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Erratum to "On groups with S^2 Bowditch boundary"

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Abstract. The purpose of this erratum is to correct the proof of Lemma 3.1 in [2].

1. The result

The following statement appears in [2, Lemma 3.1].

Theorem 1.1. *Let X be a compact metric space. Assume that there exists a surjection $\pi: X \to S^2$ such that* (i) *there exists a countable dense subset $Z \subset S^2$ so that the restriction of π to $\pi^{-1}(S^2 \setminus Z)$ is injective,* (ii) *for each $w \in Z$, the space X_w obtained from X by collapsing each $\pi^{-1}(z)$ to a point for $z \neq w$ is homeomorphic to a closed disk $\mathbb{D}^2$, and* (iii) *for each $z \in Z$, $\pi^{-1}(z)$ is an embedded circle. Then X is homeomorphic to the Sierpiński curve.*

The proof in [2] is not complete, as pointed out to us by Lucas H. R. Souza, whom we kindly thank. We also thank the referee for carefully reading the erratum and for additional corrections.

About the error. The proof in [2] attempts to show that any two spaces X, X' as in the statement are homeomorphic by expressing $X = \lim X(k)$ as an inverse limit, and similarly for X', and constructing a homeomorphism $X \to X'$ by showing that the associated inverse systems $\{X(k)\}$ and $\{X'(k)\}$ are isomorphic. This is done inductively. The base case is a theorem of Bennett [1], which says that any two countable dense subsets of S^2 differ by a homeomorphism $\phi: S^2 \to S^2$. Given this, we want to obtain $\phi_k: X(k) \to X'(k)$ by a "blowup" of ϕ. However, given the non-explicit nature of Bennett's result, it is not clear that one can construct ϕ_k in this manner. In our argument, we attempt to obtain ϕ_k as an extension of a map $\phi_{k-1}|$ that is claimed to be uniformly continuous, but this assertion is not justified.

We also point out that the hypothesis (iii) in Theorem 1.1 is omitted in [2, Lemma 3.1], where it is incorrectly asserted that (ii) implies (iii). One can give an example of X satisfying (i) and (ii), and such that $\pi^{-1}(z)$ is an annulus for some $z \in Z$. Such an X will not be a Sierpiński carpet as it is not 1-dimensional.

The fix. We provide a different approach that is closer to Whyburn's classical result [3, Theorem 3] that characterizes the Sierpiński curve as the unique locally-connected, 1-dimensional continuum in S^2 whose complement is a union of open disks whose boundaries are disjoint.

2. Setup for the proof

Let (X, π, Z) be as in the Theorem 1.1. We call X (or more precisely the tuple (X, π, Z)) an $\mathcal{S}$-*space*. The main step in the proof of Theorem 1.1 is to show that any two $\mathcal{S}$-spaces are homeomorphic. In this section we collect some basic facts about $\mathcal{S}$-spaces that we use to prove the Theorem 1.1 in Section 3.

Given (X, π, Z), we denote $\mathcal{C} = \{\pi^{-1}(z) \colon z \in Z\}$. By assumption, each $C \in \mathcal{C}$ is an embedded circle in X. We call these circles *peripheral*.

Lemma 2.1 (diameter of peripheral circles). *Let X be a $\mathcal{S}$-space. For any $d > 0$, there are only finitely many peripheral circles with diameter $> d$.*

Proof. Suppose for a contradiction that there are infinitely many $C_1, C_2, \dots$ of diameter $> d$. Choose $x_i, y_i \in C_i$ of distance $> d$. After passing to a subsequences, we may assume that $x_i \to x$ and $y_i \to y$ with $x \neq y$.

