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Narrow equidistribution and counting of closed geodesics on noncompact manifolds

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Abstract. We prove the equidistribution of (weighted) periodic orbits for the geodesic flow on noncompact negatively curved manifolds toward equilibrium states in the narrow topology, i.e. in the dual of bounded continuous functions. We deduce an exact asymptotic counting for periodic orbits (weighted or not), which was previously known only for geometrically finite manifolds.

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1. Introduction

A well-known feature of compact hyperbolic dynamics is the abundance of periodic orbits: they have a positive exponential growth rate, equal to the topological entropy. Moreover, the periodic measures supported by these orbits become equidistributed towards the measure of maximal entropy of the system. A weighted version of this property also holds: given any Hölder potential F, the periodic measures weighted by the periods of the potential become equidistributed towards the (unique) equilibrium state of the potential, see the classical works of Bowen [2], Parry [13] and Parry and Pollicott [14].

A typical geometric example is the geodesic flow of a compact negatively curved manifold, which is an Anosov flow, and therefore satisfies the above properties. In this geometric context, it has been proved in [18, 15] that similar equidistribution properties also hold in a noncompact setting. Let M be a negatively curved manifold, T^1M be its unit tangent bundle, and (g^t) be the geodesic flow. As soon as it admits a finite invariant measure maximizing entropy, called the *Bowen–Margulis–Sullivan measure* m_{BMS}, the average of all orbital measures supported by periodic orbits of length at most T converge to the normalized measure $\bar{m}_{\mathrm{BMS}}$, in the *vague topology*, i.e. in the dual of continuous functions *with compact support*.

A weighted version of this result also holds: given a Hölder continuous map $F: T^1M \to \mathbb{R}$, as soon as it admits a finite equilibrium state m_F the orbital measures supported by periodic orbits of length at most T, suitably weighted by the periods of the potential F, converge to the normalized measure $\bar{m}_F$ in the vague topology.

A major motivation for proving such equidistribution results is to get asymptotic counting estimates for the number of (weighted) periodic orbits of length at most T. However, it turns out that the equidistribution property required to get such counting estimates is a stronger convergence, in the *narrow topology*, i.e. the dual of continuous *bounded* functions. Until now, such *narrow equidistribution* or such *asymptotic counting* for periodic orbits have been proven only when M is *geometrically finite* in [18, 15].

In this note, inspired by work done in [16] and [19], we remove this assumption and show this narrow equidistribution as soon as the Bowen–Margulis–Sullivan (or the equilibrium state) is finite.

Let us precise some notations, mostly coming from [15]. We denote by $\mathcal{P}'$ the set of primitive periodic orbits of the geodesic flow and $\mathcal{P}'(t)$ the subset of those primitive periodic orbits with length $\ell(p)$ at most t. Given a compact set $\mathcal{W} \subset T^1M$, we denote by $\mathcal{P}'_{\mathcal{W}}$ (resp. $\mathcal{P}'_{\mathcal{W}}(t)$) the set of primitive periodic orbits (resp. of length at most t) which intersect $\mathcal{W}$. If p is an oriented periodic orbit, let $\mathcal{L}_p$ be the periodic measure of mass $\ell(p)$ supported on p, so that $\frac{1}{\ell(p)}\mathcal{L}_p$ is a probability measure.

If $F: T^1M \to \mathbb{R}$ is a Hölder continuous potential, we denote by δ_F its critical exponent, and m_F the Gibbs measure associated to F, given by the Patterson–Sullivan–Gibbs construction (see [15]). Under some additional geometric assumptions (pinched negative curvature, bounded derivatives of the curvature), one knows that δ_F is also the topological pressure of F (see [12] when $F = 0$ and [15] for general F), and that when m_F is finite, the normalized probability measure $\bar{m}_F = m_F/\|m_F\|$ is the unique equilibrium state for F. We will not need this characterization here.

Our main result is the following.

Theorem 1.1. *Let M be a manifold with negative curvature satisfying $\kappa \le -a^2 < 0$, whose geodesic flow is topologically mixing. Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous map, with finite critical exponent δ_F, which admits a finite Gibbs measure m_F. Assume without loss of generality that its topological pressure δ_F is positive. Let $W \subset M$ be a compact set and $\mathcal{W} = T^1W$. Assume that the interior of $\mathcal{W}$ intersects at least a periodic orbit of (g^t). Then*

$$\delta_F T e^{-\delta_F T} \sum_{p \in \mathcal{P}'_{\mathcal{W}}(T)} e^{\int_p F} \frac{1}{\ell(p)} \mathcal{L}_p \longrightarrow \frac{m_F}{\|m_F\|} \quad \textit{when } T \to +\infty \tag{1}$$

in the narrow topology, i.e. in the dual of continuous bounded functions.

Integrating the constant map equal to 1 gives the following corollary. ([1])

Corollary 1.2. *Under the same assumptions, we have*

$$\sum_{p \in \mathcal{P}'_{\mathcal{W}}(T)} e^{\int_p F} \sim \frac{e^{\delta_F T}}{\delta_F T} \quad \textit{when } T \to +\infty. \tag{2}$$

When $F = 0$, the exponent δ_0 is exactly the critical exponent δ_Γ of the group Γ acting on $\widetilde{M}$, and the Gibbs measure m_0 is known as the Bowen–Margulis–Sullivan measure. As already mentioned above, when the curvature is pinched negative and the derivatives of the curvature are bounded, it is also the topological entropy of the geodesic flow, see [12]. Since our main result with $F = 0$ is valid in the more general geometric setting of CAT(-1)-metric spaces, we restate it in this context.

Theorem 1.3. *Let X be a* CAT(-1)*-metric space, and Γ a discrete group of isometries acting properly discontinuously on X. Assume that the geodesic flow of X/Γ is topologically mixing and admits a finite Bowen–Margulis–Sullivan measure. Let $W \subset X/\Gamma$ be a compact set and $\mathcal{W} = T^1W$. Assume that its interior intersects at least a periodic orbit of (g^t). Then*

$$\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{p \in \mathcal{P}'_{\mathcal{W}}(T)} \frac{1}{\ell(p)} \mathcal{L}_p \longrightarrow \frac{m_{\mathrm{BMS}}}{\|m_{\mathrm{BMS}}\|} \quad \textit{when } T \to +\infty \tag{3}$$

in the narrow topology.

