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Algorithmic aspects of branched coverings III/V. Erasing maps, orbispaces, and the Birman exact sequence

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Abstract. Let $\tilde{f}: (S^2, \tilde{A}) \circlearrowleft$ be a Thurston map and let $M(\tilde{f})$ be its mapping class biset: isotopy classes rel $\tilde{A}$ of maps obtained by pre- and post-composing $\tilde{f}$ by the mapping class group of $(S^2, \tilde{A})$. Let $A \subseteq \tilde{A}$ be an $\tilde{f}$-invariant subset, and let $f: (S^2, A) \circlearrowleft$ be the induced map. We give an analogue of the Birman short exact sequence: just as the mapping class group $\mathbf{Mod}(S^2, \tilde{A})$ is an iterated extension of $\mathbf{Mod}(S^2, A)$ by fundamental groups of punctured spheres, $M(\tilde{f})$ is an iterated extension of $M(f)$ by the dynamical biset of f.

Thurston equivalence of Thurston maps classically reduces to a conjugacy problem in mapping class bisets. Our short exact sequence of mapping class bisets allows us to reduce in polynomial time the conjugacy problem in $M(\tilde{f})$ to that in $M(f)$. In case $\tilde{f}$ is geometric (either expanding or doubly covered by a hyperbolic torus endomorphism) we show that the dynamical biset $B(f)$ together with a "portrait of bisets" induced by $\tilde{A}$ is a complete conjugacy invariant of $\tilde{f}$.

Along the way, we give a complete description of bisets of $(2,2,2,2)$-maps as a crossed product of bisets of torus endomorphisms by the cyclic group of order 2, and we show that non-cyclic orbisphere bisets have no automorphism.

We finally give explicit, efficient algorithms that solve the conjugacy and centralizer problems for bisets of expanding or torus maps.

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Contents

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1. Introduction

A *Thurston map* is a branched covering $f\colon S^2 \circlearrowleft$ of the sphere whose *post-critical set* $P_f := \bigcup_{n\ge 1} f^n(\text{critical points of } f)$ is finite.

Extending [21], we developed in [2, 3, 4, 5] an algebraic machinery that parallels the topological theory of Thurston maps: one considers the *orbisphere* $(S^2, P_f, \operatorname{ord}_f)$, with $\operatorname{ord}_f\colon P_f \to \{2,3,\dots,\infty\}$ defined by local orders

$$\operatorname{ord}_f(p) = \operatorname{l.c.m.}\{\deg_q(f^n) \mid n \ge 0, q \in f^{-n}(p)\},$$

and the *orbisphere fundamental group* $G = \pi_1(S^2, P_f, \operatorname{ord}, *)$, which has for each point $p \in P_f$ a generator (a "lollipop": a small loop around p, connected to the basepoint) of order $\operatorname{ord}_f(p)$, and one additional relation. Then f is encoded in the structure of a *G-G-biset* $B(f)$: a set with commuting left and right actions of G. By [5, Theorem 8.9] (see also Corollary 3.7), the isomorphism class of $B(f)$ is a complete invariant of f up to isotopy.

There is a missing element to this description, that of "extra marked points." In the process of a decomposition of spheres into smaller pieces, one is led to consider Thurston maps with extra marked points, such as periodic cycles or preimages of post-critical points. They can be added to P_f, but they have order 1 under ord_f so are invisible in $\pi_1(S^2, P_f, \operatorname{ord}_f)$. The orbisphere orders can be made artificially larger, but then other properties, such as the characterization of expanding maps as those having a "contracting" biset (see [4, Theorem A]) are lost.

We resolve this issue, in this article, by introducing *portraits of bisets* and exhibiting their algorithmic properties. As an outcome, the conjugacy and centralizer problems for Thurston maps with extra marked points reduces to that of the underlying map with only P_f marked.

This allows us, in particular, to understand algorithmically maps doubly covered by torus endomorphisms.

1.1. Maps and bisets. The natural setting is an orbisphere $(S^2, \widetilde{A}, \widetilde{\operatorname{ord}})$ and a sub-orbisphere $(S^2, A, \operatorname{ord})$; namely one has $A \subseteq \widetilde{A}$ and $\operatorname{ord}(a) \mid \widetilde{\operatorname{ord}}(a)$ for all $a \in A$. There is a corresponding morphism of fundamental groups $\pi_1(S^2, \widetilde{A}, \widetilde{\operatorname{ord}}, *) =: \widetilde{G} \twoheadrightarrow G := \pi_1(S^2, A, \operatorname{ord}, *)$, called a *forgetful morphism*. We considered in [5, §8] the *inessential* case in which $\operatorname{ord}(a) > 1 \iff \widetilde{\operatorname{ord}}(a) > 1$, so that only the type of singularities are changed by the forgetful functor. Here we are more interested in the *essential* case, in which $A \subsetneq \widetilde{A}$.

In this introduction, we restrict ourselves to self-maps of orbispheres; the general non-dynamical case is covered in §2. An orbisphere self-map

$$f\colon (S^2, A, \operatorname{ord}) \circlearrowleft$$

is a branched covering $f\colon S^2 \circlearrowleft$ with $\operatorname{ord}(p)\deg_p(f) \mid \operatorname{ord}(f(p))$ for all $p \in S^2$.

Given a map $f\colon (S^2, A) \circlearrowleft$, there may exist different orbisphere structures on (S^2, A) turning f into an orbisphere self-map; in particular, the maximal one, in which $\operatorname{ord}(a) = \infty$ for every $a \in A$, and the minimal one ord_f, in which $\operatorname{ord}(a) = \text{l. c. m.}\{\deg_q(f^n) \mid n \geq 0, q \in f^{-n}(a)\}$. The self-map f is encoded by the biset

$$B(f) := \{\beta\colon [0,1] \longrightarrow S^2 \setminus A \mid \beta(0) = * = f(\beta(1))\}/{\approx_{A,\mathrm{ord}}}, \tag{1}$$

where $\approx_{A,\mathrm{ord}}$ denotes homotopy rel (A, ord), and the commuting left and right $\pi_1(S^2, A, \operatorname{ord}, *)$-actions are given respectively by concatenation of paths or their appropriate f-lift.

Bisets form a convenient category with products, detailed in [3]. A *self-conjugacy* of a G-G-biset B is a pair of maps $(\phi\colon G \circlearrowleft, \beta\colon B \circlearrowleft)$ with $\beta(hbg) = \phi(h)\beta(b)\phi(g)$; and an *automorphism* of B is a self-conjugacy with $\phi = \mathbb{1}$. Cyclic bisets (that of maps $z \mapsto z^d\colon (\widehat{\mathbb{C}}, \{0,\infty\}) \circlearrowleft$) have a special status; for the others,

Proposition A (= Corollary 4.30). *If ${}_G B_G$ is a non-cyclic orbisphere biset then $\operatorname{Aut}(B) = 1$.*

Consider $\widetilde{A} = A \sqcup \{d\}$, obtained by adding a point to the orbisphere $(S^2, A, \operatorname{ord})$, resulting in an orbisphere $(S^2, \widetilde{A}, \widetilde{\operatorname{ord}})$. Let us denote by $\mathbf{Mod}(S^2, A)$ the pure mapping class group of $S^2 \setminus A$; these are homeomorphisms of S^2 fixing A pointwise, considered up to isotopy. Note that the relation of isotopy $\approx_A$ does not make use of the orbisphere structure. There is a short exact sequence, called the *Birman exact sequence*,

$$1 \longrightarrow \pi_1(S^2 \setminus A, d) \longrightarrow \mathbf{Mod}(S^2, \widetilde{A}) \longrightarrow \mathbf{Mod}(S^2, A) \longrightarrow 1. \tag{2}$$

Let $f\colon (S^2, A, \operatorname{ord}) \circlearrowleft$ be an orbisphere map, and recall that its *mapping class biset* is the set

$$M(f) := \{m' f m'' \mid m', m'' \in \mathbf{Mod}(S^2, A)\}/{\approx_A},$$

with natural left and right actions of $\mathbf{Mod}(S^2, A)$. Assume first that the extra point d is fixed by f, and write $\tilde{f}\colon (S^2, \widetilde{A}, \widetilde{\operatorname{ord}}) \circlearrowleft$ for the map f acting on $(S^2, \widetilde{A}, \widetilde{\operatorname{ord}})$. There is then a natural map $M(\tilde{f}) \to M(f)$, and we will see that it is an *extension of bisets* (see Definition 2.10):

Theorem B (= Corollary 2.23). *Subordinate to the exact sequence of groups (2), there is a short exact sequence of bisets*

$${}_{\pi_1(S^2\setminus A,d)}\Big(\bigsqcup_{f' \in M(f)} B(f')\Big)_{\pi_1(S^2\setminus A,d)} \hookrightarrow M(\tilde{f}) \twoheadrightarrow M(f),$$

where $B(f')$ denotes the biset of $f'\colon (S^2, A) \circlearrowleft$ rel the base point d.

Let us call a class $m \in \mathbf{Mod}(S^2, \widetilde{A})$ *knitting* if the image of m is trivial in $\mathbf{Mod}(S^2, A \sqcup \{a\})$ for every $a \in \widetilde{A} \setminus A$. Our statement covers the case of $\widetilde{A} = A \sqcup \{\text{periodic points}\}$, up to knitting elements: for a cycle of length ℓ, the fibres in the short exact sequence are of the form $B(f)^\ell$ with left- and right-action twisted along an ℓ-cycle. This case is at the heart of our reduction of the conjugacy problem from $M(\widetilde{f})$ to $M(f)$. Approximately the same picture applies if D contains preimages of points in A, but there are subtle complications which are taken care of in §2, see Theorem 2.13. In essence, the presence of preperiodic points imposes a finite index condition on centralizers, and splits conjugacy classes into finitely many pieces, see the remark after Lemma 4.24.

1.2. Portraits of bisets. Portraits of bisets emerge from a simple remark: fixed points of f naturally yield conjugacy classes in $B(f)$. Indeed if $f(p) = p$ then choose a path $\ell\colon [0,1] \to S^2 \setminus A$ from $*$ to p, and consider $c_p := \ell \# \ell^{-1} \uparrow_f^p \in B(f)$, the concatenation of ℓ with the reverse of its lift starting at p. It is well defined up to conjugation by G, namely a different choice of ℓ would yield $g^{-1} c_p g$ for some $g \in G$. Conversely, if f expands a metric, then every conjugacy class in $B(f)$ corresponds to a unique repelling f-fixed point.

A *portrait of bisets* in $B(f)$, see Definition 2.15, consists of a map $f_*\colon \widetilde{A} \circlearrowleft$ extending $f\restriction_A$; a collection of *peripheral* subgroups $(G_a)_{a \in \widetilde{A}}$ of G: they are such that every G_a is cyclic and generated by a “lollipop” around a; and a collection of cyclic G_a-$G_{f_*(a)}$-bisets $(B_a)_{a \in \widetilde{A}}$. Two portraits of bisets $(G_a, B_a)_{a \in \widetilde{A}}$ and $(G'_a, B'_a)_{a \in \widetilde{A}}$ parameterized by the same $f_*\colon \widetilde{A} \circlearrowleft$ are *conjugate* if there exist $(\ell_a)_{a \in \widetilde{A}}$ in $G^{\widetilde{A}}$ such that $G_a = \ell_a^{-1} G'_a \ell_a$ and $B_a = \ell_a^{-1} B'_a \ell_{f_*(a)}$. The $(\ell_a)_{a \in \widetilde{A}}$ itself is called a *conjugator*, and the set of self-conjugators of a portrait is called its *centralizer*. We will show in Lemma 2.17 that every conjugacy class contains a unique representative with specified G_a.

In case $A = \widetilde{A}$, every biset admits a unique *minimal* portrait up to conjugacy, which may be understood geometrically as follows. Consider a branched covering $f\colon (S^2, A) \circlearrowleft$. For every $a \in A$ choose a small disk neighbourhood $\mathcal{D}_a$ of it; up to isotopy we may assume that $f\colon \mathcal{D}_a \setminus \{a\} \to \mathcal{D}_{f(a)} \setminus \{f(a)\}$ is a covering. A choice of embeddings $\pi_1(\mathcal{D}_a \setminus \{a\}) \hookrightarrow \pi_1(S^2, A)$ yields a family $(G_a)_{a \in A}$ of peripheral subgroups; and the corresponding embeddings $B(f\colon \mathcal{D}_a \setminus \{a\} \to \mathcal{D}_{f(a)} \setminus \{f(a)\}) \hookrightarrow B(f)$ yields a minimal portrait of bisets $(G_a, B_a)_{a \in A}$ in $B(f)$.

In case $\widetilde{A} = A \sqcup \{e_1, \dots, e_n\}$ with $(e_1, \dots, e_n)$ a periodic cycle, the bisets B_{e_i} consist of single points, and almost coincide with Ishii and Smillie’s notion of *homotopy pseudo-orbits*, see [16] and §4.2: imagine that $(e_1, \dots, e_n) \subset S^2$ is almost a periodic cycle, in that $f(e_i)$ is so close to e_{i+1} that there is a well-defined “shortest” path ℓ_i from the $f(e_i)$ to e_{i+1}, indices being read modulo n. Choose for each i a path m_i from $*$ to e_i. Set then $B_i := \{m_i \# (\ell_{i+1} \# m_{i+1}^{-1}) \uparrow_f^{e_i}\}$, the portrait of bisets encoding $(e_1, \dots, e_n)$.

Portraits of bisets may be defined algebraically in an orbisphere biset B, without reference to a map f. From the biset B, it is easy to compute the *local dynamics* $f_*\colon A \circlearrowleft$ and *local degree* $\deg\colon A \to \mathbb{N}$ on its peripheral classes, and not much harder to reconstruct a map f with $B \cong B(f)$, see [5, Theorem 8.9]. Theorem B is proven via portraits of bisets. There is a natural *forgetful intertwiner of bisets*

$$ {}_{\widetilde G}\widetilde B_{\widetilde G} := B(\tilde f\colon (S^2, \widetilde A, \widetilde{\mathrm{ord}}) \circlearrowleft) \longrightarrow B(f\colon (S^2, A, \mathrm{ord}) \circlearrowleft) =: {}_G B_G \qquad (3) $$

given by $b \mapsto 1 \otimes b \otimes 1$ where $B \cong G \otimes_{\widetilde G} \widetilde B \otimes_{\widetilde G} G$. Every portrait in $\widetilde B$, for example its minimal one, induces a portrait in B via the forgetful map. We prove:

Theorem C ($\le$ Theorem 2.19). *Let ${}_G B_G$ be an orbisphere biset with local dynamics* $f_*\colon A \circlearrowleft$ *and* $\deg\colon A \to \mathbb{N}$, *and let* $\widetilde G \twoheadrightarrow G$ *and* $f_*\colon \widetilde A \circlearrowleft$ *and* $\deg\colon \widetilde A \to \mathbb{N}$ *be compatible extensions.*

There is then a bijection between, on the one hand, conjugacy classes of portraits of bisets $(G_a, B_a)_{a\in\widetilde A}$ *in* B *parameterized by* f_* *and* $\deg$ *and, on the other hand,* $\widetilde G$-$\widetilde G$*-bisets* $\widetilde B$ *projecting to* B *under* $\widetilde G \twoheadrightarrow G$ *considered up to composition with the biset of a knitting element* (*see the remark after Theorem* B). *This bijection maps a minimal portrait of bisets of* $\widetilde B$ *to* $(G_a, B_a)_{a\in\widetilde A}$.

1.3. Geometric maps. A homeomorphism $f\colon (S^2, A) \circlearrowleft$ is *geometric* if f is either of finite order ($f^n = \mathbb{1}$ for some $n > 0$) or pseudo-Anosov (there are two transverse measured foliations preserved by f such that one foliation is expanded by f while another is contracted). In both cases, f preserves a geometric structure on S^2, and by a theorem of Thurston [28] every surface homeomorphism decomposes, up to isotopy, into geometric pieces.

Consider now a non-invertible sphere map $f\colon (S^2, A) \circlearrowleft$, and let $A^\infty \subseteq A$ denote the forward orbit of the periodic critical points of f. The map f is *Böttcher expanding* if there exists a metric on $S^2 \setminus A^\infty$ that is expanded by f, and such that f is locally conjugate to $z \mapsto z^{\deg_a(f)}$ at every $a \in A^\infty$. The map f is *geometric* if f is either

{Exp} Böttcher expanding; or

{GTor/2} a quotient of a torus endomorphism $z \mapsto Mz + q\colon \mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$ by the involution $z \mapsto -z$, for a 2×2 matrix M whose eigenvalues are different from ±1.

The two cases are not mutually exclusive. A map $f \in$ {GTor/2} is expanding if and only if the absolute values of the eigenvalues of M are greater than 1. Note also that if f is non-invertible and covered by a torus endomorphism then either $f \in$ {Exp} or the minimal orbisphere of f satisfies $\#P_f = 4$ and $\mathrm{ord}_f \equiv 2$.

In that last case, we show that ${}_G B_G$ is a crossed product of an Abelian biset with an order-2 group. Let us fix a $(2,2,2,2)$-orbisphere (S^2, A, ord) and let us set $G := \pi_1(S^2, A, \mathrm{ord}) \cong \mathbb{Z}^2 \rtimes \{\pm 1\}$. We identify A with the set of all order-2 conjugacy classes of G. By Euler characteristic, every branched covering $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ is a self-covering. Therefore, the biset of f is right principal.

We denote by $\mathbf{Mat}_2^+(\mathbb{Z})$ the set of 2×2 integer matrices M with $\det(M) > 0$. For a matrix $M \in \mathbf{Mat}_2^+(\mathbb{Z})$ and a vector $v \in \mathbb{Z}^2$ there is an injective endomorphism $M^v\colon \mathbb{Z}^2 \rtimes \{\pm 1\} \circlearrowleft$ given by the following "crossed product" structure:

$$M^v(n, 1) = (Mn, 1) \quad \text{and} \quad M^v(n, -1) = (Mn + v, -1). \tag{4}$$

Note that M^v induces a map $(M^v)_*\colon A \circlearrowleft$ on the four order-2 conjugacy classes in G. Recall that with a group H and an endomorphism $\phi\colon H \circlearrowleft$ are associated a biset B_ϕ, which is H set with actions $h\cdot b\cdot h' := \phi(h)bh'$. We then have a "crossed product" decomposition $B_{M^v} = B_M \rtimes \{\pm 1\}$ of the biset of M^v.

Proposition D (= Propositions 4.5 and 4.6). The biset of M^v from (4) is an orbisphere biset, and conversely every $(2,2,2,2)$-orbisphere biset B is of the form $B = B_{M^v}$ for some $M \in \mathbf{Mat}_2^+(\mathbb{Z})$ and some $v \in \mathbb{Z}^2$. Two bisets B_{M^v} and B_{N^w} are isomorphic if and only if $M = \pm N$ and $v \equiv w \pmod{2\mathbb{Z}^2}$. The biset B_{M^v} is geometric if and only if both eigenvalues of M are different from ± 1.

A distinguished property of a geometric map is *rigidity*: two geometric maps are Thurston equivalent (namely, conjugate up to isotopy) if and only if they are topologically conjugate.

An orbisphere biset ${}_G B_G$ is *geometric* if it is the biset of a geometric map, and {GTor/2} and {Exp} bisets are defined similarly. If B is a geometric biset, then by rigidity there is a geometric map $f_B\colon (S^2, A, \mathrm{ord}) \circlearrowleft$, unique up to conjugacy, with $B(f_B) \cong B$.

If ${}_G B_G$ is geometric and ${}_{\widetilde G}\widetilde B_{\widetilde G} \to {}_G B_G$ is a forgetful intertwiner as in (3), then elements of $\widetilde A \setminus A$ (which *a priori* do not belong to any sphere) can be interpreted dynamically as extra marked points on $S^2 \setminus A$. More precisely, if $\widetilde B$ is itself geometric, and B is the biset of the geometric map $f\colon (S^2, A) \circlearrowleft$, then there is an embedding of $\widetilde A$ in S^2 as an f-invariant set, unique unless G is cyclic, in such a manner that $\widetilde B$ is isomorphic to $B(f\colon (S^2, \widetilde A) \circlearrowleft)$.

Since furthermore geometric maps have only finitely many periodic points of given period, we obtain a good understanding of conjugacy and centralizers of geometric bisets:

Theorem E (= Theorem 4.41). *Let $\widetilde G \to G$ be a forgetful morphism of groups and let ${}_{\widetilde G}\widetilde B_{\widetilde G} \to {}_G B_G$ and ${}_{\widetilde G}\widetilde B'_{\widetilde G} \to {}_G B'_G$ be two forgetful biset morphisms as in* (3). *Suppose furthermore that $\widetilde B$ is geometric of degree > 1. Denote by $(G_a, B_a)_{a\in\widetilde A}$ and $(G'_a, B'_a)_{a\in\widetilde A}$ the portraits of bisets induced by $\widetilde B$ and $\widetilde B'$ in B and B' respectively.*

$\widetilde B, \widetilde B'$ are conjugate under $\mathbf{Mod}(\widetilde G)$ if and only if there exists $\phi \in \mathbf{Mod}(G)$ such that $B^\phi \cong B'$ and the portraits $(G_a^\phi, B_a^\phi)_{a\in\widetilde A}$ and $(G'_a, B'_a)_{a\in\widetilde A}$ are conjugate.

Furthermore, the centralizer of the portrait $(G_a, B_a)_{a\in\widetilde{A}}$ *is trivial, and the centralizer* $Z(\widetilde{B})$ *of* $\widetilde{B}$ *is isomorphic, via the forgetful map* $\mathbf{Mod}(\widetilde{G}) \to \mathbf{Mod}(G)$*, to*

$$\{\phi \in Z(B) \mid (G_a^\phi, B_a^\phi)_{a\in\widetilde{A}} \sim (G_a, B_a)_{a\in\widetilde{A}}\}$$

and is a finite-index subgroup of $Z(B)$*.*

Let us call an orbisphere map $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ *weakly geometric* if its minimal quotient on $(S^2, P_f, \mathrm{ord}_f)$ is geometric; an orbisphere biset B is weakly geometric if its minimal quotient orbisphere biset is geometric. In Theorem 4.35 we characterize weakly geometric maps as those decomposing as a tuning by homeomorphisms: starting from a geometric map, some points are blown up to disks which are mapped to each other by homeomorphisms.

1.4. Algorithms. An essential virtue of the portraits of bisets introduced above is that they are readily usable in algorithms. Previous articles in the series already highlighted the algorithmic aspects of bisets; let us recall below some salient points.

From the definition of orbisphere bisets in [5, Definition 2.6], it is clearly decidable whether a given biset ${}_HB_G$ is an orbisphere biset; the groups G and H may be algorithmically identified with orbisphere groups $\pi_1(S^2, A, \mathrm{ord}, *)$ and $\pi_1(S^2, C, \mathrm{ord}, \dagger)$ respectively, and the induced map $f_*\colon C \to A$ is computable. In particular, if ${}_GB_G$ is a G-G-biset, then the dynamical map $f_*\colon A \circlearrowleft$ is computable, the minimal orbisphere quotient $\overline{G} := \pi_1(S^2, P_f, \mathrm{ord}_f, *)$ is computable, and the induced $\overline{G}$-$\overline{G}$-biset $\overline{B}$ is computable. It is also easy (see [4, §5]) to determine from an orbisphere biset whether it is {GTor/2} (and then to determine an affine map $Mz + q$ covering it) or {Exp}.

We shall show that recognizing conjugacy of portraits is decidable, and give efficient (see below) algorithms realizing it, as follows:

Algorithm 1.1 (= Algorithms 5.6 and 5.10).

GIVEN a minimal geometric orbisphere biset ${}_GB_G$, an extension $f_*\colon \widetilde{A} \to \widetilde{A}$ of the dynamics of B on its peripheral classes, and two portraits of bisets $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$ with dynamics f_*,

DECIDE whether $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$ are conjugate, and COMPUTE the centralizer of $(G_a, B_a)_{a\in\widetilde{A}}$, which is a finite group.

Algorithm 1.2 (= Algorithms 5.7 and 5.11).

GIVEN a minimal geometric orbisphere biset ${}_GB_G$ and an extension $f_*\colon \widetilde{A} \to \widetilde{A}$ of the dynamics of B on its peripheral classes,

PRODUCE A LIST of representatives of all conjugacy classes of portraits of bisets $(G_a, B_a)_{a\in\widetilde{A}}$ in B with dynamics f_*.

Thurston equivalence to a map $f : (S^2, A) \circlearrowleft$ reduces to the conjugacy problem in the mapping class biset $M(f)$. Let X be a basis of $M(f)$ and let N be a finite generating set of $\mathbf{Mod}(S^2, A)$; so $M(f) = \bigcup_{n\geq 0} N^n X$. We call an algorithm with input in $M(f) \times M(f)$ *efficient* if for $f, g \in N^n X$ the running time of the algorithm is bounded by a polynomial in n.

We deduce that conjugacy and centralizer problems are decidable for geometric maps, as long as they are decidable on their minimal orbisphere quotients:

Corollary F (= Algorithm 5.12). *There is an efficient algorithm with oracle that, given two orbisphere maps f, g by their bisets and such that f is geometric, decides whether f, g are conjugate, and computes the centralizer of f.*

The oracle must answer, given two geometric orbisphere maps f, g on their minimal orbisphere $(S^2, P_f, \mathrm{ord}_f)$ respectively $(S^2, P_g, \mathrm{ord}_g)$, whether they are conjugate and what the centralizer of f is.

Algorithms for the oracle itself will be described in details in the last article of the series [6]. Furthermore, we have the following oracle-free result, proven in §5.3:

Corollary G. *There is an efficient algorithm that decides whether a rational map is equivalent to a given twist of itself, when only extra marked points are twisted.*

1.5. Historical remarks: Thurston equivalence and its complexity. The conjugacy problem is known to be solvable in mapping class groups [14]. The state of the art is based on the Nielsen–Thurston classification: decompose maps along their canonical multicurve; then a complete conjugacy invariant of the map is given by the combinatorics of the decomposition, the conjugacy classes of return maps, and rotation parameters along the multicurve. For general surfaces, the cost of computing the decomposition is at most exponential time in n, see [17, 27], and so is the cost of comparing the pseudo-Anosov return maps [20]. Margalit, Yurttas, Strenner recently announced polynomial-time algorithms for all the above. At all rates, for punctured spheres the cost of computing the decomposition is polynomial [9].

Kevin Pilgrim developed, in [23], a theory of decompositions of Thurston maps extending the Nielsen–Thurston decomposition of homeomorphisms. This theory brings a deep understanding of Thurston equivalence of maps, without any claims on complexity.

In [7], a general strategy is developed, along with computations of the mapping class biset for the three degree-2 polynomials with three finite post-critical points and period respectively $1, 2, 3$. The Douady–Hubbard "twisted rabbit problem" is solved in this manner (it asks to determine for $n \in \mathbb{Z}$ the conjugacy class of $r \circ t^n$ with $r(z) \approx z^2 - 0.12256 + 0.74486i$ the "rabbit" polynomial and $t(z)$ the Dehn twist about the rabbit's ears). Russell Lodge computed in [19] the solution to an array of other “twisted rabbit” problems, by finding explicitly the mapping class biset structure.

If $M(f)$ is a contracting biset, then its conjugacy problem may be solved in quasi-linear time. In terms of a twist parameter such as the n above, the solution has $\mathcal{O}(\log n)$ complexity. However, this bound is not uniform, in that it requires e.g. the computation of the nucleus of $M(f)$, which cannot *a priori* be bounded. Nekrashevych showed in [22, Theorem 7.2] that the mapping class biset of a hyperbolic polynomial is contracting. Conversely, if $M(f)$ contains an obstructed map then it is not contracting.

In the case of polynomials, however, even more is possible: bisets of polynomials admit a particularly nice form by placing the basepoint close to infinity (this description goes hand-in-hand with Poirier's “supporting rays” [24]). As a consequence, the “spider algorithm” of Hubbard and Schleicher [15] can be implemented directly at the level of bisets and yields an efficient algorithm, also in practice since it does not require the computation of $M(f)$'s nucleus. This algorithm was implemented in the GAP package Img [1] available from the GAP website, and will be described in [6].

Selinger and Yampolsky showed in [26] that the canonical decomposition is computable. In this manner, they solve the conjugacy problem for maps whose canonical decomposition has only rational maps with hyperbolic orbifold: a complete conjugacy invariant of the map is given by the combinatorics of the decomposition, the conjugacy classes of its return maps, and rotation parameters along the canonical obstruction.

