Amenability and measure of maximal entropy
for semigroups of rational maps

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Abstract. In this article we discuss relations between algebraic and dynamical properties of non-cyclic semigroups of rational maps.

1. Introduction

In a series of works J. Ritt (see [29, 30]) studied non-trivial relations and functional equations on the semigroup of rational maps. Specifically, Ritt was interested in the solution of equations of the type

$$A \circ B = C \circ D,$$

where A, B, C, and D are rational maps. Ritt gave solutions to these equations for polynomials, but there were obstacles in the case of rational maps. Ritt's theory for rational maps is still under investigation and presents many open questions. The paper [28] contains a short survey on the modern research in this area as well an ample list of references. In particular, the references also include a series of Pakovich's works on recent developments of Ritt's theorems for rational maps.

In the paper [13], D. Ghioca, T. J. Tucker, and M. E. Zieve proved the following interesting result.

> *If, for polynomials P and Q, there exists a point $z_0 \in \bar{\mathbb{C}}$ such that the intersection of the forward orbits of z_0, with respect to P and Q, is an infinite set, then P and Q share a common iterate. That is, there are natural numbers n, m such that $P^n = Q^m$.*

In other words, the dynamical intersection property implies an algebraic equation on P and Q.

Our first theorem generalizes the theorem above. Recall that a polynomial (rational map) Q is *exceptional* if either $\deg(Q) = 1$ or Q is affinely (Möbius) conjugated either to z^n (with $n \in \mathbb{Z}$) or to a Chebyshev polynomial (or a Lattès map). We say that a family of polynomials (rational maps) $\mathcal{F}$ is *non-exceptional* if $\mathcal{F}$ contains a non-exceptional polynomial (rational map). Given a family of maps $\mathcal{F}$, we denote by $S(\mathcal{F})$ the semigroup generated by the family $\mathcal{F}$.

Theorem 1. *Given a finite non-exceptional family $\mathcal{F}$ of polynomials, the following conditions are equivalent.*

(1) *For every pair $P, Q \in \mathcal{F}$, there exists a point $z_0 \in \bar{\mathbb{C}}$ such that*

$$\#\left(\left\{\bigcup P^j(z_0) \cap \bigcup Q^k(z_0)\right\}\right) = \infty.$$

(2) *For every pair $P, Q \in S(\mathcal{F})$, there are integers m, n such that $P^m = Q^n$.*

(3) *The semigroup $S(\mathcal{F})$ is amenable with $\mathrm{RIM}(S(\mathcal{F})) \subset \mathrm{LIM}(S(\mathcal{F}))$, where $\mathrm{RIM}(S(\mathcal{F}))$ and $\mathrm{LIM}(S(\mathcal{F}))$ stand for the sets of right and left invariant means, respectively.*

(4) *$S(\mathcal{F})$ is embeddable into a virtually cyclic group.*

Recall that a *virtually-cyclic group* is a group containing a cyclic subgroup of finite index. As shown in Example 1 below, a semigroup of polynomials may be embeddable into a virtually-cyclic group, but not into a *metacyclic group.* Those are virtually-cyclic groups for which the cyclic subgroup is normal.

For families of rational maps we have the following.

Theorem 2. *Let $\mathcal{F}$ be a non-exceptional collection of rational maps containing an element not Möbius conjugated to a polynomial. Then, the following conditions are equivalent.*

(1) *The semigroup $S(\mathcal{F})$ is right amenable and, for every pair $P, Q \in \mathcal{F}$, there is a point $z_0 \in \bar{\mathbb{C}}$ such that*

$$\#\left\{\bigcup P^j(z_0) \cap \bigcup Q^k(z_0)\right\} = \infty.$$

(2) *For every pair $P, Q \in S(\mathcal{F})$, there exist natural numbers m, n such that $P^m = Q^n$.*

(3) *The semigroup $S(\mathcal{F})$ is right amenable and embeddable into a group.*

(4) *The semigroup $S(\mathcal{F})$ is ρ-right amenable, where ρ denotes the Lyubich representation (definitions below), and, for every pair $P, Q \in \mathcal{F}$, there is a point $z_0 \in \bar{\mathbb{C}}$ such that*

$$\#\left(\left\{\bigcup P^j(z_0) \cap \bigcup Q^k(z_0)\right\}\right) = \infty.$$

If a semigroup of rational maps is finitely generated and satisfies condition (3) of Theorem 2, then the associated group is virtually cyclic.

The following two theorems describe the right-amenable semigroups of rational maps.

Theorem 3. *Let ρ be the Lyubich representation. Assume S is a ρ-right-amenable semigroup of rational maps containing a non-exceptional rational map which is not Möbius conjugated to a polynomial. The following statements hold true.*

(1) *There exists a probability measure μ invariant under S.*

(2) *If $P \in S$ and $\deg(P) > 1$, then μ is the measure of maximal entropy of P.*

Since right amenability implies ρ-right-amenability for every bounded representation ρ (definitions and discussion in the next section), then Theorem 3 holds for right-amenable semigroups of rational maps.

For polynomials, we have the following result.

Theorem 4. *Given a finite non-exceptional collection of polynomials $\mathcal{F}$, the following conditions are equivalent.*

(1) *The semigroup $S(\mathcal{F})$ is ρ-right amenable for the Lyubich representation ρ.*

(2) *The semigroup $S(\mathcal{F})$ is right amenable.*

(3) *There exists a probability measure μ invariant under $S(\mathcal{F})$ which coincides with the measure of maximal entropy for every element in $S(\mathcal{F})$.*

The equivalence of (1) and (2) is rare even among groups. According to M. Day (see Theorem 2 in [7]), a semigroup S is right amenable if and only if S is ρ-right amenable for every bounded representation ρ.

Hence the theorem above presents an interest from this point of view.

For rational maps, Theorem 4 is still an open question. We propose the following:

Conjecture. *If a semigroup S of rational maps have the same measure of the maximal entropy, then S is a right-amenable semigroup.*

Theorem 4 gives a partial answer to this conjecture (see also Theorem 20 and Corollary 23).

In fact, Theorems 1–4 hold in more general settings, see the discussion in Section 3.

In the last section, we discuss amenability properties for another important representation in holomorphic dynamics, namely the Ruelle representation. The Ruelle representation is closely related to quasiconformal deformations of rational maps. In Proposition 37 and Proposition 38, we show that a right amenable semigroup S is quasiconformally deformable whenever S contains a hyperbolic

element which admits a non-trivial quasiconformal deformation. Moreover, a semigroup S of rational maps is structurally stable whenever S satisfies the Levin relations, is finitely generated, and contains a structurally stable element.

To every rational map R of degree at least 2, we associate a right-amenable finitely generated semigroup of operators $D(R)$ acting on the space $L_1(A)$, for every Lebesgue measurable subset A completely invariant with respect to R. In Proposition 39, we observe that if A does not possess a Beltrami differential, invariant under R, then the action of $D(R)$ on $L_1(A)$ is left amenable. The main theorem of the last section is the following.

Theorem 5. *If R is a rational map and the action $D(R)$ on $L_1(\overline{\mathbb{C}})$ is left-amenable, then the following conditions are equivalent*:

(1) *the Ruelle operator R_* has non-zero fixed points in $L_1(\overline{\mathbb{C}})$*;

(2) *R is Möbius conjugated to a flexible Lattés map.*

The dynamics of non-cyclic semigroups of rational maps first treated by A. Hinkkanen and G. Martin in [15] is now an active area of research in holomorphic dynamics. Yet another approach is presented in [5] and [12]. In particular, in [15] the authors adapt the Klein–Maskit combination theorem to construct free semigroups of rational maps. The arguments in [15] allow to show the following statement.

> *If the polynomials P and Q have mutually disjoint filled Julia sets, then there are integers $m, n > 0$ such that the semigroup $\langle P^m, Q^n \rangle$ is free.*

So, the semigroup $\langle P, Q \rangle$ contains a free two-generated subsemigroup, which is neither right nor left amenable. This observation is another motivation for considering amenability conditions.

We have the following conjecture:

Conjecture. *A finitely generated semigroup S not containing a rank-2 free subsemigroup admits an invariant probability measure which is the measure of maximal entropy of every non-injective element of S.*

In this work, we will use standard notations and concepts from holomorphic dynamics which can be found, for instance, in [23].

Let us describe the structure of the article as follows. In Section 2, we give some basic facts and notions of amenability of semigroups, holomorphic dynamics, measure or maximal entropy, and Ruelle and Lyubich representations. We also introduce three intersection properties for semigroups of rational maps. The comparison between them is one of the main motivations of this work leading to Theorem 1, which is proved in a slightly more general version in Section 3 as Theorem 36.

Theorem 2 is the version of Theorem 1 for arbitrarily semigroups of rational maps containing an non-exceptional map not Möbius conjugated to a polynomial. Theorem 2 is proved in Section 3 as Theorem 27.

The proofs of Theorem 1 and Theorem 2 are based on Theorem 3 and Theorem 4, respectively, which describe (and characterize for polynomials) right-amenable semigroups of rational maps. These theorems are proven in Section 3 as Theorem 19 and Corollary 26, respectively. The gap between the stronger case of polynomials and arbitrary rational maps of semigroups is contained in the conjectures above. In Section 3, we consequently develop all ingredients and combine them together to form the proofs.

Presenting independent interest, Proposition 33 and Corollary 34 describe quotients and representations of rational semigroups for an equivalence relation motivated by Proposition 11. Also, Theorem 31 and Theorem 35 can be regarded as a characterization of amenable semigroups of polynomials.

Finally, last section is devoted to the proof of Theorem 5 which is presented as Theorem 40.

While this paper was under revision, we learned that some results presented here were generalized in [26] and later the conjecture about rank-2 free semigroups was answered in [3].

2. Preliminaries

2.1. Semigroup amenability. Let S be a semigroup and let $L_\infty(S)$ be the linear space of bounded complex valued functions equipped with the supremum norm. A continuous linear functional M on $L_\infty(S)$ is called a *mean* if M satisfies the following properties:

(1) M is positive, that is, if $\phi \in L_\infty(S)$ and $\phi \geq 0$, then $M(\phi) \geq 0$;

(2) $\|M\| = M(\chi_S) = 1$, where χ_S denotes the characteristic function of S.

The right and left actions of S onto itself generate right and left actions on the space $L_\infty(S)$ given by the formulas

$$r_s(\phi)(x) = \phi(xs),$$
$$l_s(\phi)(x) = \phi(sx),$$

for every $s, x \in S$, and $\phi \in L_\infty(S)$, respectively.

These actions induce right and left representations of the semigroup S into the semigroup $\mathrm{End}(L_\infty(S))$ of linear continuous endomorphisms of $L_\infty(S)$ given by $s \mapsto r_s$ and $s \mapsto l_s$.

The semigroup S is called *right amenable*, or shortly an *RA-semigroup,* if there exists a mean which is invariant for the right action of S on $L_\infty(S)$, that is, if $M(r_s(\phi)) = M(\phi)$ for every $s \in S$ and $\phi \in L_\infty(S)$. We denote by RIM(S) the set of all right invariant means on the semigroup S. Note that RIM(S) is a convex, closed subset of $L_\infty^*(S)$ which does not contains the 0 functional. Moreover, RIM(S) is compact in the $*$-weak topology.

Analogously, S is called *left amenable*, or an LA semigroup for short, if there is a mean invariant under the left action. We denote by LIM(S) the set of all left invariant means.

Finally, the semigroup S is an *amenable semigroup,* if RIM$(S) \cap$ LIM$(S) \neq \emptyset$.

Let us mention some basic facts about amenable semigroups. Further details and proofs may be found in the papers of M. Day [6] and [8].

Basic facts on amenability.   (1) Every abelian semigroup is amenable.

(2) Every finite group is amenable, but not every finite semigroup is amenable. For example, the finite semigroup $\langle a, b : ab = a^2 = a, ba = b^2 = b \rangle$ is not an LA-semigroup.

(3) Every semigroup is a subsemigroup of an amenable semigroup. For groups, every subgroup of an amenable group is amenable.