As a corollary, integrating the constant map equal to 1, we get the following striking consequence of our work.

Corollary 1.4. *Let M be a manifold with pinched negative curvature or a quotient of a* CAT(-1)*-space, whose geodesic flow is topologically mixing. Assume that there exists a (finite) measure of maximal entropy, or equivalently that the Bowen–Margulis–Sullivan measure is finite. Let $W \subset M$ be a compact set and $\mathcal{W} = T^1W$. Assume that its interior intersects at least a periodic orbit of (g^t). Then*

$$\# \mathcal{P}'_{\mathcal{W}}(T) \sim \frac{e^{\delta_\Gamma T}}{\delta_\Gamma T} \quad \textit{when } T \to +\infty. \tag{4}$$

[1] Recall that $u(T) \sim v(T)$ when $T \to \infty$ means $u(T)/v(T) \to 1$.

The results above are completely new in the geometrically infinite setting. We refer to [19] (resp. [9]) for several classes of geometrically infinite manifolds (resp. potentials) satisfying the so-called *SPR property*, which implies the finiteness of the Bowen–Margulis–Sullivan measure (resp. of the associated Gibbs measure).

The asymptotic counting given in our last corollary is due to Margulis [10] on compact negatively curved manifolds. The exact multiplicative constant in the asymptotic is due to Toll [21] in his Ph.D. On geometrically finite spaces, it is due to [18] when $F = 0$ in the CAT(-1)-setting. Let us mention also the very recent work of Ricks [17] for compact quotients of rank one CAT(0) spaces. For general potentials on geometrically finite manifolds with pinched negative curvature, it had been shown in [15].

Theorem 1.1 and Corollary 1.2 are also announced in [22] under the (more restrictive) assumption that F is a Hölder potential satisfying the SPR property and converging to 0 at infinity. It seems to us that his (interesting) approach cannot work in general.

The restriction to $\mathcal{P}'_W$ instead of $\mathcal{P}'$ is intrinsic to the noncompact geometrically infinite case. Indeed, except in the geometrically finite case, where both sets typically coincide for W large enough (containing the compact part of the manifold), $\mathcal{P}'_W(T)$ is a finite set, whereas $\mathcal{P}'(T)$ could easily often be infinite.

The assumption of finiteness of the measure in Theorems 1.1 and 1.3 is unavoidable. Indeed, it is proven in [18, 15] that the sum over all periodic orbits of $\mathcal{P}'(t)$ vaguely converges to 0, and the sums considered in both Theorems 1.1 and 1.3 are obviously smaller, and therefore converge also vaguely to 0. However, in some situations, it can happen that counting results similar to Corollary 1.4 with different asymptotics hold, as in [23]. This infinite measure situation is still widely unexplored.

In Section 2, we provide the geometric background. In Section 3, we recall important facts on thermodynamic formalism in the Riemannian setting, and the proof of Theorem 1.1 is detailed in Section 4. In Section 5, we explain how the whole discussion extends to the CAT(-1)-setting, in the specific case $F \equiv 0$, and prove Theorem 1.3.

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2. Geodesic flows of spaces with negative curvature

2.1. Riemannian manifolds with negative curvature. Let M be a non compact Riemannian manifold, with negative sectional curvature satisfying $\kappa \leq -a^2 < 0$ everywhere. In the sequel, we assume M be nonelementary, i.e. there are at least two distinct closed geodesics on M (and therefore an infinity).

Let $\Gamma = \pi_1(M)$ be its fundamental group, $\widetilde{M}$ be its universal cover, and $\partial\widetilde{M}$ its boundary at infinity. Denote by $\pi\colon T^1\widetilde{M} \to \widetilde{M}$ the canonical projection, and $p_\Gamma\colon \widetilde{M} \to M$ the quotient map.

The Busemann cocycle is defined on $\partial\widetilde{M} \times \widetilde{M} \times \widetilde{M}$ by

$$\beta_\xi(x, y) = \lim_{z\to\xi} d(x, z) - d(y, z).$$

The unit tangent bundle $T^1\widetilde{M}$ of $\widetilde{M}$ is homeomorphic to $\partial^2\widetilde{M} \times \mathbb{R}$, where

$$\partial^2\widetilde{M} := \partial\widetilde{M} \times \partial\widetilde{M} \setminus \text{Diagonal}$$

through the well-known *Hopf coordinates*:

$$v \longmapsto (v^-, v^+, \beta_{v^+}(o, \pi(v))),$$

where o is an arbitrary fixed point chosen once for all.

The geodesic flow on $T^1\widetilde{M}$ or T^1M is denoted by $(g^t)_{t\in\mathbb{R}}$. In the above coordinates, it acts by translation on the real factor.

The action of Γ can be expressed in these coordinates as follows:

$$\gamma(v^-, v^+, t) = (\gamma v^-, \gamma v^+, t + \beta_{v^+}(\gamma^{-1}o, o)).$$

Therefore, any invariant measure m under the geodesic flow on T^1M can be lifted into an invariant measure $\tilde{m}$ which, in these coordinates, can be written $\tilde{m} = \mu \times dt$ where μ is a Γ-invariant measure on $\partial^2\widetilde{M}$ and dt is the Lebesgue measure on $\mathbb{R}$.

As said in the introduction, we denote by

- $\mathcal{P}$ (resp. $\mathcal{P}'$) the set of periodic orbits (resp. primitive periodic orbits) of (g^t) on T^1M;
- $\mathcal{P}(t)$ (resp. $\mathcal{P}'(t)$) the set of (primitive) periodic orbits of length at most t;
- $\mathcal{P}(t_1, t_2)$ (resp. $\mathcal{P}'(t_1, t_2)$) the set of (primitive) periodic orbits of length in $(t_1, t_2]$,

When M is noncompact, all these sets can be infinite. We therefore consider only periodic orbits intersecting a given compact set $\mathcal{W}$. We denote by $\mathcal{P}_\mathcal{W}, \mathcal{P}'_\mathcal{W}, \mathcal{P}_\mathcal{W}(t), \mathcal{P}'_\mathcal{W}(t), \mathcal{P}_\mathcal{W}(t_1, t_2), \mathcal{P}'_\mathcal{W}(t_1, t_2)$ the corresponding sets.

