtorus $(S^1)^{|\sigma|}$. The *Salvetti complex* for A_L is the union of the subtori $T(\sigma)$ over simplices σ in L:

$$S(L) := \bigcup_{\sigma \in L} T(\sigma).$$

If a finite group H acts on a flag complex L, then H acts on A_L by permuting the generators of A_L. Therefore, we can form the semidirect product $A_L \rtimes H$. Suppose now that L is a flag H-complex which is an equivariant $(n+1)$-obstructor.

Theorem 6.2. *If L is a d-dimensional flag-complex which is an equivariant $(2d + 1)$-obstructor, then*

$$\mathrm{eqobdim}(A_L \rtimes H) \geq 2d + 3.$$

Proof. Fix a point $*$ in the universal cover $\widetilde{S}(L)$ which is a lift of the unique vertex of $S(L)$. Inside $\widetilde{S}(L)$, there is a unique lift $\mathbb{R}_\sigma$ of T_σ containing $*$. For each T_σ, let $\mathbb{R}_\sigma^+$ be the points with nonnegative coordinates. Then the union of the boundaries of the $\mathbb{R}_\sigma^+$ is homeomorphic to L. Furthermore, the action of H on $\widetilde{S}(L)$ permutes these lifts, and stabilizes this copy of L in $\partial_\infty \widetilde{S}(L)$. Furthermore, the restriction of the action to this copy is precisely the original H-action. Since L is an equivariant $(n + 1)$-obstructor, we conclude from Lemma 5.8 that $\mathrm{eqobdim}(A_L \rtimes H) = n + 3$. $\square$

Remark 6.3. If L is a d-dimensional complex, then

$$\mathrm{actdim}(A_L) = 2d + 2 = 2\,\mathrm{gd}(A_L),$$

see [1]. Since equivariant $(n + 1)$-obstructors have $H_n(L; \mathbb{Z}/2) \neq 0$, we are making quite a strong assumption on L (for example, in [1] L could be a triangulation of an n-sphere whereas we require L to be more like a n-fold join of m points).

6.2. Products of virtually 3-manifold groups.

We recall the examples of virtually 3-manifold groups constructed in [15] (the examples in [17] have nearly identical proofs). We start with two closed surfaces S_a and S_b of genus ≥ 2, and a choice of essential simple closed curves γ_a and γ_b on S_a and S_b respectively. We build a 2-complex X_{mn} by attaching an annulus to $S_a \sqcup S_b$. We glue one boundary component of the annulus to γ_a by a degree m-map, and the other boundary component to γ_b along a degree n-map, see Figure 2. Let $G_{mn} = \pi_1(X_{mn})$. Note that X_{mn} naturally admits a locally CAT(0) metric, so $\widetilde{X}_{mn}$ has a CAT(0) metric. Hruska, Stark, and Tran show the following:

Theorem 6.4 (Theorem 5.6, [15]). *For all m and n, G_{mn} is virtually a 3-manifold group. It is a 3-manifold group if and only if*

- $m = n = 1$;
- $m = 1$, $n = 2$ and γ_b is non-separating;
- $m = 2$, $n = 2$, and γ_a and γ_b are non-separating.

In fact, they show G_{mn} virtually embeds as a subgroup of a right-angled Coxeter group W with planar boundary. The Davis complex of W can be W-equivariantly thickened to a 3-manifold.
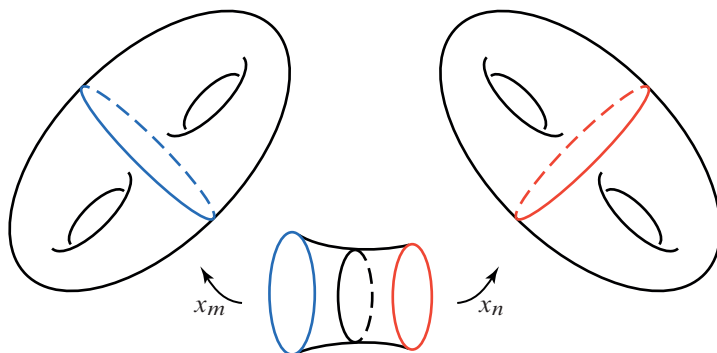

Figure 2. The space X_{mn} as in [15]. Each end of the cylinder is glued onto the corresponding curve with a positive degree map.

Theorem 6.5. *If m or n is divisible by* 4, *then* eqobdim$(G_{mn}) = 4$.

Proof. First assume that $m = 4$ and $n > 1$. Let A_{mn} denote the quotient space of the cylinder $S^1 \times I$ under the identifications $(z, 0) \cong (e^{2\pi i/m}z, 0)$ and $(z, 1) \cong (e^{2\pi i/n}z, 1)$. The universal cover of A_{mn} is $T_{mn} \times \mathbb{R}$, where T_{mn} is the biregular tree of valence m and n. The fundamental group of A_{mn} has presentation

$$\pi_1(A_{mn}) = \langle a, b \mid a^m = b^n \rangle.$$

There is a natural totally geodesic embedding $A_{mn} \to X_{mn}$. We will identify $\pi_1(A_{mn})$ with its image in G_{mn}. Inside $EG_{mn} = \tilde{X}_{mn}$, choose a copy of $T_{mn} \times \mathbb{R}$ which the group $\langle a, b \rangle$ acts geometrically on. Let $\tilde{\gamma}_a$ denote the axis of a inside this copy of $T_{mn} \times \mathbb{R}$. Then $\tilde{\gamma}_a$ is $v \times \mathbb{R}$, where v is a valence m-vertex in T_{mn}. The element a cyclically permutes the m-edges emanating from v and translates n-units in the $\mathbb{R}$-direction. Furthermore, if P_0 is a geodesic ray based at $v \in T_{mn}$, then $\langle a \rangle$ cyclically permutes the collection of m-half planes $\{a^i(P_0 \times \mathbb{R})\}_{i=0}^m$. The universal cover of S_a is glued to this union of half planes along $\tilde{\gamma}_a$; $\langle a \rangle$ acts on this universal cover by a hyperbolic translation. Let H be one of the half planes in $\tilde{S}_a$ that γ_a bounds.