(4) Given a semigroup S, let us consider the antiproduct $*$ on the set S defined by $a * b = ba$. The set S equipped with the product $*$ is a semigroup anti-isomorphic to S and thus the space RIM(S) coincides with LIM$(S, *)$. In the same way, the left action of S on $L_\infty(S)$ is the right action of $(S, *)$ on $L_\infty(S)$.

(5) Let M be an element of either LIM(S) or RIM(S). If $M(\chi_{S_0}) > 0$ for a subsemigroup $S_0 < S$, then S_0 is itself either an LA- or an RA-semigroup, respectively.

(6) Let $\phi : S_1 \to S_2$ be an epimorphism of semigroups, then S_2 is either RA or LA whenever S_1 is RA or LA, respectively.

Let us consider a weaker version of amenability, namely *ρ-amenability* which was introduced by M. Day in [8].

First, we say that a proper right (left) S-invariant subspace $X \subset L_\infty(S)$ is either *right* or *left amenable*, again RA or LA for short, if X contains constant functions and there exists a mean M such that, when M is restricted to X, it induces a functional which is invariant for either the right or left actions of S on X, respectively. In other words, X is invariant and admits an invariant functional for the associated action. Note that every semigroup admits an amenable subspace; for example, the subspace of constant functions is always amenable.

Now, let ρ be a bounded homomorphism from S into End(B), where B is a Banach space, and End(B) is the semigroup of linear continuous endomorphisms.

Given a pair, $(b, b^*) \in B \times B^*$ consider the function $f_{(b,b^*)} \in L_\infty(S)$ given by

$$f_{(b,b^*)}(s) = b^*(\rho(s)(b)).$$

Let $Y_\rho \subset L_\infty(S)$ be the closure of the linear span of the family of functions $\{f_{(b,b^*)}\}$ for all pairs $(b, b^*) \in B \times B^*$. Finally, let X_ρ be the space generated by Y_ρ and the constant functions. Note that X_ρ and Y_ρ are both right and left invariant.

Definition. We say that ρ is either *RA* or *LA* whenever X_ρ is either a right or a left amenable subspace of $L_\infty(S)$, respectively. Also we say that S is ρ-RA or ρ-LA whenever ρ has the respective property, Equivalently, we say that the ρ-action of S on B is either RA or LA, respectively.

To show that, in general, amenability is different from ρ-amenability, let us recall Day's theorem (see Theorem 2 in [7]).

Theorem 6 (Day Theorem). *A semigroup S is either RA or LA if and only if S is either ρ-RA or ρ-LA, respectively, for every bounded representation ρ.*

Roughly speaking, the existence of an invariant functional on a proper subspace does not always implies the existence of an invariant functional on $L_\infty(S)$.

For example, consider a free group G which is neither RA nor LA. Let h be a homomorphism from G onto a non trivial abelian group Γ. Then, the space $h^*(L_\infty(\Gamma)) \subset L_\infty(G)$ is amenable, where $h^*(\phi) = \phi \circ h$ is the pull-back operator. We do not know examples of semigroups which are neither ρ-RA nor ρ-LA for every bounded representation ρ, even in the case when the associated Banach space is infinitely dimensional.

2.2. Maximal entropy and representations. In this article we consider two important representations of semigroups of rational maps: namely, the Lyubich and the Ruelle representations. These are push-forward actions of rational maps on the spaces $C(\bar{\mathbb{C}})$ and $L_1(\bar{\mathbb{C}})$ of continuous and Lebesgue integrable functions on the Riemann sphere $\bar{\mathbb{C}}$, respectively.

Let us first discuss the Lyubich representation. Every rational map R induces an operator given by

$$L_R(\phi)(y) = \frac{1}{\deg(R)} \sum_{R(x)=y} \phi(x),$$

where the sum is taken with multiplicities. The operator L_R is a continuous endomorphism of $C(\bar{\mathbb{C}})$ with the unit norm. L_R was firstly considered by M. Lyubich in [21]. We call it the *Lyubich operator* of the rational map R. Now we reformulate the main results of [22] as follows:

Theorem 7. *For every rational map R with $\deg(R) > 1$, there exist an invariant non-atomic probability measure μ_R which represents an invariant functional on $C(\mathbb{C})$ with respect to the Lyubich operator L_R. The measure μ_R is of maximal entropy, ergodic, and unique in the following sense: if an L_R-invariant functional is generated by a non-atomic measure ν, then ν is a multiple of μ_R.*

The support of μ_R coincides with the set $J(R)$, the *Julia set* of R. Observe that the Lyubich operator is well defined for every branched self-covering of the Riemann sphere of finite degree.

Definition. Let $f \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a branched covering of finite degree. We call the correspondence $\rho \colon f \mapsto L_f$ the *Lyubich representation*.

The Lyubich representation gives a homomorphism from the whole semigroup of finite degree branched self-coverings of $\overline{\mathbb{C}}$ into $\mathrm{End}(C(\overline{\mathbb{C}}))$, the semigroup of continuous linear endomorphisms of the space of continuous functions on $\overline{\mathbb{C}}$.

Note that the uniqueness statement in Theorem 7 is, in general, false for non-holomorphic branched self-coverings (see discussion after Theorem 16). We call a complex valued measure ν a *Lyubich measure* for a semigroup S generated by a collection of finite degree branched self-coverings of $\overline{\mathbb{C}}$ whenever ν induces an L_f-invariant functional for every $f \in S$.

Let us discuss the Ruelle representation of rational maps.

Definition. Let R be a rational map. The operator

$$R_*(\phi)(y) = \sum_{R(x)=y} \frac{\phi(x)}{(R'(x))^2}$$

is called the *Ruelle transfer operator* or the *Ruelle operator*.

Ruelle operator acts on the space $L_1(\overline{\mathbb{C}})$ with $\|R_*\| \leq 1$. The operator $B_R(\phi) = \phi(R)\frac{\overline{R'}}{R'}$ is called the *Beltrami operator*, which is a continuous endomorphism of $L_\infty(\overline{\mathbb{C}})$ with unitary norm. The space $\mathrm{Fix}(B_R)$ of fixed points of B_R is called the space of *invariant Beltrami differentials* of R. In other words, the form $\phi(z)\frac{\bar{\partial}z}{\partial z}$ is invariant under the pull-back action of R whenever $\phi \in \mathrm{Fix}(B_R)$. By the Ahlfors–Bers theorem, the space of invariant Beltrami differentials generates all quasiconformal deformations of the map R.

The relevance of Ruelle operator comes from the following lemma (see for example [4] and [24]).

Lemma 8. *The Beltrami operator B_R is dual to the Ruelle operator R_*.*

Let us note that both the Beltrami and the Ruelle operators can be extended to almost-everywhere differentiable branched self-coverings of the Riemann sphere.

2.3. Relations and functional equations on rational maps. The following theorem was proven by Ritt in [29] and completed by Eremenko in [11].

Theorem 9. *Let $S \subset$ Rat be an abelian semigroup of rational maps. Assume that S contains a non-exceptional element R with $\deg(R) \geq 2$. Then, for every pair of elements $P, Q \in S$ with $\deg(P), \deg(Q) \geq 2$, there are numbers m, n such that $P^m = Q^n$.*

Definition. We say that the rational maps Q, R satisfy the *Levin relations* if

$$Q \circ R = Q \circ Q$$

and

$$R \circ Q = R \circ R.$$

The following theorem is proved in [19] and [20]. We present it as formulated by H. Ye in [31].

Theorem 10. *Two non-exceptional rational maps Q and R share the same measure of maximal entropy if and only if there are numbers m, n such that Q^m and R^n satisfy the Levin relations.*

The following theorem is a consequence of Ritt's results given in [30].

Proposition 11 (Ritt). *Let F, A, B be rational maps satisfying the equation*

$$FA = FB.$$

Then, either $\deg(F) > \deg(A) = \deg(B)$ or A and B share a common right factor, that is, there are rational maps $X, Y,$ and Z such that

$$A = X \circ Z$$

and

$$B = Y \circ Z.$$

By Proposition 11, if Q and R satisfy the Levin relations, then Q and R share a right common factor. Moreover, if either Q or R is an indecomposable rational map, then the rational maps X and Y given in Proposition 11 must be Möbius transformations. Recall that a map R is called *indecomposable* if, whenever we have $R = P \circ T$, then one of the factors, either P or T, must be a Möbius transformation.

Also note that the relations given in Proposition 11 pose an obstacle to the left cancellation property (definitions and discussions are given below).

2.4. Intersection properties. Let us introduce three intersection properties which will be discussed in this work.

Definition (dynamical intersection property). Let DIP $\subset$ Rat $\times$ Rat be the set consisting of the pairs of rational maps Q, R for which there exists a point $z_0 \in \overline{\mathbb{C}}$ with

$$\#\{\mathcal{O}_Q(z_0) \cap \mathcal{O}_R(z_0)\} = \infty,$$

where $\mathcal{O}_R(z) = \bigcup_{n \geq 0} R^n(z)$ denotes the forward orbit of z.

Definition (algebraic intersection property). Let AIP $\subset$ Rat $\times$ Rat be the set of all pairs (Q, R) sharing a common iteration.

Note that AIP $\subset$ DIP.

Definition (ideal intersection property). The semigroup S satisfies the *left* (resp. *right*) *ideal intersection* property whenever every pair of principal left (resp. right ideals) I, J in S have non-empty intersection.

The last property is closely related to the problem of embedding a semigroup into a group, that is, to specify under what circumstances a given semigroup S is "half" of a group.

Let Γ be a countable group with a minimal set of generators $\langle \gamma_1, \ldots, \gamma_n, \ldots \rangle$. Consider the subset Γ_+ of all words in the alphabet $\{\gamma_1, \ldots \gamma_n, \ldots\}$. Then, Γ_+ forms a countable semigroup which is called the *positive part* of Γ. Note that Γ is generated by Γ_+ and $(\Gamma_+)^{-1}$. A countable semigroup S is *embeddable into a group* if S is isomorphic to the positive part of a group.

Recall that a semigroup S is *left cancellative* if for $a, b, c \in S$ the equation $ca = cb$ implies $a = b$. An analogous definition applies for a *right cancellative* semigroup S. For example, every semigroup generated by a set of surjective endomorphisms of a set A is always right cancellative. In particular, every semigroup of rational maps is always right cancellative.

If S is both left and right cancellative, then S is called a *cancellative* semigroup. For instance, any finitely generated free semigroup S is cancellative and, moreover, S is embeddable into a finitely generated free group.

The following theorem due to Ø. Ore provides sufficient conditions for a semigroup to be embeddable into a group (see [25]).

Theorem 12 (Ore Theorem). *Let S be a cancellative semigroup. S is embeddable into a group whenever S satisfies either the left or right ideal intersection property.*

In fact, the theorem does not need the countability condition. As a consequence of the Ore Theorem, we have that every abelian semigroup S is embeddable into a

group if and only if S is cancellative. Hence, every abelian semigroup of rational maps is embeddable into a group.

In order to apply the Ore Theorem, we need either the right or the left ideal intersection property which is known for RA semigroups (see for example [16]). For sake of completeness, we include it in the following lemma.

Lemma 13. *If S is an RA semigroup, then S satisfies the left ideal intersection property.*

Proof. If for $P, Q \in S$ we have $SP \cap SQ = \emptyset$, then for every r-mean ν

$$\nu(\chi_S) \geq \nu(\chi_{SP} + \chi_{SQ}) = \nu(\chi_{SP}) + \nu(\chi_{SQ}) = 2\nu(\chi_S),$$

which is a contradiction. $\qquad\square$

The following corollary is an immediate consequence of the Ore Theorem and Lemma 13.

Corollary 14. *A semigroup S of rational maps is embeddable into group whenever S is left cancellative and RA.*

2.5. Ergodic actions. Given an operator T on a Banach space X, the *n-Cesàro averages* of T are the operators $A_n(T)$ defined for $x \in X$ by

$$A_n(T)(x) = \frac{1}{n} \sum_{i=0}^{n-1} T^i(x).$$

An operator T on a Banach space X is called *mean-ergodic* if T is power-bounded, that is, $\|T^n\| \leq M$ for some number M independent of n, and the Cesàro averages $A_n(T)(x)$ converge in norm for every $x \in X$.

The following fact can be found, for example, in Krengel's book [18].