If p is a periodic orbit of the geodesic flow on T^1M, we denote by $\ell(p)$ its period, and $\mathcal{L}_p$ the Lebesgue measure along p.

2.2. CAT(−1)-spaces. We refer to Roblin's monograph [18] for more details on what follows.

A metric space (X, d) is Gromov-hyperbolic if there exists a constant $\delta > 0$ such that every geodesic triangle of X is δ-thin: any side of the triangle is in the δ-neighbourhood of the union of the other sides.

A metric space (X, d) is a CAT(-1)-space if it is *geodesic* (there exists a geodesic segment joining two points of X), and it satisfies the following comparison inequality with the hyperbolic plane. Given any geodesic triangle (a, b, c) in X and the comparison triangle (a', b', c') with sides of same length in $\mathbb{H}^2$, if a_t (resp. b_t) is the point of (a, c) (resp. (b, c) at distance t from c, then $d(a_t, b_t) \le d(a'_t, b'_t)$. In particular, (X, d) is Gromov-hyperbolic and there is a unique geodesic between any two points of X.

The role of the unit tangent bundle will be played by the space $\mathcal{G}X$ of biinfinite geodesics (with origin) of X, that is the set of isometric embeddings $c\colon \mathbb{R} \to X$. The canonical projection $T^1M \to M$ in the Riemannian case is replaced here by the projection $c \to c(0)$ from $\mathcal{G}X$ to X. If W is some subset of X, by abuse of notation, we denote by $\mathcal{W} = T^1W$ the set of geodesics $c \in \mathcal{G}X$, such that $c(0) \in W$.

As in the Riemannian setting, the geometric boundary ∂X is the set of equivalence classes of asymptotic geodesic rays, and the Busemann cocycle is defined as in the above section. The space $\mathcal{G}X$ is homeomorphic to $\partial^2 X = \partial X \times \partial X \setminus \text{Diagonal} \times \mathbb{R}$ through the Hopf coordinates exactly as above. The *geodesic flow* acts on $\mathcal{G}X$ by translation on the source. In the Hopf coordinates, it acts by translation on the real factor.

If Γ is a discrete group of isometries of (X, d) acting properly discontinuously on X, its action can be read in the Hopf coordinates by the same formula as in the previous subsection. It commutes with the geodesic flow, which induces on the quotient the *geodesic flow on* $\mathcal{G}X/\Gamma$.

The sets of periodic orbits $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{P}(t)$, $\mathcal{P}'(t)$, $\mathcal{P}(t_1, t_2)$, $\mathcal{P}'(t_1, t_2)$, $\mathcal{P}_{\mathcal{W}}$, $\mathcal{P}'_{\mathcal{W}}$, $\mathcal{P}_{\mathcal{W}}(t)$, $\mathcal{P}'_{\mathcal{W}}(t)$, and $\mathcal{P}_{\mathcal{W}}(t_1, t_2)$, $\mathcal{P}'_{\mathcal{W}}(t_1, t_2)$ are defined exactly as above.

The following sections are devoted to the Riemannian setting, and the proof of Theorem 1.3 is postponed to Section 5.

3. Thermodynamic formalism in the Riemannian setting

In this section, M is a Riemannian manifold, with negative sectional curvature satisfying $\kappa \le -a^2 < 0$ everywhere.

3.1. Pressure, Gibbs measures. We recall briefly here the necessary background on Gibbs measures in this *Riemannian setting*. We refer to [15] and [16] for more details.

Let $F\colon T^1M \to \mathbb{R}$ be a Hölder continuous map, and $\widetilde{F}$ its Γ-invariant lift to $T^1\widetilde{M}$. The following property, called *Bowen property* in many references as [5] and [4], and *(HC)-type property* in [3] is crucial in all estimates. It is a direct consequence of [15, Lemma 3.2].

Lemma 3.1. *Let $F : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder map. For all $D > 0$ and $x, y \in \widetilde{M}$, there exists $C > 0$ depending on D, on the upperbound of the curvature, on the Hölder constants of F and on $\sup_{B(x,D)} |F|$ and $\sup_{B(y,D)} |F|$ such that for all x', y' in $\widetilde{M}$ with $d(x, x') \leq D$, $d(y, y') \leq D$,*

$$\left| \int_x^y F - \int_{x'}^{y'} F \right| \leq C.$$

The series $\sum_{\gamma\in\Gamma} e^{\int_o^{\gamma o} F - s d(o,\gamma o)}$ has a critical exponent δ_F. This exponent, when finite, coincides with the pressure of F when the manifold M has pinched negative curvature and bounded derivatives of the curvature, see [12, 15].

By the obvious relation $\delta_{F+c} = \delta_F + c$ for any constant $c \in \mathbb{R}$, we can easily assume that $\delta_F > 0$ as soon as it is finite.

A shadow $\mathcal{O}_x(B(y, R))$, for $x, y \in \widetilde{M}$, is the set of points $z \in \widetilde{M} \cup \partial\widetilde{M}$, such that the geodesic line from x to z intersects the ball $B(y, R)$.

The Patterson–Sullivan–Gibbs construction gives a measure ν_o^F on the boundary $\partial\widetilde{M}$, satisfying the following *Sullivan Shadow Lemma*. It was first shown on hyperbolic manifolds for $F = 0$ by Sullivan in [20], and is due to Mohsen [11] for general potential when Γ is cocompact. See [15, Lemma 3.10] for a proof in general.

Lemma 3.2 (Shadow Lemma). *Let $F : T^1M \to \mathbb{R}$ be a Hölder-continuous map with finite critical exponent δ_F, $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ its lift, and ν_o^F be the measure on $\partial\widetilde{M}$ given by the Patterson–Sullivan–Gibbs construction. There exists $R > 0$, such that for all $r \geq R$, there exists $C > 0$ such that for all $\gamma \in \Gamma$,*

$$\frac{1}{C} e^{\int_o^{\gamma o}(F-\delta_F)} \leq \nu_o^F(\mathcal{O}_o(B(\gamma o, r)) \leq C e^{\int_o^{\gamma o}(F-\delta_F)}.$$

A nice consequence of the Shadow Lemma is the following proposition, that we will use in the proof of Theorem 1.1 and which can be useful for other purposes.