We showed in [5] that the conjugacy problem is decidable in general: a complete conjugacy invariant of a Thurston map is given by the combinatorics of the decomposition, together with conjugacy classes of return maps, rotation parameters along the canonical obstruction, together with the induced action of the centralizer groups of return maps.

We finally mention a different path towards understanding Thurston maps, in the case of maps with four post-critical points: the “nearly Euclidean Thurston maps” from [10]. There, the restriction on the size of the post-critical set implies that the maps may be efficiently encoded via linear algebra; and as a consequence, conjugacy of NET maps is efficiently decidable. We are not aware of any direct connections between their work and ours.

1.6. Notations. Throughout the text, some letters keep the same meaning and are not always repeated in statements. The symbols A, C, D, E denote finite subsets of the topological sphere S^2. There is a sphere map $\tilde{f}\colon (S^2, C \sqcup E) \to (S^2, A \sqcup D)$, which restricts to a sphere map $f\colon (S^2, C) \to (S^2, A)$. Implicit in the definition, we have $f(C) \cup \{\text{critical values of } f\} \subseteq A$. We write $\widetilde{C} = C \sqcup E$ and $\widetilde{A} = A \sqcup D$. If there are, furthermore, orbisphere structures on the involved spheres, we denote them by (S^2, A, ord) etc., with the same symbol “ord”. We also abbreviate $X := (S^2, A, \mathrm{ord})$ and $Y := (S^2, C, \mathrm{ord})$.

For sphere maps $f_0, f_1\colon (S^2, C) \to (S^2, A)$, we mean by $f_0 \approx_C f_1$ that f_0 and f_1 are isotopic, namely there is a path $(f_t)_{t\in[0,1]}$ of sphere maps $f_t\colon (S^2, C) \to (S^2, A)$ connecting them. Note that f_t does not move the critical values.

For γ a path and f a (branched) covering, we denote by $\gamma{\uparrow}_f^x$ the unique f-lift of γ that starts at the preimage x of $\gamma(0)$.

We denote by $f\downharpoonright_Z$ the restriction of a function f to a subset Z of its domain. Finally, for a set Z we denote by $Z\lightning$ the group of all permutations of Z.

2. Forgetful maps

We recall a minimal amount of information from [5]: a *marked orbisphere* is (S^2, A, ord) for a finite subset $A \subset S^2$ and a map $\mathrm{ord}\colon A \to \{2, 3, \dots, \infty\}$, extended to S^2 by $\mathrm{ord}(S^2 \setminus A) \equiv 1$. For a choice of basepoint $* \in S^2 \setminus A$, its *orbispace fundamental group* is generated by "lollipop" loops $(\gamma_a)_{a\in A}$ based at $*$ that each encircle once counterclockwise a single point of A, and with $A = \{a_1, \dots, a_n\}$ has presentation

$$G = \pi_1(S^2, A, \mathrm{ord}, *) = \langle \gamma_{a_1}, \dots, \gamma_{a_n} \mid \gamma_{a_1}^{\mathrm{ord}(a_1)}, \dots, \gamma_{a_n}^{\mathrm{ord}(a_n)}, \gamma_{a_1} \cdots \gamma_{a_n} \rangle. \quad (5)$$

Abstractly, i.e. without reference to a sphere, an orbisphere group is a group G as in (5) together with the conjugacy classes $\Gamma_1, \dots, \Gamma_n$ of $\gamma_{a_1}, \dots, \gamma_{a_n}$ respectively.

An *orbisphere map* $f\colon (S^2, C, \mathrm{ord}_C) \to (S^2, A, \mathrm{ord}_A)$ between orbispheres is an orientation-preserving branched covering between the underlying spheres, with $f(C) \cup \{\text{critical values of } f\} \subseteq A$, and with $\mathrm{ord}_C(p)\deg_p(f) \mid \mathrm{ord}_A(f(p))$ for all $p \in S^2$.

To avoid special cases, we *make, throughout this article except in* §6.3, *the assumption*

$$\#A \geq 3 \quad \text{and} \quad \#C \geq 3. \quad (6)$$

For $\#A = 2$, many things go wrong: one must require ord to be constant; the fundamental group has a non-trivial centre; and the degree-d self-covering of (S^2, A, ord) has an extra symmetry of order $d-1$. All our statements can be modified to take into account this special case, see §6.3.

Fix basepoints $* \in S^2 \setminus A$ and $\dagger \in S^2 \setminus C$. The *orbisphere biset* of an orbisphere map

$$f\colon (S^2, C, \mathrm{ord}_C) \longrightarrow (S^2, A, \mathrm{ord}_A)$$

is the $\pi_1(S^2, C, \mathrm{ord}_C, \dagger)$-$\pi_1(S^2, A, \mathrm{ord}_A, *)$-biset

$$B(f) = \{\beta\colon [0,1] \to S^2 \setminus C \mid \beta(0) = \dagger,\ f(\beta(1)) = *\} / \approx_{C, \mathrm{ord}_C},$$

with "$\approx_{C,\mathrm{ord}}$" denoting homotopy in the orbispace (S^2, C, ord). An orbisphere biset ${}_H B_G$ can also be defined purely algebraically, see [5, §8 and Definition 2.6]. By [5, Theorem 8.9], there is an orbisphere map $f\colon (S^2, C, \mathrm{ord}_C) \to$

(S^2, A, ord_A), unique up to isotopy, such that B is isomorphic to $B(f)$. We denote by $B_*\colon C \to A$ the induced map on the peripheral conjugacy classes of H and G – they are identified with the associated punctures.

Let (S^2, A, ord) and $(S^2, A \sqcup D, \widetilde{\mathrm{ord}})$ be orbispheres, and suppose that $\mathrm{ord}(a) \mid \widetilde{\mathrm{ord}}(a)$ for all $a \in A$. We then have a natural *forgetful* homomorphism

$$\mathcal{F}_D\colon \pi_1(S^2, A \sqcup D, \widetilde{\mathrm{ord}}, *) \longrightarrow \pi_1(S^2, A, \mathrm{ord}, *)$$

given by $\gamma_a \mapsto \gamma_a$ for $a \in A$ and $\gamma_d \mapsto 1$ for $d \in D$. We write the forgetful map $(S^2, A \sqcup D, \widetilde{\mathrm{ord}}) \dashrightarrow (S^2, A, \mathrm{ord})$ with a dashed arrow, because even though $\mathcal{F}_D$ is a genuine group homomorphism, the corresponding map between orbispheres is only densely defined. Note, however, that its inverse is a genuine orbisphere map.

Consider forgetful maps

$$(S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \dashrightarrow (S^2, C, \mathrm{ord})$$

and

$$(S^2, A \sqcup D, \widetilde{\mathrm{ord}}) \dashrightarrow (S^2, A, \mathrm{ord});$$

in the sequel we shall keep the notations

$$X := (S^2, A, \mathrm{ord}), \quad Y := (S^2, C, \mathrm{ord}), \quad \widetilde{A} := A \sqcup D, \quad \widetilde{C} := C \sqcup E.$$

Let $\tilde{f}\colon (S^2, \widetilde{C}, \widetilde{\mathrm{ord}}) \to (S^2, \widetilde{A}, \widetilde{\mathrm{ord}})$ be an orbispace map, with such that $\tilde{f}$ restricts to an orbisphere map $f\colon (S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$. We thus have

$$\begin{array}{ccc} (S^2, \widetilde{C}, \widetilde{\mathrm{ord}}) & \xrightarrow{\tilde{f}} & (S^2, \widetilde{A}, \widetilde{\mathrm{ord}}) \\ \downarrow{\scriptstyle \mathcal{F}_E} & & \downarrow{\scriptstyle \mathcal{F}_D} \\ X = (S^2, C, \mathrm{ord}) & \xrightarrow{f} & (S^2, A, \mathrm{ord}) = Y \end{array} \tag{7}$$

(In particular, $\tilde{f}(C) \subseteq A$ and A contains all critical values of $\tilde{f}$.) We are concerned, in this section, with the relationship between $B(\tilde{f})$ and $B(f)$; we shall show that $B(\tilde{f})$ may be encoded by $B(f)$ and a *portrait of bisets*, and that this encoding is unique up to a certain equivalence. The algebraic counterpart of (7) is

$$\begin{array}{ccccc} \widetilde{H} := \pi_1(S^2, \widetilde{C}, \widetilde{\mathrm{ord}}, \dagger) & \curvearrowright & B(\tilde{f}) & \curvearrowleft & \pi_1(S^2, \widetilde{A}, \widetilde{\mathrm{ord}}, *) =: \widetilde{G} \\ \downarrow{\scriptstyle \mathcal{F}_E} & & \downarrow{\scriptstyle \mathcal{F}_{E,D}} & & \downarrow{\scriptstyle \mathcal{F}_D} \\ H := \pi_1(S^2, C, \mathrm{ord}, \dagger) & \curvearrowright & B(f) & \curvearrowleft & \pi_1(S^2, A, \mathrm{ord}, *) =: G \end{array} \tag{8}$$

where we denote by

$$\mathcal{F}_{E,D}\colon B(\tilde{f}) \longrightarrow B(f) \cong H \otimes_{\widetilde{H}} B(\tilde{f}) \otimes_{\widetilde{G}} G \tag{9}$$

the natural map given by $b \mapsto 1 \otimes b \otimes 1$.

2.1. Braid groups and knitting equivalence. We recall that $\mathbf{Mod}(S^2, A)$ and $\mathbf{Mod}(X)$ are by definition the same groups, and coincide with the groups of outer automorphisms of both $\pi_1(S^2, A)$ and $\pi_1(X)$ that preserve peripheral conjugacy classes, see [5, Theorem 8.3]. We shall define a subgroup $\mathbf{Mod}(X|D)$ intermediate between $\mathbf{Mod}(X) \cong \mathbf{Mod}(S^2, A)$ and $\mathbf{Mod}(S^2, \widetilde{A})$ which will be useful to relate $B(\tilde{f})$ and $B(f)$.

Definition 2.1 (pure braid group). Let D be a finite set on a finitely punctured sphere $S^2 \setminus A$. The *pure braid group* $\mathbf{Braid}(S^2 \setminus A, D)$ is the set of continuous motions $m\colon [0,1] \times D \to S^2 \setminus A$ considered up to isotopy so that

- $m(t, -)\colon D \hookrightarrow S^2 \setminus A$ is an inclusion for every $t \in [0,1]$;
- $m(0, -) = m(1, -) = \mathbb{1}\restriction_D$.

The product in $\mathbf{Braid}(S^2 \setminus A, D)$ is concatenation of motions, and the inverse is time reversal.

Note that in the special case $D = \{*\}$ we have

$$\mathbf{Braid}(S^2 \setminus A, \{*\}) = \pi_1(S^2 \setminus A, *).$$

Theorem 2.2 (Birman [8]). *For every $m \in \mathbf{Braid}(S^2 \setminus A, D)$ there is a unique mapping class* $\operatorname{push}(m) \in \ker(\mathbf{Mod}(S^2, A \sqcup D) \to \mathbf{Mod}(S^2, A))$ *such that* $\operatorname{push}(m)$ *is isotopic rel A to the identity via an isotopy moving D along m^{-1}. The map*

$$\operatorname{push}\colon \mathbf{Braid}(S^2 \setminus A, D) \longrightarrow \ker(\mathbf{Mod}(S^2, A \sqcup D) \longrightarrow \mathbf{Mod}(S^2, A)) \tag{10}$$

is an isomorphism (*it would be merely an epimorphism if $\#A \le 2$*).

From now on we identify $\mathbf{Braid}(S^2 \setminus A, D)$ with its image under (10).

Definition 2.3 (knitting group). Let $X = (S^2, A, \operatorname{ord})$ be an orbisphere and let D be a finite subset of $S^2 \setminus A$. The *knitting braid group* $\mathbf{knBraid}(X, D)$ is the kernel of the forgetful morphism

$$\mathcal{E}_D\colon \mathbf{Braid}(S^2 \setminus A, D) \longrightarrow \prod_{d \in D} \pi_1(X, d);$$

it is the set of D-strand braids in $S^2 \setminus A$ all of whose strands are homotopically trivial in X.

In case $X = (S^2, A)$, knitting elements are the "$(\#D - 1)$-decomposable braids" from [18]. The terminology states that the strands of the braid may be knitted among themselves, while each strand is individually trivial in X.

Lemma 2.4. **knBraid**(X, D) *is a normal subgroup of* **Mod**$(S^2, A \sqcup D)$.

Proof. We show that for every $m \in$ **knBraid**(X, D) and $h \in$ **Mod**$(S^2, A \sqcup D)$ we have $h^{-1}mh \in$ **knBraid**(X, D). Indeed, h restricts to an orbisphere map $h\colon X \circlearrowleft$ fixing D pointwise. Thus, $m(d, -)$ is a trivial loop in $\pi_1(X, d)$ if and only if $h(m(d, -))$ is a trivial loop in $\pi_1(X, d)$ for every $d \in D$. $\square$

Define

$$\mathbf{Mod}(X|D) := \mathbf{Mod}(S^2, A \sqcup D)/\mathbf{knBraid}(X, D). \tag{11}$$

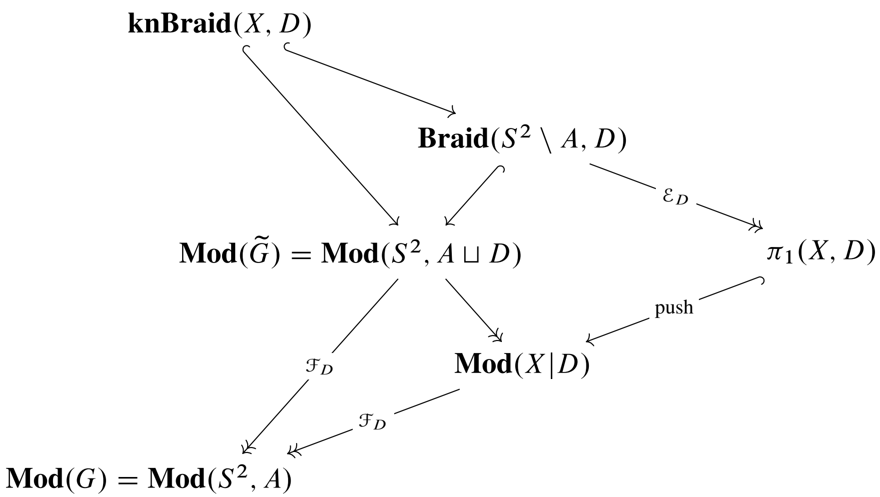
With $G = \pi_1(X, *)$, we write

$$\mathbf{Mod}(G|D) = \mathbf{Mod}(X|D) \quad \text{and} \quad \mathbf{Mod}(G) = \mathbf{Mod}(X).$$

We interpret elements of **Mod**(G) as outer automorphisms of G, as mapping classes and as biprincipal bisets. We also introduce the notation

$$\pi_1(X, D) := \prod_{d \in D} \pi_1(X, d) \cong G^D.$$

Note the following four exact sequences:



(12)

Exactness follows by definition except surjectivity in the top sequence. Given a sequence of loops $(\gamma_d)_{d \in D} \in \pi_1(X, D)$, we may isotope them slightly to obtain $\gamma_d(t) \neq \gamma_{d'}(t)$ for all $t \in [0, 1]$ and all $d \neq d'$. Then $b(d, t) := \gamma_d(t)$ is a braid in $S^2 \setminus A$ and defines via "push" an element of **Braid**$(S^2 \setminus A, D)$ mapping to (γ_d).

2.2. $\tilde{f}$-impure mapping class groups. As in (7), consider an orbisphere map $\tilde{f}\colon(S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \to (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$ that projects to $f\colon(S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$. We will enlarge the groups in (12) to "$\tilde{f}$-impure mapping class groups" so that exact sequences analogous to (16) hold.

Let $\mathbf{Mod}^*(S^2, C \sqcup E)$ be the group of homeomorphisms $m\colon(S^2, C \sqcup E)\circlearrowleft$ considered up to isotopy rel $C \sqcup E$ such that $m\restriction_C$ is the identity and for every $e \in E$ we have $\tilde{f}(m(e)) = \tilde{f}(e)$; i.e., m may permute points in $\tilde{f}^{-1}(\tilde{f}(e)) \cap E$. There is a natural forgetful morphism

$$\mathcal{F}_E\colon \mathbf{Mod}^*(S^2, C \sqcup E) \longrightarrow \mathbf{Mod}(S^2, C).$$

As in Definition 2.1, the *braid group* $\mathbf{Braid}^*(S^2 \setminus C, E)$ is the set of continuous motions $m\colon[0,1] \times E \to S^2 \setminus C$ considered up to isotopy so that

- $m(t, -)\colon E \hookrightarrow S^2 \setminus C$ is an inclusion for every $t \in [0, 1]$;
- $m(0, -) = \mathbb{1}\restriction_E$;
- $m(1, e) \in \tilde{f}^{-1}(\tilde{f}(e))$ for every $e \in E$.

Every $m \in \mathbf{Braid}^*(S^2 \setminus C, E)$ induces a permutation $\pi_m\colon e \mapsto m(1, e)$ of E. The pure braid group consists of those permutations with $\pi_m = \mathbb{1}$. The product in $\mathbf{Braid}^*(S^2 \setminus C, E)$ is $m \cdot m' = m \# (m' \circ (\mathbb{1} \times \pi_m))$, with as usual "$\#$" standing for concatenation of motions. Birman's theorem (a slight generalization of Theorem 2.2) still holds: the group $\mathbf{Braid}^*(S^2 \setminus C, E)$ is isomorphic to the kernel of $\mathbf{Mod}^*(S^2, C \sqcup E) \to \mathbf{Mod}(S^2, C)$ via the push operator.

Let $\pi_1^*(Y, E)$ be the group of motions $m\colon[0, 1] \times E \to Y = (S^2, C, \mathrm{ord})$, considered up to homotopy, such that

- $m(0, -) = \mathbb{1}\restriction_E$;
- $m(1, e) \in \tilde{f}^{-1}(\tilde{f}(e))$ for every $e \in E$;

here $m, m'\colon E \hookrightarrow Y$ are *homotopic* if $m(-, e)$ and $m'(-, e)$ are homotopic curves (relative to their endpoints) in Y for all $e \in E$. The product in $\pi_1^*(Y, E)$ is again $m \cdot m' = m \# (m' \circ (\mathbb{1} \times \pi_m))$. We have

$$\pi_1^*(Y, E) \cong \prod_{e \in E} \pi_1(Y, e) \rtimes \prod_{d \in \tilde{f}(E)} (\tilde{f}^{-1}(d) \cap E)\nleftarrow, \tag{13}$$

the isomorphism mapping m to its restrictions $m(-, e)$ and its permutation π_m.

Lemma 2.5. *The following sequence is exact*:

$$\mathbf{knBraid}(Y, E) \hookrightarrow \mathbf{Braid}^*(S^2 \setminus C, E) \xrightarrow{\mathcal{E}_E} \pi_1^*(Y, E), \tag{14}$$

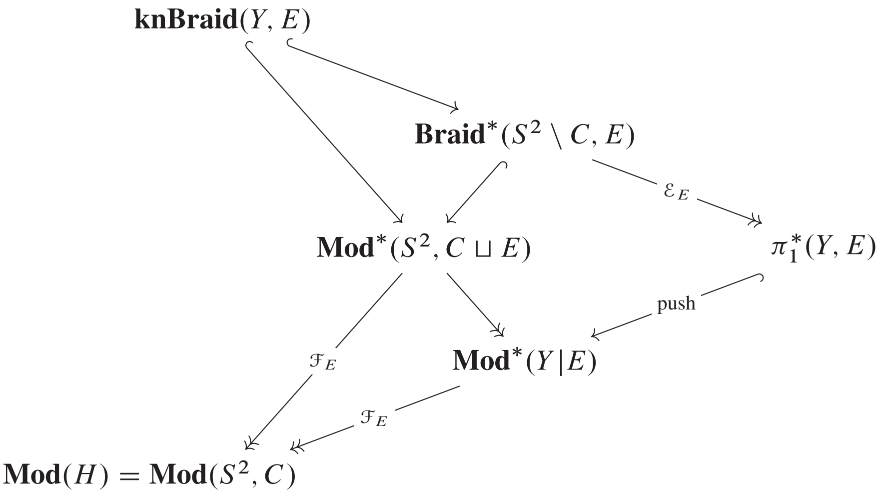
where $\mathcal{E}_E$ is the natural forgetful morphism.

Proof. Suppose than $\mathcal{E}_E(m) = \mathcal{E}_E(m')$. Then $m^{-1}m'$ is pure and so one has $m^{-1}m' \in \mathbf{knBraid}(Y, E)$. The converse is also obvious. □

The same argument as in Lemma 2.4 shows that $\mathbf{knBraid}(Y, E)$ is a normal subgroup of $\mathbf{Mod}^*(S^2, C \sqcup E)$. We may thus define

$$\mathbf{Mod}^*(Y|E) := \mathbf{Mod}^*(S^2, C \sqcup E)/\mathbf{knBraid}(Y, E). \tag{15}$$

As in (12) we have the following exact sequences



(16)

2.3. Branched coverings.

Recall that, for orbisphere maps

$$f_0, f_1 \colon (S^2, C, \mathrm{ord}) \longrightarrow (S^2, A, \mathrm{ord}),$$

we write

$$f_0 \approx_C f_1,$$

and call them *isotopic*, if there is a path $(f_t)_{t\in[0,1]}$ of orbisphere maps

$$f_t \colon (S^2, C, \mathrm{ord}) \longrightarrow (S^2, A, \mathrm{ord}).$$

Equivalently,

Lemma 2.6 ([5, §2.12]). *$f_0 \approx_C f_1$ if and only if $h f_0 = f_1$ for a homeomorphism $h \colon (S^2, C) \circlearrowleft$ that is isotopic to the identity.*

Proof. If there exists an isotopy (h_t) witnessing $\mathbb{1} \approx_C h$, then $f_t := h_t f_0$ witnesses $f_0 \approx_C f_1$. Conversely, since all critical values of f_t are frozen in A, the set $f_t^{-1}(y)$ moves homeomorphically for every $y \in S^2$ (equivalently, no critical points collide). Therefore, we may factor $f_t = h_t f_0$, with $h_t(z)$ the trajectory of $z \in f_t^{-1}(y)$; this defines an isotopy from $\mathbb{1}$ to $h := h_1$. □

Consider orbisphere maps $\tilde{f}, \tilde{g}\colon (S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \to (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$ as in (7). We write $\tilde{f} \approx_{C|E} \tilde{g}$, and call $\tilde{f}, \tilde{g}$ *knitting-equivalent*, if $\tilde{f} = h\tilde{g}$ for a homeomorphism $h\colon (S^2, C \sqcup E) \circlearrowleft$ in $\mathbf{knBraid}(Y, E)$; we have

$$\tilde{f} \approx_{C \sqcup E} \tilde{g} \implies \tilde{f} \approx_{C|E} \tilde{g} \implies \tilde{f} \approx_{C} \tilde{g}.$$

For $m \in \mathbf{Braid}(S^2 \setminus A, D)$ we define its *pullback* $(\tilde{f})^* m\colon [0,1] \times E \to S^2 \setminus C$ by

$$((\tilde{f})^* m)(-, e) := \begin{cases} m(-, \tilde{f}(e)){\uparrow}^e_{\tilde{f}} & \text{if } \tilde{f}(e) \in D, \\ e & \text{if } \tilde{f}(e) \in A. \end{cases}$$

This defines a motion of E; note that $(\tilde{f})^* m(1, e)$ need not equal e:

Lemma 2.7. *If* $\mathrm{push}(m) \in \mathbf{Braid}(S^2 \setminus A, D)$, *then* $\mathrm{push}((\tilde{f})^* m)$ *defines an element of* $\mathbf{Braid}^*(S^2 \setminus C, E)$ *and we have the following commutative diagram*:

$$\begin{array}{ccc} (S^2, C \sqcup E) & \xrightarrow{\mathrm{push}((\tilde{f})^* m)} & (S^2, C \sqcup E) \\ \tilde{f}\downarrow & \approx_{C \sqcup E} & \downarrow\tilde{f} \\ (S^2, A \sqcup D) & \xrightarrow[\mathrm{push}(m)]{} & (S^2, A \sqcup D) \end{array} \tag{17}$$

Proof. Let us discuss in more detail the operator "push". Consider a simple arc $\gamma\colon [0,1] \to S^2 \setminus A$ and let $\mathcal{U} \subset S^2 \setminus A$ be a small disk neighborhood of γ. We can define (in a non-unique way) a homeomorphism $\mathrm{push}(\gamma)\colon (S^2, A) \circlearrowleft$ that maps $\gamma(0)$ to $\gamma(1)$ and is identity away from $\mathcal{U}$. Let $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_d$ be the preimages of $\mathcal{U}$ under $\tilde{f}$, where $d = \deg(\tilde{f})$. Each $\mathcal{U}_i$ contains a preimage γ_i of γ. Let $\mathrm{push}(\gamma_i)\colon \mathcal{U}_i \circlearrowleft$ be the lift of $\mathrm{push}(\gamma) {\downharpoonleft}_{\mathcal{U}}$ under $\tilde{f}\colon \mathcal{U}_i \to \mathcal{U}$, extended by the identity on $S^2 \setminus \mathcal{U}_i$. Then

$$\mathrm{push}(\gamma_1)\mathrm{push}(\gamma_2) \cdots \mathrm{push}(\gamma_d) \cdot \tilde{f} = \tilde{f} \cdot \mathrm{push}(\gamma). \tag{18}$$

For $m \in \mathbf{Braid}(S^2 \setminus A, D)$, we can define $\mathrm{push}(m)$ as a composition of pushes along finitely many simple arcs β_i. Using (18) we lift all $\mathrm{push}(\beta_i)$ through $\tilde{f}$; considering the equation rel $C \sqcup E$ we obtain (17). □

We note that $\mathrm{push}(m)$ does not necessarily lift to a "push" if m moves a critical value. Indeed if γ is a simple loop then $\mathrm{push}(\gamma)$ is the quotient of two Dehn twists about the boundary curves of an annulus surrounding γ, see [12, §4.2.2]; however the lift of γ will not be a union of simple closed curves if γ contains a critical value; an annulus around γ will not lift to an annulus, but rather to a more complicated surface F; and the quotient of Dehn twists about boundary components of F will not be a quotient of Dehn twists about boundary components of annuli.

Proposition 2.8. *Let $\tilde{f}\colon (S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \to (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$ be an orbisphere map as in (7). Then every element in* $\mathbf{knBraid}(X, D)$ *lifts through $\tilde{f}$ to an element of* $\mathbf{knBraid}(Y, E)$.

If $\tilde{g}\colon (S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \to (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$ is another orbisphere map, then $\tilde{f} \approx_{C|E} \tilde{g}$ if and only if one has $\tilde{f} \approx_C h\tilde{g}k$ for some $h \in$ $\mathbf{knBraid}(Y, E)$, *$k \in$* $\mathbf{knBraid}(X, D)$.

Proof. Consider $h \in \mathbf{knBraid}(X, D)$. By Theorem 2.2 we may write $h = \mathrm{push}(b)$. Since $b(-, d)$ is homotopically trivial in X for every $d \in D$, the curve $b(-, \tilde{f}(e))\!\uparrow^e_{\tilde{f}}$ ends at e for all $e \in E$ with $\tilde{f}(e) \in D$, because this curve is in Y. Therefore, $(\tilde{f})^* b(1, -) = \mathbb{1}\!\downarrow_E$, and $\mathrm{push}((\tilde{f})^* b) \in \mathbf{Braid}(S^2 \setminus C, E)$. Since the lifts $b(-, \tilde{f}(e))\!\uparrow^e_{\tilde{f}}$ are homotopically trivial in Y for all $e \in E$, we have $\mathrm{push}((\tilde{f})^* b) \in \mathbf{knBraid}(Y, E)$ and Lemma 2.7 concludes the first claim.

The second claim is a direct consequence of the first. □

As a consequence, we may detail a little bit more the map $\mathcal{E}_D$ in (12). Choose for every $d \in D$ a path ℓ_d in $S^2 \setminus A$ from $*$ to d. This path defines an isomorphism $\pi_1(X, d) \to \pi_1(X, *) = G$ by $\gamma \mapsto \ell_d \# \gamma \# \ell_d^{-1}$. We thus have a map

$$\mathcal{E}_D\colon \mathbf{Braid}(S^2 \setminus A, D) \twoheadrightarrow G^D, \quad m \longmapsto (\ell_d \# (\mathrm{push}^{-1}(m)\!\downarrow_d) \# \ell_d^{-1})_{d \in D}, \tag{19}$$

and $\ker(\mathcal{E}_D) = \mathbf{knBraid}(X, D)$.