Separation principle. An operator T is mean-ergodic if and only if T satisfies the principle of separation of fixed points:

if x^* is a fixed point of T^*, where T^* denotes the dual operator of T, then there exist $y \in X$ a fixed point of T such that $\langle x^*, y \rangle \neq 0$.

Recall that an operator T acting on a Banach space $\mathcal{B}$ is called *weakly almost periodic* if $\{T^n(f)\}$ is weakly sequentially precompact for every $f \in \mathcal{B}$. The following theorem is due to I. Kornfeld and M. Lin [17].

Theorem 15. *Let T be a positive operator with $\|T\| \leq 1$ acting on $L_1(X, \mu)$ space. The operator T is weakly almost periodic if and only if T is mean-ergodic.*

3. Lyubich representation

We start with the following theorem.

Theorem 16. *Let S be a semigroup of branched self-coverings of the sphere. If the Lyubich representation of S is right amenable, then there exists a Lyubich probability measure for S.*

Proof. Let $\rho: S \to \operatorname{End}(C(\bar{\mathbb{C}}))$ be the Lyubich representation. Let σ be a probability measure. Let $H: C(\bar{\mathbb{C}}) \to L_\infty(S)$ be the map defined for $s \in S$ by

$$H(\phi)(s) = \int_{\bar{\mathbb{C}}} \rho(s)(\phi(z))d\sigma(z).$$

Since the characteristic function $\chi_{\bar{\mathbb{C}}}$ is a fixed element for every Lyubich operator $\rho(s)$, then the closure of the image of H is a subspace $L_\infty(S)$ containing the constant functions on S. The space $X = \overline{\operatorname{im}(H)}$ is invariant under the right action of S. By assumption, X admits a non-zero r-invariant mean L. Therefore, the functional ℓ given by

$$\ell(\phi) = L(H(\phi))$$

is continuous and positive on $C(\bar{\mathbb{C}})$. Let us show that ℓ is invariant with respect to $\rho(S)$. Indeed, for $t \in S$,

$$\begin{aligned}
r_t(H(\phi))(s) &= H(\phi)(st) \\
&= \int \rho(st)(\phi)d\sigma \\
&= \int \rho(s)(\rho(t)(\phi))d\sigma \\
&= H(\rho(t)(\phi))(s).
\end{aligned}$$

Since L is r-invariant, it follows that

$$\begin{aligned}
\ell(\phi) = L(H(\phi)) &= L(r_t(H(\phi))) \\
&= L(H(\rho(t)(\phi))) = \ell(\rho(t)(\phi)).
\end{aligned}$$

By the Riesz representation theorem, there exists a probability measure μ satisfying $\ell(\phi) = \int \phi d\mu$, since ℓ is $\rho(t)$ invariant then μ is a Lyubich measure. $\qquad\square$

The measure μ depends on the choice of the measure σ. For instance, consider $\sigma = \delta_{z_0}$ the delta measure on a suitable point $z_0 \in \bar{\mathbb{C}}$. If the cardinality of $\mathcal{O}_-(S)(z_0)$ is finite, then μ is atomic and, in fact, is a linear combination of delta measures based on $\overline{\mathcal{O}_-(S)(z_0)}$, where

$$\mathcal{O}_-(S)(z_0) = \bigcup_{\substack{n \geq 0 \\ s \in S}} s^{-n}(z_0).$$

If $S <$ Pol is a polynomial semigroup, then, choosing $z_0 = \infty$, the measure δ_∞ is an atomic Lyubich measure.

If $\overline{\mathcal{O}_-(S)(z_0)}$ is infinite, then μ may be non-atomic, as in the case of cyclic semigroups of rational maps.

Moreover, for semigroups of non-holomorphic branched coverings of the sphere, a non-atomic Lyubich measure may not be unique even for cyclic semigroups. For example, if f is a formal mating of two polynomials, say P and Q, then the conformal copies of the measures of maximal entropy for P, Q, and $z^{\deg P}$ generate a three-dimensional space of Lyubich measures for f. One can use the tuning procedure to construct a map with a two-dimensional space of Lyubich measures. Hence, by repeating the procedures of mating and tuning, we can produce a multidimensional space of Lyubich measures. The definitions of tuning and mating were given by A. Douady in Section 2 and Section 3 of [9].

From Theorem 16 we have the following corollary.

Corollary 17. *Let S be an RA-semigroup of branched self-coverings of the Riemann sphere. Then, there exists a Lyubich probability measure for S.*

For semigroups of rational maps, we have Theorem 3 which gives a stronger conclusion. For this, we need the following.

Definition. We say that a collection of rational maps $\mathcal{F}$ is *admissible* if it contains a non exceptional element g and either

- g is not Möbius conjugated to a polynomial, or
- g is conjugated to a polynomial, but there exists another element $g_1 \in \mathcal{F}$ such that there is no Möbius map γ simultaneously conjugating g and g_1 to polynomials, or
- $\mathcal{F}$ consists of rational maps simultaneously Möbius conjugated to a family of polynomials $\mathcal{F}'$, and there exists a disk $D_{\mathcal{F}}$ centered at ∞ such that $P(D_{\mathcal{F}}) \subset D_{\mathcal{F}}$ for every $P \in \mathcal{F}'$.

In other words, a family $\mathcal{F}$ containing a non-exceptional map is admissible if either there is no point $z_0 \in \overline{\mathbb{C}}$ with $R^{-1}(z_0) = \{z_0\}$ for every $R \in \mathcal{F}$, or, otherwise, there exists a common R-invariant topological disk around such z_0 for every $R \in \mathcal{F}$.

Every finite collection of non-linear polynomials containing a non-exceptional polynomial is admissible. Moreover, every collection of monic non-linear polynomials with bounded coefficients and containing a non-exceptional polynomial is admissible. Finally, a collection $\mathcal{F}$ is admissible if and only if the semigroup $S(\mathcal{F})$ is also admissible. Indeed, if $\mathcal{F}$ is admissible, then $S(\mathcal{F})$ is also admissible. Reciprocally, if $S(\mathcal{F})$ is admissible, but $\mathcal{F}$ is not, then $\mathcal{F}$ is Möbius conjugated to polynomials and there is no disk D such that $P(D) \subset D$ for all $P \in S(\mathcal{F})$, which contradicts the hypothesis that $S(\mathcal{F})$ is admissible.

We also need the following lemma.

Lemma 18. *Let R be a non-injective rational map and ν be a probability Lyubich measure for R. Then, $\nu = s\mu_R + (1-s)\omega$, with $s \in [0, 1]$, where μ_R is the measure of maximal entropy and either $\omega = \delta_a$, with $R^{-1}(a) = a$, or $\omega = t\delta_a + (1-t)\delta_b$, with $t \in [0, 1]$ and $R^{-1}(\{a, b\}) = \{a, b\}$.*

In other words, an L_R-invariant functional is presented by an atomic measure ν if and only if either R or R^2 is Möbius conjugated to a polynomial and the support of ν contains the point associated to ∞ as an atom of ν.

Proof. Let us observe that the support of a measure ν is a completely invariant closed set. Indeed, let $A = \mathrm{supp}(\nu)$ and χ_A be its characteristic function. Since $\int \chi_A d\nu = \int L_R(\chi_A)d\nu$ and $L_R(\chi_A) \le \chi_A$ ν-almost everywhere, then $L_R(\chi_A) = \chi_A$ ν-almost everywhere. Hence $R^{-1}(A) = A$.

Now, if

$$W = F(R) \setminus [\{\text{periodic points}\} \cup \{\text{rotational domains}\}],$$

then $A \cap W = \emptyset$. Otherwise, there are a point $z_0 \in A \cap W$ and a disk D centered at z_0 with $\nu(D) > 0$ so that $R^{-n}(D)$ forms a pairwise disjoint family of open subsets of W, for $n = 1, 2, 3, \ldots$. But this contradicts the hypothesis that ν is invariant. Since $R^{-1}(A \cap F(R)) = A \cap F(R)$, the intersection $A \cap F(R)$ consists of periodic points and, by a normal families argument, contains at most two points $\{a, b\}$. Therefore, ω, which is the atomic part of ν, consists of delta measures based on the set $\{a, b\}$.

If $A \cap J(R) = \emptyset$, then we are done. Otherwise, $A \cap J(R) = J(R)$ and, by Theorem 7, the restriction of ν on $J(R)$ is a multiple of the measure of maximal entropy, which finishes the proof. $\qquad\square$

Theorem 19. *Let $\mathcal{F}$ be an admissable family of rational maps. Assume that $S = S(\mathcal{F})$ admits a right amenable Lyubich representation. There exists a unique non-atomic Lyubich measure m_S. Moreover, $m_S = m_s$ for any $s \in S$ with $\deg(s) > 1$, where m_s is the measure of maximal entropy of s.*

Proof. By Theorem 16 and Lemma 18, there is a Lyubich measure m_S given by a measure $\sigma = \delta_{y_0}$ for some $y_0 \in \bar{\mathbb{C}}$. Since $\mathcal{F}$ contains a non-exceptional map g, by Lemma 18, the measure $m_S = \alpha(y_0)m_g + \beta(y_0)\delta_{z_0}$, $\alpha(y_0) + \beta(y_0) = 1$ for a suitable $z_0 \in \bar{\mathbb{C}}$ and m_g, is the measure of maximal entropy of g. If $\beta(y_0) \ne 0$, then z_0 is determined by the condition $g^{-1}(z_0) = z_0$.

If $\alpha(y_0) \ne 0$ for a $y_0 \in \bar{\mathbb{C}}$, then we are done by Theorem 7, since m_g is the claimed measure.

Let us show that $\alpha(y_0) \ne 0$ for some y_0. Otherwise, $\beta(y_0) = 1$ for every $y_0 \in \bar{\mathbb{C}}$ and $\mathcal{F}$ consists of polynomials, since, by conjugation, we can assume that

$z_0 = \infty$, because $s^{-1}(z_0) = z_0$ for every $s \in S$. Hence, $\mathcal{F}$ satisfies the third case of the definition of admissability.

Thus, there exists a disk $D_{\mathcal{F}}$ such that $P(D_{\mathcal{F}}) \subset D_{\mathcal{F}}$ for every $P \in S$. Let $K = \overline{\mathbb{C}} \setminus D_{\mathcal{F}}$ and take a compact subdisk $D \subset D_{\mathcal{F}}$ centered at ∞ and consider a continuous function ϕ such that $\phi(z)$ is 1 for $z \in K$ and 0 for $z \in D$. We have

$$\int_{\overline{\mathbb{C}}} \phi(z) dm_S(z) = 0.$$

Let $x_0 \in K$. Since $s^{-1}(K) \subset K$ for any $s \in S$, then, by construction of ϕ, for every $s \in S$ we have

$$H(\phi)(s) = \int_{\overline{\mathbb{C}}} \rho(s)(\phi(z)) d\delta_{x_0}(z) = (\rho(s)(\phi))(x_0) = 1,$$

where ρ is the Lyubich representation. Thus, $H(\phi)$ is the constant function 1 and $M(H(\phi)) = 1$ for every mean M. Besides, for a suitable r-invariant mean L,

$$L(H(\phi)) = \ell(\phi) = \int_{\overline{\mathbb{C}}} \phi(z) dm_S(z) = \phi(\infty) = 0,$$

which is a contradiction. $\qquad\qquad\square$

Let R be a rational map with $\deg(R) \geq 2$. We define $E(R)$ to be the set of all rational maps Q such that

$$L_Q^*(m_R) = m_R,$$

where L_Q^* is the dual operator to the Lyubich operator L_Q and m_R is the measure of maximal entropy of R. The set $E(R)$ is a semigroup under composition.

Define $G(R) = E(R) \cap \text{Mob}$. Also, $G(R)$ can be characterized as the maximal subgroup of $E(R)$.

Theorem 20. *Let P be a non-exceptional polynomial of degree at least* 2*, and S be a subsemigroup of $E(P)$ then S is right amenable.*

This theorem is reciprocal of Theorem 19.