Proposition 3.3. *With the above notations, for all $r \geq R$ and all $c > 0$, there exists a constant $k > 0$ such that for all $\alpha \in \Gamma$ and all $T > 0$, one has*

$$\sum_{\substack{\gamma\in\Gamma, \gamma o\in\mathcal{O}_o(B(\alpha o,r)) \\ d(o,\gamma o)\in[T,T+c]}} e^{\int_o^{\gamma o}(F-\delta_F)} \leq k e^{\int_o^{\alpha o}(F-\delta_F)}.$$

The reverse inequality (with different constant) holds when Γ acts cocompactly on $\widetilde{M}$.

Proof. By the Shadow Lemma (Lemma 3.2), the above sum is comparable, up to constants, to

$$\sum_{\substack{\gamma\in\Gamma,\gamma o\in\mathcal{O}_o(B(\alpha o,r))\\ d(o,\gamma o)\in[T,T+c]}} \nu_o^F(\mathcal{O}_o(B(\gamma o,r))).$$

As Γ acts properly on $\widetilde{M}$, the multiplicity of an intersection of such shadows is uniformly bounded. Therefore, the latter sum is comparable, up to constants, to

$$\nu_o^F\Big(\bigcup_{\substack{\gamma\in\Gamma,\gamma o\in\mathcal{O}_o(B(\alpha o,r))\\ d(o,\gamma o)\in[T,T+c]}} \mathcal{O}_o(B(\gamma o,r))\Big).$$

As this union is included in $\mathcal{O}_o(B(\alpha o,2r))$, we deduce that it is bounded from above by $\nu_o^F(\mathcal{O}_o(B(\alpha o,r)))$. A final application of the Shadow Lemma 3.2 gives the desired upper bound.

When Γ acts cocompactly on $\widetilde{M}$, the above union covers $\mathcal{O}_o(B(\alpha o,r))$ so that, once again, the Shadow Lemma gives the desired lower bound. □

3.2. Finiteness criterion for Gibbs measures. Through the Hopf coordinates, one defines a measure $\tilde{m}_F$ equivalent to $\nu_o^{\check{F}}\times\nu_o^F\times dt$ on $\partial^2\widetilde{M}\times\mathbb{R}\simeq T^1\widetilde{M}$, where $\check{F}(v):=F(-v)$, which is Γ-invariant and invariant under the geodesic flow; see [15, Chapter 3] for a precise construction. The induced measure m_F on the quotient, when finite, is the *Gibbs measure associated to* F involved in Theorem 1.1.

It is well known (Hopf–Tsuji–Sullivan–Gibbs Theorem) that m_F is ergodic and conservative if and only if the series $\sum_{\gamma\in\Gamma} e^{\int_o^{\gamma o} F-sd(o,\gamma o)}$ diverges at the critical exponent δ_F, see [15, Theorem 5.4].

Let us recall the finiteness criterion shown in [16]. For geometrically finite manifolds, it had been previously shown in [8] for $F=0$ and in [6, 15] for general potentials.

If W is a compact subset of $\widetilde{M}$, we define Γ_W as

$$\Gamma_W=\{\gamma\in\Gamma,\exists x,y\in W,[x,\gamma y]\cap\Gamma W\subset W\cup\gamma W\} \tag{5}$$

Theorem 3.4 ([16] and [7]). *Let M be a negatively curved manifold with sectional curvature satisfying $\kappa\le -a^2<0$. Let $F\colon T^1M\to\mathbb{R}$ be a Hölder continuous map with finite critical exponent δ_F. The measure m_F is finite if and only if it is ergodic and conservative, and there exists some compact subset $W\subset\widetilde{M}$ whose interior intersects at least a closed geodesic, such that*

$$\sum_{\gamma\in\Gamma_W} d(o,\gamma o)\, e^{\int_o^{\gamma o}(F-\delta_F)}<+\infty.$$

Note that in [16] it is assumed that M has pinched negative curvature $-b^2 \leq \kappa \leq -a^2 < 0$, but the lower bound is not used in the proof of this finiteness criterion.

3.3. Equidistribution with respect to the vague convergence. There are many variants of equidistribution of weighted closed orbits with respect to the vague convergence, which are essentially all equivalent. See [15, Chapter 9] for several versions.

The statement which is the closest to our Theorem 1.1 is the following.

Theorem 3.5 (Paulin, Policott, and Schapira [15, Theorem 9.11]). *Let M be a manifold with pinched negative curvature, whose geodesic flow is topologically mixing. Let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map with finite pressure δ_F, which admits a finite equilibrium state m_F. Assume without loss of generality that δ_F is positive.*

Then

$$\delta_F T e^{-\delta_F T} \sum_{p \in \mathcal{P}'(T)} e^{\int_p F} \frac{1}{\ell(p)} \mathcal{L}_p \longrightarrow \frac{m_F}{\|m_F\|}, \tag{6}$$

in the vague topology, i.e. the dual of continuous maps with compact support on T^1M.

To get narrow equidistribution of periodic orbits, we will rather use the following statement.

Theorem 3.6 (Paulin, Policott, and Schapira [15, Theorem 9.14]). *Let M be a manifold with pinched negative curvature, whose geodesic flow is topologically mixing. Let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map with finite nonzero pressure δ_F. Let $c > 0$ be fixed. Assume that F admits a finite equilibrium state m_F. Then*

$$\frac{\delta_F}{1 - e^{-\delta_F c}} T e^{-\delta_F T} \sum_{p \in \mathcal{P}'(T-c,T)} e^{\int_p F} \frac{1}{\ell(p)} \mathcal{L}_p \longrightarrow \frac{m_F}{\|m_F\|}, \tag{7}$$

and

$$\frac{\delta_F}{1 - e^{-\delta_F c}} e^{-\delta_F T} \sum_{p \in \mathcal{P}'(T-c,T)} e^{\int_p F} \mathcal{L}_p \longrightarrow \frac{m_F}{\|m_F\|}, \tag{8}$$

in the vague topology, i.e. the dual of continuous maps with compact support on T^1M.