2.4. Mapping class bisets. We introduce some notation parallel to that in (12) and (16) for mapping class bisets. Let $\mathcal{F}_E\colon \widetilde{H} \to H$ and $\mathcal{F}_D\colon \widetilde{G} \to G$ be forgetful morphisms of orbisphere groups as in (8), and let $\widetilde{B}$ be an orbisphere biset. Let $B := H \otimes_{\widetilde{H}} \widetilde{B} \otimes_{\widetilde{G}} G$ be the induced H-G-biset. We have forgetful morphisms of groups $\mathcal{F}_D\colon \mathbf{Mod}(\widetilde{G}) \to \mathbf{Mod}(G)$ and $\mathcal{F}_E\colon \mathbf{Mod}(\widetilde{H}) \to \mathbf{Mod}(H)$. Corresponding mapping class bisets are written respectively, with $\tilde{f}$ and f the orbisphere maps associated with $\widetilde{B}$ and B,

$$\begin{aligned} M(\widetilde{B}) = M(\tilde{f}) &:= \{n \otimes \widetilde{B} \otimes m \mid n \in \mathbf{Mod}(\widetilde{H}), m \in \mathbf{Mod}(\widetilde{G})\}/{\cong} \\ &= \{n \tilde{f} m \mid n \in \mathbf{Mod}(S^2, \widetilde{C}), m \in \mathbf{Mod}(S^2, \widetilde{A})\}/{\approx_{\widetilde{C}}}, \\ M(B) = M(f) &:= \{n \otimes B \otimes m \mid n \in \mathbf{Mod}(H), m \in \mathbf{Mod}(G)\}/{\cong} \\ &= \{n f m \mid n \in \mathbf{Mod}(S^2, C), m \in \mathbf{Mod}(S^2, A)\}/{\approx_C} \end{aligned}$$

together with the natural forgetful intertwiner

$$\mathcal{F}_{E,D}\colon M(\widetilde{B}) \longrightarrow M(B), \quad \widetilde{B}' \longrightarrow \mathcal{F}_{E,D}(B') = H \otimes_{\widetilde{H}} \widetilde{B}' \otimes_{\widetilde{G}} G. \tag{20}$$

We may also define the following mapping class biset, sometimes larger than $M(\widetilde{B})$: assume first that $\widetilde{\mathrm{ord}}$ is constant on E, possibly ∞, and set

$$M^*(\widetilde{B}) = M^*(\tilde{f}) := \left\{ \widetilde{B}' \text{ an } \widetilde{H}\text{-}\widetilde{G}\text{-orbisphere biset} \;\middle|\; \begin{matrix} \mathfrak{F}_{E,D}(\widetilde{B}') \in M(B) \\ (\widetilde{B}')_* = \widetilde{B}_* \end{matrix} \right\} /\cong, \tag{21}$$

where $(\widetilde{B}')_*\colon \widetilde{C} \to \widetilde{A}$ denotes the induced map on marked conjugacy classes. It is an $\mathbf{Mod}^*(S^2,\widetilde{C})$-$\mathbf{Mod}(S^2,\widetilde{A})$-biset; note indeed that we have $n \otimes \widetilde{B}' \in M^*(\widetilde{B})$ for $\widetilde{B}' \in M^*(\widetilde{B})$ and $n \in \mathbf{Mod}^*(S^2,\widetilde{C})$, because $\mathfrak{F}_{E,D}(n \otimes \widetilde{B}) = \mathfrak{F}_E(n) \otimes B$. Again there is a natural forgetful intertwiner

$$\mathfrak{F}_{E,D}\colon {}_{\mathbf{Mod}^*(S^2,\widetilde{C})}M^*(\tilde{f})_{\mathbf{Mod}(S^2,\widetilde{A})} \longrightarrow {}_{\mathbf{Mod}(S^2,C)}M(f)_{\mathbf{Mod}(S^2,A)}. \tag{22}$$

We note that the left action of $\mathbf{Mod}(S^2,\widetilde{C})$ on $M(\tilde{f})$ does not necessarily extend to an action of $\mathbf{Mod}^*(S^2,\widetilde{C})$ on $M(\tilde{f})$, because the result of the action is in general in $M^*(f)$ and not in $M(f)$, see Example 6.2.

In case $\widetilde{\mathrm{ord}}$ is not constant on E, we should be careful, because permutation of points in E does not leave $\widetilde{H}$ invariant; rather, the image of $\widetilde{H}$ under such a permutation gives an orbisphere group isomorphic to $\widetilde{H}$. However, $M^*(\widetilde{B})$ and $\mathbf{Mod}^*(S^2,\widetilde{C})$ do not depend on the orbisphere structure, so the definition may be applied with $\widetilde{H}$ and $\widetilde{G}$ replaced by orbisphere groups with larger orders.

Let us call the set of extra marked points E *saturated* if

$$\tilde{f}^{-1}(\tilde{f}(E)) \subseteq C \sqcup E.$$

Lemma 2.9. (1) *The mapping class biset $M^*(\tilde{f})$ is left-free.*

(2) *Suppose that E is saturated and that*

$$g_0, g_1\colon (S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \longrightarrow (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$$

are orbisphere maps coinciding on $C \sqcup E$ and such that $\mathfrak{F}_{E,D}(g_0)$ and $\mathfrak{F}_{E,D}(g_1)$ are isotopic through maps $(S^2,C) \to (S^2,A)$. Then $g_0, g_1 \in M^*(\tilde{f})$, and there is an $m \in \mathbf{Braid}^*(S^2 \setminus C, E)$ such that $mg_0 = g_1$ holds in $M^*(\tilde{f})$.

(3) *If E is saturated, then*

$$M^*(\tilde{f}) = \{m\tilde{f}n \mid m \in \mathbf{Mod}^*(S^2, C \sqcup E), n \in \mathbf{Mod}(S^2, A \sqcup D)\}/{\approx_{\widetilde{C}}}. \tag{23}$$

Proof. The proof of the first claim follows the lines of [5, Proposition 6.4]: suppose that g, mg are isotopic through maps $(S^2,\widetilde{C}) \to (S^2,\widetilde{A})$ for some $m \in \mathbf{Mod}^*(S^2, C \sqcup E)$ and $g \in M^*(\tilde{f})$. By Lemma 2.6, we may assume $g = mg$ as maps; then the homeomorphism m is a deck transformation of the covering induced by g, so m has finite order because $\deg(g) < \infty$. Recall that $\#C \geq 3$ by our standing assumption (6). Since m fixes at least 3 points in C and m has finite order, we deduce that m is the identity. This shows that $M^*(\tilde{f})$ is left-free.

For the second claim, let $(g_t\colon (S^2, C) \to (S^2, A))_{t\in[0,1]}$ be an isotopy between g_0 and g_1. By Lemma 2.6, we may write g_t at $m_t g_0$ for $m_t\colon (S^2, C) \circlearrowleft$. Then m_1 preserves E because E is saturated, so $m_1 \in \mathbf{Braid}^*(S^2 \setminus C, E)$ as required.

The third claim directly follows from the second. □

Definition 2.10 (extensions of bisets, see [5, Definition 5.3]). Let ${}_{G_1}B_{G_2}$ be a G_1-G_2-biset and let N_1, N_2 be normal subgroups of G_1 and G_2 respectively, so that for $i = 1, 2$ we have short exact sequences

$$1 \longrightarrow N_i \longrightarrow G_i \overset{\pi}{\longrightarrow} Q_i \longrightarrow 1. \tag{24}$$

If the quotient Q_1-Q_2-biset $N_1\backslash B/N_2$, consisting of connected components of ${}_{N_1}B_{N_2}$, is left-free, then the sequence

$${}_{N_1}B_{N_2} \hookrightarrow {}_{G_1}B_{G_2} \overset{\pi}{\twoheadrightarrow} {}_{Q_1}(N_1\backslash B/N_2)_{Q_2} \tag{25}$$

is called an *extension of left-free bisets*.

Definition 2.11 (inert biset morphism). Let $\widetilde{H} \twoheadrightarrow H$ and $\widetilde{G} \twoheadrightarrow G$ be surjective group homomorphisms, and let $\widetilde{B}$ be a left-free $\widetilde{H}$-$\widetilde{G}$-biset. Recall that the tensor product

$$B := H \otimes_{\widetilde{H}} \widetilde{B} \otimes_{\widetilde{G}} G$$

is isomorphic to the double quotient $\ker(\widetilde{H} \to H)\backslash\widetilde{B}/\ker(\widetilde{G} \to G)$ with natural H-G-actions. The natural map $\mathcal{F}\colon \widetilde{B} \to B$ is called *inert* if B is a left-free biset and the natural map $\{\cdot\} \otimes_{\widetilde{G}} \widetilde{B} \to \{\cdot\} \otimes_G B$ is a bijection. In particular, B has the same number of left orbits as $\widetilde{B}$. In other words, assuming that the groups $\widetilde{H}$ and H have similarly-written generators and so do $\widetilde{G}$ and G, the wreath recursions of $\widetilde{B}$ and B are identical.

Yet said differently, in the extension ${}_{\ker}\widetilde{B}_{\ker} \hookrightarrow {}_{\widetilde{H}}\widetilde{B}_{\widetilde{G}} \twoheadrightarrow {}_H B_G$ the kernel is a disjoint union of left-principal bisets. If $\widetilde{G} = \widetilde{H}$ and $G = H$ so that the bisets can be iterated, then $\widetilde{B} \to B$ is inert precisely when we have a factorization $\widetilde{G} \to G \to \mathrm{IMG}_{\widetilde{G}}(\widetilde{B})$, the latter group being the quotient of G by the kernel of the right action on the rooted tree $\{\cdot\} \otimes_H \bigsqcup_{n\ge 0} B^{\otimes n}$, see [5, §8.3].

Define

$$M^*(B|E, D) := \mathbf{knBraid}(Y, E)\backslash M^*(\widetilde{B})/\mathbf{knBraid}(X, D); \tag{26}$$

this is naturally a $\mathbf{Mod}^*(Y|E)$-$\mathbf{Mod}(X|D)$-biset, and Proposition 2.8 implies in particular that it is left-free:

Proposition 2.12. *The natural forgetful maps*

$$_{\mathbf{Mod}(\tilde{H})}M(\widetilde{B})_{\mathbf{Mod}(\tilde{G})} \longrightarrow {}_{\mathbf{Mod}(H|E)}M(B|E,D)_{\mathbf{Mod}(G|D)}$$

and

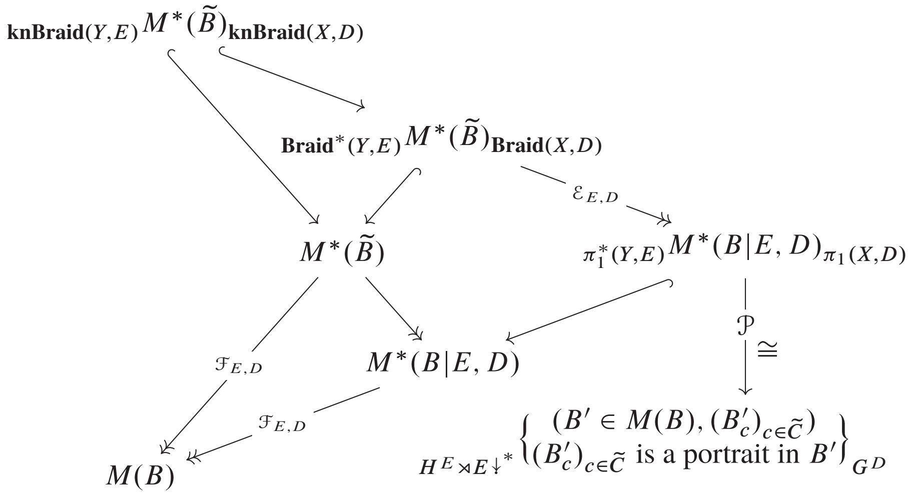
$$_{\mathbf{Mod}^*(S^2,C\sqcup E)}M^*(\widetilde{B})_{\mathbf{Mod}(S^2,A\sqcup D)} \longrightarrow {}_{\mathbf{Mod}^*(Y|E)}M^*(B|E,D)_{\mathbf{Mod}(X|D)}$$

are inert.

Let $E{\downarrow}^*$ denote the group of all permutations $t\colon E \circlearrowleft$ such that $\tilde{f}(t(e)) = \tilde{f}(e)$. We denote by $H^E \rtimes E{\downarrow}^*$ the semidirect product where $E{\downarrow}^*$ acts on H^E by permuting coordinates; compare with (13). We have $\pi_1^*(Y,E) \cong H^E \rtimes E{\downarrow}^*$.

We denote by $_{\mathbf{Braid}^*(Y,E)}M^*(\widetilde{B})_{\mathbf{Braid}(X,D)}$ and $_{\pi_1^*(Y,E)}M^*(B|E,D)_{\pi_1(X,D)}$ the restrictions of $M^*(\widetilde{B})$ and $M^*(B|E,D)$ to braid and fundamental groups.

Theorem 2.13. *If E is saturated, then the following sequences are extensions of bisets*:



(27)

(*The isomorphism on the right is the topic of Theorem 2.19, and will be proven there.*)

Proof. By Lemma 2.9 (2), we know that $\mathcal{F}_{E,D}^{-1}(B)$ is a connected subbiset of $_{\mathbf{Braid}^*(Y,E)}M^*(\widetilde{B})_{\mathbf{Braid}(X,D)}$; thus, the central-to-left sequence is an extension of bisets. Exactness of other sequences follows from Proposition 2.12. □

Note that, if E were not saturated or if we replaced $M^*(\widetilde{B})$ by $M(\widetilde{B})$ in (27), then we wouldn't have exact sequences of bisets anymore, because the fibres of

$M^*(\widetilde{B}) \twoheadrightarrow M(B)$ wouldn't have to be connected; see Example 6.1. The failure of transitivity is described by Lemma 2.20. There are similar exact sequences in case $f_*\colon E \to D$ is a bijection, see Theorem 2.22.

2.5. Portraits of bisets. First, a *portrait of groups* amounts to a choice of representative in each peripheral conjugacy class of an orbisphere group:

Definition 2.14 (portraits of groups). Let G be an orbisphere group with marked conjugacy classes $(\Gamma_a)_{a\in A}$ and let $\widetilde{A}$ be a finite set containing A. A *portrait of groups* $(G_a)_{a\in\widetilde{A}}$ in G is a collection of cyclic subgroups $G_a \le G$ so that

$$G_a = \begin{cases} \langle g\rangle & \text{for some } g \in \Gamma_a, \text{ if } a \in A,\\ \langle 1\rangle & \text{otherwise.}\end{cases}$$

If $\widetilde{A} = A$, then $(G_a)_{a\in A}$ is called a *minimal* portrait of groups.

Definition 2.15 (portraits of bisets). Let H, G be orbisphere groups with peripheral classes indexed by C, A respectively, let ${}_H B_G$ be an orbisphere biset, and let $f_*\colon C \to A$ be its portrait. We also have a "degree" map $\deg\colon C \to \mathbb{N}$. A *portrait of bisets* relative to these data consists of

- portraits of groups $(H_c)_{c\in\widetilde{C}}$ in H and $(G_a)_{a\in\widetilde{A}}$ in G;
- extensions $f_*\colon \widetilde{C} \to \widetilde{A}$ of f_* and $\deg\colon \widetilde{C} \to \mathbb{N}$ of deg;
- a collection $(B_c)_{c\in\widetilde{C}}$ of subbisets of B such that every B_c is an H_c-$G_{f_*(c)}$-biset that is right-transitive and left-free of degree $\deg(c)$, and such that if $f_*(c) = f_*(c')$ and $HB_c = HB_{c'}$ qua subsets of B then $c = c'$.

If $\widetilde{C} = C$ and $\widetilde{A} = A$, then $(B_c)_{c\in C}$ is called a *minimal* portrait of bisets.

Note in particular that if $c \in \widetilde{C} \setminus C$ then H_c is trivial and the subbiset B_c consists of $\deg(c)$ elements permuted by $G_{f_*(c)}$. If moreover $f_*(c) \notin A$, then $\deg(c) = 1$. For simplicity the portrait of bisets is sometimes simply written $(B_c)_{c\in\widetilde{C}}$, the other data $f_*, \deg, (H_c)_{c\in\widetilde{C}}, (G_a)_{a\in\widetilde{A}}$ being implicit.

Here is a brief motivation for the definition. Consider an expanding Thurston map f and its associated biset B. Recall from [21] that bounded sequences in B represent points in the Julia set of f; in particular constant sequences represent fixed points of f and vice versa. On the other hand, fixed Fatou components of f are represented by local subbisets of B with same cyclic group acting on the left and on the right. All of these are instances of portraits of bisets in B. Furthermore, (pre)periodic Julia or Fatou points are represented by portraits of bisets with (pre)periodic map f_*. E.g., closures of Fatou components intersect if and only if they admit portraits that intersect; and similarly for inclusion in the closure of a Fatou component of a (pre)periodic point in the Julia set.

Let $\mathcal{F}_D\colon \widetilde{G} \to G$ be a marked forgetful morphism of orbisphere groups as in (8). For all $a \in \widetilde{A}$ choose a small disk neighbourhood $\mathcal{U}_a \ni a$ that avoids all other points in $\widetilde{A}$, so that $\pi_1(\mathcal{U}_a \setminus \widetilde{A}) \cong \mathbb{Z}$. We call a curve γ *close to* a if $\gamma \subset \mathcal{U}_a$.

Lemma 2.16. *Every minimal portrait of groups* $(\widetilde{G}_a)_{a\in\widetilde{A}}$ *in* $\widetilde{G}$ *is uniquely characterized by a family of paths* $(\ell_a)_{a\in\widetilde{A}}$ *with*

$$\ell_a\colon [0,1] \to S^2, \quad \ell_a(t) \notin \widetilde{A} \text{ for } t < 1, \quad \ell_a(0) = *, \quad \ell_a(1) = a, \tag{28}$$

considered up to homotopy rel $\widetilde{A}$, *and such that for any sufficiently small* $\epsilon > 0$

$$\widetilde{G}_a = \{\ell_a\restriction_{[0,1-\epsilon]} \#\alpha_a \# (\ell_a\restriction_{[0,1-\epsilon]})^{-1} \mid \alpha_a \text{ is close to } a\}/{\approx_{\widetilde{A}}}. \tag{29}$$

Conversely, every collection of curves (28) *defines by* (29) *a minimal portrait of groups for every sufficiently small* $\epsilon > 0$.

Proof. This follows immediately from the definition of "lollipop" generators, see (5). ☐

It follows that every $\widetilde{G}_a$ is self-normalizing in $\widetilde{G}$: conjugating $\widetilde{G}_a$ by an element $g \notin \widetilde{G}_a$ amounts to precomposing the path ℓ_a with g, resulting in a different path.

Lemma 2.17. *Let* ${}_H B_G$ *be an orbisphere biset. Then every pair of minimal portraits of groups* $(H_c)_{c\in C}$ *in* H *and* $(G_a)_{a\in A}$ *in* G *can be uniquely completed to a minimal portrait of bisets* $(B_c)_{c\in C}$ *in* B.

As a consequence, the minimal portrait of bisets is unique up to conjugacy. We note that the lemma is also true in case ${}_H B_G$ is a cyclic biset, namely if $H, G \cong \mathbb{Z}$; in this case all B_c are equal to B.

Proof. Since by Assumption (6) the sets A and C contain at least 3 elements, H and G are non-cyclic, and in particular have trivial centre.

Let $f_*\colon C \to A$ be the portrait of B. Choose generators $g_a \in G_a$ and $h_c \in H_c$ in marked conjugacy classes. Recall from [3, §2.6] or [5, Definition 2.5] that there are $b_c \in B$ and $k_c \in H$ such that $k_c^{-1} h_c k_c b_c = b_c g_{f(c)}^{\deg(c)}$ for all $c \in C$. Set then $B_c := H_c k_c b_c G_{f(c)}$, and note that these subbisets satisfy Definition 2.15.

Suppose next that $(B'_c)_{c\in C}$ is another portrait of bisets, and choose elements $b'_c \in B'_c$. Then by [5, Definition 2.6(SB$_3$)] the conjugacy class Δ_c appears exactly once among the lifts of $\Gamma_{f(c)}$, so $Hb'_c \cap B_c \neq \emptyset$, so we may choose $k_c \in H$ with $k_c b'_c \in B_c$. Then $k_c B'_c = k_c b'_c G_{f(c)} = B_c G_{f(c)} = B_c$. We have $k_c H_c = H_c k_c$, so $k_c \in H_c$ because H_c is self-normalizing in H, and therefore $B_c = B'_c$. ☐

Consider next a forgetful morphism $\mathcal{F}_{E,D}\colon {}_{\widetilde{H}}\widetilde{B}_{\widetilde{G}} \to {}_H B_G$ of orbisphere bisets, and let $(\widetilde{B}_c)_{c\in\widetilde{C}}$ be the minimal portrait of bisets given by Lemma 2.17.

Lemma 2.18. *Let $(m_c)_{c\in\widetilde{C}}$ and $(\ell_a)_{a\in\widetilde{A}}$ be the paths (see Lemma 2.16) associated with the respective portraits of bisets $(\widetilde{H}_c)_{c\in\widetilde{C}}$ and $(\widetilde{G}_a)_{a\in\widetilde{A}}$. The portrait $(\mathfrak{F}_{E,D}(\widetilde{B}_c))_{c\in\widetilde{C}}$ admits the following description: for every sufficiently small $\epsilon > 0$,*

$$B_c = \{m_c \downharpoonright_{[0,1-\epsilon]} \# \beta_c \# (\ell_{f_*(c)} \downharpoonright_{[0,1-\epsilon]})^{-1} \uparrow_f^{\beta_c(1)} \mid \beta_c \text{ is close to } c\}/{\approx_C}. \quad (30)$$

Proof. Since B_c is obtained from $\widetilde{B}_c$ by the intertwiner $\mathfrak{F}_{E,D}$ (see (9)), it suffices to consider the case $E = D = \emptyset$; and in that case, by Lemma 2.17 it suffices to note that B_c is indeed an H_c-$G_{f_*(c)}$-biset. □

Let ${}_H B_G$ be an orbisphere biset; let $f_*\colon \widetilde{C} \to \widetilde{A}$ be an abstract portrait extending B_*; let $\deg\colon \widetilde{C} \to \mathbb{N}$ be an extension of $\deg_B\colon C \to \mathbb{N}$; and let $(B_c)_{c\in\widetilde{C}}$ be a portrait of bisets in B with portraits of groups $(H_c)_{c\in\widetilde{C}}$ in H and $(G_a)_{a\in\widetilde{A}}$ in G. A *congruence of portraits* is defined by a choice of $(h_c)_{c\in\widetilde{C}}$ in H and $(g_a)_{a\in\widetilde{A}}$ in G, and modifies the portrait of bisets by replacing

$$H_c \rightsquigarrow h_c^{-1} H_c h_c, \quad B_c \rightsquigarrow h_c^{-1} B_c g_{f_*(c)}, \quad G_a \rightsquigarrow g_a^{-1} G_a g_a.$$

By Lemma 2.17, any two minimal portraits of bisets are congruent.

2.6. Main result. Consider an orbisphere map

$$f\colon (S^2, C, \mathrm{ord}) \longrightarrow (S^2, A, \mathrm{ord}).$$

We call it *compatible* with

$$\mathfrak{F}_E\colon (S^2, \widetilde{C}, \widetilde{\mathrm{ord}}) \dashrightarrow (S^2, C, \mathrm{ord})$$

and

$$\mathfrak{F}_D\colon (S^2, \widetilde{A}, \widetilde{\mathrm{ord}}) \dashrightarrow (S^2, A, \mathrm{ord})$$

and $f_*\colon \widetilde{C} \to \widetilde{A}$ and $\deg\colon \widetilde{C} \to \mathbb{N}$ if $\deg(e) = 1$ for all $e \in E$ with $f_*(e) \in D$, and $\widetilde{\mathrm{ord}}(c)\deg(c) \mid \widetilde{\mathrm{ord}}(f_*(c))$ for all $c \in \widetilde{C}$, and $\{\deg_{f^{-1}(a)}(f)\} = \{\deg(f_*^{-1}(a)\}$ for all $a \in A$. Equivalently, there is an orbisphere map $\tilde{f}\colon (S^2, \widetilde{C}, \widetilde{\mathrm{ord}}) \to (S^2, \widetilde{A}, \widetilde{\mathrm{ord}})$ that can be isotoped within maps $(S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$ to a map making (7) commute and such that $\deg\colon \widetilde{C} \to \mathbb{N}$ and $f_*\colon \widetilde{C} \to \mathbb{N}$ are induced by $\tilde{f}$.

Compatibility of an orbisphere biset ${}_H B_G$ with $\mathfrak{F}_E\colon \widetilde{H} \to H$, $\mathfrak{F}_D\colon \widetilde{G} \to G$, $f_*\colon \widetilde{C} \to \widetilde{A}$, and $\deg\colon \widetilde{C} \to \mathbb{N}$ is defined similarly, and is equivalent to the existence of a biset $\widetilde{B}$ making (8) commute.

We are now ready to relate the mapping class bisets M^* to portraits of bisets:

Theorem 2.19. *Let $\mathfrak{F}_E\colon \widetilde{H} \to H$ and $\mathfrak{F}_D\colon \widetilde{G} \to G$ be forgetful morphisms of orbisphere groups as in (8), and let ${}_H B_G$ be an orbisphere biset compatible with $\mathfrak{F}_E$, $\mathfrak{F}_D$, $f_*\colon \widetilde{C} \to \widetilde{A}$, and* $\deg\colon \widetilde{C} \to \mathbb{N}$.

Then for every portrait of bisets $(B_c)_{c\in\widetilde C}$ *in* B *parameterized by* f_* *and* deg *there exists an orbisphere biset* ${}_{\widetilde H}\widetilde B_{\widetilde G}$ *mapping to* B *under* $\mathcal F_{E,D}$ *with a minimal portrait mapping to* $(B_c)_{c\in\widetilde C}$.

Furthermore, $\widetilde B$ *is unique up to pre- and post-composition with bisets of knitting mapping classes and we have a congruence of bisets (see (19))*

$$\begin{array}{c} {}_{\pi_1^*(Y,E)}M^*(B|E,D)_{\pi_1(X,D)} \\ \Big\downarrow \mathcal P \\ {}_{H^E\rtimes E\downarrow^*}\{(B'\in M(B),(B'_c)_{c\in\widetilde C})\mid B'\in M(B),(B'_c)_{c\in\widetilde C}\text{ a portrait in }B'\}_{G^D} \end{array} \tag{31}$$

given by $\mathcal P(\widetilde B') = (\mathcal F_{E,D}(\widetilde B'), \textit{induced portrait of } \widetilde B')$.

The H^E-G^D-action on $\{(B',(B'_c)_{c\in\widetilde C})\}$ is given by

$$(h_c)_{c\in E}(B',(B'_c)_{c\in\widetilde C})(g_d)_{d\in D} = (B',(h_cB'_cg_{f_*(c)})_{c\in\widetilde C}),$$

with the understanding that $h_c = 1$ if $c\notin E$ and $g_{f_*(c)} = 1$ if $f_*(c)\notin D$, and the action of $E\downarrow^*$ is given by permutation of the portrait of bisets.

In the dynamical situation (i.e. when $H = G$ and $\widetilde H = \widetilde G$), Theorem 2.19 proves Theorem C.

Proof of Theorem 2.19. Clearly, (31) is an intertwiner: firstly, the actions of $E\downarrow^*$ are compatible; we may ignore them in the sequel. Secondly, let $(B'_c)_{c\in\widetilde C}$ be the induced portrait of bisets of $\widetilde B'$. For $e\in E$ we may write

$$B'_e = \{m_e\,\#\,p^{-1} \mid p\colon[0,1]\longrightarrow Y,\ p^{-1}(0)=e,\ f\circ p=\ell_{f_*(e)}\},$$

see (30). Consider $\psi\in\mathbf{Mod}(Y|E)$; the action of ψ on B'_e replaces m_e by $\psi\circ m_e$; and if $\mathcal E_E(\psi) = (h_e)_{e\in E}$ then $\psi\circ m_e = h_e m_e$ by the very definition of "push" and (19). The same argument applies to the right action.

Let us now show that $\mathcal P$ is a congruence. Since $\mathcal P$ is an intertwiner between left-free bisets with isomorphic acting groups, it is sufficient to show that $\mathcal P$ preserves left orbits.