We start with the following. Let U and T be semigroups with a homomorphism $\rho \colon T \to \text{End}(U)$. Let $U \rtimes_\rho T$ denote the semidirect product of the semigroups U and T which is the set $U \times T$ equipped with the multiplication

$$(u_1, t_1) \cdot (u_2, t_2) = (u_1 \cdot \rho(t_1)(u_2), t_1 t_2).$$

Next, proposition puts together two results of M. Klawe (see [16]).

Proposition 21. *Let U and T be semigroups and $\rho\colon T \to \mathrm{End}(U)$ be a homomorphism.*

- *If U and T are RA, then $U \rtimes_\rho T$ is RA.*
- *If U and T are amenable semigroups and $\rho(t)$ is surjective for every $t \in T$, then $U \rtimes_\rho T$ is amenable.*

Proof. These results appear as Proposition 3.10 and Corollary 3.11 in [16]. $\qquad\square$

Theorem 22. *Let P be a polynomial map such that $E(P)$ is not abelian. There are an abelian subsemigroup $\Gamma < E(P)$ and a homomorphism $\Phi\colon \Gamma \to \mathrm{End}(G(P))$ such that the semidirect product $G(P) \rtimes_\Phi \Gamma$ is isomorphic to $E(P)$. Moreover, if P is not Möbius conjugated to z^n, then Γ can be chosen as a cyclic semigroup.*

Proof. First, assume that P is not affinely conjugated to z^n for some n. Since the elements of $G(P)$ leave the Julia set $J(P)$ invariant, then $G(P)$ is a finite group of rotations around a common center (see for example Lemma 4 of [2]).

Let $Q \in E(P)$ be a non-linear polynomial of minimal degree. Since $E(P)$ is not abelian, then by the Atela–Hu theorem in [1], for every element $\widetilde{Q} \in E(P)$ there is $n \geq 0$ and a $\gamma \in G(P)$ such that $\widetilde{Q} = \gamma \circ Q^n$. But the degree of Q is minimal and $Q \circ \gamma \in E(P)$; therefore $Q \circ \gamma = \gamma' \circ Q$. Thus, the correspondence $\gamma \mapsto \gamma'$ defines a homomorphism Φ from $\langle Q \rangle$ to $\mathrm{End}(G(P))$. With Φ, construct the semidirect product $G(P) \rtimes_\Phi \langle Q \rangle$ so that the correspondence $(u, g) \mapsto u \circ g$ defines a surjective isomorphism $G(P) \rtimes_\Phi \langle Q \rangle \to E(P)$.

Now, assume P is affinely conjugated to z^n for some n. Then, $E(P)$ is not abelian and after a suitable conjugation $G(P)$ becomes $G(z^n)$ which is generated by the group of all rotations around 0 and the element $1/z$. In this situation, we can choose a subsemigroup $\Gamma \subset E(P)$ conjugated to all powers of z. Therefore, Γ is an infinitely-generated abelian semigroup acting on $G(P)$ by semiconjugacy as a semigroup of surjective endomorphisms of $G(P)$. Again, the correspondence $(u, T) \to u \circ T$ generates an isomorphism $G(P) \rtimes_\Phi \Gamma \to E(P)$. $\qquad\square$

As an immediate consequence, we have the following result.

Corollary 23. *Let P be a polynomial. Then, $E(P)$ is RA. Moreover, if P is conjugated to z^n, then $E(P)$ is amenable.*

Proof. If $E(P)$ is abelian, then it is amenable. Otherwise, the corollary follows from Proposition 21 and Theorem 22. $\qquad\square$

Given a rational map R, let

$$\mathrm{Deck}(R) = \{\gamma \in \mathrm{Mob}\colon R \circ \gamma = R\}$$

and

$$\mathrm{Aut}(R) = \{\gamma \in \mathrm{Mob}\colon R \circ \gamma = \gamma \circ R\}.$$

Corollary 24. *Let P be a polynomial with finite G(P). Let $Q \in E(P)$ be a non-injective polynomial of minimal degree. Then, there exist natural numbers m, n such that G(P) is generated by $\mathrm{Deck}(Q^m)$ and $\mathrm{Aut}(Q^n)$.*

Proof. By Theorem 22, the map Q defines an endomorphism $\Phi(Q): G(P) \to G(P)$ by the semiconjugacy $Q \circ \gamma = \Phi(Q)(\gamma) \circ Q$. Since $G(P)$ is finite, the map $\Phi(Q)$ is surjective if and only if $\Phi(Q)$ is an automorphism of $G(P)$. In this case, there exists n such that $\Phi^n(Q) = \mathrm{Id}$, and thus $Q^n \circ \gamma = \Phi^n(Q)(\gamma) \circ Q = \gamma \circ Q^n$.

If $\Phi(Q)$ is not an automorphism, then, as $G(P)$ is finite, there exists k so that Φ acts on $\Phi^k(G(P))$ as an automorphism and

$$G(P) \simeq \mathrm{Ker}(\Phi^k) \otimes \mathrm{Im}(\Phi^k).$$

Let $m > 0$ be the minimal number satisfying the equation above. Let n be the minimal natural number such that $\Phi^n: \mathrm{Im}(\Phi^m) \to \mathrm{Im}(\Phi^m) = \mathrm{Id}$. Then, every $\gamma \in \mathrm{Im}(\Phi^m)$ commutes with Q^n. On the other hand, if $\gamma \in \mathrm{Ker}(\Phi^m)$, then $Q^m(\gamma) = Q^m$. $\qquad\square$

Example 1. Let $P(z) = z^5 + z^2$. Then,

$$G(P) = \{\lambda z : \lambda^3 = 1\}$$

and

$$E(P) = \{\lambda^k P^l, \text{ for } k = 0, 1, 2, \text{ and } l = 0, 1, \dots\}$$

is amenable by Theorem 22.

Since $G(P) = \mathrm{Aut}(P^2)$, then by Corollary 24 we have $m = 0$ and $n = 2$. Therefore, the polynomial $Q = \lambda P$ commutes with P^2, but does not commutes with P. In particular, amenability does not implies commutativity even for polynomials.

Lemma 25. *Let P be a polynomial with finite G(P). There exists $\mu \in \mathrm{RIM}(E(P))$ so that $\mu(\chi_{\langle Q \rangle}) > 0$ for every $Q \in E(P)$.*

Proof. We follow Klawe's proof of Proposition 21 (Proposition 3.10 in [16]). We summarize Klawe's construction of a RIM for a semidirect product $S = U \rtimes_\rho T$ of two RA-semigroups U and T with representation $\rho: T \to \mathrm{End}(U)$ as follows.

- Choose both a RIM ϕ for U and a RIM ν for T.
- For each $f \in L_\infty(S)$, construct the function $\tilde{f} \in L_\infty(T)$ as follows. Given $a \in T$, let $f_a(u) = f(u, a)$ for $u \in U$. The family of functions $\{f_a\}$ belongs to $L_\infty(U)$. Let $\tilde{f}(a) = \phi(f_a)$.
- The mean μ on S given by

$$\mu(f) = \nu(\tilde{f})$$

  is a RIM for S.

By Theorem 22, the semigroup $E(P)$ contains a polynomial map R such that $E(P)$ is isomorphic to the semidirect product of $G(P)$ and $\langle R \rangle$. Choose a RIM ϕ for $G(R)$ and a RIM ν for $\langle R \rangle$. Since $G(R)$ is finite, $\phi(\chi_A) \geq \frac{1}{|G(P)|}$ for each subset $A \subset G(P)$. Let $Q \in E(P)$. Then, there exists a number $m \geq 0$ such that, for every $n > 0$, $Q^n = \gamma_n R^{mn}$ with $\gamma_n \in G(P)$.

If $f = \chi_{\langle Q \rangle}$ is the characteristic function of $\langle Q \rangle$ in $L_\infty(E(P))$, then the family of functions $f_{R^n}(\gamma) = f(\gamma R^n)$ belongs to $L_\infty(G(P))$. Thus, $\tilde{f}(R^n) = \phi(f_{R^n}) \in L_\infty(\langle R \rangle)$. By construction, $\tilde{f} \geq \frac{1}{|G(P)|}\chi_{\langle R^m \rangle}$. Since ν is finitely additive and r-invariant, we conclude that $\mu(f) = \nu(\tilde{f}) \geq \frac{1}{|G(P)|m} > 0$.   □

We are ready to prove Theorem 20.

Proof of Theorem 20. Let $Q \in S < E(P)$ with $Q \neq \mathrm{Id}$. By Corollary 23, the semigroup $E(P)$ is RA. By Lemma 25, there exists $\mu \in \mathrm{RIM}(E(P))$ such that $\mu(\chi_{\langle Q \rangle}) > 0$. Hence, $\mu(\chi_S) \geq \mu(\chi_Q) > 0$. We finish the proof by applying the basic fact on amenability (5) of Section 2.1: let S_0 be a subsemigroup of S; if there is $\mu \in \mathrm{RIM}(S)$ such that $\mu(\chi_{S_0}) > 0$, then S_0 is right amenable itself.   □

The following Corollary implies the proof of Theorem 4.

Corollary 26. *For an admissable collection of polynomials $\mathcal{P}$, the following conditions are equivalent*:

(1) *the semigroup $S(\mathcal{P})$ is right amenable*;

(2) *the semigroup $S(\mathcal{P})$ is Lyubich right amenable*;

(3) *there exists $P \in \mathcal{P}$ such that $\mathcal{P} \subset E(P)$.*

Proof. Part (1) implies (2) by Theorem 6. Part (2) implies (3) by Theorem 19. Part (3) implies (1) by Theorem 20.   □

Theorem 27. *For an admissable collection $\mathcal{F}$ of non-injective rational maps, the following affirmations are equivalent*:

(1) *the semigroup $S(\mathcal{F})$ is RA and $\mathcal{F} \times \mathcal{F} \subset \mathrm{DIP}$;*

(2) $S(\mathcal{F}) \times S(\mathcal{F}) \subset \mathrm{AIP}$;

(3) *the semigroup $S(\mathcal{F})$ is RA and embeddable into a group.*

To prove Theorem 27 we need the following facts.

Theorem 28. *Let $S = \langle s_1, \ldots, s_m \rangle$ be a finitely generated right cancellative semigroup satisfying the Levin relations $s_i \circ s_j = s_i^2$ for all i, j. Then,*

• *the semigroup S is right amenable;*

• *if S is left amenable then S is cyclic.*

Proof. For the first point, the proof uses standard ergodic arguments. Let m be the number of generators of S. By the Levin relations, we have the following dichotomy for any pair of elements s_i and s_j in the generating set. Namely, either

$$s_i = s_j$$

or

$$\langle s_i \rangle \cap \langle s_j \rangle = \emptyset.$$

Indeed, if there are numbers k, q such that $s_i^k = s_j^q$, then by the Levin relations $k = q$ and $s_i^{k+1} = s_j^{k+1}$ but by right cancellation we have $s_i = s_j$.

Let $\mathcal{L}(s)$ be the length function that is the infimum of the length of s as a word in the letters $\{s_1, \ldots, s_m\}$. Since every element of S is the iteration of a generator, then $\#\{s : \mathcal{L}(s) \leq n\} = mn + 1$.

The semigroup S acts by the right on $L_\infty(S)$. The spherical average Θ of this action is given by

$$\Theta_n(\phi) = \frac{1}{mn + 1} \sum_{\mathcal{L}(s) \leq n} r_s(\phi).$$

Note that, for each n, the operator Θ_n is positive with

$$\|\Theta_n\| = 1 \quad \text{and} \quad \Theta_n(\chi_S) = \chi_S,$$

where χ_S is the characteristic function on S.

We claim that, if h is a generator of S, then

$$\|r_h(\Theta_n(\phi)) - \Theta_n(\phi)\| \leq \frac{2m\|\phi\|}{mn + 1} \quad \text{for every } \phi \in L_\infty(S).$$

Indeed, by the Levin relations for the right action of S,

$$r_h(\Theta_n(\phi)) - \Theta_n(\phi) = \frac{1}{mn + 1} \left[\sum_{i=1}^m r_{s_i}^{n+1}(\phi) - \sum_{s_i \neq h} r_{s_i}(\phi) - \phi \right],$$

but the right action is a contraction, that is, $\|r_h\| \leq 1$. So the claim follows.