4. Equidistribution in the narrow topology

In this section, M is a Riemannian manifold, with negative sectional curvature satisfying $\kappa \le -a^2 < 0$ everywhere.

4.1. An equidistribution statement on annuli. Choose any compact subset $W \subset M$ whose interior intersects a closed geodesic, and $\mathcal{W} = T^1W$. Denote by m_T the (locally finite and possibly infinite) measure

$$m_T = \frac{\delta_F T}{1 - e^{-c\delta_F}} e^{-\delta_F T} \sum_{p \in \mathcal{P}'(T-c,T)} e^{\int_p F} \frac{1}{\ell(p)} \mathcal{L}_p$$

and by $m_{T,W}$ the (finite) measure

$$m_{T,W} = \frac{\delta_F T}{1 - e^{-c\delta_F}} \sum_{p \in \mathcal{P}'_{\mathcal{W}}(T-c,T)} e^{\int_p F} \frac{1}{\ell(p)} \mathcal{L}_p$$

We will first prove the following theorem, and then deduce Theorem 1.1 from it.

Theorem 4.1. *Under the assumptions of Theorem 1.1, the measures $m_{T,W}$ converge to $\bar{m}_F = \frac{m_F}{\|m_F\|}$ with respect to the narrow convergence, i.e. in the dual of bounded continuous maps on T^1M.*

4.2. Vague convergence. In this section, we prove the following.

Proposition 4.2. *Under the assumptions of Theorem 1.1, $m_T - m_{T,W}$ converges to 0 in the vague topology, when $T \to +\infty$.*

By Theorem 3.6, we deduce the immediate corollary.

Corollary 4.3. *Under the assumptions of Theorem 1.1, the measure $m_{T,W}$ converges to $\bar{m}_F = \frac{m_F}{\|m_F\|}$ in the vague topology.*

Proof. Let $\varphi \in C_c(T^1M)$ be a continuous compactly supported map. Without loss of generality, we can assume that $\|\varphi\|_\infty \le 1$ Choose $R > 0$ such that the R-neighbourhood W_R of W in M contains the projection on M of the support of φ in T^1M. Choose some $\varepsilon > 0$ small enough so that the set $W_{-\varepsilon}$ of points of W at distance at least ε to the boundary is nonempty and intersects at least a closed geodesic.

Let $\widetilde{W}_R \supset \widetilde{W} \supset \widetilde{W}_{-\varepsilon}$ be three compact subsets of $\widetilde{M}$ which project respectively onto W_R, W and $W_{-\varepsilon}$. Choose a point $o \in \widetilde{W}_{-\varepsilon}$ once for all.

We begin with the following elementary inequality. For $p \in \mathcal{P}'(T-c,T)$, we have $|\delta_F T - \delta_F \ell(p)| \le c\delta_F$, so that

$$\left|m_{T,W_R}(\varphi) - m_{T,W}(\varphi)\right| \le \frac{\delta_F}{1-e^{-c\delta_F}} T e^{\delta_F c} \sum_{p\in\mathcal{P}'_{W_R}(T-c,T)\setminus\mathcal{P}'_W(T-c,T)} e^{\int_p (F-\delta_F)} \frac{\ell(p\cap W_R)}{\ell(p)}. \quad (9)$$

Now, we will compare the latter sum with the sum appearing in Theorem 3.4.

Given $p \in \mathcal{P}'_{W_R}(T-c,T) \setminus \mathcal{P}'_W(T-c,T)$, choose arbitrarily one isometry $\gamma_p \in \Gamma$, whose translation axis intersects $\widetilde{W}_R \setminus \widetilde{W}$ and projects on M on the closed geodesic associated to the periodic orbit p, and whose translation length is $\ell(p)$.

Consider the geodesic from o to $\gamma_p o$, parametrized as $(c(t))_{0\le t\le d(o,\gamma_p o)}$. It stays at bounded distance $\mathrm{diam}(\widetilde{W}_R)$ from the axis of γ_p. Therefore, these geodesics stay very close one from another, except at the beginning and at the end. More precisely, given any $\varepsilon > 0$, there exists τ depending only on ε, R, and the upper bound of the curvature, such that for t in the interval $[\tau, d(o,\gamma_p o)-\tau]$, $c(t)$ is ε-close to the axis of γ_p. In particular, as this axis does not intersect $\Gamma.\widetilde{W}$, the geodesic $(c(t))_{\tau\le t\le d(o,\gamma_p o)-\tau}$ does not intersect $\widetilde{W}_{-\varepsilon}$, whereas the full segment $(c(t))_{0\le t\le d(o,\gamma_p o)}$ starts and ends in $\Gamma.o \subset \Gamma \widetilde{W}_{-\varepsilon}$.

Denote by x^- the last point of $c([0,\tau])$ (resp. x^+ the first point of $c([d(o,\gamma_p o)-\tau, d(o,\gamma_p o)]))$) in $\Gamma\widetilde{W}_{-\varepsilon}$ and γ^- (resp. γ^+) an element of Γ such that $x^\pm \in \gamma^\pm \widetilde{W}_{-\varepsilon}$. Observe that $d(o,\gamma^- o) \le \tau + \mathrm{diam}(\widetilde{W}_R)$, and similarly $d(\gamma_p o, \gamma^+ o) \le \tau + \mathrm{diam}(\widetilde{W}_R)$. Moreover, by definition of $\Gamma_{\widetilde{W}_{-\varepsilon}}$, the element $g_p := (\gamma^-)^{-1} \circ \gamma^+$ belongs to $\Gamma_{\widetilde{W}_{-\varepsilon}}$.