If x, y belong to the same peripheral circle $C = \pi^{-1}(w)$, we consider the quotient X_w (collapsing each $\pi^{-1}(z)$ to a point for $z \neq w$) and observe that x, y cannot be separated by open sets in X_w, which contradicts the assumption that $X_w \cong \mathbb{D}^2$. Similarly, if $\pi(x) \neq \pi(y)$, we consider the quotient of X by collapsing each $C \in \mathcal{C}$ to a point, and observe that this space is not Hausdorff; on the other hand this quotient is S^2 by assumption, a contradiction. □

Lemma 2.2 (quotients of $\mathcal{S}$-spaces). *Let X be an $\mathcal{S}$-space, and let $\mathcal{C}_0 \subset \mathcal{C}$ be a finite collection of k peripheral circles. The space $X(\mathcal{C}_0)$ obtained by collapsing each $C \in \mathcal{C} \setminus \mathcal{C}_0$ to a point is homeomorphic to the compact surface of genus* 0 *with k boundary components.*

Proof. This is explained in [2] in the proof of Lemma 3.1 (this argument is independent of the aforementioned error). □

For an $\mathcal{S}$-space X, we say that a finite, connected graph $G \hookrightarrow X$ is *nicely embedded* if (i) each peripheral circle is either contained in or disjoint from G, (ii) G contains finitely many peripheral circles, and (iii) denoting $\mathcal{C}_0 \subset \mathcal{C}$ the peripheral circles contained in G, the image of G in the quotient space $X(\mathcal{C}_0)$ (defined in Lemma 2.2) is the 1-skeleton of a triangulation of $X(\mathcal{C}_0)$.

Lemma 2.3 (subdividing an $\mathcal{S}$-space). *Let X be an $\mathcal{S}$-space.*

(i) *If $S \subset X$ is an embedded circle disjoint from the peripheral circles, then the closure of each component of $X \setminus S \subset X$ is an $\mathcal{S}$-space.*

(ii) *More generally, if $G \subset X$ is a nicely embedded graph, then G decomposes X into a union of $\mathcal{S}$-spaces, one for each component of $X \setminus G$.*

Proof. (i) By assumption, $\pi(S) \subset S^2$ is an embedded circle. By the Jordan curve theorem, this circle separates S^2 into two closed disks D_1, D_2 with common boundary $\pi(S)$. Then $X \setminus S$ has two components with respective closures $X_1 = \pi^{-1}(D_1)$ and $X_2 = \pi^{-1}(D_2)$. Observe that the quotient map $X_i \to D_i/\partial D_i = S^2$ induces an $\mathcal{S}$-space structure on X_i.

(ii) Let $\mathcal{C}_0 \subset \mathcal{C}$ be the collection of peripheral circles contained in G, and consider the quotient $X(\mathcal{C}_0)$. By Lemma 2.2, $X(\mathcal{C}_0)$ is a genus-0 surface. By assumption, G embeds in $X(\mathcal{C}_0)$, and subdivides $X(\mathcal{C}_0)$ into a collection of closed disks. The preimage of each disk in X has a natural $\mathcal{S}$-space structure, similar to (i). □

Given a graph $G \subset X$ as in Lemma 2.3, we say that G *subdivides* X into the $\mathcal{S}$-spaces provided by Lemma 2.3, which we call the *components* of the subdivision. We define the *mesh* of G as the maximum diameter of the components of its subdivision.

The following lemma is analogous to [3, Lemma 1]. This lemma may be viewed as the main tool used in the proof Theorem 1.1.

Lemma 2.4. *Let X, X' be $\mathcal{S}$-spaces with peripheral circles $\mathcal{C}, \mathcal{C}'$, respectively. Given $C_0 \in \mathcal{C}$ and $C_0' \in \mathcal{C}'$, a homeomorphism $h_0\colon C_0 \to C_0'$, and $\epsilon > 0$, there exist nicely embedded graphs G and G' with $C_0 \subset G \subset X$ and $C_0' \subset G' \subset X'$, each with mesh $< \epsilon$ and a homeomorphism $h\colon G \to G'$ extending h_0.*

Proof. The proof is nearly identical to the proof of [3, Lemma 1], even though our setup is slightly different. Take $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathcal{C}_0' \subset \mathcal{C}'$ equal-sized collections of peripheral circles containing all the peripheral circles with diameter $\geq \epsilon$. We can choose $\mathcal{C}_0, \mathcal{C}_0'$ finite by Lemma 2.1. By Lemma 2.2, there is a homeomorphism $f\colon X(\mathcal{C}_0) \to X'(\mathcal{C}_0')$ that extends the given homeomorphism $h_0\colon C_0 \to C_0'$ (here we are abusing notation slightly by identifying the $C_0 \subset X$ with its homeomorphic image in $X(\mathcal{C}_0)$).