We proceed by adding new points to D and E. If $E=\emptyset$, then the forgetful maps $M^*(B|E,D)\to M(B)$ and

$$\{(B'\in M(B),(B'_c)_{c\in\widetilde C})\mid B'\in M(B),(B'_c)_{c\in\widetilde C}\text{ a portrait in }B'\}\longrightarrow M(B)$$

are bijections and the claim follows. Therefore, it is sufficient to assume that $D=\emptyset$.

By [5, Theorem 8.9], the biset B may be written as $B(f)$ for a branched covering $f\colon (S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$, unique up to isotopy rel C. We lift f to a branched covering $f^+\colon (S^2, f^{-1}(A)) \to (S^2, A)$. Let $\widetilde{B}^+$ be its biset and let $(\widetilde{B}^+_c)_{c\in f^{-1}(A)}$ be its minimal portrait of bisets, which is unique by Lemma 2.17.

Let us show that for every portrait of bisets $(B_c)_{c\in\widetilde{C}}$ in B there is an orbisphere biset $\widetilde{B}$ whose minimal portrait of bisets maps to (B_c). This $\widetilde{B}$ will be of the form $B(m) \otimes \mathfrak{F}_{f^{-1}(A)\setminus\widetilde{C},\emptyset}(\widetilde{B}^+)$ for a homeomorphism $m\colon (S^2, f^{-1}(A)) \circlearrowleft$.

First, consider the images of all $\widetilde{B}^+_c$ in $\{\cdot\} \otimes_{\widetilde{H}} \widetilde{B}^+ \cong \{\cdot\} \otimes_H B$, and compare them to the images of all $\widetilde{B}_c$. The condition that as c' ranges over $f_*^{-1}(f_*(c))$ the $B_{c'}$ lie in different H-orbits of B lets us select which preimages of A correspond to the marked points in $\widetilde{C}$, and thus produces a well-defined map $\widetilde{C} \to f^{-1}(A)$. Let m' be an isotopy of (S^2, C) which maps $\widetilde{C}$ to $f^{-1}(A)$ in the specified manner; then $m' f^+\colon (S^2, \widetilde{C}, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$ has biset $\widetilde{B}^0$ and portrait of bisets $(\widetilde{B}^0_c)_{c\in\widetilde{C}}$; and $\mathfrak{F}_{E,\emptyset}(\widetilde{B}^0_c) \subseteq HB_c$ for all $c \in \widetilde{C}$, so we may write $h_c\mathfrak{F}_{E,\emptyset}(\widetilde{B}^0_c) = B_c$ for some $h_c \in H$. (We recall that B_c consists of $d(c)$ elements permuted by $G_{f_*(c)}$, where $d(c)$ is the local degree of f at c. We have $h_c\mathfrak{F}_{E,\emptyset}(\widetilde{B}^0_c) = B_c$ if and only if $h_c\mathfrak{F}_{E,\emptyset}(\widetilde{B}^0_c) \cap B_c \neq \emptyset$.) We set $\widetilde{B} = \prod_{c\in E} \mathrm{push}(h_c)\widetilde{B}^0$, and note that the minimal portrait of bisets of $\widetilde{B}$ maps to $(B_c)_{c\in\widetilde{C}}$ under $\mathfrak{F}_{E,\emptyset}$.

The only choice involved is that of a mapping class that yields $\mathrm{push}(h_c)$ when restricted to $C \cup \{c\}$, namely that of knitting mapping classes. □

2.7. Fibre bisets. Consider an orbisphere map

$$\tilde{f}\colon (S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \longrightarrow (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$$

as above and define the saturation of E as

$$\bar{E} := \bigsqcup_{e\in E} \tilde{f}^{-1}(\tilde{f}(e)) \setminus C.$$

Lemma 2.20. *Let $\tilde{f}\colon (S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \to (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$ be an orbisphere map and let $m\colon (S^2, C \sqcup \bar{E}) \circlearrowleft$ be a homeomorphism such that $m \restriction_C = \mathbb{1}$ and for every $e \in \bar{E}$ we have $\tilde{f}(m(e)) = \tilde{f}(e)$. If the isotopy class of m is not in $\mathbf{Mod}^*(S^2, C \sqcup E)$, namely if m moves at least one point in E to $\bar{E} \setminus E$, then $m\tilde{f} \not\approx_{C\sqcup E} \tilde{f}$.*

For every $\tilde{g} \in M^*(\tilde{f})$ there is a homeomorphism $m\colon (S^2, C \sqcup \bar{E}) \circlearrowleft$ as above (i.e. $m \restriction_C = \mathbb{1}$ and $\tilde{f}(m(e)) = \tilde{f}(e)$ for $e \in \bar{E}$) such that $\tilde{g} \approx_{C\sqcup E} m\tilde{f}$.

Note that $\mathbf{Mod}^*(S^2, C \sqcup \bar{E})$ does *not* act on $M^*(\tilde{f})$: there are orbisphere maps $\tilde{f}_1, \tilde{f}_2\colon (S^2, C \sqcup E, \widetilde{\mathrm{ord}}) \to (S^2, A \sqcup D, \widetilde{\mathrm{ord}})$ such that $\tilde{f}_1 \approx_{C\sqcup E} \tilde{f}_2$ but $m\tilde{f}_1 \not\approx_{C\sqcup E} m\tilde{f}_2$ for a homeomorphism $m\colon (S^2, C \sqcup \bar{E}) \circlearrowleft$ as above.

Proof. Suppose $m\tilde{f} \approx_{C\sqcup E} \tilde{f}$. Since $\bar{E}$ is saturated, by Lemma 2.9(2) there is an $n\colon (S^2, C \sqcup \bar{E})\circlearrowleft$ with $n\restriction_{C\sqcup E} = \mathbb{1}$ and $\tilde{f}(n(e)) = \tilde{f}(e)$ for $e \in \bar{E} \setminus E$ such that $nm\tilde{f} \approx_{C\sqcup\bar{E}} \tilde{f}$. This contradicts Lemma 2.9(1): the biset

$$M^*(\tilde{f}\colon (S^2, C \sqcup \bar{E}) \longrightarrow (S^2, A \sqcup D)) \tag{32}$$

is left-free while $nm \neq \mathbb{1}$.

The second claim follows from Lemma 2.9(2) applied to the biset (32). □

We are interested in the fibre biset ${}_{H^E}\mathcal{F}^{-1}_{E,D}(B')_{G^D}$ under the forgetful map $\mathcal{F}_{E,D}\colon M^*(B|E,D) \to M(B)$. For every $a \in \widetilde{A}$ define $E_a := E \cap f_*^{-1}(a)$, where $f_*\colon \widetilde{C} \to \widetilde{A}$ is the portrait.

Proposition 2.21. *We have*

$${}_{H^E}\mathcal{F}^{-1}_{E,D}(B')_{G^D} \cong \prod_{a\in A} {}_{H^{E_a}} M^*(B'|E_a,\emptyset)_1 \times \prod_{d\in D} {}_{H^{E_d}} M^*(B'|E_d,\{d\})_{G^{\{d\}}}. \tag{33}$$

Suppose that B' is the biset of $g\colon (S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$; then

$${}_{H^{E_a}} M^*(B'|E_a,\emptyset)_1 \cong H^{E_a} \times \{\iota\colon E_a \hookrightarrow g^{-1}(a) \setminus C\}.$$

The biset ${}_{H^{E_d}} M^*(B'|E_d,\{d\})_{G^{\{d\}}}$ is congruent to the biset

$$\{(b_e)_{e\in E_d} \in B'^{E_d} \mid Hb_e \neq Hb_{e'} \text{ if } e \neq e'\}$$

endowed with the actions

$$(h_e)_{e\in E_d} \cdot (b_e)_{e\in E_d} \cdot g = (h_e b_e g)_{e\in E_d}.$$

Proof. By Theorem 2.19, the biset ${}_{H^E}\mathcal{F}^{-1}_{E,D}(B')_{G^D}$ is isomorphic to the biset of portraits $(B_c)_{c\in\widetilde{C}}$ in B'. Recall that $(B_c)_{c\in C}$ is fixed (by Lemma 2.17) while $(B_e)_{e\in E}$ varies; this shows (33).

The second claim follows from Lemma 2.20.

For $d \in D$ and $e \in E_d$ we have $B_e = \{b_e\}$; i.e. the choice of $(B_e)_{e\in E_d}$ is equivalent to the choice of $(b_e)_{e\in E_d} \in B'^{E_d}$ subject to the condition stated above. □

Note that ${}_{H^{E_a}} M^*(B'|E_a,\emptyset)_1$ will not be transitive, as soon as there are at least two maps $\iota\colon E_a \to g^{-1}(a) \setminus C$.

Let us define

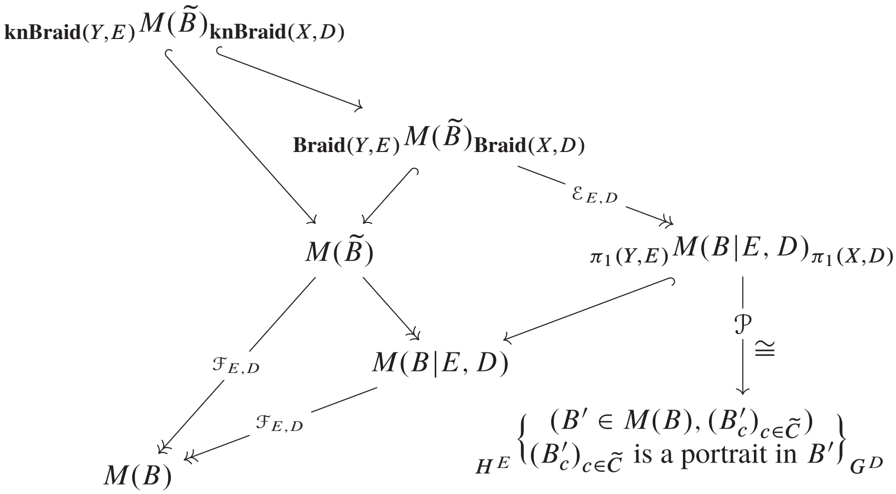
$$M(B|E,D) := \mathbf{Mod}(Y|E) \otimes M(\widetilde{B}) \otimes \mathbf{Mod}(X|D).$$

Theorem 2.22. *Suppose $f_*(E) \subset D$ and, moreover, that $f_*\colon E \to D$ is a bijection. We then have a congruence*

$$ {}_{H^E}(\mathcal{F}_{E,D}^{-1}(B'))_{G^D} \longrightarrow ({}_H B'_G)^E \tag{34}$$

mapping the portrait $(\widetilde{B}_c)_{e\in\widetilde{C}}$ in B' to $(b_e)_{e\in E}$ where $\widetilde{B}_e = \{b_e\}$. The group G^D is identified with G^E via the bijection $f_*\colon E \to D$.

Moreover, $M^*(\widetilde{B}) = M(\widetilde{B})$, $M^*(B|E,D) = M(B|E,D)$, and exact sequences similar to (27) hold:



(35)

The bottom sequence in (35) can be written (using (34)) as

$$\bigsqcup_{B'\in B}(B')^E \hookrightarrow M(B|E,D) \twoheadrightarrow M(B). \tag{36}$$

Proof. The first claim follows from Proposition 2.21. Since $(B')^E$ is a transitive biset, we obtain $M(B|E,D) = M^*(B|E,D)$ and the exact sequences (36) and (35) hold because the fibres are connected. □

Corollary 2.23. *Let $f\colon(S^2,A,\mathrm{ord})\circlearrowleft$ be an orbisphere map and let $D = \bigsqcup_{i\in I} D_i$ be a finite union of periodic cycles D_i of f. Then (36) takes the form*

$$\bigsqcup_{g\in M(f)}\bigsqcup_{i\in I} B(g)^{\otimes(\#D_i)} \hookrightarrow M(f|D,D) \twoheadrightarrow M(f). \tag{37}$$

2.8. Centralizers of portraits. Let us now consider the dynamical situation: $H = G$ and $\widetilde{G} = \widetilde{H}$; we abbreviate $\mathcal{F}_D = \mathcal{F}_{D,D}$ and $M(B|D) = M(B|D,D)$.

Given an orbisphere biset $\widetilde{B}$ we denote by $Z(\widetilde{B}) \le \mathbf{Mod}(\widetilde{G})$ the centralizer of $\widetilde{B}$ in $M(\widetilde{B})$ and by $Z(\widetilde{B}|D) \le \mathbf{Mod}(G|D)$ the centralizer of the image of $\widetilde{B}$ in $M(B|D)$, see Theorem 2.19. We have a natural forgetful map

$$Z(\widetilde{B}) \longrightarrow Z(\widetilde{B}|D) \tag{38}$$

which is, in general, neither injective nor surjective. However, we will show in Corollary 4.28 that (38) is an isomorphism if $\widetilde{B}$ is geometric non-invertible.

Recall from (12) the short exact sequence

$$\pi_1(X, D) \hookrightarrow \mathbf{Mod}(G|D) \twoheadrightarrow \mathbf{Mod}(G).$$

We have the corresponding sequence for centralizers:

$$1 \longrightarrow Z(\widetilde{B}|D) \cap \pi_1(X, D) \longrightarrow Z(\widetilde{B}|D) \longrightarrow Z(B). \tag{39}$$

If $\widetilde{B}$ is geometric non-invertible, then $Z(\widetilde{B}|D) \cap \pi_1(X, D)$ is trivial, so $Z(\widetilde{B}|D) \cong Z(\widetilde{B})$ is naturally a subgroup of $Z(B)$, and in Theorem 4.41 we will show that it has finite index.

Definition 2.24. The *relative centralizer* $Z_D((B_a)_{a\in\widetilde{A}})$ of a portrait of bisets $(B_a)_{a\in\widetilde{A}}$ is the set of $(g_d) \in G^D$ such that

$$B_d = g_d^{-1} B_d g_{f_*(d)} \quad \text{for all } d \in D,$$

with the understanding that $g_{f_*(d)} = 1$ if $f_*(d) \notin D$.

We remark that we could also have defined the "full" normalizer, consisting of all $(g_a) \in G^{\widetilde{A}}$ with $G_a^{g_a} = G_a$ and $B_a = g_a^{-1} B_a g_{f_*(a)}$ for all $a \in \widetilde{A}$, and its subgroup the "full" centralizer, in which g_a centralizes G_a and $g_a^{-1} \cdot - \cdot g_{f_*(a)}$ is the identity on B_a; but we will make no use of these notions. The "full" normalizer is the direct product of $\prod_{a\in A} G_a$ and the relative centralizer.

We also note that, if $(g_d)_{d\in D}$ belongs to the relative centralizer of $(B_a)_{a\in\widetilde{A}}$ and $f^n(d) \in A$ for some $n \in \mathbb{N}$, then $g_d = 1$.

Applying Theorem 2.19 to the dynamical situation we obtain:

Proposition 2.25. *Let* ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}} \twoheadrightarrow {}_G B_G$ *be a forgetful morphism and let* $(B_a)_{a\in\widetilde{A}}$ *be the induced portrait of bisets in* B. *Then any choice of isomorphisms* $\pi_1(X, d) \cong G$ *gives an isomorphism*

$$Z(\widetilde{B}|D) \cap \pi_1(X, D) \xrightarrow{\cong} Z_D((B_a)_{a\in\widetilde{A}}).$$

3. G-spaces

Let us recall that we prefer the algebraic order of compositions: the composition of $f: X \to Y$ and $g: Y \to Z$ is written as $fg: X \to Z$. With this convention, functions have a natural right action on points:

$$x^{fg} := (fg)(x) = (g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X.$$

We start by general considerations. Let Y be a right H-space, and let X be a right G-space. For every map $f: Y \to X$ there exists a natural H-G-biset $M(f)$, defined by

$$M(f) := \{hfg: x \longmapsto x^{hfg} \mid h \in H, g \in G\}, \tag{40}$$

namely the set of maps $Y \to X$ obtained by precomposing with the H-action and post-composing with the G-action. Note that $M(f)$ is right-free if the action of G is free on X. We have a natural G-equivariant map $Y \otimes_H M(f) \to X$ given by evaluation: $y \otimes b \mapsto y^b$. Define

$$H_f := \{h \in H \mid \text{there exists } g \in G \text{ with } hf = fg \text{ in } M(f)\}$$

the stabilizer in H of $fG \in M(f)/G$. Then f descends to a continuous map $\bar{f}: Y/H_f \to X/G$.

Lemma 3.1. *Suppose that X and Y are simply connected and that G, H act freely with discrete orbits. Then $M(f)$ is isomorphic to the biset of the correspondence $Y/H \leftarrow Y/H_f \xrightarrow{\bar{f}} X/G$ as defined in* [3].

Proof. Let us define the following subbiset of $M(f)$:

$$M'(f) := \{hfg \mid h \in H_f, g \in G\}. \tag{41}$$

Since Y, X are simply connected, the biset of $\bar{f}: Y/H_f \to X/G$ is isomorphic to $M'(f)$. The isomorphism is explicit: choose basepoints $\dagger \in Y$ and $* \in X$ so that $\pi_1(Y/H, \dagger H) \cong H$ and $\pi_1(X/G, *G) \cong G$. Given $b \in B(\bar{f})$, represent it as a path $\bar{\beta}: [0,1] \to X/G$ with $\bar{\beta}(0) = \bar{f}(\dagger H_f)$ and $\bar{\beta}(1) = *G$, and lift it to a path $\beta: [0,1] \to X$. We have $\beta(0) = f(\dagger)h$ and $\beta(1) = *g$ for some $h \in H_f, g \in G$, and we map $b \in B(\bar{f})$ to $h^{-1}fg \in M'(f)$. This map is a bijection because both $B(\bar{f})$ and $M'(f)$ are right-free. We finally have

$$M(f) \cong H \otimes_{H_f} M'(f) \cong B(Y/H \longleftarrow Y/H_f \longrightarrow X/G). \qquad \square$$

In case the actions of G, H are discrete but not free, there still is a surjective morphism $B(X/G \leftrightarrow Y/H) \twoheadrightarrow M(f)$, when X/G and Y/H are treated as orbispaces.

3.1. The modular correspondence. We discuss briefly here an application of the previous section to X, Y Teichmüller spaces: for a marked sphere (S^2, A) recall that the Teichmüller space $\mathcal{T}_A$ is the space of homeomorphisms $h\colon (S^2, A) \to \widehat{\mathbb{C}}$ considered up to isotopy rel A and post-composition by Möbius transformations. The modular group $\mathbf{Mod}(S^2, A)$ naturally acts on the left on $\mathcal{T}_A$ by precomposition (pullback), and this action is free. Taking quotients, we obtain the modular space of the associated marked sphere:

$$\mathcal{M}_A = \mathbf{Mod}(S^2, A)\backslash\mathcal{T}_A.$$

Consider now a sphere map $f\colon (S^2, C) \to (S^2, A)$. It induces a pullback map between Teichmüller spaces $\sigma_f\colon \mathcal{T}_A \to \mathcal{T}_C$. Letting

$$H_f = \{g \in \mathbf{Mod}(S^2, A) \mid \text{there exists } h \in \mathbf{Mod}(S^2, C) \text{ with } fg = hf\}$$

denote the subgroup of liftable elements and setting $\mathcal{W}_f := H_f\backslash\mathcal{T}_A$, we obtain the *modular correspondence* [11]:

$$\begin{array}{ccccc} \mathcal{T}_C & & \xleftarrow{\quad\sigma_f\quad} & & \mathcal{T}_A \\ \downarrow & & & \swarrow & \downarrow \\ & & \mathcal{W}_f & & \\ \downarrow & \swarrow_{\bar{\sigma}_f} & & \searrow & \downarrow \\ \mathcal{M}_C & & & & \mathcal{M}_A \end{array} \tag{42}$$

in which all non-labeled arrows represent covering maps. Since the pullback action of $\mathbf{Mod}(S^2, C)$ on $\mathcal{T}_C$ is free, the mapping class biset $M(f)$ from §2.4 is isomorphic to

$$M(\sigma_f) = \{\sigma_h \circ \sigma_f \circ \sigma_g \mid h \in \mathbf{Mod}(S^2, C), g \in \mathbf{Mod}(S^2, A)\} \tag{43}$$

(where in $\sigma_h \circ \sigma_f \circ \sigma_g$ the composition is from right to left). Lemma 3.1 now identifies $M(\sigma_f)$ with the biset of the modular correspondence, yielding the

Theorem 3.2 ([5, Theorem 9.1]). *The mapping class biset $M(f)$ from §2.4 is naturally isomorphic to the biset of the correspondence*

$$\mathcal{M}_C \xleftarrow{\overline{\sigma_f}} \mathcal{W}_f \longrightarrow \mathcal{M}_A.$$

An explicit isomorphism between these bisets is given in [5].

3.2. Universal covers. Let us now generalize Lemma 3.1 by dropping the requirement that X, Y be simply connected. Choose basepoints $\dagger, *$, write

$$\begin{aligned} Q &= \pi_1(Y, \dagger), & P &= \pi_1(X, *), \\ \widetilde{H} &= \pi_1(Y/H, \dagger H), & \widetilde{G} &= \pi_1(X/G, *G); \end{aligned}$$

so we have exact sequences

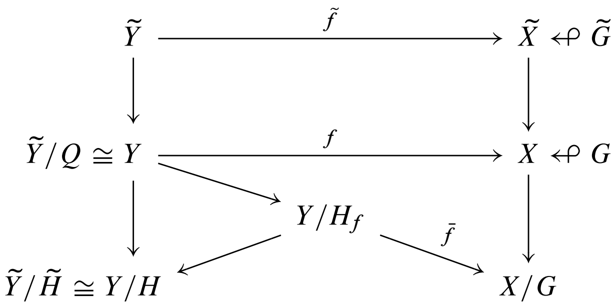
$$1 \longrightarrow Q \longrightarrow \widetilde{H} \xrightarrow{\pi} H \longrightarrow 1, \quad 1 \longrightarrow P \longrightarrow \widetilde{G} \xrightarrow{\pi} G \longrightarrow 1. \tag{44}$$

The *universal cover* of X may be defined as

$$\widetilde{X} := \{\beta\colon [0,1] \longrightarrow X \mid \beta(1) = *\}/\approx; \tag{45}$$

it has a natural basepoint $\widetilde{*}$ given by the constant path at $*$, and admits a right P-action by right-concatenation of loops at $*$. The projection $\widetilde{X} \to X$ is a covering, and is given by $\beta \mapsto \beta(0)$. We denote by $\widetilde{X}^\vee$ the left P-set structure on $\widetilde{X}$, with action $g \cdot \beta^\vee := (\beta \cdot g^{-1})^\vee$. We may naturally identify $\widetilde{X}^\vee$ with $\{\beta^{-1} \mid \beta \in \widetilde{X}\}$ and its natural left P-action.

Let us consider the universal covers $\widetilde{X}, \widetilde{Y}$ of X, Y respectively and a lift $\tilde{f}$ of f. We have the following situation, with the acting groups represented on the right, and omitted in the left column:



Let $\widetilde{H}_f$ be the full preimage of H_f in $\widetilde{H}$. Note that $\widetilde{H}_f$ is the stabilizer of $\tilde{f}\widetilde{G}$ in $M(\tilde{f})/\widetilde{G}$: we have $\tilde{h}\tilde{f} \in \tilde{f}\widetilde{G}$ if and only if $\pi(\tilde{h})f \in fG$ by the unique lifting property. By Lemma 3.1, we have

$$\begin{aligned} M(\tilde{f}) &\cong B(\widetilde{Y}/\widetilde{H} \longleftarrow \widetilde{Y}/\widetilde{H}_f \longrightarrow \widetilde{X}/\widetilde{G}) \\ &\cong B(Y/H \longleftarrow Y/H_f \longrightarrow X/G), \end{aligned}$$

and we shall see that $M(\tilde{f})$ is an extension of bisets. We have a natural map $\pi\colon M(\tilde{f}) \to M(f)$, given by $h\tilde{f}g \mapsto \pi(h)f\pi(g)$ for all $h \in \widetilde{H}, g \in \widetilde{G}$.

Lemma 3.3. *There is a short exact sequence of bisets*

$$ {}_Q M(\tilde{f})_P \hookrightarrow M(\tilde{f}) \twoheadrightarrow M(f), \tag{46}$$

in which every fibre $\pi^{-1}(hfg)$ is isomorphic to a twisted form $B(hfg)$ of the biset of f.

Proof. This follows immediately from Lemma 3.1 applied to $\widetilde{Y}, \widetilde{X}$ with actions of Q, P respectively. □

Let us now assume that the short exact sequences (44) are split, so $\pi_1(X/G) \cong P \rtimes G$ and $\pi_1(X/H) \cong Q \rtimes H$. We shall see that the sequence (46) is split, and that some additional structure on $M(f)$ and $B(f)$ allow the extension $M(f) \cong B(Y/H \leftrightarrow X/G)$ to be reconstructed as a crossed product.

The splitting of the map $\pi\colon \pi_1(X/G, *G) \to G$ means that there is a family $\{\alpha_g\}_{g\in G}$ of curves in X such that α_g connects $*g$ to $*$ and $\alpha_{g_1g_2} \approx (\alpha_{g_1}g_2)\#\alpha_{g_2}$. Similarly there is a family $\{\beta_h\}_{h\in H}$ of curves in Y such that β_h connects $\dagger h$ to $\dagger$ and $\beta_{h_1h_2} \approx (\beta_{h_1}h_2)\#\beta_{h_2}$.

For every $h \in H_f$ there is a unique element of G, written $h^f \in G$, such that $h \cdot f = f \cdot h^f$ in $M'(f)$. For every $h \in H_f$ and every $b \in B(f)$, represented as a path $b\colon [0,1] \to X$ with $b(0) = f(\dagger)$ and $b(1) = *$, define

$$ b^h := (f \circ \beta_h^{-1})\#(b \cdot h^f)\#\alpha_{h^f}. $$

We clearly have $(q \cdot b \cdot p)^h = q^h \cdot b^h \cdot p^{h^f}$, so H_f acts on $B(f)$ by congruences. We convert that right action to a left action by ${}^h b := b^{h^{-1}}$.

For every $c \in M'(f)$ and every $p \in P$, write $c = fg$ and define ${}^c p := {}^g p = gpg^{-1}$. We clearly have ${}^{cg} p = {}^c({}^g p)$.

Lemma 3.4. *The biset of $\bar{f}\colon Y/H_f \to X/G$ is isomorphic to the crossed product $B(f) \rtimes M'(f)$, which is $B(f) \times M'(f)$ as a set, with actions of $\widetilde{H}_f \cong Q \rtimes H_f$ and $\widetilde{G} \cong P \rtimes G$ given by*

$$ (q,h)\cdot(b,c)\cdot(p,g) = (q \cdot {}^h(b \cdot {}^c p), h \cdot c \cdot g). $$

As a consequence,

$$ B(Y/H \longleftarrow Y/H_f \xrightarrow{\bar{f}} X/G) \cong \widetilde{H} \otimes_{\widetilde{H}_f} B(f) \rtimes M'(f). $$

Proof. This is almost a tautology. The short exact sequence (46) splits, with section $h \cdot f \cdot g \mapsto (1,h)\cdot \tilde{f} \cdot (1,g)$, and the actions of $\widetilde{G}, \widetilde{H}$ on $\widetilde{X}, \widetilde{Y}$ can be identified with concatenation of lifts of the paths α_g, β_h. □

3.3. Self-similarity of G-spaces. We return in more detail the situation in which Y, X are universal covers; we rename them to $\widetilde{Y}, \widetilde{X}$ so as to keep Y, X for the original space.

Consider two pointed spaces $(Y, \dagger)$ and $(X, *)$ with $H := \pi_1(Y, \dagger)$ and $G := \pi_1(X, *)$, and let $f : Y \to X$, be a continuous map. Recall that its biset is defined by

$$ {}_H B(f)_G := \{\beta : [0,1] \longrightarrow X \mid \beta(0) = f(\dagger), \beta(1) = *\}/{\approx}, \tag{47}$$

with the natural actions by left- and right-concatenation. We thus naturally have $B(f) \subseteq \widetilde{X}$, see (45), with corresponding right actions, and left action given by composing loops via f.

The map $f : Y \to X$ naturally lifts to a map $\widetilde{Y} \to Y \to X$, and every choice of $\beta \in B(f)$ defines uniquely, by the lifting property of coverings, a lift $\tilde{f}_\beta : \widetilde{Y} \to \widetilde{X}$ with the property that $\tilde{\dagger} \mapsto \beta$. Furthermore, we have the natural identities $\tilde{f}_\beta(- \cdot h) \cdot g = \tilde{f}_{h\beta g}$, so that the biset $B(f)$ as defined in (47) is canonically isomorphic to every biset $M(\tilde{f}_\beta)$ as defined in (40), when an arbitrary $\beta \in B(f)$ is chosen, and to $B(\tilde{f})$, when an arbitrary lift $\tilde{f} : \widetilde{Y} \to \widetilde{X}$ of f is chosen.