Using Lemma 3.1, we see easily that there exists some constant C depending on the upperbound of the curvature, on the Hölder constant of F and $\|F_{|W_R}\|_\infty$ and on the diameter of $\widetilde{W}_R$, such that

$$\left|\ell(p) - d(o, g_p o)\right| \le C \quad \text{and} \quad \left|\int_p F - \int_o^{g_p o} F\right| \le C. \quad (10)$$

We have hence defined a procedure which, given any $p \in \mathcal{P}'_{W_R}(T-c,T) \setminus \mathcal{P}'_W(T-c,T)$ and any choice of an isometry γ_p in the conjugacy class corresponding to p whose axis intersects $\widetilde{W}_R$, determines a unique pair (γ_-, γ_+) where $d(o,\gamma^- o) \le \mathrm{diam}(\widetilde{W}_R) + \tau$ and $g_p = (\gamma^-)^{-1} \circ \gamma^+$ is an element of $\Gamma_{\widetilde{W}_{-\varepsilon}}$, satisfying (10) and $d(\gamma^- g_p o, \gamma_p o) \le \mathrm{diam}(\widetilde{W}_R) + \tau$. Moreover, a coarse bound gives $\ell(p \cap \widetilde{W}_R) \le \ell(p) \le d(o, g_p o) + C$.

We want to control from above (9) by a sum involving $\Gamma_{\widetilde{W}_{-\varepsilon}}$. To do that, it is enough to control the multiplicity of the "map" $p \to g_p$.

Let p_1 be a periodic orbit leading to an element $g_p \in \Gamma_{\widetilde{W}_{-\varepsilon}}$ by the above construction, by some arbitrary choice of an axis of an isometry γ_1 intersecting $\widetilde{W}_R$.

If another periodic orbit p_2 leads to the same element g_p, it means that there exists an isometry γ_2 and elements $\gamma_i^- \in \Gamma$, with $d(o, \gamma_i^- o) \leq \mathrm{diam}(\widetilde{W}_R) + \tau$, such that

$$d(o, (\gamma_2)^{-1}\gamma_2^-(\gamma_1^-)^{-1}\gamma_1 o) \leq 2\mathrm{diam}(\widetilde{W}_R) + 2\tau.$$

In particular, as Γ is discrete, there are finitely many possibilities, for γ_i^- and therefore for γ_2, and p_2. Denote by N the maximal multiplicity of this map $p \to g_p$.

Now, using (10), we bound from above the right hand side of (9) by

$$\frac{\delta_F}{1 - e^{-c\delta_F}} T e^{c\delta_F} N \sum_{\substack{g \in \Gamma_{\widetilde{W}-\varepsilon} \\ d(o,go) \in [T-c-C, T+C]}} e^{C+\delta_F} e^{\int_o^{go}(F-\delta_F)} \frac{d(o, go)}{T-c}.$$

This sum, up to constants, is bounded from above by

$$\sum_{\substack{g \in \Gamma_{\widetilde{W}-\varepsilon} \\ d(o,go) \geq T-c-C}} d(o, go)\, e^{\int_o^{go}(F-\delta_F)}.$$

Theorem 3.4 ensures us that this is the rest of a convergent series, whence it goes to zero as $T \to +\infty$. □

4.3. Tightness. Let W be a compact subset of M, W_R its R-neighbourhood, for R large enough, and $\widetilde{W} \subset \widetilde{W}_R \subset \widetilde{M}$ compact sets which project onto $W \subset W_R$. Choose some fixed point $o \in \widetilde{W}$. As above, we denote by $\mathcal{W}$ the unit tangent bundle of W and by abuse of notation, set $\mathcal{W}_R = T^1 W_R$.

Proposition 4.4. *Under the assumptions of Theorem 1.1, for all $\varepsilon > 0$, there exists $R > 0$ and $T > 0$, such that for $t \geq T$,*

$$m_{t,W}((\mathcal{W}_R)^c) \leq \varepsilon.$$

Proof. By definition of $m_{t,W}$, we have

$$m_{t,W}((\mathcal{W}_R)^c) \leq \frac{\delta_F}{1 - e^{-c\delta_F}} \frac{t}{t-c} \sum_{p \in \mathcal{P}'_W(t-c,t)} e^{\int_p (F-\delta_F)} \ell(p \cap W_R^c). \tag{11}$$

If p is a periodic orbit appearing in the above sum with $\ell(p \cap W_R^c) \neq 0$, there exists an hyperbolic isometry $\gamma_p \in \Gamma$ whose axis projects onto the closed geodesic associated to p, which intersects $\widetilde{W}$ and $\gamma_p \widetilde{W}$, but also $\Gamma.(\widetilde{W}_R)^c$. Denote by $\Gamma_{\widetilde{W}}(p, W_R)$ the set of elements $\alpha \in \Gamma_{\widetilde{W}}$ such that some axis of some isometry γ_p associated to p as above intersects $\widetilde{W}$, and $\alpha\widetilde{W}$ and goes outside $\Gamma\widetilde{W}_R$ between

$\widetilde{W}$ and $\alpha\widetilde{W}$. In particular, for each $\alpha \in \Gamma_{\widetilde{W}}(p, W_R)$, we have $d(o, \alpha o) \geq 2R$, so that

$$\ell(p \cap (W_R)^c) \leq \sum_{\substack{\alpha\in\Gamma_{\widetilde{W}}(p,W_R)\\ d(o,\alpha o)\geq 2R}} (d(o, \alpha o) + 2\mathrm{diam}(\widetilde{W})).$$

We deduce that (11) is bounded from above, up to some constants, by

$$\sum_{p\in\mathcal{P}'_W(t-c,t)} e^{\int_p (F-\delta_F)} \sum_{\substack{\alpha\in\Gamma_{\widetilde{W}}(p,W_R)\\ d(o,\alpha o)\geq 2R}} (d(o, \alpha o) + 2\mathrm{diam}(\widetilde{W}))$$

Observe now that if $\alpha \in \Gamma_{\widetilde{W}}(p, W_R)$ and γ_p is an isometry whose axis intersects $\widetilde{W}$, and $\alpha\widetilde{W}$, then $\gamma_p o$ belongs to the shadow $\mathcal{O}_o(B(\alpha o, r))$ for $r = \mathrm{diam}(\widetilde{W})$. As in (10) in the proof of Proposition 4.2, we know that $\int_p (F - \delta_F)$ is uniformly close to $\int_o^{\gamma_p o}(F - \delta_F)$, which, by the Shadow Lemma 3.2, is comparable to $\log \nu_o^F(\mathcal{O}_o(B(\gamma o, r))$ for r large enough. Up to some constants, the above sum is bounded from above by

$$\sum_{\substack{\alpha\in\Gamma_W,\\ d(o,\alpha o)\geq 2R}} d(o, \alpha o) \sum_{\substack{\gamma\in\Gamma\\ t-c-C\leq d(o,\gamma o)\leq t+C}} \mathbf{1}_{\mathcal{O}_o(B(\alpha o,r)}(\gamma o)\nu_o^F(\mathcal{O}_o(B(\gamma o, r)).$$