Let $\mathcal{M}$ be an L_1-mean on $L_\infty(S)$; that is, $\mathcal{M}$ is induced by a non-negative function $\omega : S \rightarrow \mathbb{C}$ with L_1-norm $\|\omega\| = \sum_{s \in S} \omega(s) = 1$ and $\mathcal{M}(\phi) = \sum_{s \in S} \phi(s)\omega(s)$. Consider the family of means $\mathcal{M}_n = \Theta_n^*(\mathcal{M})$, where Θ_n^* is the dual operator of Θ_n. Then, $\mathcal{M}_n$ forms a precompact family in the $*$-weak topology. Note that $\mathcal{M}_n(\chi_S) = 1$, since $\Theta(\chi_S) = \chi_S$, so we get that any accumulation point of $\{\mathcal{M}_n\}$ is a mean. If $\mathcal{M}_0$ is an accumulation point, then by the claim $\mathcal{M}_0$ is invariant by the right action of any generator of S. Hence, $\mathcal{M}_0 \in \mathrm{RIM}(S)$, which finishes the first part of the proof.

The last part is by contradiction. Assume that $S = \langle s_1, \ldots, s_m \rangle$ is left amenable with $m > 1$ and $\{s_i\}_{i=1}^m$ forms a minimal set of generators. By the dichotomy

above, $S = \bigsqcup_{i=1}^m \langle s_i \rangle$. Take $L \in \mathrm{LIM}(S)$. Since $L(\chi_S) = 1$, there exists a generator s_i such that $L(\chi_{\langle s_i \rangle}) > 0$. As $m > 1$, let s_j be a generator with $i \neq j$ and let $\Gamma = \langle s_i, s_j \rangle$. Then, $L(\chi_\Gamma) = L(\chi_{\langle s_i \rangle} + \chi_{\langle s_j \rangle}) > 0$. By the basic fact on amenability (5), the semigroup Γ is a non-cyclic LA-semigroup with two generators.

Let us show that $\langle s_i \rangle \cap \langle s_j \rangle \neq \emptyset$. Indeed, if $\langle s_i \rangle \cap \langle s_j \rangle = \emptyset$, then, by the Levin relations, $s_j \circ s \in \langle s_j \rangle$ for every $s \in \Gamma$. Hence,

$$l_{s_j}(\chi_{\langle s_i \rangle})(s) = \chi_{\langle s_i \rangle}(s_j \circ s) = 0.$$

By left invariance, $L(\chi_{\langle s_i \rangle}) = L(l_{s_j}(\chi_{\langle s_i \rangle})) = 0$, which contradicts the choice of s_i. Hence, $\langle s_i \rangle \cap \langle s_j \rangle \neq \emptyset$. Therefore, by the dichotomy above, $s_i = s_j$, which is again a contradiction with the choice of s_j. $\qquad\square$

Theorem 29. *Let $S = S(\mathcal{F})$ be an amenable semigroup satisfying* $\mathrm{RIM}(S) \subset \mathrm{LIM}(S)$, *where $\mathcal{F}$ is an admissable collection of rational maps. For every f and h in S with $\deg(f), \deg(h) > 1$, there are numbers m, n satisfying $f^m = h^n$.*

Proof. Fix two arbitrary elements $f, h \in S$ with $\deg(f), \deg(h) > 1$. By Theorem 19, the maps f and h have the same measure of maximal entropy, and, by Theorem 10, there are iterates F and H of f and h, respectively, satisfying the Levin relations. Let $\Gamma = \langle F, H \rangle$ be the semigroup generated by F and H. By Theorem 28, the semigroup Γ is RA. If Γ is LA, then, again by Theorem 28, $F = H$ and we are done.

Let us show that indeed Γ is a left amenable semigroup. We follow a theorem of Granirer (see Theorem E2 in [14]) as well as the arguments of the proof of this theorem. The theorem states the following.

> *Let S be an LA-semigroup with left cancellation and let $S_0 \subset S$ be an LA-subsemigroup. There exists a linear isometry T from the subspace of left invariant elements of $L_\infty^*(S_0)$ into the subspace of left invariant elements of $L_\infty^*(S)$ with $T(\mathrm{LIM}(S_0)) \subset \mathrm{LIM}(S)$.*

More precisely, using the left cancellation and the left cosets of S_0 in S, Granirer constructs an isometric linear section $j: L_\infty(S_0) \to L_\infty(S)$ to the restriction $\rho: L_\infty(S) \to L_\infty(S_0)$, which is a positive linear map, so that, for every left invariant functional $\nu_0 \in L_\infty^*(S_0)$, the following formula holds (see p. 55 of [14]):

$$T(\nu_0)(x) = \nu_0(\rho(x)), \quad \text{for all } x \in j(L_\infty(S_0)).$$

Let $s \in S_0$ and $r_s, \tilde{r}_s$ be the right action of s on $L_\infty(S)$ and $L_\infty(S_0)$ respectively. For every $x \in L_\infty(S)$,

$$\rho(r_s(x)) = \tilde{r}_s \rho(x).$$

If $T(\nu_0) \in \mathrm{RIM}(S)$ and $r_s(j(x_0)) - j(\tilde{r}_s(x_0)) \in \mathrm{Ker}(T(\nu_0))$ for every $x_0 \in L_\infty(S_0)$, then $\nu_0 \in \mathrm{RIM}(S_0)$. Indeed,

$$\nu_0(\tilde{r}_s(x_0)) = T(\nu_0(j(\tilde{r}_s(x_0)))) = T(\nu_0)(r_s(j(x_0))) = T(\nu_0)(j(x_0)) = \nu_0(x_0).$$

Suppose that $T(\nu_0) \in \mathrm{RIM}(S)$. We claim

$$r_s(j(x_0)) - j(\tilde{r}_s(x_0)) \in \mathrm{Ker}(T(\nu_0)).$$

Indeed, if $f = j(\chi_{S_0})$, then

$$T(\nu_0)(f) = \nu_0(\rho(f)) = \nu_0(\chi_{S_0}) = 1.$$

Moreover, $T(\nu_0)(\chi_S) = 1$, thus $\chi_S - f \in \mathrm{Ker}(T(\nu_0))$. But $T(\nu_0)$ is a positive functional and $\chi_S - f \geq \chi_{S \setminus \mathrm{supp}(f)}$. Then, $\phi \in \mathrm{Ker}(T(\nu_0))$ whenever $\mathrm{supp}(\phi) \in S \setminus \mathrm{supp}(f)$. Since j is a positive isometric section of the restriction map ρ, for every $s \in S_0$ and $x_0 \in L_\infty(S_0)$ we have

$$\mathrm{supp}(r_s(j(x_0)) - j(\tilde{r}_s(x_0))) \subset S \setminus \mathrm{supp}(j(\chi_{S_0}))$$

as claimed.

By assumption, $T(\nu_0) \in \mathrm{LIM}(S)$ whenever $\nu_0 \in \mathrm{LIM}(S_0)$. Hence, by the claim, if $\mathrm{LIM}(S) \subset \mathrm{RIM}(S)$, then $\nu_0 \in \mathrm{RIM}(S_0)$ and, in particular, $\mathrm{LIM}(S_0) \subset \mathrm{RIM}(S_0)$.

To apply Granirer Theorem and the discussion above, we consider S^* to be the semigroup S endowed with the antiproduct. Since S is amenable with right cancellation and $\mathrm{RIM}(S) \subset \mathrm{LIM}(S)$, then S^* is an amenable semigroup with left cancellation and $\mathrm{LIM}(S^*) \subset \mathrm{RIM}(S^*)$, hence $\Gamma^* \subset S^*$ is left amenable. It follows that Γ^* is an RA semigroup and hence Γ is left amenable. This finishes the proof. $\qquad\square$

As a corollary, we get the following statement.

Corollary 30. *Let S be a right cancellative amenable semigroup satisfying* $\mathrm{RIM}(S) \subset \mathrm{LIM}(S)$. *Consider an RA subsemigroup $S_0 < S$. Then,* $\mathrm{RIM}(S_0) \subset \mathrm{LIM}(S_0)$.

Theorem 31. *Assume $\mathcal{F} \times \mathcal{F} \subset \mathrm{AIP}$ for a non-exceptional collection of rational maps $\mathcal{F}$. Then, $S = S(\mathcal{F})$ is amenable.*

Proof. The proof of the theorem is a consequence of a theorem due to M. Day [6] which states the following.

> *Let $S = \bigcup S_n$, where S_n are semigroups such that for every m, n there exists k with $S_m \cup S_n \subset S_k$. Then, S is amenable whenever the semigroups S_n are amenable for every n.*

Fix a non-exceptional element $Q \in \mathcal{F}$. For $n > 0$, let S_n be the subsemigroup of all elements in S commuting with Q^n. By assumption, $S = \bigcup_n S_n$. Moreover, for every finite collection of indexes $n_1, \ldots, n_k$, there exists N such that $\bigcup_{i=1}^{k} S_{n_i} \subset S_N$; for instance, take $N = \prod n_i$. To finish the proof we, have to show that the semigroups S_n are amenable.

Indeed, for each n, let

$$M_n(Q) = \{\mu \in L_\infty^*(S_n) : l_{Q^n}^*(\mu) = r_{Q^n}^*(\mu) = \mu, \mu \geq 0, \mu(\chi_{S_n}) = \|\mu\| = 1\}.$$

$M_n(Q)$ is a non-empty, closed, convex, and compact set with respect to the $*$-weak topology of $L_\infty^*(S_n)$. In fact, $M_n(Q)$ is a subset of the unit sphere in $L_\infty^*(S_n)$ so does not contains the zero element. Fix n. Since every element $s \in S_n$ commutes with Q^n, then r_s^* and l_s^* leave $M_n(Q)$ invariant. Hence, we constructed left and right representations ρ_l and ρ_r of S_n into the semigroup $\mathrm{End}(M_n)$ of continuous endomorphisms of M_n.

By Theorem 9, every element of S_n shares a common iteration with Q^n. Thus, the images $\Gamma_l := \mathrm{Im}(\rho_l)$ and $\Gamma_r := \mathrm{Im}(\rho_r)$ are groups in $\mathrm{End}(M_n)$.

If Γ_l and Γ_r are finite groups, then S_n is amenable for every n. Indeed, as M_n is convex, the averages defined by $A_r(\nu) = \frac{1}{\#\{\Gamma_r\}} \sum_{\gamma \in \Gamma_r} \gamma(\nu)$ leaves M_n invariant. This means that $A_r(M_n) = \mathrm{RIM}(S_n)$. Similarly, if $A_l(\nu) = \frac{1}{\#\Gamma_l} \sum_{\gamma \in \Gamma_l} \gamma(\nu)$, then $A_l(M_n) = \mathrm{LIM}(S_n)$. But we have $A_r^2 = A_r$, $A_l^2 = A_l$ and $A_r \circ A_l = A_l \circ A_r$. Therefore, $A_r \circ A_l(M_n) \subset \mathrm{LIM}(S_n) \cap \mathrm{RIM}(S_n)$. Thus, S_n is amenable.

To finish, we need the following result of F. Pakovich ([28]).

Let f be a non-exceptional rational map of degree at least 2 and let $C(f)$ be the semigroup of all rational maps commuting with f. There are finitely many rational maps $x_1, \ldots, x_k \in C(f)$ such that every $g \in C(f)$ has the form $g = x_i \circ f^l$ for some i and $l \geq 0$.

Hence, Γ_r and Γ_l belong to the image of a finite set of elements. So, these groups are indeed finite. □

In the proof of the previous theorem, it is enough that Γ_l and Γ_r are amenable. While we were completing this work, Pakovich kindly inform us about his theorem in [28] which significantly shortened our original proof of Theorem 31.

We are ready to prove Theorem 27.

Proof of Theorem 27. Let us show that (1) implies (2). Since S is RA, then, by Theorem 10 and Theorem 19, for every P and Q in $\mathcal{F}$ there are numbers m and n such that $P^m \circ Q^n = P^{2m}$ and $Q^n \circ P^m = Q^{2n}$. Let us show that $P^m = Q^n$. Indeed, the pair (P^m, Q^n) satisfies the dynamical intersection property. So, there exist z_0 and sequences k_i, l_i of natural numbers such that $P^{mk_i}(z_0) = Q^{nl_i}(z_0)$.