If $f : Y \to X$ is a covering, then we may assume $\widetilde{Y} = \widetilde{X}$; choosing $\tilde{f} = \mathbb{1}$ gives the simple description $B(\tilde{f}) = {}_H G_G$. Recall that the biset of a correspondence $(f, i) : Y \leftarrow Z \to X$ is defined as $B(f, i) = B(i)^\vee \otimes B(f)$. In the case of a covering correspondence, in which f is a covering, we therefore arrive at $B(f, i) = B(i)^\vee \otimes_{\pi_1(Z)} G$. We shall give now a more explicit description of this biset using covering spaces, just as we had in [3, eq. (15)]

$$ B(f, i) = \{(\delta : [0,1] \longrightarrow Y, p \in Z) \mid \delta(0) = \dagger, \delta(1) = i(p), f(p) = *\}/{\approx} $$

and the special case, when $i : Z \to Y$ is injective, of

$$ B(f, i) = \{\delta : [0,1] \longrightarrow Y \mid \delta(0) = \dagger, f(i^{-1}(\delta(1))) = *\}/{\approx}. $$

Still assuming that $f : Z \to X$ is a covering map, define

$$ \widetilde{Z}_H := \{(\delta, p) \in \widetilde{Y} \times Z \mid i(p) = \delta(0)\}, \tag{48}$$

the fibre product of $\widetilde{Y}$ with Z above Y, see Diagram (50) left. (Note that $\widetilde{Z}_H$ is *not* the universal cover of Z.) In case $i : Z \to Y$ is injective, we may write

$$ \widetilde{Z}_H = \{\delta \in \widetilde{Y} \mid \delta(0) \in i(Z)\}, $$

so $\widetilde{Z}_H$ is the full preimage of $i(Z)$ under the covering map $\widetilde{Y} \to Y$.

Proposition 3.5. *If (f, i) is a covering correspondence, the following map defines an isomorphism of H-spaces*:

$$\Phi\colon {}_H B(f,i)_G \otimes \widetilde{X}^\vee \to (\widetilde{Z}_H)^\vee \text{ given by } \Phi((\delta, p)\otimes\alpha^\vee) = ((\delta \# (i \circ \alpha^{-1}{\uparrow}_f^p))^{-1}, p).$$

For every $b = (\delta, p) \in B(f, i)$ the map

$$\tilde{f}_b^{-1}\colon \widetilde{X}^\vee \to (\widetilde{Z}_H)^\vee \quad \text{given by } \alpha^\vee \mapsto \Phi(b \otimes \alpha^\vee)$$

is the unique lift of the inverse of the correspondence $Y \leftarrow Z \to X$ mapping $\tilde{*}$ to b; *we have the equivariance properties*

$$\tilde{f}_{hbg}^{-1} = h \cdot \tilde{f}_b^{-1}(g \cdot -). \tag{49}$$

The inverses and contragredients may seem unnatural in the statements above; but are necessary for the actions to be on the right sides, and are justified by the fact that we construct a lift of the *inverse* of the correspondence, rather than the correspondence itself:

$$\begin{array}{ccccc} \widetilde{Y}^\vee & \xleftarrow{\tilde{\imath}} & (\widetilde{Z}_H)^\vee & \xleftarrow{\tilde{f}_b^{-1}} & \widetilde{X}^\vee \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xleftarrow{i} & Z & \xrightarrow{f} & X \end{array} \tag{50}$$

Proof. It is obvious that Φ is H-equivariant, and it is surjective: given $(\delta, p) \in \widetilde{Z}_H$, choose a path $\alpha \in \widetilde{X}$ with $f(p) = \alpha(0)$, and write $\delta = (\alpha^{-1}{\uparrow}_f^p)^{-1} \# \alpha^{-1}{\uparrow}_f^p \# \delta$, expressing in this manner $(\delta, p)^\vee = \Phi(((\alpha^{-1}{\uparrow}_f^p \# \delta)^{-1}, \alpha^{-1}{\uparrow}_f^p(1)) \otimes \alpha^\vee)$.

If $\Phi((\delta, p) \otimes \alpha^\vee) = \Phi((\delta', p') \otimes (\alpha')^\vee)$, then α and α' start at the same point, so $\alpha = \alpha' g$ for some $g \in G$, and we have then $(\delta, p) = (\delta', p')g^{-1}$ so $(\delta, p) \otimes \alpha^\vee = (\delta', p') \otimes (\alpha')^\vee$ and Φ is injective.

It is easy to see that $\tilde{f}_b^{-1}$ is a lift of f^{-1}. Conversely, every lift $\tilde{f}^{-1}$ of f^{-1} maps $\tilde{*}$ to an element $b \in B(f, i)$ because $\tilde{f}^{-1}(\tilde{*}^\vee)$ ends at an f-preimage of $*$ by construction; and then $\tilde{f}^{-1} = \tilde{f}_b^{-1}$ by unicity of lifts. Finally, equivariance follows from $\tilde{f}_b^{-1}(g\alpha^\vee) = \Phi(b \otimes g\alpha^\vee) = \Phi(bg \otimes \alpha^\vee) = \tilde{f}_{bg}^{-1}(\alpha^\vee)$. □

3.4. Planar discontinuous groups. A *planar discontinuous group* $\widetilde{X} \looparrowleft G$ is a group acting properly discontinuously on a plane, which will be denoted by $\widetilde{X}$: for every bounded set V the set $\{g \in G \mid Vg \cap V \neq \emptyset\}$ is finite.

Let $X := (S^2, A, \mathrm{ord})$ be an orbifold with non-negative Euler characteristic, consider $* \in S^2 \setminus A$ a base-point, and write $G := \pi_1(X, *)$. Then the universal cover $\widetilde{X}_G$ of X is a plane endowed with a properly discontinuous action of G. We denote by $\pi: \widetilde{X} \to X$ the covering map.

By the classification of surfaces, there are only two cases to consider: $X = \mathbb{C}$ with G a lattice in the affine group $\{z \mapsto az + b \mid a, b \in \mathbb{C}, |a| = 1\}$, and $X = \mathbb{H}$ the upper half plane, with G a lattice in $\mathrm{PSL}_2(\mathbb{R})$.

Consider another orbisphere $Y := (S^2, C, \mathrm{ord})$ and a branched covering $f: Y \to X$, and fix basepoints $\dagger \in Y$ and $* \in X$ with corresponding fundamental groups $H = \pi_1(Y, \dagger)$ and $G = \pi_1(X, *)$. Let ${}_H B(f)_G$ be the biset of f. As usual, we view f as a correspondence $Y \leftarrow Z \to X$ with $Z = (S^2, f^{-1}(A), \mathrm{ord})$ and i a homeomorphism $S^2 \to S^2$ mapping a subset of $f^{-1}(A)$ to C. The fibre product $\widetilde{Z}_H$ constructed in (48) is naturally a subset of the plane $\widetilde{Y}$, with orbispace points and punctures at all H-orbits of $f^{-1}(A) \setminus C$. We need the following classical result.

Theorem 3.6 (Baer [29, Theorem 5.14.1]). *Every orientation-preserving homeomorphism of a plane commuting with a properly discontinuous group action is isotopic to the identity along an isotopy commuting with the action.*

Let us reprove [5, Theorem 8.9] using our language of group actions:

Corollary 3.7. *Let two orbisphere maps $f, g: (S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$ have isomorphic orbisphere bisets. Then $f \approx_C g$.*

In other words, the orbisphere biset of $f: (S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$ is a complete invariant of f up to isotopy rel C.

Proof. Let us write $X = (S^2, A, \mathrm{ord})$ and $Y = (S^2, C, \mathrm{ord})$ and $G = \pi_1(X, *)$ and $Y = \pi_1(Y, \dagger)$. We may represent f, g respectively by covering pairs (f, i) and (g, i), with coverings $f, g: (S^2, P, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$ and $i: (S^2, P, \mathrm{ord}) \to (S^2, C, \mathrm{ord})$. Let us furthermore write $Z = (S^2, P, \mathrm{ord})$ and $\widetilde{Z}_H$ its fibre product with $\widetilde{Y}$. Identifying $B(f, i)$ and $B(g, i)$, choose $b \in B(f, i) = B(g, i)$, and let

$$\tilde{f}_b^{-1}, \tilde{g}_b^{-1}: \widetilde{X}^\vee \longrightarrow (\widetilde{Z}_H)^\vee$$

be the corresponding lifts as in Proposition 3.5.

Since i is injective, the map $(\widetilde{Z}_H)^\vee \to Z$ is a covering, so $\tilde{f}_b^{-1}$ and $\tilde{g}_b^{-1}$ are coverings. We may therefore consider their quotient $\tilde{f}_b \circ \tilde{g}_b^{-1}$, which is a well-defined map $(\widetilde{X})^\vee \circlearrowleft$, and is normalized to preserve the base point $\tilde{*}$. By (49) it is a homeomorphism commuting with the action of G.

By Theorem 3.6 there is an isotopy $(\tilde{h}_{b,t}^{-1})_{t \in [0,1]}$ of maps satisfying (49) between $\mathbb{1}$ and $\tilde{f}_b \circ \tilde{g}_b^{-1}$.

Since the set of fixed points of G is discrete, $\tilde{h}_{b,t}^{-1}$ preserves all fixed points of G and therefore projects to an isotopy $(h_t)_{t\in[0,1]}$ in X. We have $h_0 = \mathbb{1}$ and $h_1 \circ g = f$, so the maps f and g are isotopic rel B. □

4. Geometric maps

Let M be a matrix with integer entries and with $\det(M) > 1$. We call M *exceptional* if one of the eigenvalues of M lies in $(-1, 1)$; so the dynamical system $M\colon \mathbb{R}^2 \circlearrowleft$ has one attracting and one repelling direction.

For $r \in \mathbb{R}^2$, denote by $\langle \mathbb{Z}^2, -z + r\rangle$ the group of affine transformations of $\mathbb{R}^2$ generated by translations $z \mapsto z + v$ with $v \in \mathbb{Z}^2$ and the involution $z \mapsto -z + r$. The quotient $\mathbb{R}^2/\langle \mathbb{Z}^2, -z + r\rangle$ is a topological sphere, with cone singularities of angle π at the four images of $\frac{1}{2}(r + \mathbb{Z}^2)$.

We call a branched covering $f\colon S^2 \circlearrowleft$ a *geometric exceptional* map if $f\colon S^2 \circlearrowleft$ is conjugate to a quotient of an exceptional linear map $M\colon \mathbb{R}^2 \circlearrowleft$ under the action of $\langle \mathbb{Z}^2, -z + r\rangle$ for some $r \in \mathbb{R}^2$; in particular, $(\mathbb{1} - M)r \in \mathbb{Z}^2$. A Thurston map $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ is called *geometric* if the underlying branched covering $f\colon S^2 \circlearrowleft$ is either Böttcher expanding (see [4, Definition 4.1]; there exists a metric on S^2 that is expanded everywhere except at critical periodic cycles), a geometric exceptional map, or a pseudo-Anosov homeomorphism. We refer to the first two types as *non-invertible geometric maps*.

We may consider more generally affine maps $z \mapsto Mz+q$ on $\mathbb{R}^2$, and then their quotient by the group $\langle \mathbb{Z}^2, -z\rangle$; the map $z \mapsto Mz$ on $\mathbb{R}^2/\langle \mathbb{Z}^2, -z+r\rangle$ is converted to that form by setting $q = (M - \mathbb{1})r$. Conversely, if 1 is not an eigenvalue of M, then we can always convert an affine map into a linear one, at the cost of replacing the reflection $-z$ by $-z + r$ in the acting group.

Lemma 4.1. *Let $M\colon \mathbb{R}^2 \circlearrowleft$ be exceptional. Then for every bounded set $D \subset \mathbb{R}^2$ containing $(0,0)$ there is an $n > 0$ such that for every $m \geq n$ we have $M^{-m}(D) \cap \mathbb{Z}^2 = \{(0,0)\}$. Moreover, $n = n(D)$ can be taken with $n(D) \preccurlyeq \log\operatorname{diam}(D)$.*

Proof. Let λ_1, λ_2 be the eigenvalues of M, and let e_1 and e_2 be unit-normed eigenvectors associated with λ_1 and λ_2. It is sufficient to assume that D is a parallelogram centered at $(0,0)$ with sides parallel to e_1 and e_2:

$$D = \{v \in \mathbb{R}^2 \mid v = t_1 e_1 + t_2 e_2 \text{ with } |t_1| \leq x \text{ and } |t_2| \leq y\}. \tag{51}$$

In particular, the area of D is comparable to xy. Then $M^{-n}(D)$ is again a parallelogram centered at $(0,0)$ with sides parallel to e_1 and e_2.

We claim that there is $\delta > 0$ such that if D is a parallelogram of the form (51) with $\operatorname{area}(D) < \delta$, then $D \cap \mathbb{Z}^2 = \{(0,0)\}$. This will prove the lemma because $\operatorname{area}(M^{-n}(D)) = \operatorname{area}(D)/(\det M)^n$ and $\log\operatorname{diam}(D) \asymp \log\operatorname{area}(D)$.

Without loss of generality we assume $x > y$, so D is close to $\mathbb{R}e_1$. Let μ_1 be the slope of $\mathbb{R}e_1$. Since M is exceptional, the numbers $\lambda_1, \lambda_2, \mu_1$ are quadratic irrational, so they are not well approximated by rational numbers: there is a positive constant C such that $|\mu_1 - \frac{p}{q}| > \frac{C}{q^2}$ for all $\frac{p}{q} \in \mathbb{Q}$.

Suppose that D contains a non-zero integer point $w = (p, q)$; so $x \geq |q|$. Also, w is close to $\mathbb{R}e_1$, and in particular $q \neq 0$ if δ is sufficiently small; we may suppose $q > 0$. It also follows that μ_1 is close to $\frac{p}{q}$. The distance from w to $\mathbb{R}e_1$ is

$$d(w, \mathbb{R}e_1) \leq y \asymp \frac{\operatorname{area}(D)}{x}.$$

On the other hand,

$$d(w, \mathbb{R}e_1) = |w| \sin \sphericalangle(w, e_1) \succcurlyeq q \left|\mu_1 - \frac{p}{q}\right|.$$

Combining, we get

$$\left|\mu_1 - \frac{p}{q}\right| \preccurlyeq \frac{d(w, \mathbb{R}e_1)}{q} \preccurlyeq \frac{\operatorname{area}(D)}{q \cdot x} \preccurlyeq \frac{\delta}{q^2},$$

and the claim follows for $\delta \lll C$. □

Corollary 4.2. *Let $f\colon (S^2, A) \circlearrowleft$ be a geometric exceptional map, let $p \in S^2 \setminus A$ be a periodic point with period n, and let $\gamma \in \pi_1(S^2 \setminus A, p)$ be a loop starting and ending at p. Let $|\gamma|$ be the length of γ with respect to the Euclidean metric of the minimal orbifold structure $(P_f, \operatorname{ord}_f)$. If γ is trivial rel $(P_f, \operatorname{ord}_f)$ and $m \succ \log|\gamma|$, then the lift $\gamma{\uparrow}^p_{f^{nm}}$ is a trivial loop rel A.*

Proof. By passing to an iterate, we may assume that p is a fixed point of f. Since f is geometric exceptional, we have a branched covering map $\pi\colon \mathbb{R}^2 \to S^2$ under which f lifts to an exceptional linear map M, and we may assume $\pi(0,0) = p$. By assumption, $\gamma{\uparrow}^{(0,0)}_\pi$ is a trivial loop in $\mathbb{R}^2$. By Lemma 4.1, the sets $M^{-m}(\gamma{\uparrow}^{(0,0)}_\pi)$ and $\pi^{-1}(A)$ are disjoint for $m \succ \log|\gamma|$; hence $\gamma{\uparrow}^p_{f^{nm}}$ is a trivial loop rel A. □

Geometric exceptional maps all admit a minimal orbifold modeled on the quotient $\mathbb{R}^2/\langle\mathbb{Z}^2, -z\rangle$, which has cone singularities of angle π at the images of $\{0, \frac{1}{2}\} \times \{0, \frac{1}{2}\}$. We consider this class in a little more detail:

4.1. (2, 2, 2, 2)-maps. A branched covering $f\colon (S^2, P_f, \operatorname{ord}_f) \circlearrowleft$ is *of type* $(2,2,2,2)$ if $|P_f| = 4$ and $\operatorname{ord}_f(x) = 2$ for every $x \in P_f$. In this case, f is isotopic to a quotient of an affine map $z \mapsto Mz + q$ under the action of $\langle\mathbb{Z}^2, -z\rangle$, see Proposition 4.6(B).

Lemma 4.3. *Let M be a matrix with integer entries and* $\det(M) > 1$. *Denote by λ_1 and λ_2 the eigenvalues of M, ordered as $|\lambda_2| \geq |\lambda_1| > 0$. Then the following are all mutually exclusive possibilities*:

- *if M is exceptional, that is $0 < |\lambda_1| < 1 < |\lambda_2|$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then $M\colon \mathbb{R}^2 \circlearrowleft$ is expanding in the direction of the eigenvector corresponding to λ_2 and $M\colon \mathbb{R}^2 \circlearrowleft$ is contracting in the direction of the eigenvector corresponding to λ_1;*
- *if $\lambda_1 \in \{\pm 1\}$, then $M^2\colon \mathbb{R}^2 \circlearrowleft$ preserves the rational line $\{z \in \mathbb{R}^2 \mid Mz = z\}$;*
- *the map $M\colon \mathbb{R}^2 \circlearrowleft$ is expanding in all other cases; that is $\lambda_1 = \overline{\lambda_2} \notin \mathbb{R}$, or $\lambda_1, \lambda_2 \in \mathbb{R}$ but $|\lambda_1|, |\lambda_2| > 1$.*

Proof. If M's eigenvalues are non-real, then $\lambda_1 = \overline{\lambda_2}$ and $|\lambda_1| = |\lambda_2|$, so M is expanding. If λ_1 and λ_2 are real, then they are of the same sign and their product is greater than 1. The lemma follows. □

The following lemma follows from [26, Main Theorem II]:

Lemma 4.4. *If $f\colon (S^2, P_f, \mathrm{ord}_f) \circlearrowleft$ is doubly covered by a torus endomorphism $z \mapsto Mz + q$, then f is geometric if and only if f is Levy-free.*

Proof. We consider the exclusive cases of Lemma 4.3. In the first case, f is geometric and $z \mapsto Mz + q$ preserves transverse irrational laminations (given by the eigenvectors of M), so $z \mapsto Mz + q$ admits no Levy cycle and *a fortiori* neither does f.

In the second case, f is not geometric and admits as Levy cycle the projection of the 1-eigenspace of M.

In the third case, f is expanding, and admits no Levy cycle by [4, Theorem A]. □

We shall mainly study $(2,2,2,2)$-maps algebraically: we write the torus as $\mathbb{R}^2/\mathbb{Z}^2$ and the $(2,2,2,2)$-orbisphere as $\mathbb{R}^2/\langle \mathbb{Z}^2, -z\rangle$. Its fundamental group G is identified with $\langle \mathbb{Z}^2, -z\rangle \cong \mathbb{Z}^2 \rtimes \{\pm 1\}$. The orbifold points are the images on the orbisphere of $\{0, \frac{1}{2}\} \times \{0, \frac{1}{2}\}$. We start with some basic properties of G:

Proposition 4.5. (A) *Every injective endomorphism of G is of the form*

$$M^v\colon (n,1) \longmapsto (Mn, 1) \quad \text{and} \quad (n,-1) \longmapsto (Mn + v, -1) \tag{52}$$

for some $v \in \mathbb{Z}^2$ and some non-degenerate matrix M with integer entries. We have

$$N^w \circ M^v = (NM)^{w+Nv}.$$

There are precisely 4 order-2 conjugacy classes in G, each of the form

$$(a,-1)^G = \{(a + 2w, -1) \mid w \in \mathbb{Z}^2\} \text{ for some } a \in \{0,1\} \times \{0,1\} \subset \mathbb{Z}^2.$$

The set of order-2 conjugacy classes of G is preserved by M^v.

(B) *The automorphism group of* G *is* $\{M^v \mid \det M = \pm 1\}$ *and is naturally identified with* $\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$*. The inner automorphisms of* G *are identified with* $2\mathbb{Z}^2 \rtimes \{\pm\mathbb{1}\}$*, and the outer automorphism group of* G *is identified with* $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathrm{PGL}_2(\mathbb{Z})$*. The index-*2 *subgroup of positively oriented outer automorphisms is* $\mathrm{Out}^+(G) = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathrm{PSL}_2(\mathbb{Z})$*.*

(C) *The modular group* $\mathbf{Mod}(G)$ *is free of rank* 2*, and we have*

$$\begin{aligned}\mathbf{Mod}(G) &= \{M^v \mid \det(M) = 1, M \equiv \mathbb{1} \pmod 2, v \in 2\mathbb{Z}^2\}/(\pm\mathbb{1})^{2\mathbb{Z}^2}\\ &\cong \{M^0 \mid M \equiv \mathbb{1} \pmod 2\}/\{\pm\mathbb{1}\}.\end{aligned}$$

(D) *Two bisets* B_{M^v} *and* B_{N^w} *are isomorphic if and only if* $M = \pm N$ *and* $(M^v)_* = (N^w)_*$ *as maps on order-*2 *conjugacy classes, if and only if* $M = \pm N$ *and* $v \equiv w \pmod{2\mathbb{Z}^2}$*.*

We remark that (52) also follows from Lemma 3.4.

Proof. (A) It is easy to check that M^v defines an injective endomorphism. Conversely, let $M'\colon G \to G$ be an injective endomorphism. Then $M'(0,-1) = (v,-1)$ for some $v \in \mathbb{Z}^2$ because all $(w,-1)$ have order 2. On the other hand, $M' \downharpoonleft_{\mathbb{Z}^2\times\{1\}} = M \downharpoonleft_{\mathbb{Z}^2}$ for a non-degenerate matrix M with integer entries because M' is injective. It easily follows that $M' = M^v$ as given by (52).

The claims on composition and order-2 conjugacy classes of G follow from direct computation.

(B) Follows directly from (A) and the identification of G with $\{\pm\mathbb{1}\}^{\mathbb{Z}^2}$.

(C) We use (B); the modular group of G is the subgroup of $\mathrm{Out}^+(G) = (\mathbb{Z}/2)^2 \rtimes \mathrm{PSL}_2(\mathbb{Z})$ that fixes the order-2 conjugacy classes. The action of $(\mathbb{Z}/2\mathbb{Z})^2$ on these classes is simply transitive, so the set of order-2 classes may be put in bijection with $(\mathbb{Z}/2)^2$ under the correspondence $(a + 2\mathbb{Z}^2, -1) \leftrightarrow a$; then the action of $\mathrm{PSL}_2(\mathbb{Z})$ on order-2 conjugacy classes is identified with the natural linear action (noting that $-\mathbb{1}$ acts trivially mod 2). It follows that $\mathbf{Mod}(G)$ is the congruence subgroup $\{M \in \mathrm{PSL}_2(\mathbb{Z}) \mid M \equiv \mathbb{1} \pmod{\pm 2}\}$, and it is classical that it is a free group of rank 2.

(D) The bisets B_{M^v} and B_{N^w} are isomorphic if and only the maps M^v, N^w are conjugate by an inner automorphism; so the claimed description follows from (B). □

We turn to (2, 2, 2, 2)-maps, and their description in terms of the above; we use throughout $G = \mathbb{Z}^2 \rtimes \{\pm 1\}$:

Proposition 4.6. *Let f be a $(2,2,2,2)$-map with biset ${}_G B_G = B(f, P_f, \mathrm{ord}_f)$. Then,*

(A) *the biset B is right principal, and for any choice of $b_0 \in B$ there exists an injective endomorphism M^v of G satisfying $g b_0 = b_0 M^v(g)$ for all $g \in G$;*

(B) *the map f is Thurston equivalent to a quotient of $z \mapsto Mz + \frac{1}{2}v \colon \mathbb{R}^2 \circlearrowleft$ under the action of G;*

(C) *the map f is Levy obstructed if and only if M has an eigenvalue in $\{\pm 1\}$;*

(D) *if f is not Levy obstructed then for every $b_0 \in B$, writing M^v as in* (A) *and* (52)*, the map f is Thurston equivalent to the quotient of $z \mapsto Mz \colon \mathbb{Z}^2 \circlearrowleft$ by the action of $\langle \mathbb{Z}^2, z \mapsto -z + r \rangle \curvearrowright \mathbb{R}^2$ for a vector $r \in \mathbb{R}^2$ satisfying $Mr = r + v$. The G-equivariant map of Proposition 4.17 takes the form*

$$\Phi \colon {}_G B_G \otimes_{\langle \mathbb{Z}^2, -z+r \rangle} \mathbb{R}^2 \longrightarrow {}_{\langle \mathbb{Z}^2, -z+r \rangle} \mathbb{R}^2, \qquad b_0 \otimes z \longmapsto M^{-1} z. \tag{53}$$

Proof. (A) Since f is a self-covering of orbifolds, ${}_G B_G$ is right principal. The claim then follows from Proposition 4.5(A).

(B) We claim that the quotient map, denoted by $\bar{f}$, has a biset isomorphic to ${}_G B_G$. Indeed, for $g \in G$ it suffices to verify that

$$\left(Mz + \tfrac{1}{2}v\right) \circ g(z) = M^v(g) \circ \left(Mz + \tfrac{1}{2}v\right).$$

If $g = (t, 1)$ then both parts are $Mz + Mt + \frac{1}{2}v$, and if $g = (0, -1)$ then both parts are $-\left(Mz + \frac{1}{2}v\right)$. Therefore, f and $\bar{f}$ have isomorphic orbisphere bisets because marked conjugacy classes are preserved automatically by Proposition 4.5 (A). By Corollary 3.7 the maps f and $\bar{f}$ are isotopic.

(C) Suppose that M has an eigenvalue in $\{1, -1\}$; let w be the eigenvector of M associated with this eigenvalue. Consider the foliation F_w of $\mathbb{R}^2$ parallel to w. Then F_w is invariant under $z \mapsto Mz + \frac{1}{2}v$ as well as under the action of $\langle \mathbb{Z}^2, z \mapsto -z \rangle$. Therefore, the quotient map has invariant foliation $\bar{F}_w = F_w / G$. There are two leaves in $\bar{F}_w$ connecting points in pairs in the post-critical set of the quotient map; any other leave is a Levy cycle.

(D) Since $\det(M - \mathbb{1}) \neq 0$, there is a unique r such that $Mr = r + v$. It is easy to see (as in (C)) that the quotient map in (E) has a biset isomorphic to ${}_G B_G$. By Corollary 3.7 the quotient map is conjugate to f, and (53) is immediate. □

Corollary 4.7. *Let f be a $(2,2,2,2)$-map. Then its biset $B(f \colon (S^2, P_f, \mathrm{ord}_f) \circlearrowleft)$ is of the form B_{M^v} for an endomorphism M^v of $G := \mathbb{Z}^2 \rtimes \{\pm 1\}$, namely it is G qua right G-set, with left action given by $g \cdot h = M^v(g) h$.*

Let us also recall how the biset of a $(2,2,2,2)$-map is converted to the form B_{M^v}. First, the fundamental group G is identified with $\mathbb{Z}^2 \rtimes \{\pm 1\}$. The group G has a unique subgroup H of index 2 that is isomorphic to $\mathbb{Z}^2$, so H is easy to find. The restriction of the biset to H yields a 2×2-integer matrix M; and the translation part v is found by tracking the peripheral conjugacy classes. Note that this procedure applies as well to non-invertible maps as to homeomorphisms; and that the map is orientation-preserving precisely if $\det(M) > 0$.

4.2. Homotopy pseudo-orbits and shadowing. Let $f\colon S^2 \circlearrowleft$ be a self-map, and let I be an index set together with an index map also written $f\colon I \circlearrowleft$. An *$I$-poly-orbit* $(x_i)_{i\in I}$ is a collection of points in S^2 such that $f(x_i) = x_{f(i)}$. If all points x_i are distinct, then $(x_i)_{i\in I}$ is an *I-orbit*.