By Proposition 3.3, (11) is then dominated (up to multiplicative constants) by

$$\sum_{\substack{\alpha\in\Gamma_W\\ d(o,\alpha o)\geq 2R}} d(o, \alpha o) e^{\int_o^{\alpha o}(F-\delta_F)}.$$

This is the rest of the convergent series appearing in Theorem 3.4. Therefore, it goes to 0 when $R \to +\infty$, so that for R large enough, it is smaller than ε. It is the desired result. □

4.4. Conclusion

4.4.1. Proof of Theorem 4.1. Corollary 4.3 ensures that $m_{t,W}$ converges vaguely to $\overline{m_F}$. Getting narrow convergence from vague convergence and tightness is very classical, see for example [1]. We recall it for the comfort of the reader.

Given a continuous bounded function φ and $\varepsilon > 0$, by Proposition 4.4, one can find a compact set $\mathcal{W}_R$ such that $m_{t,W}((\mathcal{W}_R)^c) \leq \varepsilon$ for all $t \geq T$, $\overline{m_F}(\mathcal{W}_R) \geq 1-\varepsilon$, and $\left|\int_{\mathcal{W}_R^c} \varphi \, d\overline{m_F}\right| \leq \varepsilon$. Choose $\psi \in C_c(T^1M)$ with $\psi = \varphi \in W_R$, and $|\psi| \leq \|\varphi\|$, and $\left|\int(\varphi - \psi)\, d\overline{m_F}\right| \leq 2\varepsilon$.

By the above choices,

$$m_{t,W}(\varphi) - m_{t,W}(\psi)| \leq \|\varphi\|_\infty m_{t,}(W_R^c) \leq \varepsilon\|\varphi\|_\infty.$$

By Proposition 4.2, $m_{t,W}(\psi) \to \int \psi \, d\overline{m_F}$, which is 2ε-close to $\int \varphi \, d\overline{m_F}$. The result follows.

4.4.2. Proof of Theorem 1.1. The proof is elementary and similar to the deduction of [15, Theorem 9.14] (see Theorem 3.6) from [15, Theorem 9.11] (see Theorem 3.5), but in the other direction. Let us begin with an elementary lemma, which is a reformulation of [15, Lemma 9.5].

Lemma 4.5. *Let I be a discrete set and $f, g\colon I \to [0, +\infty[$ be maps with f proper. For all $c > 0$ and $\delta, \kappa \in \mathbb{R}$, with $\delta + \kappa > 0$, the following are equivalent*:

(1) *as* $t \to +\infty$, $\displaystyle\sum_{\substack{i\in I\\ t-c<f(i)\le t}} g(i) \sim \frac{1-e^{-c\delta}}{\delta} e^{\delta t}$;

(2) *as* $t \to +\infty$, $\displaystyle\sum_{\substack{i\in I\\ f(i)\le t}} e^{\kappa f(i)} g(i) \sim \frac{e^{(\delta+\kappa)t}}{\delta+\kappa}$.

Proof. Note that even though only one implication of the above lemma is stated in [15, Lemma 9.5], its proof gives indeed the equivalence. We refer the reader to [15, p. 182] for details. □

Now, let us conclude the proof of Theorem 1.1. By linearity, it is enough to prove it for nonnegative maps φ that satisfy $\int \varphi d\overline{m_F} \neq 0$. The desired result follows then from Theorem 4.1 and applying the above lemma with $I = \mathcal{P}'_{\mathcal{W}}$, and for $p \in I = \mathcal{P}'_{\mathcal{W}}$, $f(p) = \ell(p)$ and $g(p) = e^{\int_p F} \dfrac{\int_p \varphi}{\int \varphi d\overline{m_F}}$.

5. Narrow equidistribution on CAT(−1) metric spaces

5.1. Proof of Theorem 1.3. The proof of Theorem 1.3 is exactly the same as the above proof in the Riemannian case. Nevertheless, in this generality, the basic ingredients that we use (vague equidistribution for periodic orbits and finiteness criterion for Gibbs measure) are only known for the potential $F = 0$. We just mention in this section which parts of our proof have to be adapted, and how.

First, the definition of the geodesic flow and its invariant measures is now well-known, with many properties established by Roblin [18], see Section 2.2.

Lemma 3.1 in the case $F = 0$ follows from the triangular inequality. Shadow Lemma 3.2 holds without difficulty in the CAT(−1)-setting when $F = 0$ (see Lemma 1.3 of [18]). The proof of Proposition 3.3 only uses the metric structure and the CAT(−1)-property, so that it also holds in the CAT(−1)-setting.

The finiteness criterion for Gibbs measures, Theorem 3.4 above ([16]), has been extended in [7, Theorem 4.16] to the Gromov-hyperbolic setting (which includes CAT(−1) spaces) when $F = 0$.

The vague equidistribution result that we use, that is Theorem 3.6 above (Theorem 9.14 of [15]), had been established earlier in the case $F = 0$ in the CAT(−1)-setting in [18, Theorem 9.1.1].

The arguments of the proofs of Propositions 4.2, Corollary 4.3 and 4.4 do not use the Riemannian structure, and hold in the CAT(−1)-setting.

Theorem 1.3 follows.

5.2. Other generalizations? It is natural to ask whether our main results holds

- for more general (nonconstant) Hölder-continuous potentials on CAT(−1)-spaces?
- for the zero potential on Gromov-hyperbolic spaces?
- or more general potentials on Gromov-hyperbolic spaces?

Some ingredients of a proof would hold probably without difficulty, but it is unclear whether everything would work or not.

Let us mention what would be the ingredients of a proof.