First, assume that $k_i = l_i$. Then, by the Levin relations,

$$Q^n \circ Q^{n(k_i-1)}(z_0) = P^m \circ P^{m(k_i-1)}(z_0) = P^m \circ Q^{n(k_i-1)}(z_0).$$

So, P^m and Q^n coincide on the infinite set $\{Q^{n(k_i-1)}(z_0)\}$, and thus $P^m = Q^n$.

If $k_i \neq l_i$, then, again using the Levin relations,

$$P^{m(k_i+1)}(z_0) = P^m \circ Q^{nl_i}(z_0) = P^{m(l_i+1)}(z_0).$$

Hence, z_0 has a finite orbit, which contradicts that $(P, Q) \in$ DIP. For a non-exceptional $Q \in \mathcal{F}$ and for every $P \in \mathcal{F}$, there exists n such that P commutes with Q^n. So, for every element $R \in S$, there exists a number $m = m(R)$ such that R commutes with Q^m. Thus, $\langle R, Q^m \rangle$ is abelian and, by Theorem 9, R and Q share a common iteration. Therefore, every pair of elements in S also share a common iteration.

Theorem 31 gives the implication from (2) to (1).

Let us show the equivalence of (2) and (3). First, let us show that (3) implies (2). Since S is RA, by Theorem 10 and Theorem 19, if R and Q are non-identity elements in S, then there exist numbers n and m such that R^m and Q^n satisfy the Levin relations, but in this case $R^m = Q^n$, since S is embeddable into a group.

(2) implies (3). First let us show that S is a cancellative semigroup. We already know that S is right cancellative, so let us show that is also left cancellative. Assume there exist three elements A, X, Y in S with $AX = AY$. Then, $P = XA$, and $Q = YA$ satisfy the Levin relations

$$P \circ Q = P \circ P,$$
$$Q \circ P = Q \circ Q.$$

By assumption, there are numbers m and n such that $P^m = Q^n$. Therefore, $m = n$ since $\deg(P) = \deg(Q)$. Again, by the Levin relations, $P^{m+1} = Q^{m+1}$. Therefore,

$$XA = P = Q = YA$$

and $X = Y$ by right-cancellation. By Lemma 13, S has the left ideal intersection property. We have fulfilled the conditions of the Ore Theorem, which finishes the proof. □

Remark. Let us note that in the proof of Theorem 27 it is enough that the intersection $\mathcal{O}_+(P^m, z_0) \cap \mathcal{O}_+(Q^n, z_0)$ is sufficiently large. For instance, if

$$\#\{\mathcal{O}_+(P^m, z_0) \cap \mathcal{O}_+(Q^n, z_0)\} > \deg(P^m)\deg(Q^n),$$

then the arguments still follow. Since bounds are invariants of the semigroup, it is interesting to find precise bounds on the intersection of the orbits.

Another conclusion that follows from Theorem 27 is that r-amenability is necessary to compare the intersection property with the algebraic property of sharing a common iterate. As an immediate corollary, we have:

Corollary 32. *Let R be a non exceptional rational map and let $S(R)$ be the semigroup of rational maps commuting with R. Then, $S(R)$ is an embeddable semigroup.*

Proof. By Theorem 9 and Theorem 31, $S(R) \subset$ AIP and $S(R)$ is amenable. By Corollary 14 it is enough to show that $S(R)$ is left cancellative. Let $\Gamma(R) = S(R) \cap$ Mob. Then, $S(R)$ is generated by $\Gamma(R)$ and $S(R) \setminus \Gamma(R)$. But, by Theorem 9, $S(R) \setminus \Gamma(R)$ is a semigroup satisfying the conditions of Theorem 27 (2), so $S(R) \setminus \Gamma(R)$ is cancellative.

Finally, if there exist $Q \in S(R) \setminus \Gamma(R)$ and $a, b \in \Gamma(R)$ with $a \neq b$ and $Q \circ a = Q \circ b$, then $\text{Deck}(Q)$ contains $a \circ b^{-1}$ which belongs to $\Gamma(R)$. Therefore, Q and $ab^{-1}Q$ satisfy the Levin relations, since Q and $ab^{-1}Q$ share a common iterate. So, $ab^{-1} = \text{Id}$ which is a contradiction. $\qquad\square$

In the theorems above, we used the equation $X \circ A = X \circ B$ to study the left-cancellation of semigroups. The maps $P = A \circ X$ and $Q = B \circ X$ satisfy the Levin relations and, by Theorem 10, P and Q have the same measure of maximal entropy. In [27], Pakovich proved the reciprocal theorem, that is:

> *if $P \circ Q = P \circ P$ and $Q \circ P = Q \circ Q$, then there are rational maps X, A, B such that $P = A \circ X$ and $Q = B \circ X$ and $X \circ A = X \circ B$. So, if P and Q satisfy the Levin relation, then, by the Pakovich theorem above, we get $X \circ A = X \circ B$.*

Definition. Given a semigroup $S < $ Rat, we write $A \approx B$ if there exists $X \in G$ so that $X \circ A = X \circ B$.

In general, $\approx$ is not an equivalence relation. However, for instance, if S satisfies the left ideal intersection property, then $\approx$ is an equivalence relation in S. It is interesting to characterize when the equation $X \circ A = X \circ B$ defines an equivalence relation on semigroups of rational maps.

Proposition 33. *Let $S = S(\mathcal{F})$ be an RA-semigroup, where $\mathcal{F}$ is an admissable collection of rational maps. Then, $\approx$ defines an equivalence relation and the quotient semigroup $S_1 = S/\approx$ is embeddable into a group. Moreover, if $\pi : S \to S_1$ is a projection homomorphism, then, for every $P, Q \in S$ with $\deg(P), \deg(Q) \geq 2$, there are numbers m and n so that $\pi(P)^m = \pi(Q)^n$.*

Proof. The proof that $\approx$ is an equivalence relation relies on standard amenability arguments (see for example [16]).

To verify that $\approx$ is an equivalence relation, it is enough to check that it is transitive. Indeed, assume that there are $a, b, c \in S$ such that $a \approx b$ and $b \approx c$. Then, there are $f_1, f_2 \in S$ with $f_1 a = f_1 b$ and $f_2 b = f_2 c$. By Lemma 13, the semigroup S has the left ideal intersection property, so there are $\alpha, \beta \in S$ such that $\alpha f_1 = \beta f_2$, and

$$\alpha f_1 a = \alpha f_1 b = \beta f_2 b = \beta f_2 c = \alpha f_1 c.$$

The multiplication induced over representative classes endows $S_1 = S/_{\approx}$ with a semigroup product. By the basic fact on amenability (6), S_1 is an RA-semigroup. In particular, S_1 has the left ideal intersection property. To show that S_1 is embeddable into a group, by the Ore Theorem we need to verify that S_1 is cancellative. First, S_1 is r-cancellative since S is also r-cancellative and, by construction, S_1 is l-cancellative.

Finally, since S is RA, by Theorem 19 and Theorem 10, for every $P, Q \in S$ there are numbers m, n such that P^m and Q^n satisfy the Levin relations. Since S is embeddable into a group, then $\pi(P^m) = \pi(Q^n)$ as claimed. $\qquad\square$

The following corollary produces, in the polynomial case, a realization for semigroups of the type of S_1 in the proposition above.

Corollary 34. *Let $S = S(\mathcal{F})$ be an RA semigroup, where $\mathcal{F}$ is an admissable collection of polynomials. There exist a polynomial P and an isomorphism*

$$\phi: S/_{\approx} \longrightarrow E(P).$$

Proof. Since S is RA, by Theorem 19, Theorem 20, and Theorem 22, there exist a polynomial P such that $S \subset E(P)$ and numbers r, s, and t so that every element $Q \in S$ has the form $Q = \gamma \circ h \circ P^t$, where $\gamma \in \mathrm{Deck}(P^s)$ and $h \in \mathrm{Aut}(P^r)$. Thus, the class $[Q]$ contains a unique element $h \circ P^t$ and then the correspondence $[Q] \mapsto h \circ P^t$ induces the desired representation. $\qquad\square$

For arbitrary semigroups of rational maps, the previous corollary is still an open question. For polynomials, we have the following theorem.

Theorem 35. *Let $\mathcal{F}$ be a non-exceptional family of polynomials with $\mathcal{F} \times \mathcal{F} \subset$ AIP. Then, the semigroup $S(\mathcal{F})$ is amenable with $\mathrm{RIM}(S(\mathcal{F})) \subset \mathrm{LIM}(S(\mathcal{F}))$.*

Proof. Let $P \in \mathcal{F}$ be non-exceptional. Then, by the conditions, $S(\mathcal{F}) \subset E(P)$. If $E(P)$ is abelian, then $S(\mathcal{F})$ is abelian, and hence $S(\mathcal{F})$ is amenable with $\mathrm{RIM}(S(\mathcal{F})) = \mathrm{LIM}(S(\mathcal{F}))$. Otherwise, by Theorem 22, there are a polynomial $T \in E(P)$, a finite group $G(P) = E(P) \cap \mathrm{Mob}$, and a representation $\rho: \langle T \rangle \to \mathrm{End}(G(P))$ by semiconjugation so that $E(P) \cong G(P) \rtimes_\rho \langle T \rangle$. By Corollary 24, there are numbers r, s such that the group $G(P)$ is the direct product of $K(P) = \mathrm{Ker}(\rho(T^r))$ and $A(P) = \mathrm{Aut}(T^s)$. Let $AE(P)$ be the subsemigroup of $E(P)$

generated by T and $A(P)$. We claim that every subsemigroup Γ in $AE(P)$ is amenable with RIM(Γ) $\subset$ LIM(Γ).

To prove the claim, first let us show that

$$\sum_{\gamma \in A(P)} r_\gamma(\psi) = \sum_{\gamma \in A(P)} l_\gamma(\psi)$$

for every $\psi \in L_\infty(AE(P))$. Indeed, if $s \in AE(P)$, then $s = h \circ T^k$ for a suitable $h \in A(P)$ and $k \geq 0$. Since $\rho(T)$ is an automorphism of $A(P)$,

$$\sum_{\gamma \in A(P)} r_\gamma(\psi)(s) = \sum_{\gamma \in A(P)} \psi(h \circ T^k \circ \gamma)$$
$$= \sum_{\gamma \in A(P)} \psi((\rho(T))^k(\gamma) \circ h \circ T^k)$$
$$= \sum_{\gamma \in A(P)} l_\gamma(\psi)(h \circ T^k).$$

Therefore, for every $\phi \in \text{RIM}(AE(P))$, the averages

$$A_r = \frac{1}{\#\{A(P)\}} \sum_{\gamma \in A(P)} r_\gamma$$

and

$$A_l = \frac{1}{\#\{A(Q)\}} \sum_{\gamma \in A(P)} l_\gamma$$

satisfy

$$\phi = A_r^*(\phi) = A_l^*(\phi).$$

Since $l_\gamma^* \circ A_l^* = A_l^* \circ l_\gamma^* = A_l^*$, we conclude that $l_\gamma^* \phi = \phi$ for every $\gamma \in A(P)$. In other words, every right invariant mean ϕ is invariant by the left action of $A(P)$.

Second, let us show that

$$l_T^*(\phi) = \phi.$$

For every $\psi \in L_\infty(AE(P))$,

$$A_l(l_T(\psi))(s) = A_l(r_T(\psi))(s).$$

Indeed, let $s = h \circ T^k$. Then,

$$A_l(\psi(T \circ h \circ T^k)) = A_l(\psi[(\rho(T))(h) \circ h^{-1}(h \circ T^k \circ T)])$$
$$= A_l(\psi(h \circ T^k \circ T))$$
$$= A_l(r_T(\psi))(s).$$

By duality and the fact that ϕ is left invariant under $A(P)$, we get

$$l_T^*(A_l^*(\phi)) = r_T^*(A_l^*(\phi)) = \phi,$$

which implies $l_T^*(\phi) = \phi$. Hence, $\phi \in \mathrm{LIM}(AE(P))$.