Thus, a poly-orbit differs from an orbit only in that it allows repetitions. Every poly-orbit has a unique associated orbit, whose index set is obtained from I by identifying i and j whenever $x_i = x_j$. Note that we allow $I = \mathbb{N}$, $I = \mathbb{Z}$ and $I = \mathbb{Z}/n\mathbb{Z}$ as index sets, with $f(i) = i+1$, as well as $I = \{0,\dots,m,\dots,m+n\}$ with $f(i) = i+1$ except $f(m+n) = m$, an orbit with period n and preperiod m.

We shall consider a homotopical weakening of the notion of orbits. Our treatment differs from [16] in a subtle manner (see below); recall that $\beta \approx_{A,\mathrm{ord}} \gamma$ means that the curves β,γ are homotopic in the orbispace (S^2, A, ord):

Definition 4.8 (homotopy pseudo-orbits). Let $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ be an orbisphere self-map, and let I be a finite index set together with an index map also written $f\colon I \circlearrowleft$.

An *I-homotopy pseudo-orbit* is a collection of paths

$$(\beta_i)_{i\in I} \text{ with } \beta_i\colon [0,1] \longrightarrow S^2 \setminus A \text{ satisfying } \beta_{f(i)}(0) = f(\beta_i(1)).$$

Two homotopy pseudo-orbits $(\beta_i)_{i\in I}$ and $(\beta'_i)_{i\in I}$ are *homotopic*, written

$$(\beta_i)_{i\in I} \approx_{A,\mathrm{ord}} (\beta'_i)_{i\in I},$$

if $\beta_i \approx_{A,\mathrm{ord}} \beta'_i$ for all $i \in I$. In particular, $\beta_i(0) = \beta'_i(0)$ and $\beta_i(1) = \beta'_i(1)$.

Two homotopy pseudo-orbits $(\beta_i)_{i\in I}$ and $(\gamma_i)_{i\in I}$ are *conjugate*, written

$$(\beta_i)_{i\in I} \sim (\gamma_i)_{i\in I},$$

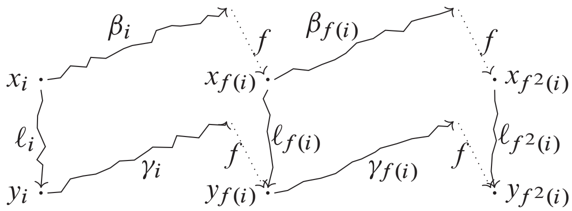
if there is a collection of paths $(\ell_i)_{i\in I}$ with

$$\ell_i(0) = \beta_i(0), \quad \ell_i(1) = \gamma_i(0), \quad \beta_i \# \ell_{f(i)}{\uparrow}_f^{\beta_i(1)} \approx_{A,\mathrm{ord}} \ell_i \# \gamma_i.$$

Poly-orbits are special cases of homotopy pseudo-orbits, in which the paths β_i are all constant.

Remark 4.9. A homotopy pseudo orbit can also be defined for an infinite I with the assumption that $(\ell_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ in Definition 4.8 have uniformly bounded length. Then Theorem 4.22 still holds.

In a homotopy pseudo-orbit, the curves $(\beta_i)_{i\in I}$ encode homotopically the difference between $x_i := \beta_i(0)$ and a choice of preimage of $x_{f(i)}$. Note that Ishii-Smillie define in [16, Definition 6.1] homotopy pseudo-orbits by paths $\overline{\beta_i}$ connecting $f(x_i)$ to $x_{f(i)}$; their $\overline{\beta_i}$ may be uniquely lifted to paths β_i from x_i to an f-preimage of $x_{f(i)}$ as in Definition 4.8, but our definition does not reduce to theirs, since our β_i are defined rel (A, ord), not rel $(f^{-1}(A), f^*(\mathrm{ord}))$.



Choose a length metric on S^2, and define the distance between homotopy pseudo-orbits by

$$d((\beta_i)_{i\in I}, (\gamma_i)_{i\in I}) := \max_{i\in I, t\in[0,1]} d(\gamma_i(t), \beta_i(t)). \tag{54}$$

The following result states that the set of conjugacy classes of homotopy pseudo-orbits is discrete:

Lemma 4.10 (discreteness). *Let us consider a homotopy pseudo-orbit $(\beta_i)_{i\in I}$ of $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$. Then there is an $\varepsilon > 0$ such every homotopy pseudo-orbit at distance less than ε from $(\beta_i)_{i\in I}$ is conjugate to it.*

Proof. Set $\varepsilon = \min_{a\neq b\in A} d(a,b)$. Consider a homotopy pseudo-orbit $(\gamma_i)_{i\in I}$ at distance $\delta < \varepsilon$ from $(\beta_i)_{i\in I}$. Connect $\beta_i(0)$ to $\gamma_i(0)$ by a path ℓ_i of length at most δ. Since $S \setminus A$ is locally contractible space, the curve ℓ_i is unique up to homotopy and $\beta_i \# \ell_{f(i)}\!\uparrow_f^{\beta_i(1)} \approx_{A,\mathrm{ord}} \ell_i \# \gamma_i$. □

We recall that A^∞ denotes the union of all periodic cycles containing critical points.

Definition 4.11 (shadowing). A homotopy pseudo-orbit $(\beta_i)_{\in I}$ *shadows rel* (A, ord) a poly-orbit $(p_i)_{i\in I}$ in $S \setminus A^\infty$ if they are conjugate; namely if there are curves ℓ_i connecting $\beta_i(0)$ to p_i that lie in $S^2 \setminus A$ except possibly their endpoints and such that for every $i \in I$ we have

$$\ell_i \approx_{(A,\mathrm{ord})} \beta_i \# \ell_{f(i)}\!\uparrow_f^{\beta_i(1)}.$$

Lemma 4.12. *The homotopy pseudo-orbit $(\beta_i)_{\in I}$ shadows the poly-orbit $(p_i)_{i\in I}$ if and only if every neighbourhood $(\mathcal{U}_i)_{i\in I}$ of $(p_i)_{i\in I}$ contains a homotopy pseudo-orbit $(\beta'_i)_{i\in I}$ conjugate to $(\beta_i)_{i\in I}$; namely, $\beta'_i \subset \mathcal{U}_i \ni p_i$ for all $i \in I$.*

Proof. If (β_i) can be conjugated into a small enough neighbourhood of $(p_i)_{i\in I}$, then it is conjugate to $(p_i)_{i\in I}$ by Lemma 4.10. The converse is obvious. □

Proposition 4.13. *Suppose $f:(S^2, A, \mathrm{ord}) \circlearrowleft$ is a geometric non-invertible map. Then a periodic pseudo-orbit $(\beta_i)_{i\in I}$ shadows $(p_i)_{i\in I}$ rel (A, ord) if and only if $(\beta_i)_{i\in I}$ shadows $(p_i)_{i\in I}$ rel (P_f, ord_f).*

Proof. Clearly if $(\beta_i)_{i\in I}$ shadows an orbit $(p_i)_{i\in I}$ rel (A, ord) then $(\beta_i)_{i\in I}$ shadows $(p_i)_{i\in I}$ rel (P_f, ord_f). Conversely, suppose $(\beta_i)_{i\in I}$ shadows an orbit $(p_i)_{i\in I}$ rel (P_f, ord_f), so there are paths ℓ_i connecting $\beta_i(0)$ to p_i with $\ell_i \approx_{P_f,\mathrm{ord}_f} \beta_i \# \ell_{f(i)}{\uparrow}_f^{\beta_i(1)}$. Thus, $\ell_i^{-1} \# \beta_i \# \ell_{f(i)}{\uparrow}_f^{\beta_i(1)}$ is a loop which is trivial rel (P_f, ord_f), but may not be trivial rel A.

Consider the following pullback iteration. Set

$$(\beta_i^0)_{i\in I} := (\beta_i)_{i\in I} \quad \text{and} \quad (\ell_i^0)_{i\in I} := (\ell_i^0)_{i\in I}.$$

Define

$$(\beta_i^n)_{i\in I} := (\beta_{f(i)}^{n-1}{\uparrow}_f^{\beta_i^{n-1}(1)})_{i\in I} \quad \text{and} \quad (\ell_i^n)_{i\in I} := (\ell_{f(i)}^{n-1}{\uparrow}_f^{\beta_i(1)})_{i\in I}.$$

Clearly the $(\beta_i^n)_{i\in I}$ are all conjugate. Observe that $(\ell_i^n)^{-1} \# \beta_i \# \ell_{f(i)}^n{\uparrow}_f^{\beta_i(1)}$ is a loop passing through p_i and that $(\ell_i^n)^{-1} \# \beta_i \# \ell_{f(i)}^n{\uparrow}_f^{\beta_i(1)}$ is a degree-1 preimage of $(\ell_i^{n-1})^{-1} \# \beta_i \# \ell_{f(i)}^{n-1}{\uparrow}_f^{\beta_i(1)}$.

If f is expanding, then the diameter of $(\ell_i^n)^{-1} \# \beta_i \# \ell_{f(i)}^n{\uparrow}_f^{\beta_i(1)}$ tends to 0 exponentially fast, hence $(\ell_i^n)^{-1} \# \beta_i \# \ell_{f(i)}^n{\uparrow}_f^{\beta_i(1)}$ is trivial rel (A, ord) for all sufficiently large n, and the claim follows.

If f is exceptional, then $(\ell_i^n)^{-1} \# \beta_i \# \ell_{f(i)}^n{\uparrow}_f^{\beta_i(1)}$ is trivial rel (A, ord) for all sufficiently large n by Corollary 4.2. □

At one extreme, homotopy pseudo-orbits include poly-orbits, represented as constant paths. At the other extreme, homotopy pseudo-orbits may be assumed to consist of paths all starting at the basepoint $*$. As such, these paths represent elements of the biset $B(f)$ of f, see 1.

4.3. Symbolic orbits.

We shall be interested in marking periodic orbits of regular points. These are conveniently encoded in the following simplification of portraits of bisets (in which the subbisets are singletons and therefore represented simply as elements):

Definition 4.14 (symbolic orbits). Let I be a finite index set with self-map $f: I \circlearrowleft$, and let B be a G-biset. An *I-symbolic orbit* is a sequence $(b_i)_{i\in I}$ of elements of B, and two I-symbolic orbits $(b_i)_{i\in I}$ and $(c_i)_{i\in I}$ are *conjugate* if there exists a sequence $(g_i)_{i\in I}$ in G with $g_i c_i = b_i g_{f(i)}$ for all $i \in I$.

Lemma 4.15. *Every homotopy pseudo-orbit can be conjugated to a symbolic orbit, unique up to conjugacy, in $B(f, A, \operatorname{ord})$.*

Proof. Given $(\beta_i)_{i\in I}$ a homotopy pseudo-orbit, choose paths ℓ_i from $\beta_i(0)$ to $*$ and define $\gamma_i = \ell_i^{-1} \# \beta_i \# \ell_{f(i)}\uparrow_f^{\beta_i(1)}$. Then $\gamma_i \in B(f, A, \operatorname{ord})$ and $(\beta_i)_{i\in I} \sim (\gamma_i)_{i\in I}$. Furthermore another choice of paths $(\ell'_i)_{i\in I}$ differs from $(\ell_i)_{i\in I}$ by $\ell'_i = \ell_i g_i$ for some g_i, so the symbolic orbits $(\gamma_i)_{i\in I}$ and $((\ell'_i)^{-1} \# \beta_i \# \ell'_{f(i)}\uparrow_f^{\beta_i(1)})_{i\in I}$ are conjugate. □

Let $f\colon (S^2, P_f, \operatorname{ord}_f) \circlearrowleft$ be an expanding map, and let ${}_G B_G$ be its biset. Recall that the Julia set $\mathcal{J}(f)$ of f is the accumulation set of preimages of a generic point $*$,

$$\mathcal{J}(f) := \overline{\bigcap_{n\geq 0} \bigcup_{m\geq n} f^{-m}(*)}.$$

Every bounded sequence $b_0 b_1 \cdots \in B^{\otimes\infty}$ defines an element of $\mathcal{J}(f)$ as follows: set $c_0 = b_0$ and $c_i = c_{i-1} \# b_i \uparrow_{f^i}^{c_{i-1}(1)}$ for all $i \geq 1$; then $\lim_{n\to\infty} c_n(1)$ converges to a point $\rho(b_0 b_1 \cdots) \in \mathcal{J}(f)$. The following proposition directly follows from the definition:

Proposition 4.16 (expanding case). *Suppose $f\colon (S^2, P_f, \operatorname{ord}_f) \circlearrowleft$ is an expanding map with orbisphere biset ${}_G B_G$, and let $(b_i)_{i\in I}$ be a finite symbolic orbit. Let Σ be a generating set of ${}_G B_G$ containing all b_i and let $\rho\colon \Sigma^{+\infty} \to \mathcal{J}(f)$ be the symbolic encoding defined above. Then $(b_i)_{i\in I}$ shadows $(\rho(b_i b_{f(i)} b_{f^2(i)} \dots))_{i\in I}$.*

This proposition is useful to solve shadowing problems (namely, determining when two symbolic orbits shadow the same poly-orbit) using language of automata.

It also follows from the proposition that every orbit homotopy pseudo-orbit shadows a unique orbit in $\mathcal{J}(f)$.

Proposition 4.17 (shadowing and universal covers). *Let $f\colon (S^2, A, \operatorname{ord}) \circlearrowleft$ be an orbisphere map with biset ${}_G B_G$, let $\pi\colon {}_G\widetilde{U} \to (S^2, A, \operatorname{ord})$ be the universal covering map of $(S^2, A, \operatorname{ord})$, and let $\Phi\colon {}_G B_G \otimes {}_G\widetilde{U} \hookrightarrow {}_G\widetilde{U}$ be the G equivariant map defined by Proposition 3.5.*

Then there is a completion ${}_G\widetilde{U}^+$ of ${}_G\widetilde{U}$ such that π and Φ extend to continuous maps $\pi\colon {}_G\widetilde{U}^+ \to S^2 \setminus A^\infty$ and $\Phi\colon {}_G B_G \otimes {}_G\widetilde{U}^+ \to {}_G\widetilde{U}^+$ with the following property: a symbolic orbit $(b_i)_{i\in I}$ shadows an orbit $(x_i)_{i\in I}$ in $S^2 \setminus A^\infty$ if and only if

$$\Phi(b_i \otimes \tilde{x}_{f(i)}) = \tilde{x}_i \quad \textit{for some } \tilde{x}_i \in \widetilde{U}^+ \textit{ with } \pi(\tilde{x}_i) = x_i.$$

Furthermore, if $\operatorname{ord}(a) < \infty$ for every $a \in A \setminus A^\infty$ then we may take $\widetilde{U}^+ = \widetilde{U}$.

Proof. To define $\widetilde{U}^+$, add to $\widetilde{U}$ all limit points of parabolic elements corresponding to small loops around punctures $a \in A \setminus A^\infty$ with $\operatorname{ord}(a) = \infty$. The extension of π and Φ is given by continuity; π is a branched covering with branch locus $\widetilde{U}^+ \setminus \widetilde{U}$.

If $(b_i)_{i\in I}$ shadows $(x_i)_{i\in I}$, then there are curves

$$\tilde{x}_i \colon [0,1] \longrightarrow (S^2, A, \operatorname{ord}) \quad \text{with } \tilde{x}_i(0) = * \text{ and } \tilde{x}_i(1) = x_i$$

such that $\tilde{x}_i^{-1} \# b_i \# \tilde{x}_{f(i)} \uparrow_f^{b_i(1)}$ is a homotopically trivial loop. This exactly means that $\Phi(b_i \otimes \tilde{x}_{f(i)}) = \tilde{x}_i$. Conversely, if $\Phi(b_i \otimes \tilde{x}_{f(i)}) = \tilde{x}_i$ for all $i \in I$ then $(b_i)_{i\in I}$ shadows $(\pi(\tilde{x}_i))_{i\in I}$. □

Proposition 4.18 (shadowing in the $(2,2,2,2)$ case). *Using notations of Proposition 4.6, suppose f is a $(2,2,2,2)$ geometric non-invertible map and*

$$\Phi \colon {}_G B_G \otimes {}_{\langle \mathbb{Z}^2, -z+r\rangle}\mathbb{R}^2 \longrightarrow {}_{\langle \mathbb{Z}^2, -z+r\rangle}\mathbb{R}^2, \quad (b_0, z) \longmapsto M^{-1}z$$

is as in Proposition 4.6(D). Then for every symbolic orbit $(b_i)_{i\in I}$ of ${}_G B_G$ there is a unique collection $(r_i)_{i\in I}$ of points in $\mathbb{R}^2$ such that $r_i = \Phi(b_i \otimes r_{f(i)})$. The points r_i are solutions of linear equations (55).

By Proposition 4.17, the image of $(r_i)_{i\in I}$ under the projection

$$\pi \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2/G \approx (S^2, P_f, \operatorname{ord}_f)$$

is the unique poly-orbit shadowed by $(b_i)_{i\in I}$.

The proof will use the following easy fact.

Lemma 4.19. *If M is non-invertible geometric, then $|\det(M^n + \epsilon I)| \geq 1$ for every $n \geq 1$ and every $\epsilon \in \{\pm 1\}$.*

Proof. If λ_1, λ_2 are M's eigenvalues, then Lemma 4.3 gives

$$|\det(M^n + \epsilon \mathbb{1})| = |(\lambda_1^n + \epsilon)(\lambda_2^n + \epsilon)| \geq 1. \qquad \square$$

Proof of Proposition 4.18. Write $b_i = b_0 g_i$. Recall that every g_i acts on $\mathbb{R}^2$ as $z \mapsto \epsilon_i z + t_i$ with $\epsilon_i \in \{\pm 1\}$. We get the following system of linear equations:

$$M^{-1}(\epsilon_i r_{f(i)} + t_i) = r_i, \quad i \in I.$$

By splitting $f \colon I \circlearrowleft$ into grand orbits and eliminating variables we arrive at equations of the form

$$(\mathbb{1} + \theta M^n) r_i = t \quad \text{with } \theta \in \{\pm 1\} \tag{55}$$

and n the period of an orbit and $t \in \mathbb{R}^2$ some parameters depending on ϵ_i and t_i. By Lemma 4.19, the system (55) has a unique solution. □

Consider an *I-symbolic orbit* $(b_i)_{i\in I}$ shadowing a poly-orbit $(x_i)_{i\in I}$. This means (see Definition 4.11) that there are curves $(\ell_i)_{i\in I}$ connecting $*$ to x_i such that $\ell_i^{-1} \# b_i \# \ell_{f(i)}\uparrow_f^{b_i(1)}$ is a trivial loop rel (A, ord). The *local group* $G_{x_i} \le \pi_1(X,*) = G$ consists of loops of the form

$$\ell_i[0,1-\varepsilon] \# \alpha \# \ell_i^{-1}[0,1-\varepsilon], \quad \alpha \text{ is close to } x_i. \tag{56}$$

If $x_i \notin A$, then G_{x_i} is a trivial group; otherwise G_{x_i} is an abelian group of order $\mathrm{ord}(x_i)$. If $(A,\mathrm{ord}) = (P_f, \mathrm{ord}_f)$, then G_{x_i} is a finite abelian group. Clearly, $(g_i b_i h_i)_i$ also shadows x_i for all $g_i \in G_{x_i}$ and all $h_i \in G_{x_{i+1}}$. Conversely:

Lemma 4.20. *Let $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ be a geometric non-invertible map. Suppose that the symbolic orbits $(b_i)_{i\in I}$ and $(c_i)_{i\in I}$ shadow $(x_i)_i$. Let G_{x_i} be the local group associated with $(b_i)_i$. Then there are $h_i \in G_{x_{f(i)}}$ such that $(c_i)_{i\in I}$ is conjugate to $(b_i h_i)_{i\in I}$.*

If I consists only of periodic indices, then $(c_i)_{i\in I}$ is conjugate to $(g_i b_i)_{i\in I}$ for some $g_i \in G_{x_i}$.

Proof. By conjugating $(c_i)_{i\in I}$ we can assume that $\ell_i^{-1} c_i \ell_i \uparrow_f^{b_i(1)}$ is a trivial loop; i.e. local groups associated with $(c_i)_{i\in I}$ coincide with whose associated with $(b_i)_{i\in I}$. It is now routine to check that $(b_i)_{i\in I}$ and $(c_i)_{i\in I}$ differ by the action of local groups. Indeed, b_i is of the form

$$\ell_i[0,1-\varepsilon] \# \beta_i \# (\ell_{f(i)}[0,1-\varepsilon])^{-1}\uparrow_f^{\beta_i(1)}, \quad \beta_i \text{ is close to } x_i,$$

and, similarly, c_i is of the form

$$\ell_i[0,1-\varepsilon] \# \gamma_i \# (\ell_{f(i)}[0,1-\varepsilon])^{-1}\uparrow_f^{\gamma_i(1)}, \quad \gamma_i \text{ is close to } x_i.$$

Let α_i be the image of $\beta_i^{-1} \# \gamma_i$ and define h_i to be $\ell_{f(i)}[0,1-\varepsilon] \# \alpha_i \# \ell_{f(i)}^{-1}[0,1-\varepsilon]$; compare with (56). Then $c_i = b_i h_i$ for all $i \in I$.

If I has only periodic indices, then β_i^{-1} and γ_i end at the same point and we set $\alpha_i := \gamma_i \# \beta_i^{-1}$ and $g_i := \ell_i[0,1-\varepsilon] \# \alpha_i \# \ell_i^{-1}[0,1-\varepsilon]$. We obtain $c_i = g_i b_i$ for all $i \in I$. □

Corollary 4.21. *Let $f\colon (S^2, P_f, \mathrm{ord}_f) \circlearrowleft$ be a geometric non-invertible map. For every poly-orbit there are only finitely many symbolic orbits that shadow it. Therefore, there are only finitely many conjugacy classes of portraits in $B(f)$.*

Theorem 4.22. *Let $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ be a non-invertible geometric map. Then the shadowing operation defines a map from conjugacy classes of symbolic finite orbits onto poly-orbits in $S^2 \setminus A^\infty$. If $(A,\mathrm{ord}) = (P_f, \mathrm{ord}_f)$, then the shadowing map is finite-to-one.*

Proof. Follows from Propositions 4.13, 4.16, 4.18, and Corollary 4.21. □

4.4. From symbolic orbits to portraits of bisets. The *centralizer* of a symbolic finite orbit $(b_i)_{i\in I}$ is the set of $(g_i)_{i\in I} \in G^I$ such that $g_i b_i = b_i g_{f(i)}$ for all $i \in I$.

Lemma 4.23. *If ${}_G B_G$ is the biset of a geometric non-invertible map f and $(b_i)_{i\in I}$ is a symbolic finite orbit shadowing a poly-orbit $(x_i)_{i\in I}$, then its centralizer is contained in $\prod_{i\in I} G_{x_i}$, where G_{x_i} are the local groups associated with $(b_i)_{i\in I}$.*

Proof. By Definition 4.11 there are curves $(\ell_i)_{i\in I}$ connecting $*$ to x_i such that $\ell_i^{-1} \# b_i \# \ell_i{\uparrow}_f^{b_i(1)}$ is a trivial loop rel (A, ord). Suppose that $(g_i)_{i\in I} \in G^I$ centralizes $(b_i)_{i\in I}$. Set $\tilde g_i := \ell_i^{-1} \# g_i \# \ell_i$. Then $\tilde g_{f(i)}{\uparrow}_f^{x_i}$ is isotopic to $\tilde g_i$ rel (A, ord). By the expanding property of f, or Corollary 4.2 if f is exceptional, $\tilde g_i$ is trivial rel (A, ord). This implies that $(g_i)_{i\in I} \in \prod_{i\in I} G_{x_i}$. □

Consider a portrait of bisets $(G_a, B_a)_{a\in\widetilde A}$ in ${}_G B_G$. We may decompose $\widetilde A = A \sqcup F \sqcup I$ with $f_*^n(F) \subseteq A$ for $n \gg 0$ and $f_*(I) \subseteq I$. Then for every $i \in I$ the group G_i is trivial and $B_i = \{b_i\}$ is a singleton. We obtain the symbolic orbit $(b_i)_{i\in I}$ which is the essential part of $(G_a, B_a)_{\widetilde A}$:

Lemma 4.24. *The relative centralizer $Z_D((G_a, B_a)_{a\in\widetilde A})$ (see §2.8) is isomorphic to the centralizer of $(b_i)_{i\in I}$ via the forgetful map $(g_d)_{d\in D} \to (g_i)_{i\in I}$.*

Let $(G_a, B_a)_{a\in\widetilde A}$ and $(G'_a, B'_a)_{a\in\widetilde A}$ be two portraits of bisets with associated symbolic orbits $(b_i)_{i\in I}$ and $(b'_i)_{i\in I}$. Assume that $G_a = G'_a$ and $B_a = B'_a$ for all $a \in A$. Then $(G_a, B_a)_{a\in\widetilde A}$ and $(G'_a, B'_a)_{a\in\widetilde A}$ are conjugate if and only if

(1) *$(b_i)_{i\in I}$ and $(b'_i)_{i\in I}$ are conjugate; and*

(2) *$G \otimes_{G_d} B_d = G \otimes_{G_d} B'_d$ for every $d \in F$.*

This reduces the conjugacy problem of portrait of bisets to the conjugacy problem of symbolic orbits; indeed Condition (2) is easily checkable. Note that every preperiodic point in $\widetilde A$ imposes a finite condition on conjugators and centralizers: for points attracted to A, Condition (2) imposes a congruence condition modulo the action on $\{\cdot\} \otimes_G B$ on the conjugator; for preperiodic points in I, conjugacy again amounts (by Condition (1) and Definition 4.14) to a congruence condition modulo the action on $\{\cdot\} \otimes_G B$.

Proof. If $d \in D$ with $f_*(d) \notin D$, then from $g_d B_d = B_d g_{f_*(d)} = B_d$ follows that $g_d = 1$. Induction on the escaping time to A gives $g_d = 1$ for all $d \in F$.

Similarly, if $d \in D$ with $f_*(d) \notin D$, then there is a $g_d \in G$ with $g_d B_d = B'_d = B'_d g_{f_*(d)}$ if and only if $G \otimes_{G_d} B_d = G \otimes_{G_d} B'_d$. By induction on the escaping time, $(G_a, B_a)_{a\in A\sqcup F}$ and $(G'_a, B'_a)_{a\in A\sqcup F}$ are conjugate if and only if Condition (2) holds. □

We are now ready to show that a geometric map, equipped with a portrait of bisets, yields a sphere map – possibly with some points infinitesimally close to each other. A *blowup* of a two-dimensional sphere is a topological sphere $\widetilde{S}^2$ equipped with a monotone map $\widetilde{S}^2 \to S^2$, namely a continuous map under which preimages of connected sets are connected. We remark that arbitrary countable subsets of S^2 may be blown up into disks.

Proposition 4.25. *Let $f: (S^2, A) \circlearrowleft$ be a non-invertible geometric map, let $\widetilde{A}$ be a set containing a copy of A, let $f_*: \widetilde{A} \circlearrowleft$ be a symbolic map which coincides with f on the subset $A \subseteq \widetilde{A}$, and let $(B_a)_{a\in\widetilde{A}}$ be a portrait of bisets in $B(f)$.*

Then there exists a unique map $e: \widetilde{A} \to S^2$ extending the identity on A, and a blowup $b: \widetilde{S}^2 \to S^2$, with the following properties. The locus of non-injectivity of e is disjoint from A^∞, and b blows up precisely at the grand orbits of

$$\{x \mid \text{there exist } a \neq a' \in \widetilde{A}: x = e(a) = e(a')\}$$

replacing points by disks on which the metric is identically 0 and all points in $\widetilde{A} \subset \widetilde{S}^2$ are disjoint. The maps $f_*: \widetilde{A} \circlearrowleft$ and $f: (S^2, A) \circlearrowleft$ extend to a map $\tilde{f}: (\widetilde{S}^2, \widetilde{A}) \circlearrowleft$, which is semiconjugate to f via b, and whose minimal portrait of bisets projects to $(B_a)_{a\in\widetilde{A}}$.

Proof. Let us set $G := \pi_1(S^2 \setminus A, *)$. We may decompose $\widetilde{A} = A \sqcup I \sqcup J$ with $f_*(I) = I$ and $f_*^n(J) \subseteq A \sqcup I$ for $n \gg 0$. On A, we naturally define e as the identity. If $i \in I$, then i is f_*-periodic and the bisets $B_i, B_{f_*(i)}, \dots$ determine a homotopy pseudo-orbit which shadows a unique periodic poly-orbit in S^2, by Theorem 4.22. Thus, e is uniquely defined.

We now blow up the grand orbits of all points in S^2 which are the image of more than one point in $A \sqcup I$ under e, replacing them by a disk on which the metric is identically 0. We inject $A \sqcup I$ arbitrarily in the blown-up sphere $\widetilde{S}^2$, and now identify $A \sqcup I$ with its image in the blowup.