Let $K = \text{Cone}((m + 1)$ points$)$. We define an embedding $f \colon K \to \partial_\infty EG_{mn}$. We send the cone point to $\gamma_a^{+\infty}$, one of the points to $\gamma_a^{-\infty}$, and the other m points to

the endpoints of the $a^i P_0$, see Figure 3. We extend this to the cone on K by sending one interval to the boundary of H, and the other m intervals to the boundaries of the $a_i(P_0 \times \mathbb{R}^+)$. We let the group $\mathbb{Z}_m = \langle h \rangle$ act on Cone K in the usual way by fixing one interval and permuting the other m.

We observe that the $\langle a \rangle$-action on $\partial_\infty E G_{mn}$ cyclically permutes the boundaries of $\{a^i(P_0 \times \mathbb{R})\}_{i=0}^m$ and fixes setwise $\partial_\infty \tilde{S}_a$. Since hyperbolic translations are isotopic to the identity, there is a homotopy from $a^i \circ f(\text{Cone } K)$ to $f \circ h^i(K)$ which preserves setwise the images of simplices of K, and therefore each map in the homotopy is an almost embedding. Since Cone K is an equivariant 2-obstructor, by Lemma 5.8, we have that eqobdim$(G_{mn}) = 4$.
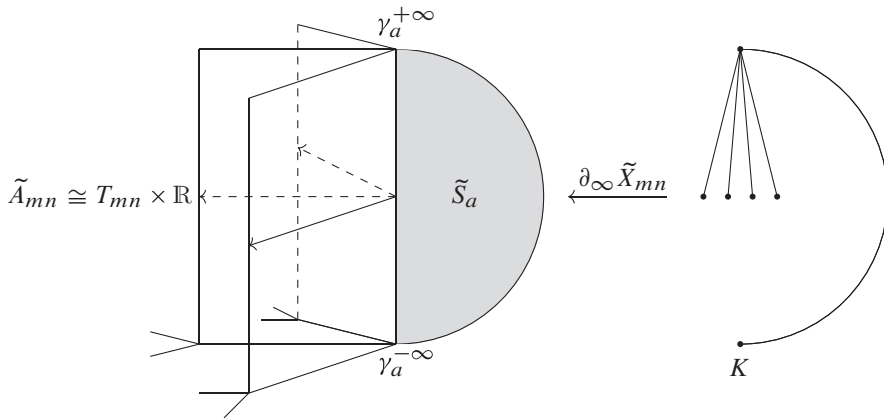

Figure 3. A piece of the universal cover of X_{mn}. The complex $T_{mn} \times \mathbb{R}$ is glued along $\tilde{\gamma}_a$ to the universal cover $\tilde{S}_a \cong \mathbb{H}^2$. For $m, n \geq 2$ we map K into $\partial_\infty X_{mn}$ by mapping one interval to the boundary of a half space in $\mathbb{H}^2$, and the other m intervals to the boundary of hyperplanes in $\tilde{A}_{mn}$.

Now, suppose that $n = 1$. The proof in this case is similar; but with a slightly different choice of hyperplanes. In this case, for each edge e_i adjacent to v in T_{mn}, $e_i \times \mathbb{R}$ intersects a lift of $\tilde{S}_b$ in $\tilde{X}_{mn}$ along a lift of γ_b. Label these lifts by $\tilde{S}_b^i$ and γ_b^i respectively. Choose a geodesic γ_i' in $\tilde{S}_b^i$ which is perpendicular to γ_b^i and which intersects $e_i \times \mathbb{R}$ in $\partial e_i \times 0$. Let Q_i denote one of the quadrants bounded by γ_b^i and γ_i' that is mapped into itself by a positive translation along γ_b^i.

Let W be the union of the Q_i along with H_a. The action of $\langle a \rangle$ fixes setwise the hyperplane H_a, and simultaneously cyclically permutes the Q_i and acts on them by a hyperbolic translation along γ_b^i. Define $f: K \to \partial_\infty W \subset \partial_\infty X$ which again sends the cone point to $\gamma_a^{+\infty}$ and sends each interval to the boundary of Q_i. Note that $a^i \partial_\infty Q_i$ is strictly contained in $h^i \partial_\infty Q_{i+1}$. Furthermore, the two embeddings of K into $\partial_\infty W$ are homotopic and the image of each simplex σ under the homotopy is contained in $f(\sigma)$. Therefore, by Lemma 5.8, eqobdim$(G_{m1}) = 4$.

The same argument obviously works for m a multiple of 4, as we can choose a $\mathbb{Z}/4$-subgroup of $\mathbb{Z}/m$. $\qquad\square$

Therefore, we have the following theorem.

Theorem 6.6. *If m or n is divisible by* 4*, then* $\mathrm{eqobdim}\left(\prod_k G_{mn}\right) = 3k + 1$*. If G is a group with* $\mathrm{obdim}_{\mathbb{Z}}(G) = l$*, then* $\mathrm{eqobdim}\left(\prod_k G_{mn}\right) = 3k + 1 + l$*.*

For example, the simplest computation that does not follow from [17] is that $\mathrm{actdim}(G_{mn} \times F_2) = 6$ if m or n is divisible by 4.

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