Third, let $\Gamma < AE(P)$. Then, Γ is right cancellative and by Corollary 23 the semigroup Γ is RA. Therefore, by Corollary 30, $\mathrm{RIM}(\Gamma) \subset \mathrm{LIM}(\Gamma)$, as claimed.

To finish the proof of the theorem, we have to show that $S(\mathcal{F})$ is isomorphic to a subsemigroup of $AE(P)$.

Let $Q \in \mathcal{F}$ be a polynomial of minimal degree. Then, Q has the expression

$$Q = h \circ \gamma \circ T^l$$

for a suitable $l \geq 1$, $h \in K(P)$, and $\gamma \in A(P)$. Fix $m \leq r$ such that one has $h \in \mathrm{Ker}(\rho(T^m))$ and put

$$\tilde{h} = h \circ \rho(T^l)(h) \circ \rho(T^{2l}) \circ \cdots \circ \rho(T^{(m-1)l})(h).$$

Then, $\tilde{h}^{-1} \circ Q \circ \tilde{h} = \gamma \circ T^l$. The family $\widetilde{\mathcal{F}} = \tilde{h}^{-1} \circ \mathcal{F} \circ \tilde{h}$ generates a semigroup $S(\widetilde{\mathcal{F}})$ isomorphic to $S(\mathcal{F})$. Note that $\tilde{Q} = \tilde{h}^{-1} \circ Q \circ \tilde{h} = \gamma \circ T^l \in \widetilde{\mathcal{F}}$.

Now, let us show that $S(\widetilde{\mathcal{F}}) < AE(P)$. It is enough to show that $\widetilde{\mathcal{F}} \subset AE(P)$. Otherwise, assume that $\widetilde{\mathcal{F}}$ contains a polynomial $R = \alpha \circ \beta \circ T^t$ for $t \geq 1$, $\alpha \in K(P) \setminus \{\mathrm{Id}\}$ and $\beta \in A(P)$. By assumption, there are numbers $d, e > 0$ such that $R^e = \tilde{Q}^d$, hence R commutes with $\tilde{Q}^d$. If $i = \#\{K(P)\}$ and $j = \#\{A(P)\}$, then, for $k = ijd$,

$$R \circ \tilde{Q}^k = \alpha \circ \beta \circ \rho(T^l)(\gamma) \circ T^t \circ T^{kl} = \tilde{Q}^k \circ R = \gamma \circ \beta \circ T^{kl} \circ T^t.$$

Therefore, $\alpha \in A(P)$, which is a contradiction by Corollary 24. $\square$

Theorem 36. *Given an admissable collection of polynomials $\mathcal{F}$, the following statements are equivalent:*

(1) $\mathcal{F} \times \mathcal{F} \subset \mathrm{DIP}$;

(2) $\mathcal{F} \times \mathcal{F} \subset \mathrm{AIP}$;

(3) $S(\mathcal{F}) \times S(\mathcal{F}) \subset \mathrm{DIP}$;

(4) $S(\mathcal{F}) \times S(\mathcal{F}) \subset \mathrm{AIP}$;

(5) $S(\mathcal{F})$ *is amenable with* $\mathrm{RIM}(S(\mathcal{F})) \subset \mathrm{LIM}(S(\mathcal{F}))$;

(6) *the semigroup $S(\mathcal{F})$ is embeddable into a virtually cyclic group.*

Proof. By the Ghioca–Tucker–Zieve Theorem in [13] (1) is equivalent to (2) and (3) is equivalent to (4). Clearly (3) implies (1). By Theorem 29 (5) implies (4). Finally, by Theorem 35 (2) implies (5).

(4) implies (6). By Theorem 27, the semigroup $S(\mathcal{F})$ is embeddable into a group. As in the proof of Theorem 35, there exist a polynomial T and a finite group $A(T) \subset$ Mob so that T acts on $A(T)$ by semiconjugacy and generates a representation $h \colon \langle T \rangle \to \mathrm{Aut}(A(T))$. The semigroup $AE(T) = \langle T, A(T) \rangle \cong A(T) \rtimes_h \langle T \rangle$ contains an isomorphic copy of $S(\mathcal{F})$. Let us show that $S(\mathcal{F})$ is a subsemigroup of a virtually cyclic group. First note that $AE(T) \cong A(T) \rtimes_{\tilde{h}} \mathbb{N}$, where $\tilde{h}(n) = h(T^n) \in \mathrm{Aut}(A(T))$, and $\mathbb{N}$ is the additive semigroup. Since $\tilde{h}(n)$ is an automorphism, we can extend $\tilde{h}$ on negative integers by the formula

$$\tilde{h}(-n) = (h(T^n))^{-1}.$$

Hence, $A(T) \rtimes_{\tilde{h}} \mathbb{N} \subset A(T) \rtimes_{\tilde{h}} \mathbb{Z}$. But $A(T) \rtimes_{\tilde{h}} \mathbb{Z}$ is a semidirect product of a cyclic group with a finite group, so it is virtually cyclic. Therefore, $AE(T)$ is the positive part of a virtually cyclic group.

Let us prove that (6) implies (4). Assume that $S(\mathcal{F})$ is embeddable into a virtually cyclic group Γ and that τ is the generator of the corresponding cyclic subgroup of finite index. Let T be an element in $S(\mathcal{F})$ corresponding to τ, let $P \in S(\mathcal{F})$ be of degree at least 2, and let $p \in \Gamma$ be the corresponding element. Then, p is an element of infinite order, so there exists k such that $p^k \in \langle \tau \rangle$, and hence $(P, T) \in \mathrm{AIP}$. □

The last section is devoted to the proof of Theorem 5.

4. Left amenability of Ruelle representation

We begin with the following definition. We say that a semigroup $S < $ Rat is *deformable* if there exists $f \colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ a quasiconformal homeomorphism so that $S_f = f \circ S \circ f^{-1} < $ Rat and S_f is not Möbius conjugated to S.

Proposition 37. *Let S be an RA semigroup of non-injective rational maps. If S contains a hyperbolic structurally stable map, then S is deformable.*

Proof. Let R be a hyperbolic structurally stable element of S. By Theorem 19, for every $Q \in S$ with $\deg(Q) > 1$ we have $J(Q) = J(R)$. By the Levin relations, we have the following facts:

- every $Q \in S$ with $\deg(Q) > 1$ is hyperbolic;
- for every periodic component D in the Fatou set $F(R) = \bar{\mathbb{C}} \setminus J(R)$,

$$Q^{-1}(\mathcal{O}_-(R, D)) = \mathcal{O}_-(R, D)$$

for every $Q \in S$ and where $\mathcal{O}_-(R, D) = \bigcup_{n=0}^{\infty} R^{-n}(D)$.

Let $K_D: \mathcal{O}_-(R, D) \to \mathbb{C}$ be the König linearizing function D for R in $\mathcal{O}_-(R, D)$, so $K_D(R) = \lambda K_D$ for some multiplier λ. Note that K_D also linearizes every $Q \in S$ with $\deg(Q) > 1$. Indeed, by the Levin relations, we have m and n so that $R^n \circ Q^m = R^n \circ R^n$ and $Q^m \circ R^n = Q^m \circ Q^m$, and so $K_D(Q^m) = \lambda^n K_D$. The differential

$$\mu = \frac{\overline{K_D'} K_D}{\overline{K_D} K_D'} \frac{d\bar{z}}{dz}$$

is invariant for Q and R. So, μ is an invariant Beltrami differential for every element in S. It follows that, for $t \in (0, 1)$, if g_t is the quasiconformal map with Beltrami coefficient $t\mu$, then g_t defines a non-trivial deformation for S. □

Remark. Let us note the following curious fact. If the semigroup S is quasiconformally deformable with Beltrami differential μ such that $\overline{\text{supp}(\mu)} \neq \overline{\mathbb{C}}$, then, for all $R, Q \in S$ with $\deg(R), \deg(Q) > 1$, we have $J(R) = J(Q)$.

For a subclass of RA semigroups, we can say more. Let $S <$ Rat be a semigroup and let $\phi: S \to$ Rat be an monomorphism preserving the degree, that is $\deg(\phi(Q)) = \deg(Q)$ for every $Q \in S$. We say that S is *structurally stable* if every monomorphism preserving degree $\phi: S \to$ Rat, which is sufficiently close to the identity on generators, is generated by a quasiconformal homeomorphism of $\overline{\mathbb{C}}$.

Proposition 38. *Let $\mathcal{F} = \{R_i\}$ be a finite collection of rational maps of degree at least 2 such that $R_i \circ R_j = R_i^2$ for every pair i, j. Then, the semigroup $S(\mathcal{F})$ is structurally stable whenever $S(\mathcal{F})$ contains a structurally stable map.*

Proof. If $g \in S(\mathcal{F})$ is structurally stable, then g is indecomposable and therefore is one of the generators, say R_1. If $\phi: S(\mathcal{F}) \to$ Rat is a representation sufficiently closed to the identity representation, then there exists a quasiconformal homeomorphism $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\phi(R_1) = f \circ R_1 \circ f^{-1}$. We claim that $\phi(Q) = f \circ Q \circ f^{-1}$ for every $Q \in S(\mathcal{F})$.

It is enough to check the latter equality holds for the generators R_i. If $\mu = \frac{\bar{\partial} f}{\partial f}$, then μ is invariant for all generators by the Levin relations.

First, let us assume that $R_i = \gamma_i \circ R_1 \circ \gamma_i^{-1} = \gamma_i \circ R_1$ and $\phi(R_i) = h_i \circ \phi(R_1) \circ h_i^{-1}$ for suitable $h_i \in \text{Deck}(\phi(R_1))$ and $\gamma_i \in \text{Deck}(R_1)$, respectively. Since γ_i leaves μ invariant, we have that $\text{Deck}(\phi(R_1)) = f \circ \text{Deck}(R_1) \circ f^{-1}$. If $T = \langle S(\mathcal{F}), \text{Deck}(R_1) \rangle$, then $\phi(S(\mathcal{F})) \subset f \circ T \circ f^{-1} \subset \text{Rat}$.

As $\phi(R_i)$ is close to R_i for all i, it follows that h_i is close to γ_i and γ_i is close to $f \circ \gamma_i \circ f^{-1}$, as f is close to the identity too.

Therefore, $h_i, f \circ \gamma_i \circ f^{-1} \in \text{Deck}(\phi(R_1))$ are sufficiently close and hence coincide since $\text{Deck}(\phi(R_1))$ is discrete. In conclusion, $\phi(S(\mathcal{F})) = f \circ S(\mathcal{F}) \circ f^{-1}$ as claimed.

It remains to show that $R_i = \gamma_i \circ R_1 \circ \gamma_i^{-1} = \gamma_i \circ R_1$ and $\phi(R_i) = h_i \circ \phi(R_1) \circ h_i^{-1}$ for suitable $h_i \in \mathrm{Deck}(\phi(R_1))$ and $\gamma_i \in \mathrm{Deck}(R_1)$ for every i.

Since $R_1 \circ R_i = R_1^2$, we have that $\mathrm{Deck}(R_i) \subset \mathrm{Deck}(R_1)$. Hence, by Theorem 9, the maps R_1 and R_i share a common right factor, that is, there are rational maps X, Y and W such that $R_1 = X \circ W$ and $R_i = Y \circ W$. But R_1 is indecomposable, so $\deg(X) = \deg(Y) = 1$. It follows that $X \circ Y^{-1} \in \mathrm{Deck}(R_1)$ and $R_i = Y \circ X^{-1} \circ R_1$. Finally, the map $\phi(R_1)$ is structurally stable as a quasiconformal deformation of a structurally stable map, so it is also indecomposable. Now we can repeat the arguments for $\phi(R_1)$. $\qquad \square$

Therefore, a semigroup S satisfying the Levin relations possesses an non-zero invariant Beltrami differential if and only if there is an element of S possessing an invariant Beltrami differential.

In what follows, for every rational map R and a every completely invariant set $A \subset \bar{\mathbb{C}}$ of positive Lebesgue measure, we construct a semigroup of operators satisfying the Levin relations and acting on $L_1(A)$ and show that the action is left amenable whenever R does not admits a non-zero Beltrami differential supported on A.