Let us next extend f to a self-map $\tilde{f}$ of $\widetilde{S}^2$ so that $(B_a)_{a\in A\sqcup I}$ is the induced portrait of bisets. We first do it by arbitrarily mapping the disks to each other by homeomorphisms restricting to f_* on $A \sqcup I$. Let $(B'_a)_{a\in A\sqcup I}$ be the projection of the minimal portrait of bisets of $\tilde{f}$ via $\pi_1(\widetilde{S}^2, A \sqcup I) \to G$. Consider $i \in I$. By Lemma 4.20 (the periodic case), we can assume that $b_i = h_i b'_i$ with $h_i \in G_{e(i)}$. (Note that if $e(i) \notin A$, then $h_i = \mathbb{1}$ and $b_i = b'_i$.) Let $m' \in \mathbf{Braid}(\widetilde{S}^2 \setminus A, I)$ be a preimage of $(h_i)_{i\in I}$ under $\mathcal{E}_I$ (see (19)) and set $m := \operatorname{push}(m')$; it is defined up to pre-composing with a knitting element. Since $h_i \in G_{e(i)}$, we can assume that m is identity away from the blown up disks. Then $(B_a)_{a\in A\sqcup I}$ is a portrait of bisets induced by $m\tilde{f}$.

Consider next $j \in J$ and assume that $f_*(j)$ has already been defined. The biset B_j, and more precisely already its image in $\{\cdot\} \otimes_G B(f)$, see Definition 2.15(B), determines the correct $\tilde{f}$-preimage of $f_*(j) \in \widetilde{S}^2$ that corresponds to j, and thus determines $e(j)$ uniquely. □

4.5. Promotions of geometric maps. Let $X := (S^2, A, \mathrm{ord})$ be an orbisphere, and consider a subset $D \subset S^2 \setminus A$. Recall from (11) the quotient $\mathbf{Mod}(X|D)$ of mapping classes of $(S^2, A \sqcup D)$ by knitting elements; there, we called two maps $f, g\colon (X, D) \circlearrowleft$ *knitting-equivalent*, written $f \approx_{A|D} g$, if they differ by a knitting element in $\mathbf{knBraid}(X, D)$. We shall show that knitting-equivalent rigid maps are conjugate rel $A \sqcup D$:

Theorem 4.26 (promotion). *Suppose $f, g\colon (X, D) \circlearrowleft$ are orbisphere maps, and assume that either*

- *g is geometric non-invertible, or*
- *$g^m(D) \subseteq A$ for some $m \geq 0$.*

Then every conjugacy $h \in \mathbf{Mod}(X|D)$ between f and g rel $A|D$ lifts to a unique conjugacy $\tilde{h} \in \mathbf{Mod}(S^2, A \sqcup D)$ between f and g rel $A \sqcup D$ such that $\tilde{h} \approx_{A|D} h$.

Proof. We begin by a

Lemma 4.27. *Under the assumption on g, consider that $b \in \mathbf{Braid}(S^2 \setminus A, D)$. If for every $m \geq 1$ the element b is liftable through g^m, then $(g^*)^m b$ is trivial for all $m \succ \log|g|$, where $|\cdot|$ is the word metric.*

Proof. Consider first the case that g is expanding. Then the lengths of curves $(g^*)^m(b)(-, a)$ tend to 0 as $m \to \infty$, so $(g^*)^m(b) = 1$ for sufficiently large m. If g is exceptional, then Corollary 4.2 replaces the expanding argument. If finally $g^m(E) \subseteq A$, then $(g^*)^m(b) = 1$ for the same m. △

We resume the proof. Let $\tilde{h}_0 \in \mathbf{pMod}(S^2, A \sqcup D)$ be any preimage of h under the forgetful map $\mathbf{Mod}(S^2, A \sqcup D) \to \mathbf{Mod}(X|D)$. Then $\tilde{h}_0 f \tilde{h}_0^{-1} \approx_{A \sqcup D} h_1^{-1} g$ for some $h_1 \in \mathbf{knBraid}(X, D)$. Setting $\tilde{h}_1 := h_1 \tilde{h}_0$ we get $\tilde{h}_1 f \tilde{h}_1^{-1} \approx_{A \sqcup D} h_2^{-1} g$, where h_2^{-1} is the lift of h_1^{-1} through g. Continuing this process we eventually get $\tilde{h}_m f \tilde{h}_m^{-1} \approx_{A \sqcup D} g$ because a the corresponding lift of h_1^{-1} is trivial by Lemma 4.27. We have shown the existence of $\tilde{h}$.

If $\tilde{h}'$ is another promotion of h, then $\tilde{h}'$ and $\tilde{h}$ differ by a knitting element commuting with f. By Lemma 4.27, that knitting element is trivial, hence $\tilde{h}' = \tilde{h}$. □

Corollary 4.28. *Let*

$$_{\mathbf{Mod}(S^2, A \sqcup D)} M(f)_{\mathbf{Mod}(S^2, A \sqcup D)} \longrightarrow {}_{\mathbf{Mod}(X|D)} M(f|D)_{\mathbf{Mod}(X|D)}$$

be the inert map forgetting the action of knitting elements (see Proposition 2.12). Suppose that either f is geometric non-invertible or $f^m(D) \subseteq A$ for some $m \geq 0$. Then f, g are conjugate in $M(f)$ if and only if their images are conjugate in $M(f|D)$. Moreover, the centralizers $Z(f)$ and $Z(f|D)$ (see §2.8) are naturally isomorphic via the projection $\mathbf{Mod}(S^2, A \sqcup D) \to \mathbf{Mod}(X|D)$.

4.6. Automorphisms of bisets. Recall that the automorphism group of a biset ${}_H B_G$ is the set $\operatorname{Aut}(B)$ of maps $\tau\colon B \circlearrowleft$ satisfying $\tau(hbg) = h\tau(b)g$ for all $h \in H, g \in G$.

Proposition 4.29. *If ${}_H B_G$ is a left-free, right-transitive biset and H is centreless, then $\operatorname{Aut}(B)$ acts freely on B, so $B/\operatorname{Aut}(B)$ is also a left-free, right-transitive H-G-biset.*

Proof. $\operatorname{Aut}(B)$ acts by permutations on $\{\cdot\} \otimes_H B$, and commutes with the right G-action, which is transitive, so $\operatorname{Aut}(B)/\ker(\text{action})$ acts freely. Consider now $\tau \in \operatorname{Aut}(B)$ that acts trivially on $\{\cdot\}\otimes_H B$, and consider $b \in B$. We have $\tau(b) = tb$ for some $t \in H$, and for all $h \in H$ there exists $g \in G$ with $hb = bg$, because B is right-transitive. We have

$$htb = h\tau(b) = \tau(hb) = \tau(bg) = \tau(b)g = tbg = thb,$$

so $t \in Z(H) = 1$ and therefore $\tau = \mathbb{1}$. It follows that $\operatorname{Aut}(B)$ acts freely on B. □

Corollary 4.30 (no biset automorphisms). *For an orbisphere biset ${}_H B_G$ with H non-cyclic, we have* $\operatorname{Aut}(B) = 1$.

In the dynamical situation ${}_G B_G$ is a cyclic biset if and only if G is a cyclic group.

Proof. Since B is an orbisphere biset, it is left-free and right-transitive; and since it is not cyclic, H is a non-abelian orbisphere group and in particular is centreless, so Proposition 4.29 applies. Now B cannot have a proper quotient, because conjugacy classes of elements of H appear exactly once in wreath recursions of peripheral elements, by [5, Definition 2.6(SB_3) of orbisphere bisets]. □

Corollary 4.31. *Suppose $f, g\colon (S^2, C, \mathrm{ord}) \to (S^2, A, \mathrm{ord})$ are isotopic orbisphere maps. Let $B(f)$ and $B(g)$ be the bisets of f and g with respect to the same base points $\dagger \in S^2 \setminus C$ and $* \in S^2 \setminus A$. Then there is a unique isomorphism between $B(f)$ and $B(g)$.*

Proof. Consider an isotopy $(f_t)_{t\in[0,1]}$ rel A from f to g. It induces a continuous motion $B(f_t, A, \mathrm{ord}, *)$ of the bisets of f_t. Therefore, all $B(f_t, A, \mathrm{ord}, *)$ are isomorphic. This shows existence, the uniqueness follows from Corollary 4.30. □

Theorem 4.32 (rigidity). *Suppose $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ is a geometric map. If $h\colon (S^2, A) \circlearrowleft$ commutes with f, is the identity in some neighbourhood of A^∞, and is isotopic to $\mathbb{1}$ rel A, then $h = \mathbb{1}$.*

Proof. Fix a basepoint $* \in S^2 \setminus A$ and the fundamental group $G = \pi_1(S^2 \setminus A, *)$. Since h is isotopic to the identity rel A, there is a path $\ell\colon [0,1] \to S^2 \setminus A$ from $*$ to $h(*)$ such that $\gamma \approx \ell \# (h \circ \gamma) \# \ell^{-1}$ for all $\gamma \in G$. Define then

$$h_*\colon {}_G B(f)_G \circlearrowleft, \quad h_*(\beta) := \ell \# (h \circ \beta) \# (\ell^{-1}){\uparrow}_f^{h(\beta(1))}.$$

Since h commutes with f, this defines an automorphism of $B(f)$. By Corollary 4.30, it is the identity on $B(f)$. It also fixes, therefore, all conjugacy classes in $B(f)^{\otimes n}$ for all $n \in \mathbb{N}$. By Theorem 4.22, these are in bijection with periodic orbits of f. It follows that h is the identity on all periodic points of f, and therefore on f's Julia set. Since furthermore h is the identity near A^∞, and every point of $S^2 \setminus A$ either belongs to $\mathcal{J}(f)$ or gets attracted to A^∞, we get $h = \mathbb{1}$ everywhere. □

4.7. Weakly geometric maps. We are now ready to show that maps whose restriction to their minimal orbisphere is geometric are of a particularly simple form.

Definition 4.33 (tunings). Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. A *tuning multicurve* for a f is an f-invariant[1] multicurve $\mathcal{C}$ such that $f^{-1}(\mathcal{C})$ does not contain nested components rel $f^{-1}(A)$; namely, the adjacency graph of components of $S^2 \setminus f^{-1}(\mathcal{C})$ is a star.

A *tuning* is an amalgam of orbisphere bisets in which the graph of bisets is a star. Equivalently, it is an amalgam of Thurston maps along a tuning multicurve.

Every tuning has a *central map*, corresponding to the centre of the star, as well as *satellite maps*, corresponding to its leaves. Furthermore, unless the star has two vertices, its central map is uniquely determined.

Definition 4.34 (weakly geometric maps). A *tuning by homeomorphisms* is a tuning in which all satellite maps are homeomorphisms.

A map is *weakly geometric* if it is a (possibly trivial) tuning by homeomorphisms in such a manner that the central map is isotopic to a geometric map.

Theorem 4.35. *Let $f\colon (S^2, A) \circlearrowleft$ be a non-invertible Thurston map. Then f is weakly geometric if and only if its restriction $f\colon (S^2, P_f, \mathrm{ord}_f) \circlearrowleft$ is isotopic to a geometric map.*

Proof. If f is weakly geometric, then its central map $\bar{f}$ is isotopic to a geometric map, and f is isotopic to $\bar{f}$ rel P_f.

[1] In the sense that $f^{-1}(\mathcal{C})$ equals $\mathcal{C}$ rel A.

Conversely, assume that the restriction $\bar{f}\colon (S^2, P_f, \mathrm{ord}_f)\circlearrowleft$ of f is geometric, and consider its portrait of bisets $(B_a)_{a\in A}$ induced from the minimal portrait of bisets of f. By Proposition 4.25, there exists a blown-up sphere $\widetilde{S}^2$ and extension $\tilde{f}$ of f, such that the portraits of bisets of f and $\tilde{f}$ are conjugate portraits of bisets in $B(\bar{f})$. The tuning multicurve we seek is the boundary of the blowup disks.

By Theorem 2.19, the bisets $B(f)$ and $B(\tilde{f})$ differ by a knitting element, so we may write $\tilde{f} = mf$ for some $m \in \mathbf{knBraid}(S^2 \setminus A, \widetilde{A} \setminus A)$. By Proposition 2.8, the class m is liftable arbitrarily often through $\bar{f}$, and by Lemma 4.27 its lift becomes eventually trivial; in other words, we have a relation $n^{-1}\tilde{f}n = mf$ for some m with trivial image in $\mathbf{Braid}(S^2 \setminus A)$, and therefore n is a product of mapping classes in the infinitesimal disks. Its restriction to these disks yields the required homeomorphisms of the tuning. □

We now turn to the algebraic aspects of geometric maps.

Definition 4.36. An orbisphere biset is *geometric* if it is the biset of a geometric map. A biset B is *weakly geometric* if its projection $\overline{B}$ to its minimal orbisphere group is geometric.

Corollary 4.37. *A Thurston map f is* [*weakly*] *geometric if and only if its biset is* (*weakly*) *geometric.*

The following result was already obtained by Selinger and Yampolsky in the case of torus maps [26]:

Corollary 4.38. *A non-invertible Thurston map is isotopic to a geometric map if and only if it is Levy-free.*

Proof. Let f be a non-invertible Thurston map. If f admits a Levy cycle, then either f shrinks no metric under which a curve in the Levy cycle is geodesic, or f is doubly covered by a torus endomorphism with eigenvalue ± 1; in all cases, f is not geometric.

Conversely, if f is Levy-free, then its restriction $\bar{f}$ to $(S^2, P_f, \mathrm{ord}_f)$ is still Levy-free. By [4, Theorem A] (for {Exp} maps) or Lemma 4.4 (for {GTor/2} maps) the map $\bar{f}$ is geometric. By Proposition 4.25, there is a map $e\colon A \to S^2$ whose image is preserved by $\bar{f}$. If e were not injective, there would be a Levy cycle for f consisting of curves surrounding elements of A with same image under e. Now if e is injective then the tuning constructed in Theorem 4.35 is trivial, so f is geometric. □

4.8. Conjugacy and centralizer problem. We are now ready to show how conjugacy and centralizer problems may be solved in geometric bisets. We review the algebraic interpretation of expanding maps: by [4, Theorem A], a map is expanding if and only if its minimal biset is *contracting*. We recall briefly that a left-free G-G-biset B is contracting if, for every basis $X \subseteq B$, there exists a finite subset $N \subseteq G$ with the following property: for every $g \in G$ there is $n \in \mathbb{N}$ such that $X^n g \in NX^n \subseteq B^{\otimes n}$.

The minimal such subset N is called the *nucleus* associated with (B, X). Note that different bases yield different nuclei, but that finiteness of N is one basis implies its finiteness for all other bases, or even for all finite sets X generating B qua left G-set.

The biset B may be given by its *structure map* $X \times G \to G \times X$, written $(x, g) \mapsto (g@x, x^g)$; it is defined by the equality $xg = (g@x)x^g$ in B. Then B is contracting with nucleus N if for every $g \in G$ every sufficiently long iteration of the maps $g \mapsto g@x$ eventually gives elements in N.

We may assume, without loss of generality, that $XN \subseteq NX$ holds. For finitely generated groups, there is perhaps more intuitive formulation of contraction: there exists a proper metric on G and constants $\lambda < 1, C$ such that $\|g@x\| \le \lambda\|g\| + C$ for all $g \in G, x \in X$.

Lemma 4.39. *Let ${}_G B_G$ be a contracting biset; choose a basis X of B, and let $N \subseteq G$ be the associated nucleus.*

Then every symbolic finite orbit $(b_i)_{i\in I}$ is conjugate to one in which $b_i \in NX$ for all $i \in I$.

Proof. Write every b_i in the form $g_i x_i$ with $g_i \in G$ and $x_i \in X$. Conjugate the symbolic orbit by $(g_i)_{i\in I}$; then each b_i becomes $g_i^{-1} b_i g_{f(i)} = x_i g_{f(i)} = g_i' x_i' \in B$ for some $g_i' \in G, x_i' \in X$. Note that each g_i' is a state of some g_i. Conjugate again by $(g_i')_{i\in E}$, etc.; after a finite number of steps, each b_i will belong to NX. □

Lemma 4.40. *Let $f\colon (S^2, A, \mathrm{ord}) \circlearrowleft$ be a geometric map. Then for every $n \in \mathbb{N}$ the number of n-periodic points of f is finite.*

Proof. If f is doubly covered by a torus endomorphism $z \mapsto Mz + q$, then its period-n points are all solutions to $z \in (\mathbb{1} - M)^{-1}q + (\mathbb{1} - M^n)^{-1}\mathbb{Z}^2$; the image of this set under $\mathbb{R}^2 \to \mathbb{R}^2/\langle \mathbb{Z}^2, -z\rangle$ is finite, of size at most $\det(\mathbb{1} - M^n)$.

If f is expanding, then consider its biset B, which is orbisphere contracting, see [4, Theorem A]. By Lemma 4.39, there are finitely many conjugacy classes of period-n finite symbolic orbits in B, and by Theorem 4.22, every symbolic orbit shadows a unique finite poly-orbit. □

Theorem 4.41. *Let $\widetilde{G} \to G$ be a forgetful morphism of groups and let*

$$ {}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}} \longrightarrow {}_G B_G \quad \textit{and} \quad {}_{\widetilde{G}}\widetilde{B}'_{\widetilde{G}} \longrightarrow {}_G B'_G $$

be two forgetful biset morphisms as in (8). Suppose furthermore that $\widetilde{B}$ is geometric of degree > 1. Denote by $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G_a', B_a')_{a\in\widetilde{A}}$ the portraits of bisets induced by $\widetilde{B}$ and $\widetilde{B}'$ in B and B' respectively.

Then $\widetilde{B}, \widetilde{B}'$ *are conjugate under* $\mathbf{Mod}(\widetilde{G})$ *if and only if there exists* $\phi \in \mathbf{Mod}(G)$ *such that* $B^\phi \cong B'$ *and the portraits* $(G_a^\phi, B_a^\phi)_{a\in\widetilde{A}}$ *and* $(G'_a, B'_a)_{a\in A\sqcup E}$ *are conjugate.*

Furthermore, the centralizer of the portrait $(G_a, B_a)_{a\in\widetilde{A}}$ *is trivial, and the centralizer* $Z(\widetilde{B})$ *of* $\widetilde{B}$ *is isomorphic, via the forgetful map* $\mathbf{Mod}(\widetilde{G}) \to \mathbf{Mod}(G)$*, to*

$$\{\phi \in Z(B) \mid (G_a^\phi, B_a^\phi)_{a\in\widetilde{A}} \sim (G_a, B_a)_{a\in\widetilde{A}}\}$$

and is a finite-index subgroup of $Z(B)$*.*

Proof. If $\widetilde{B}, \widetilde{B}'$ are conjugate, then certainly their images and subbisets are conjugate. Conversely, let $\phi \in \mathbf{Mod}(G)$ be such that $B^\phi \cong B'$ and the portraits $(G_a^\phi, B_a^\phi)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in A\sqcup E}$ are conjugate, and let $\tilde{f}, \tilde{f}'$ be maps realizing $\widetilde{B}, \widetilde{B}'$ respectively, with f geometric. By Theorem 2.19, we have $\tilde{f}' = m\tilde{f}$ for a knitting mapping class m. Since m is liftable by Proposition 2.8, we have $m\tilde{f} = fm'$ for some knitting mapping class m' and therefore $m'\tilde{f}'(m')^{-1} = m'\tilde{f}$. Repeating with m', we obtain $m^{(k)}\cdots m'\tilde{f}'(m^{(k)}\cdots m')^{-1} = m^{(k)}\tilde{f}$ for all $k \in \mathbb{N}$, and $m^{(k)} = \mathbb{1}$ when k is large enough by Lemma 4.27. Therefore, $\tilde{f}, \tilde{f}'$ are conjugate, and so are $\widetilde{B}, \widetilde{B}'$.

The description of $Z(\widetilde{B})$ follows. The centralizer of the portrait $(G_a, B_a)_{a\in\widetilde{A}}$ is trivial, because (since $\widetilde{B}$ is geometric) all points shadowed by bisets B_a are unmarked in G. The $Z(B)$-orbit of $(G_a, B_a)_{a\in\widetilde{A}}$ contains finitely many conjugacy classes by Lemma 4.40, since all of them are induced from a geometric biset, and therefore are encoded in periodic or pre-periodic points for a map realizing B. □

5. Algorithmic questions

We give some algorithms that decide whether two portraits of bisets are conjugate, thereby reducing the conjugacy and centralizer problem from orbisphere maps to their minimal orbisphere (with only the post-critical set marked).

All the algorithms we describe make use of the symbolic description of maps via bisets, and are inherently quite fast and practical. Their precise performance, and implementation details, be studied in the last paper [6] of this series.

There are two implementations of Algorithms 1.1 and 1.2, one for {GTor/2} maps and one for {Exp} maps. We describe them in separate subsections.

We already gave the following algorithms:

Algorithm 5.1 ([4, Algorithms 5.1 and 5.2]).

Given an orbisphere biset ${}_G B_G$,

decide whether B is the biset of a map double covered by a torus endomorphism, and if so

compute parameters M, q for a torus endomorphism $z \mapsto Mz + q$ covering B.

Algorithm 5.2 ([4, Algorithms 5.4 and 5.5]).

GIVEN an orbisphere biset ${}_G B_G$,

DECIDE whether B is the biset of a {GTor/2} or an {Exp} map, and in particular whether B is geometric.

5.1. {GTor/2} maps. In the case of {GTor/2} maps, we can decide, without access to an oracle, whether two such maps are conjugate, and we can compute their centralizers, as follows. We shall need the following fact:

Theorem 5.3 (Corollary of [13]). *There is an algorithm deciding whether two matrices $M, N \in \mathbf{Mat}_2^+(\mathbb{Z})$ are conjugate by an element $X \in \mathbf{SL}_2(\mathbb{Z})$, and produces such an X if it exists.*

There is an algorithm computing, as a finitely generated subgroup of $\mathbf{SL}_2(\mathbb{Z})$, the centralizer of $M \in \mathbf{Mat}_2^+(\mathbb{Z})$.

Algorithm 5.4.

GIVEN ${}_G B_G$, ${}_G C_G$ two minimal {GTor/2} bisets

DECIDE whether B and C are conjugate by an element of $\mathbf{Mod}(G)$, and if so

CONSTRUCT a conjugator; and COMPUTE the centralizer $Z(B)$,

AS FOLLOWS.

(1) If $B_* \neq C_*$ as maps on peripheral conjugacy classes, then return `fail`.

(2) Using Proposition 4.5(A), identify G with $\mathbb{Z}^2 \rtimes \{\pm 1\}$, and with Algorithm 5.1 present B and C as B_{M^v} and B_{N^w} respectively, see (52).

(3) Using Theorem 5.3 check whether M and N are conjugate. If not, return `no`; otherwise find a conjugator X and compute the centralizer $Z(M)$ of M.

(4) Check whether there is a $Y \in Z(M)$ such that $B_{(YX)^0}$ conjugates B_{M^v} to B_{N^w}, as follows. The orbit of B_{M^v} under conjugation of

$$\{B_{Y^0} \mid Y \in Z(M), B_{(YX)^0} \in \mathbf{Mod}(G)\}$$

is finite and hence computable; so is its image under X^0. Check whether B_{N^w} belongs to it; if not, return `no`, and if yes, return `yes` and the conjugator $B_{(YX)^0}$.

(5) The centralizer of B is

$$\{B_{Y^0} \mid Y \in Z(M), B_{Y^0} \in \mathbf{Mod}(G), \text{ and } B_{Y^0} \text{ centralizes } B_{M^v}\},$$

naturally embeds as a subgroup of finite index in $Z(M)$.

Here is an algorithmic version of Proposition 4.18:

Algorithm 5.5.

GIVEN a minimal {GTor/2} orbisphere biset ${}_G B_G$ with $G = \langle \mathbb{Z}^2, -z+r \rangle$, a base point $b_0 \in B$ specifying the map

$$\Phi\colon {}_G B_G \otimes_{\langle \mathbb{Z}^2, -z+r\rangle} \mathbb{R}^2 \longrightarrow {}_{\langle \mathbb{Z}^2, -z+r\rangle}\mathbb{R}^2 \colon (b_0, z) \mapsto M^{-1} z \quad (57)$$

(see Proposition 4.18), an extension $f_*\colon \widetilde{A} \to \widetilde{A}$ of the dynamics of B on its peripheral classes, and a portrait of bisets $(G_a, B_a)_{a\in\widetilde{A}}$,

COMPUTE $(r_a)_{a\in\widetilde{A}}$ shadowed by $(G_a, B_a)_{a\in\widetilde{A}}$: $r_a = \Phi(B_a \otimes r_{f_*(a)})$, the local groups $G_{\bar{r}_a}$ where $\bar{r}_a$ is the image of r_a in $\mathbb{R}^2/G$ (see (56)), and the relative centralizer $Z_D((G_a, B_a)_{a\in\widetilde{A}})$, which is a finite abelian group

AS FOLLOWS.

(1) Choose $b_a \in B_a$. Proceed through all periodic cycles E of I. Choose $e \in E$ and solve the linear equation

$$r_e = \Phi^{|E|}(b_e \otimes b_{f_*(e)} \otimes \cdots \otimes b_{f_*^{|E|-1}(e)} \otimes r_e);$$

the equation takes form $(\mathbb{1} + \theta M^n) r_i = t$ with $\theta \in \{\pm 1\}$ (see (55)) and has a unique solution by Lemma 4.19.

(2) Inductively compute $r_a = \Phi(b_a \otimes r_{f_*(a)})$ for all $a \in \widetilde{A}$.

(3) For $a \in A$ we have $G_{\bar{r}_a} = G_a$.

(4) For $a \in \widetilde{A} \setminus A$ check whether $r_a - z \in G$. If $r_a - z \notin G$, then $G_{\bar{r}_a}$ is the trivial subgroup; otherwise $G_{\bar{r}_a} = \langle r_a - z \rangle$.

(5) By a finite check compute $Z_D((G_a, B_a)_{a\in\widetilde{A}})$: by Lemma 4.24 it is the set of self-conjugators of the corresponding homotopy pseudo-orbit, and by Lemma 4.20 it is a subgroup of $\prod_{a\in\widetilde{A}} G_{\bar{r}_a}$, and is therefore an easily computable finite group.

Algorithm 5.6.

GIVEN a minimal {GTor/2} orbisphere biset ${}_G B_G$, an extension $f_*\colon \widetilde{A} \to \widetilde{A}$ of the dynamics of B on its peripheral classes, and two portraits of bisets $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$,

DECIDE whether $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$ are conjugate

AS FOLLOWS.

(1) Normalize the portraits in such a manner that $G_a = G'_a$ and $B_a = B'_a$ for all $a \in A$; by Lemma 2.17 this follows from the conjugacy for subgroups: find $(\ell_a)_{a\in A} \in G^A$ with $G_a^{\ell_a} = G'_a$ and conjugate $(G_a, B_a)_{a\in\widetilde{A}}$ by $(\ell_a)_{a\in\widetilde{A}}$ with $\ell_a = 1$ if $a \notin A$.

(2) Identify G with $\langle \mathbb{Z}^2, -z+r \rangle$ and choose $b_0 \in B$ characterizing the map Φ, see (57).

(3) Using Algorithm 5.5 compute the points r_a and r'_a shadowed by $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$ respectively.

(4) Check if $\bar{r}_a = \bar{r}'_a$ in $\mathbb{R}^2/G$. If $\bar{r}_a \neq \bar{r}'_a$ for some $a \in \widetilde{A}$, then return no. Otherwise find $\ell_a \in G$ with $\ell_a r'_a = r_a$ with $\ell_a = 1$ for $a \in A$ and conjugate (G'_a, B'_a) by $(r_a)_{a\in\widetilde{A}}$. This reduces to original problem to the case $r'_a = r_a$.

(5) Using Algorithm 5.5 compute the (finite abelian) local groups $G_{\bar{r}_a}$ and by a finite check decide if an element in $\prod_{a\in\widetilde{A}} G_{\bar{r}_a}$ conjugates $(G_a, B_a)_{a\in\widetilde{A}}$ to $(G'_a, B'_a)_{a\in\widetilde{A}}$.

If $(\bar{r}_a)_{a\in\widetilde{A}}$ is an actual orbit, then $G_{\bar{r}_a}$ are trivial groups for all $a \notin \widetilde{A} \setminus A$ and Step (5) can be omitted. This is the case when the algorithm is called from Algorithm 5.12.

Algorithm 5.7.