Let $Z_0 \subset X(\mathcal{C}_0)$ be the image of the collapsed peripheral circles under the quotient $X \to X(\mathcal{C}_0)$, and define $Z_0' \subset X'(\mathcal{C}_0')$ similarly. Then $f(Z_0) \cup Z_0' \subset X'(\mathcal{C}_0')$ is a countable collection of points, and for any $\delta > 0$, we can find a graph $\overline{G}' \subset X'(\mathcal{C}_0')$ containing $\partial X'(\mathcal{C}_0')$ of mesh $< \delta$ that is disjoint from $f(Z_0) \cup Z_0'$. We can choose $\overline{G}'$ to be the 1-skeleton of a triangulation of the surface $X'(\mathcal{C}_0')$, so that the graphs $\overline{G} := f^{-1}(\overline{G}')$ and $\overline{G}'$ lift homeomorphically to nicely embedded graphs $G \subset X$ and $G' \subset X'$. By construction, point-preimages of $X \to X(\mathcal{C}_0)$

have diameter $< \epsilon$, and there are only finitely many diameters bigger than any given size. Therefore, since X and $X(\mathcal{C}_0)$ are compact, if δ is sufficiently small, then $G \subset X$ will have mesh $< \epsilon$. See [3, Lemma 2] for a proof of this fact. The same goes for $G' \subset X'$.

Finally, observe that the map $f|: \bar{G} \to \bar{G}'$ lifts to the desired homeomorphism $h: G \to G'$. □

3. The corrected proof

The Sierpiński curve is an $\mathcal{S}$-space, as explained in [2, Proof of Lemma 3.1]. Thus to prove the theorem, it suffices to show that any two $\mathcal{S}$-spaces are homeomorphic. This argument is almost identical to the proof of [3, Theorem 3]. We sketch the argument and refer to [3] for additional details.

Let (X, π, Z) and (X', π', Z') be two $\mathcal{S}$-spaces with peripheral circles $\mathcal{C}$ and $\mathcal{C}'$, respectively. For each $n \geq 1$, we construct nicely embedded graphs $G_n \subset X$ and $G'_n \subset X'$ satisfying (1) G_n and G'_n have mesh $< \frac{1}{n}$ and (2) $G_n \subset G_{n+1}$ and $G'_n \subset G'_{n+1}$. In addition, we construct homeomorphisms $h_n: G_n \to G'_n$ with h_{n+1} extending h_n.

First we explain how to construct a homeomorphism $X \to X'$ given the existence of the maps $h_n: G_n \to G'_n$. First, these homeomorphisms induce a homeomorphism h between $G := \bigcup G_n$ and $G' := \bigcup G'_n$. Since G_n and G'_n have mesh $\to 0$, both $G \subset X$ and $G' \subset X'$ are dense. Since adjacent components of the subdivision of G_n go to adjacent components of the subdivision of G'_n, the map $h: G \to G'$ is uniformly continuous. See [3, last two paragraphs of the proof of Theorem 3] for a detailed proof. Therefore h extends to a map $X \to X'$, which is a homeomorphism.

It remains to construct G_n, G'_n, and h_n. We proceed inductively. First choose arbitrarily $C_0 \in \mathcal{C}$, $C'_0 \in \mathcal{C}'$ and a homeomorphism $h_0: C_0 \to C'_0$, and apply Lemma 2.4 with $\epsilon = 1$ to obtain $h_1: G_1 \to G'_1$. The graph G_1 subdivides X, and each component is an $\mathcal{S}$-space with a "preferred" peripheral circle, the unique circle that intersects G_1 nontrivially. Note also that there is a natural correspondence between the components of the subdivisions of $G_1 \subset X$ and $G'_1 \subset X'$. For the induction step, given G_n, G'_n, h_n, we apply Lemma 2.4 to each pair of corresponding components of the subdivisions $G_n \subset X$ and $G'_n \subset X'$, taking $\epsilon = \frac{1}{n}$ and using the preferred peripheral circles and h_n as input. □

References

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