* In the CAT(−1)-setting, thermodynamical formalism with nonzero potentials has already been considered. The construction of Gibbs measures and the variational principle are proven in Roblin [18] when $F = 0$ and Broise, Parkonnen, and Paulin [3] for some (restrictive) classes of nonconstant Hölder-continuous potentials on $\mathcal{G}X$.
* The finiteness criterion for Gibbs measures of Pit–Schapira [16] is extended in [7] in the Gromov-hyperbolic setting for the potential $F = 0$ and should probably hold also for more general potentials in the CAT(−1)-setting. As said above, thermodynamic formalism for some classes of nonconstant potentials is possible on CAT(−1)-spaces, but more delicate and completely open for the moment in general Gromov-hyperbolic spaces.
* The equidistribution theorem of periodic orbits with respect to the vague topology is proven in [18] in the CAT(−1)-setting for $F = 0$, in [15] for Riemannian negatively curved manifolds, and in [3] on real trees. Removing the restriction to manifolds and trees in the equidistribution theorem [3, Theorem 11.1] may require a different approach.

In particular, a generalization of Theorem 1.3 to the Gromov-hyperbolic setting when $F \equiv 0$, using tools of [7] seems possible. A generalization of Theorem 1.1 to some classes of potentials as in [3] could maybe also be possible. Thermodynamic formalism for general potentials in the Gromov-hyperbolic setting seems unexplored at the moment.

References

[1] P. Billingsley, *Convergence of probability measures.* Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, New York, 1999. Zbl 0944.60003 MR 1700749

[2] R. Bowen, The equidistribution of closed geodesics. *Amer. J. Math.* **94** (1972), 413–423. Zbl 0249.53033 MR 315742

[3] A. Broise-Alamichel, J. Parkkonen, and F. Paulin, Equidistribution and counting under equilibrium states in negative curvature and trees. Applications to non-Archimedean Diophantine approximation. Appendix by J. Buzzi. Progress in Mathematics, 329. Birkhäuser/Springer, Cham, 2019. Zbl 1442.37001 MR 3971204

[4] K. Burns, V. Climenhaga, T. Fisher, and D. Thompson, Unique equilibrium states for geodesic flows in nonpositive curvature. *Geom. Funct. Anal.* **28** (2018), no. 5, 1209–1259. Zbl 1401.37038 MR 3856792

[5] V. Climenhaga and D. J. Thompson, Equilibrium states beyond specification and the Bowen property. *J. Lond. Math. Soc.* (2) **87** (2013), no. 2, 401–427. Zbl 1276.37023 MR 3046278

[6] Y. Coudene, A short proof of the unique ergodicity of horocyclic flows. In I. Assani (ed.), *Ergodic theory.* Contemporary Mathematics, 485. American Mathematical Society, Providence, R.I., 2009, 85–89. Zbl 1183.37020 MR 2553211

[7] R. Coulon, R. Dougall, B. Schapira, and S. Tapie, Twisted Patterson–Sullivan measures and applications to amenability and coverings. Preprint, 2018. arXiv:1809.10881

[8] F. Dal'bo, J.-P. Otal, and M. Peigné, Séries de Poincaré des groupes géométriquement finis. *Israel J. Math.* **118** (2000), 109–124. Zbl 0968.53023 MR 1776078

[9] S. Gouezel, B. Schapira, and S. Tapie, Strongly positively recurrent potentials in negative curvature, 2020. hal-02901142v2

[10] G. A. Margulis, Applications of ergodic theory to the investigation of manifolds of negative curvature. *Funkcional. Anal. i Priložen.* **3** (1969), no. 4, 89–90. In Russian. English translation, *Functional Anal. Appl.* **3** (1969), no. 4, 335–336 Zbl 0207.20305 MR 0257933

[11] O. Mohsen, Le bas du spectre d'une variété hyperbolique est un point selle. *Ann. Sci. École Norm. Sup.* (4) **40** (2007), no. 2, 191–207. Zbl 1128.58008 MR 2339284

[12] J.-P. Otal and M. Peigné, Principe variationnel et groupes kleiniens. *Duke Math. J.* **125** (2004), no. 1, 15–44. Zbl 1112.37019 MR 2097356

[13] W. Parry, Equilibrium states and weighted uniform distribution of closed orbits. In J. C. Alexander (ed.), (College Park, MD, 1986–87.) Lecture Notes in Mathematics, 1342. Springer-Verlag, Berlin, 1988, 617–625. Zbl 0667.58056 MR 970574

[14] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque* **187–188** (1990), 268 pp. Zbl 0726.58003 MR 1085356

[15] F. Paulin, M. Pollicott, and B. Schapira, Equilibrium states in negative curvature. *Astérisque* **373** (2015), viii+281. Zbl 1347.37001 MR 3444431

[16] V. Pit and B. Schapira, Finiteness of Gibbs measures on noncompact manifolds with pinched negative curvature. *Ann. Inst. Fourier* (*Grenoble*) **68** (2018), no. 2, 457–510. Zbl 1409.37039 MR 3803108

[17] R. Ricks, Counting closed geodesics in a compact rank one locally CAT(0) space. Preprint, 2019. arXiv:1903.07635

[18] T. Roblin, Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr.* (*N.S.*) **95** (2003), vi+96. Zbl 1056.37034 MR 2057305

[19] B. Schapira and S. Tapie, Regularity of entropy, geodesic currents and entropy at infinity. *Ann. Sci. Éc. Norm. Supér.* (4) **54** (2021), no. 1, 1–68. Zbl 07360795 MR 4245867

[20] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 171–202. MR 556586

[21] C. H. Toll, *A multiplicative asymptotic for the prime geodesic theorem*. Thesis (Ph.D.)–University of Maryland, College Park. ProQuest LLC, Ann Arbor, MI, 1984. Zbl 0439.30034 MR 2634161

[22] A. Velozo, Ergodic theory of the geodesic flow and entropy at infinity. Ph.D. thesis. Princeton University. Princeton, N.J., 2018. MR 3828513

[23] P. Vidotto, Ergodic properties of some negatively curved manifolds with infinite measure. *Mém. Soc. Math. Fr.* (*N.S.*) **160** (2019), vi+132. Zbl 1455.37035 MR 3939852

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