Definition. Let R be a rational map; let σ be an analytic arc in $\mathbb{C}$ containing all critical values of R; and let $U = \bar{\mathbb{C}} \setminus \sigma$ and $D = R^{-1}(U)$. Then, $D = \bigcup_{i=1}^{\deg(R)} D_i$, $\pi_1(D_i) = 1$, and $R \colon D_i \to V$ is holomorphic homeomorphism. Set $R_i = R|_{D_i}$ and, for each i, j, define the piecewise conformal map

$$
h_{i,j} = \begin{cases}
R_j^{-1} \circ R_i, & \text{on } D_i, \\
R_i^{-1} \circ R_j, & \text{on } D_j, \\
\mathrm{Id}, & \text{otherwise.}
\end{cases}
$$

The map $h_{i,j}$ is a bijection that is piecewise conformal almost everywhere and such that $h_{i,j}^2 = \mathrm{Id}$ and $h_{i,i} = \mathrm{Id}$ everywhere. We denote by $D(R)$ the group generated by the maps $h_{i,j}$ as the full deck group of R associated to the arc σ.

Note that $D(R)$ is isomorphic to the symmetric group on $\deg(R)$ symbols. For every $\gamma \in D(R)$, we have that $R(\gamma) = R$ almost everywhere. The group $D(R)$ acts on $L_1(\mathbb{C})$ by the push-forward map

$$
\gamma_* \colon f \longrightarrow f(\gamma)\gamma'^2,
$$

with $\|\gamma_*\|_{L_1} \leq 1$ for every $\gamma \in D(R)$.

For every subgroup $\Gamma < D(R)$ and $\gamma \in \Gamma$, let $R_\gamma = \gamma \circ R \circ \gamma^{-1} = \gamma \circ R$. Define the semigroup

$$
\mathcal{S}(\Gamma) = \langle R_\gamma \rangle_{\gamma \in \Gamma}.
$$

$S(\Gamma)$ is a finitely generated semigroup of piecewise holomorphic maps which is RA by Theorem 28. For example, if $\Gamma < \text{Deck}(R)$, then $S(\Gamma)$ consists of rational maps.

Using the action of R on $L_1(\mathbb{C})$, by mean of the Ruelle operator R_*, we construct the Ruelle representation $\rho: S(D(R)) \to \text{End}(L_1(\bar{\mathbb{C}}))$ defined by the following formulas on generators:

$$\rho(R_\gamma)(\phi) = (R_\gamma)_*(\phi) = \gamma_* \circ R_*(\phi) \quad \text{for } \phi \in L_1(\mathbb{C}).$$

If A is a completely invariant positive Lebesgue measure set, that is if one has $\text{Leb}(R^{-1}(A) \setminus A) = 0$, then $\text{Leb}(\gamma(A) \setminus A) = 0$ for $\gamma \in D(R)$, where Leb denotes the Lebesgue measure.

Proposition 39. *Let R be a rational map and A be a completely invariant set of positive Lebesgue measure. Assume that A does not support a non-zero-invariant Beltrami differential of R. Then, the Ruelle representation of $S(D(R))$ on $L_1(A)$ is left amenable.*

Proof. The semigroup $S(D(R))$ is RA by Theorem 28. Therefore, the space $X_\rho \subset L_\infty(S(D(R)))$ possesses a right-invariant mean m. Recall that X_ρ is the closure of the linear span of constant functions together with the space Y_ρ.

We claim that $\text{Ker}(m)$ contains Y_ρ. Otherwise, there would be two elements $\psi \in L_1(A)$ and $\nu \in L_\infty(A)$ so that $m(\phi_{\psi,\nu}) \neq 0$. Then

$$M(f) = m(\phi_{f,\nu})$$

would be a continuous R_*-invariant functional on $L_1(A)$. But

$$M(\psi) = m(\phi_{\psi,\nu}) \neq 0,$$

and by the Riesz representation theorem there would exist an invariant Beltrami differential $\mu \neq 0$, which is a contradiction.

Since X_ρ and Y_ρ are both left-invariant, then by the claim every right mean on X_ρ is left invariant. □

Conversely we have the following theorem.

Theorem 40. *Let R be a rational map and $\Gamma < D(R)$ be a transitive subgroup. Assume that $S(\Gamma)$ is ρ-LA, where ρ is the Ruelle representation. The following conditions are equivalent*:

(1) *R_* has non-zero fixed points in $L_1(\mathbb{C})$*;

(2) *R is Möbius conjugated to a flexible Lattés map.*

Proof. (1) implies (2). Assume that R_* has a non-zero fixed point $f \in L_1(\mathbb{C})$. By Lemma A in [24], there exists an invariant Beltrami differential μ with $\mu = \frac{|f|}{f}$ almost everywhere on the support of f. We can assume that R acts ergodically on the support of μ. Then, the representation $R_*: L_1(\mathrm{supp}(\mu)) \to L_1(\mathrm{supp}(\mu))$ has fixed point $\alpha \neq 0$ if and only if α is a multiple of f. Moreover, the Beltrami operator $(R_*)^*: L_\infty(\mathrm{supp}(\mu)) \to L_\infty(\mathrm{supp}(\mu))$ has a fixed point $\beta \neq 0$ if and only if β is a multiple of μ. By the separation principle, we conclude that $R_*: L_1(\mathrm{supp}(\mu)) \to L_1(\mathrm{supp}(\mu))$ is mean-ergodic.

Moreover, R_* is weakly almost periodic. Indeed, since R_* is mean-ergodic, the conjugated operator $T(\phi) = \mu R_*(\bar{\mu}\phi)$ is also a mean-ergodic operator with the same norm. A straightforward computation shows that

$$T(\phi)(y) = \sum_{R(x)=y} \frac{\phi(x)}{|R'(x)|} = \sum \phi(\zeta_i(y))|\zeta_i'|^2(y)$$

is a positive operator which is almost weakly periodic by Theorem 15, where ζ_i is a complete local system of branches of R^{-1}. So, R_* is weakly almost periodic on $L_1(\mathrm{supp}(\mu))$.

The semigroup $\mathcal{S}(\Gamma)$ consists only of iterations of the generators and every generator is conjugated to R. Hence, $\rho(\mathcal{S}(\Gamma))$ also consists only of iteration of the generators $\rho(R_\gamma)$ and each $\rho(R_\gamma)$ is conjugated to $\rho(R)$, where ρ is the Ruelle representation. This implies that $\rho(\mathcal{S}(\Gamma))$ is a weakly almost periodic semigroup of operators on $L_1(\mathrm{supp}(\mu))$.

Since $\mathcal{S}(\Gamma)$ is ρ-LA, we claim that there exist a functional $\ell \in L_\infty^*(\mathrm{supp}(\mu))$ which is invariant for the semigroup $(\rho(\mathcal{S}(\Gamma)))^* = \{t^*: t \in \rho(\mathcal{S}(\Gamma))\}$.

Indeed, if L is a mean, we define the functional

$$\ell(h) = L(\phi_{h,f}), \quad \text{for } \phi_{h,f} \in L_\infty(\mathcal{S}(\Gamma)),$$

given by

$$\phi_{h,f}(g) = \int h\rho(g)(f)|dz|^2,$$

where $g \in \mathcal{S}(\Gamma)$, $h \in L_\infty(\mathrm{supp}(\mu))$, and $f \in L_1(\mathrm{supp}(\mu))$. Since L is left invariant, we get $\ell(t^*(h)) = \ell(h)$ for every $t \in \rho(\mathcal{S}(\Gamma))$.

Now, we continue the proof of the theorem by standard arguments of functional analysis (see for example [10]). The functional ℓ generates a finite complex valued invariant finitely additive measure α_ℓ defined by the formula

$$\alpha_\ell(A) = \ell(\chi_A),$$

where A is a measurable subset of $\mathrm{supp}(\mu)$. From the definition, it follows that α_ℓ is null on every zero Lebesgue measure subset of $\mathrm{supp}(\mu)$. Next, we show that α_ℓ

is a measure absolutely continuous with respect to Lebesgue. It is enough to show that α_ℓ is a countably additive set function, that is,

$$\alpha_\ell\left(\bigcup A_i\right) = \sum \alpha_\ell(A_i),$$

for every pairwise disjoint family of measurable subsets of $\mathrm{supp}(\mu)$.

Since $\rho(\mathcal{S}(\Gamma))$ is weakly almost periodic, we have that for every $\epsilon > 0$ and $\beta \in L_1(\mathrm{supp}(\mu))$, there exists $\delta > 0$ such that

$$\int_B |t(\beta)| \le \epsilon, \quad \text{for every } t \in \rho(\mathcal{S}(\Gamma)),$$

whenever the Lebesgue measure of B is less than δ.

Let $X \subset \mathrm{supp}(\mu)$ a finite Lebesgue measure set which has a decomposition $X = \bigcup_{i=0}^{\infty} A_i$ by a family of pairwise disjoint measurable subsets. For every k, by finite additivity,

$$\alpha_\ell(X) = \left(\sum_{i=0}^{k} \alpha_\ell(A_i)\right) + \alpha_\ell\left(\bigcup_{i=k+1}^{\infty} A_i\right).$$

Since $\rho(S(\Gamma))(f)$ is a weakly precompact set, for every $\epsilon > 0$ we get a $\delta > 0$ so that, if k_0 is such that $\mathrm{Leb}(X_k) < \delta$ for $k > k_0$, where $X_k = \bigcup_{i=k+1}^{\infty} A_i$, then,

$$|\alpha_\ell(X_k)| \le |L(\phi_{\chi_{X_k}, f})| \le \sup_{g \in \mathcal{S}(\Gamma)} \int_{X_k} |\rho(g)(f)||dz|^2 \le \epsilon.$$

It follows that α_ℓ is a finite measure which is absolutely continuous with respect to the Lebesgue measure on $\mathrm{supp}(\mu)$. Hence, there exists a non-zero $w \in L_1(\mathrm{supp}(\mu))$ so that $\ell(h) = \int hw|dz|^2$. Since ℓ is $\rho(\mathcal{S}(\Gamma))^*$ invariant, then w is $\rho(\mathcal{S}(\Gamma))$ invariant and therefore w is a multiple of f.

As $R_*(f) = f$, we conclude that f is a fixed point for γ_*, with $\gamma \in \Gamma$. Since Γ is transitive, we can choose $d = \deg(R)$ elements $\gamma_1, \ldots, \gamma_d \in \Gamma$ so that, for every fixed branch ζ_i of R^{-1} on $\overline{\mathbb{C}} \setminus \sigma$, the collection $\{\gamma_j \circ \zeta_i\}$ forms a complete collection of branches of R^{-1} on $\overline{\mathbb{C}} \setminus \sigma$. Therefore,

$$dw(\zeta_i)(\zeta_i')^2 = \sum_j (\gamma_j)_*(w) \circ (\zeta_i)(\zeta_i')^2 = \sum w(\zeta_j)(\zeta_j')^2 = w.$$

For every $z \in R^{-1}(\overline{\mathbb{C}} \setminus \sigma)$,

$$\frac{w(R(z))R'^2(z)}{\deg(R)} = w(z).$$

Hence, $|w|$ defines a continuous functional on $C(\overline{\mathbb{C}})$, via $\phi \mapsto \int_{\mathbb{C}} \phi|w||dz|^2$, which is invariant under Lyubich operator L_R, and so, by Theorem 7 and

Lemma 18, it is the density of the measure of maximal entropy for R. Thus, the map R has maximal entropy measure absolutely continuous with respect to Lebesgue. By Zdunik's Theorem (see [32]), the map R is an exceptional map. Since $\frac{|f|}{f}$ is an invariant differential for R, then R is a flexible Lattès map.

(2) implies (1). If R is a flexible Lattès map, then, again by Zdunik's Theorem, the measure m_R of maximal entropy of R is absolutely continuous with respect to Lebesgue. Therefore, $dm_R = \omega|dz|^2$ with $\omega \in L_1(\overline{\mathbb{C}})$, $\omega > 0$ and

$$\frac{\omega(R)|R'|^2}{\deg(R)} = \omega$$

almost everywhere. On the other hand, R has non-zero invariant Beltrami differential μ. Hence, the function $\bar{\mu}\omega$ is fixed by R_*, and we are done. $\qquad\square$

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