GIVEN a minimal {GTor/2} orbisphere biset ${}_GB_G$ and an extension $f_*\colon \widetilde{A} \to \widetilde{A}$ of the dynamics of B on its peripheral classes,

PRODUCE A LIST of all conjugacy classes of portraits of bisets $(G_a, B_a)_{a\in\widetilde{A}}$ in B with dynamics f_*

AS FOLLOWS.

(1) Write $G = \langle \mathbb{Z}^2, -z + r\rangle$ and $B = B_{M^v}$, using Algorithm 5.1. Choose $b_0 \in B$ characterizing the map Φ, see (57).

(2) Using Algorithm 5.5 compute the orbit $(\bar{r}_a)_{a\in A}$ of the map $M\colon \mathbb{R}^2/G \circlearrowleft$. Produce a list of all possible poly-orbits $(\bar{r}_a)_{a\in\widetilde{A}}$ extending $(\bar{r}_a)_{a\in A}$.

(3) For every poly-orbit $(\bar{r}_a)_{a\in\widetilde{A}}$ find a portrait $(G_a, B_a)_{a\in\widetilde{A}}$ that shadow $(\bar{r}_a)_{a\in\widetilde{A}}$.

(4) Using Algorithm 5.5 compute the (finite) local groups $G_{\bar{r}_a}$. By Lemma 4.20 the finite set

$$\{(G_a, B_a h_a)_{a\in\widetilde{A}} \mid h_a \in G_{\bar{r}_a}, h_a = 1 \text{ if } a \in A\}$$

contains a representative of every conjugacy class of portraits that shadows $(\bar{r}_a)_{a\in\widetilde{A}}$. Using Algorithm 5.5 produce a list of all conjugacy classes of portraits of bisets that shadow $(\bar{r}_a)_{a\in A}$.

5.2. {Exp} maps. We now turn to expanding maps, and start by a short example showing how the conjugacy problem may be solved algorithmically. We note, by following the proof of Lemma 4.39, that every portrait of bisets can be algorithmically conjugated to one in which the terms belong to NX, for X a basis and N the associated nucleus.

Example 5.8. Suppose $E = \{e\}$ with $f_*(e) = e$, and let $(G_a, B_a)_{a\in A\sqcup E}$ and $(G_a, C_a)_{a\in A\sqcup E}$ be two portraits of bisets. Then $G_e = 1$ and $B_e = \{b\}$ and $C_e = \{c\}$.

The portraits $(G_a, B_a)_{a\in A\sqcup E}$ and $(G_a, C_a)_{a\in A\sqcup E}$ are conjugate if and only if there exists $\ell \in G$ such that $\ell^{-1}b\ell = c$.

Write $B_e = \{gx\}$ in a basis X of B, with associated nucleus N; recall that N is symmetric, contains 1 and generates G. Then gx is conjugate to $xg = (g@x)x^g = g'x'$. After iterating finitely many times the process $gx \sim xg = g'x'$, we can assume $g \in N$, as in Lemma 4.39. Similarly, we may replace C_e by a conjugate biset $\{hy\}$ with $h \in N$.

Find, by direct search, a $t \in \mathbb{N}$ with $N^{t-3}X \supseteq XN^t$; such a $t \geq 4$ exists because B is contracting. Then B_e and C_e are conjugate if and only if there exists $\ell \in G$ with $b\ell = \ell c$, namely $gx\ell = \ell hy$. Let $u \in \mathbb{N}$ be such that $\ell \in N^u \setminus N^{u-1}$; if $u \geq t$ then $gx\ell \in NN^{u-3}X$ while $\ell hy \notin N^{u-1}NX$, a contradiction. Therefore the search for a conjugator ℓ is constrained to $\ell \in N^{t-1}$.

As in Example 5.8 we have

Lemma 5.9. *Let ${}_GB_G$ be a contracting biset; choose a basis X of B, and let $N \subseteq G$ be the associated nucleus. Suppose that $t \in \mathbb{N}$ is such that $N^{t-3}X \supseteq XN^t$.*

If two symbolic orbits $(b_i)_{i\in I}$ and $(c_i)_{i\in I}$ with $b_i, c_i \in NX$ are conjugate by $(g_i)_{i\in I}$, then $g_i \in N^{t-1}$ for all $i \in I$.

Proof. Suppose $g_i \in N^u \setminus N^{u-1}$ for some $u \geq t$ and $i \in I$; write $b_i = n_i x_i$ with $n_i \in N$ and $x_i \in X$. We can also assume that $g_j \in N^u$ for all $j \in I$. Then $g_i b_i = g_i n_i x_i \notin N^{u-2}X$: if $g_i n_i$ belonged to N^{u-2}, we would have $g_i \in N^{u-2}n_i^{-1} \subseteq N^{u-1}$. On the other hand, $g_i b_i = c_i g_{f_*(i)} \in NXN^u \subseteq NN^{u-3}X = N^{u-2}X$. This is a contradiction. □

Algorithm 5.10.

GIVEN a minimal {Exp} orbisphere biset ${}_GB_G$, an extension $f_*\colon \widetilde{A} \to \widetilde{A}$ of the dynamics of B on its peripheral classes, and two portraits of bisets $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$,

DECIDE whether $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$ are conjugate, and COMPUTE the centralizer of $(G_a, B_a)_{a\in\widetilde{A}}$, which is a finite group,

AS FOLLOWS.

(1) Write $\widetilde{A} = A \sqcup J \sqcup I$ with $f_*^n(J) \subseteq A$ and $f_*(I) \subseteq I$. Normalize $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G'_a, B'_a)_{a\in\widetilde{A}}$ such that $G_a = G'_a$ and $B_a = B'_a$; by Lemma 2.17 this follows from the conjugacy for subgroups: find $(\ell_a)_{a\in A} \in G^A$ with $G_a^{\ell_a} = G'_a$ and conjugate $(G_a, B_a)_{a\in\widetilde{A}}$ by $(\ell_a)_{a\in\widetilde{A}}$ with $\ell_a = 1$ if $a \notin A$.

(2) Check whether there is $(\ell_a)_{a\in A\sqcup J}$ with $\ell_a = 1$ for $a \in A$ conjugating $(G_a, B_a)_{a\in A\sqcup J}$ and $(G'_a, B'_a)_{a\in A\sqcup J}$. If not, return no. This reduces the conjugacy problem of portraits to the conjugacy problem of symbolic orbits: writing $B_i = \{b_i\}$ and $B'_i = \{b'_i\}$ for $i \in I$ solve a conjugacy problem of $(b_i)_{i\in I}$ and $(b'_i)_{i\in I}$.

(3) Write the biset B in the form GX for a basis X, and let N be its nucleus. Find, by direct search, a $t \in \mathbb{N}$ with $N^{t-3}X \supseteq XN^t$; such a $t \geq 4$ exists because B is contracting.

(4) Write $b_i = g_i x_i$ and replace b_i with $x_i g_i = g'_i x'_i$. After iterating finitely many times this process, we obtain $b_i \in NX$ by Lemma 4.39. By a similar iteration, we can assume $b'_i \in NX$.

(5) Answer whether $(b_i)_{i\in I}$ and $(b'_i)_{i\in I}$ are conjugate by elements in N^{t-1}. This is correct by Lemma 5.9.

(6) By a direct search compute the centralizer of $(G_a, B_a)_{a\in\tilde{A}}$: the centralizer of $(G_a, B_a)_{a\in A\sqcup J}$ is trivial, while elements centralizing $(G_a, B_a)_{a\in I}$ are within N^{t-1}.

Algorithm 5.11.

GIVEN a minimal {Exp} orbisphere biset ${}_G B_G$ and an extension $f_*\colon \tilde{A} \to \tilde{A}$ of the dynamics of B on its peripheral classes,

PRODUCE A LIST of all conjugacy classes of portraits of bisets $(G_a, B_a)_{a\in\tilde{A}}$ in B with dynamics f_*

AS FOLLOWS. (1) Write $\tilde{A} = A \sqcup J \sqcup I$ with $f_*^n(J) \subseteq A$ and $f_*(I) \subseteq I$. Produce a list $\mathcal{L}_{A\sqcup J}$ of all conjugacy classes of portraits of bisets $(G_a, B_a)_{a\in A\sqcup J}$.

(2) Write the biset B in the form GX for a basis X, and let N be its nucleus. Produce a list of all symbolic orbits $(b_i)_{i\in I}$ with $b_i \in NX$. Using Algorithm 5.10 produce a list $\mathcal{L}_I$ of all conjugacy classes of symbolic orbits $(b_i)_{i\in I}$ with $b_i \in NX$.

(3) Combine $\mathcal{L}_{A\sqcup J}$ and $\mathcal{L}_I$ to produce a list $\mathcal{L}_{A\sqcup J} \times \mathcal{L}_I$ of all conjugacy classes of portraits of bisets $(G_a, B_a)_{a\in\tilde{A}}$ by setting $B_i := \{b_i\}$ and $G_i := \{1\}$ for $i \in I$.

5.3. Decidability of conjugacy and centralizer problems. The statements of Theorem E can be turned into an algorithm:

Algorithm 5.12.

GIVEN ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}}$, ${}_{\widetilde{G}}\widetilde{C}_{\widetilde{G}}$ two geometric bisets,

AND GIVEN an oracle that decides whether two minimal {Exp}\{GTor/2} bisets are conjugate, and computes their centralizers,

DECIDE whether $\widetilde{B}$ and $\widetilde{C}$ are conjugate by an element of $\mathbf{Mod}(\widetilde{G})$, and if so CONSTRUCT a conjugator; and

COMPUTE the centralizer $Z(\widetilde{B})$,

AS FOLLOWS.

(1) If $\widetilde{B}_* \neq \widetilde{C}_*$ as maps on peripheral conjugacy classes, then return no.

(2) Let ${}_{\widetilde{G}}\widetilde{B}_{\widetilde{G}} \to {}_{G}B_{G}$ and ${}_{\widetilde{G}}\widetilde{C}_{\widetilde{G}} \to {}_{G'}C_{G'}$ be the maximal forgetful morphisms. If $G \neq G'$, then return no.

(3) From now on assume $G = G'$. Using Algorithm 5.4 or the oracle, decide whether B, C are conjugate; if not, return no. Using Algorithm 5.4 or the oracle compute the centralizer $Z(B)$.

(4) From now on, assume $B^\phi = C$ for some $\phi \in \mathbf{Mod}(G)$. Compute the action of $Z(B)$ on the list of all conjugacy classes of portraits of bisets (this list is finite by Corollary 4.21; Algorithms 5.7 and 5.11 compute the list) and check whether $(G_a, B_a)_{a\in\widetilde{A}}$ and $(G_a^{\phi^{-1}}, C_a^{\phi^{-1}})_{a\in\widetilde{A}}$ belong to the same orbit. Return no if the answer is negative. Otherwise replace ϕ by an element in $Z(B)\phi$ such that $(G_a^\phi, B_a^\phi)_{a\in\widetilde{A}}$ and $(G_a, C_a)_{a\in\widetilde{A}}$ are conjugate portraits.

(5) Choose an arbitrary lift $\tilde{\phi} \in \mathbf{Mod}(\widetilde{G})$ of ϕ, and compute $k \in \mathbf{Mod}(\widetilde{G})$ such that $k\widetilde{B}^{\tilde{\phi}} \cong \widetilde{C}$. It follows that k is a knitting element. Compute inductively $k^{(n)}$ with

$$k^{(n)}\widetilde{B}^{\tilde{\phi}k'k''\cdots k^{(n-1)}} \cong \widetilde{C};$$

proceed until $k^{(n)}$ is the identity (guaranteed to happen by Lemma 4.27). Return $\tilde{\phi}k'k''\cdots k^{(n-1)}$.

(6) To compute the centralizer of $\widetilde{B}$, consider first $Z(B)$. Compute the action of $Z(B)$ on the list of all conjugacy classes of portraits of bisets (this list is finite by Corollary 4.21; Algorithms 5.7 and 5.11 compute the list) and compute the stabilizer $Z_0(B)$ of the $Z(B)$-action. Note that $Z_0(B)$ is a finite-index subgroup of $Z(B)$.

(7) List a generating set S of $Z_0(B)$. For every $\phi \in S$ compute its lift $\hat{\phi} \in \mathbf{Mod}(\widetilde{G})$ with $\widetilde{B}^{\hat{\psi}} \cong \widetilde{B}$: compute first an arbitrary lift $\tilde{\phi}$ of ϕ, then inductively define $k^{(n)}$ with

$$k^{(n)} \widetilde{B}^{\tilde{\phi}k'k''\cdots k^{(n-1)}} \cong \widetilde{B}$$

until $k^{(n)}$ is identity, and set $\hat{\phi} := \tilde{\phi}k'k''\cdots k^{(n-1)}$. Return $\{\hat{\phi} \mid \phi \in S\}$.

Proof of Corollary F. We claim that Algorithms 5.5, 5.6, 5.7, 5.10, 5.11 are efficient. Indeed, the efficiency of Step 4 of Algorithm 5.10 follows from expansion. The efficiency of Steps 5 and 7 of Algorithm 5.12 follows from Lemma 4.27. All the remaining steps are obviously efficient. □

We note that, in the case of $(2,2,2,2)$ bisets, the oracle itself is efficient. Indeed the oracle reduces to solving conjugacy and centralizer problems in the group $\mathbf{SL}_2(\mathbb{Z})$, and Theorem 5.3 is efficient, because $\mathbf{SL}_2(\mathbb{Z})$ has a free subgroup of index 12.

Corollary 5.13. *There is an algorithm that, given two geometric bisets ${}_GB_G$ and ${}_HC_H$, decides whether B and C are conjugate, and computes the centralizer of B.*

Proof. We briefly sketch an algorithm that justifies an existence of an oracle: given two minimal {Exp} orbisphere bisets, whether they are conjugate and computes their centralizers; details will appear in [6].

Let B, C be two minimal {Exp} orbisphere G-G-bisets. They admit a decomposition into rational maps along the *canonical obstruction*, which is computable by [25]. The graph of bisets along this decomposition is computable, and rational maps may be computed for each of the small bisets in the decomposition, e.g. by giving their coëfficients as algebraic numbers with floating-point enclosures to distinguish them from their Galois conjugates. The bisets B, C are conjugate precisely when their respective rational maps are conjugate and the twists along the canonical obstruction agree; the first condition amounts to finite calculations with algebraic numbers, while the second is the topic of [5, Theorem B].

The centralizer of a rational map is trivial, and [5, Theorem B] shows that the centralizer of B is computable. □

Proof of Corollary G. If the rational map is $(2,2,2,2)$, then the oracle is efficient, as we noted above, so Corollary F applies. In the other case, the rational map is hyperbolic, so it has trivial centralizer and no oracle is needed in the application of Algorithm 5.12. □

6. Examples

Finally, in this brief section, we consider some examples of portraits of bisets. The first ones come from marking points on the Basilica map $f(z) = z^2 - 1$, and more generally maps with three post-critical points. The second ones from cyclic bisets (which are particularly simple, but to which our main results apply with restrictions because these bisets have automorphisms).

6.1. Twisted marked Basilica. Consider the Basilica polynomial

$$f(z) := z^2 - 1 \colon (\widehat{\mathbb{C}}, \{0, -1, \infty\}) \circlearrowleft.$$

It has two fixed points α and β, with $\alpha \in (-1, 0)$ and $\beta > 1$. Let us take α to be the basepoint and let ${}_G B_G$ be the biset of f. Denote respectively by γ_{-1} and γ_0 the loops circling around -1 and 0 as in Figure 1, and let $\gamma_\infty = (\gamma_{-1}\gamma_0)^{-1}$ be the loop around infinity. Then

$$G = \langle \gamma_{-1}, \gamma_0, \gamma_\infty \mid \gamma_\infty \gamma_{-1} \gamma_0 \rangle.$$

The basepoint α has two preimages α and $-\alpha$. Let x_1 be the constant path at α and let x_2 be a path slightly below 0 connecting α to $-\alpha$. The presentation of B in the basis $S := \{x_1, x_2\}$ is

$$\begin{aligned} \gamma_{-1} &= \langle\!\langle 1, \gamma_0 \rangle\!\rangle (1,2), \\ \gamma_0 &= \langle\!\langle \gamma_{-1}, 1 \rangle\!\rangle, \\ \gamma_\infty &= \langle\!\langle \gamma_{-1}^{-1}\gamma_0^{-1}, 1 \rangle\!\rangle (1,2). \end{aligned}$$

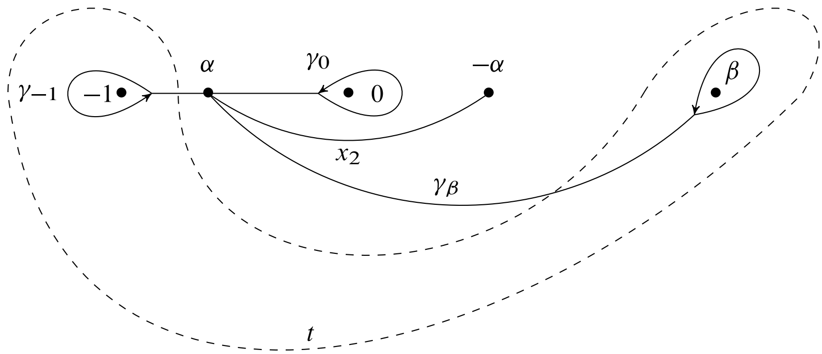


Figure 1. The dynamical plane of $z^2 - 1$. Loops $\gamma_{-1}, \gamma_0, \gamma_\beta$ circle around $0, -1, \beta$ respectively. The curve x_2 connects α to its preimage $-\alpha$. The simple closed curve t surrounds $\{-1, \beta\}$.

We shall consider first the effect of marking a fixed point, and then of marking some preimages of the post-critical set.

6.1.1. Marking β. Consider a marked Basilica

$$\tilde{f}(z) = z^2 - 1\colon (\widehat{\mathbb{C}}, \{\infty, 0, -1, \beta\}) \circlearrowleft.$$

Let γ_β be the loop around β depicted in Figure 1. The fundamental group of $(\widehat{\mathbb{C}}, \{\infty, 0, -1, \beta\})$, based at α, is

$$\widetilde{G} = \langle \gamma_{-1}, \gamma_0, \gamma_\beta, \gamma_\infty \mid \gamma_\infty \gamma_{-1} \gamma_\beta \gamma_0 \rangle$$

and the forgetful map $\widetilde{G} \twoheadrightarrow G$ sends γ_β to 1. The presentation of the biset $\widetilde{B} = B(\tilde{f})$ in the basis S is

$$\begin{aligned} \gamma_{-1} &= \langle\!\langle 1, \gamma_0 \rangle\!\rangle (1,2), \\ \gamma_\beta &= \langle\!\langle 1, \gamma_\beta \rangle\!\rangle, \\ \gamma_0 &= \langle\!\langle \gamma_{-1}, 1 \rangle\!\rangle, \\ \gamma_\infty &= \langle\!\langle \gamma_{-1}^{-1} \gamma_0^{-1}, \gamma_\beta^{-1} \rangle\!\rangle (1,2). \end{aligned}$$

Write $A = \{\infty, -1, 0\}$ and $\widetilde{A} = \{\infty, -1, 0, \beta\}$; then $\widetilde{B}$ has a minimal portrait of bisets

$$\begin{aligned} \widetilde{G}_{-1} &= \langle \gamma_{-1} \rangle, & \widetilde{B}_{-1} &= \widetilde{G}_{-1} x_1, \\ \widetilde{G}_\beta &= \langle \gamma_\beta \rangle, & \widetilde{B}_\beta &= \widetilde{G}_{-1} x_2, \\ \widetilde{G}_0 &= \langle \gamma_0 \rangle, & \widetilde{B}_0 &= \widetilde{G}_0 x_1 \sqcup \widetilde{G}_0 x_2 = x_1 \widetilde{G}_{-1}, \\ \widetilde{G}_\infty &= \langle \gamma_\infty \rangle, & \widetilde{B}_\infty &= \widetilde{G}_\infty (\gamma_{-1} x_1) \sqcup \widetilde{G}_\infty (\gamma_{-1} x_2 \gamma_\beta). \end{aligned}$$

Under the forgetful intertwiner $\widetilde{B} \twoheadrightarrow B$ the portrait $(\widetilde{G}_a, B_a)_{a \in \widetilde{A}}$ projects to

$$\begin{aligned} G_{-1} &= \langle \gamma_{-1} \rangle, & B_{-1} &= G_{-1} x_1, \\ G_\beta &= 1, & B_\beta &= \{x_2\}, \\ G_0 &= \langle \gamma_0 \rangle, & B_0 &= G_0 x_1 \sqcup G_0 x_2 = x_1 G_{-1}, \\ G_\infty &= \langle \gamma_\infty \rangle, & B_\infty &= G_\infty (\gamma_{-1} x_1) \sqcup G_\infty (\gamma_{-1} x_2). \end{aligned}$$

Note that $B_\beta = \{x_2\}$ encodes β in the sense that x_2 shadows β. This can be seen directly as follows: set $x_2^0 = x_2$ and let x_2^{i+1} be the f-lift of x_2^i that starts at $x_2^i(1)$. Then the sequence of endpoints $x_2^i(1)$ converges to β, and the infinite concatenation $x_2^0 \# x_2^1 \# \cdots$ is a path from α to β.

Let now T be the clockwise Dehn twist around the simple closed curve t surrounding an interval 0 and β as in Figure 1. The action of T on $\widetilde{G}$ is given by

$$T_*\colon \gamma_{-1} \longrightarrow \gamma_{-1}^{(\gamma_{-1}\gamma_\beta)^{-1}}, \quad \gamma_\beta \longrightarrow \gamma_\beta^{(\gamma_{-1}\gamma_\beta)^{-1}}, \quad \gamma_0 \longrightarrow \gamma_0, \gamma_\infty \to \gamma_\infty.$$

The biset $\widetilde{C}$ of $g := T \circ f$ is $\widetilde{B} \otimes B_{T_*}$. Let us show that g is conjugate to $z^2 - 1 \colon (\widehat{\mathbb{C}}, \{\infty, -1, 0, \alpha\}) \circlearrowleft$. By Theorem 2.19,

$$(\widetilde{C}_a, \widetilde{G}_a)_{a \in \widetilde{A}} = (\widetilde{B}_a, \widetilde{G}_a)_{a \in \widetilde{A}} \mathcal{E}(T).$$

Note that we have $C_a = B_a$ for all $a \in A$, so it remains to compute C_β.

Let $\bar{t} \in \pi_1(\mathbb{C} \setminus \{0, -1\}, \beta)$ be the simple loop below 0 circling around -1; then $T = \operatorname{push}(\bar{t})$. Let $\ell_\beta \colon [0,1] \to \mathbb{C} \setminus \{-1, 0\}$ (using notations of Lemma 2.16) be an arc from α to β slightly below 0 so that γ_β may be homotoped to a small neighborhood of ℓ_β. We have $C_\beta = B_\beta \gamma_{-1}^{-1}$; the claim then follows from $x_2 \gamma_{-1}^{-1} = x_1$ and the fact that x_1 shadows α.

More generally, suppose that $\tilde{g} = \operatorname{push}(\bar{s}) \tilde{f} \operatorname{push}(\bar{t})$ for some motions $\bar{s}, \bar{t} \in \pi_1(\mathbb{C} \setminus \{0, -1\}, \beta)$ of β; then

$$C_\beta = (\ell_\beta \# \bar{s} \# \ell_\beta^{-1}) B_\beta (\ell_\beta \# \bar{t} \# \ell_\beta^{-1}).$$

The process

$$(\ell_\beta \# \bar{s} \# \ell_\beta^{-1}) x_2 (\ell_\beta \# \bar{t} \# \ell_\beta^{-1}) = g_1 x_{i(1)} \sim x_{i(1)} g_1 = g_2 x_{i(2)} \sim x_{i(2)} g_2 = \dots$$

eventually terminates in either x_1 or x_2.

In the former case, $\tilde{g}$ is conjugate to $z^2 - 1 \colon (\widehat{\mathbb{C}}, \{\infty, 0, -1, \alpha\}) \circlearrowleft$. In the latter case, $\tilde{g}$ is conjugate to $z^2 - 1 \colon (\widehat{\mathbb{C}}, \{\infty, 0, -1, \beta\}) \circlearrowleft$.

6.1.2. Marking 1 and $\sqrt{2}$. Consider now

$$\tilde{f}(z) = z^2 - 1 \colon (\widehat{\mathbb{C}}, \{\infty, -1, 0, 1, \sqrt{2}\}) \circlearrowleft$$

with $\widetilde{A} \setminus A = \{1, \sqrt{2}\}$ and $A = \{\infty, -1, 0\}$. The dynamics are $\sqrt{2} \mapsto 1 \mapsto 0 \leftrightarrow -1$. As in §6.1.1 we readily compute a presentation of $\widetilde{B} = B(\tilde{f})$ in the basis S:

$$\begin{aligned}
\gamma_{-1} &= \langle\!\langle 1, \gamma_0 \rangle\!\rangle (1,2),\\
\gamma_{\sqrt{2}} &= \langle\!\langle 1, 1 \rangle\!\rangle,\\
\gamma_1 &= \langle\!\langle 1, \gamma_{\sqrt{2}} \rangle\!\rangle,\\
\gamma_0 &= \langle\!\langle \gamma_{-1}, \gamma_1 \rangle\!\rangle,\\
\gamma_\infty &= \langle\!\langle \gamma_{-1}^{-1} \gamma_0^{-1}, \gamma_1^{-1} \gamma_{\sqrt{2}}^{-1} \rangle\!\rangle (1,2),
\end{aligned}$$

with $\widetilde{G} = \langle \gamma_\infty, \gamma_{-1}, \gamma_{\sqrt{2}}, \gamma_1, \gamma_0 \mid \gamma_\infty \gamma_{-1} \gamma_{\sqrt{2}} \gamma_1 \gamma_0 \rangle$.

As in Lemma 2.16, let ℓ_{-1} and $\ell_0 \subset \mathbb{R}$ respectively be simple arcs from α to -1 and to 0 so that γ_{-1} and γ_0 may be homotoped to small neighborhoods of γ_{-1} and γ_0 respectively. Let ℓ_1 and $\ell_{\sqrt{2}}$ be simple arcs from α to 1 and $\sqrt{2}$ slightly below 0 as in Figure 2.

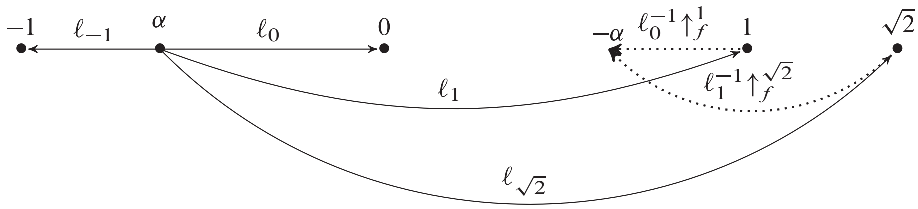


Figure 2. Curves $\ell_{-1}, \ell_0, \ell_1, \ell_{\sqrt{2}}$ and their lifts.

Denote by $(G_a, B_a)_{a\in\tilde{A}}$ the portrait of bisets in B induced by $\widetilde{B} = B(\tilde{f})$. Then G_a and B_a are the same as in §6.1.1 for all $a \in A$, while (by Lemma 2.18)

$$B_1 = \{\ell_1 \# \ell_0^{-1}\uparrow_f^1\} = \{x_2\} \quad \text{and} \quad B_{\sqrt{2}} = \{\ell_{\sqrt{2}} \# \ell_1^{-1}\uparrow_f^{\sqrt{2}}\} = \{x_2\}.$$

We consider again some twists of $\tilde{f}$. Consider $\tilde{g} = m_1 \tilde{f} m_2$ with m_1, m_2 trivial rel A. Suppose that m_1 moves 1 and $\sqrt{2}$ along $s_1 \in \pi_1(\mathbb{C} \setminus \{-1,0\}, 1)$ and $s_{\sqrt{2}} \in \pi_1(\mathbb{C} \setminus \{-1,0\}, \sqrt{2})$ respectively while m_2 moves 1 and $\sqrt{2}$ along $t_1 \in \pi_1(\mathbb{C} \setminus \{-1,0\}, 1)$ and $t_{\sqrt{2}} \in \pi_1(\mathbb{C} \setminus \{-1,0\}, \sqrt{2})$ respectively. If $(G_a, C_a)_{a\in\tilde{A}}$ is the portrait of bisets in B induced by $B(\tilde{g})$, then

$$C_1 = \{(\ell_1 \# s_1 \# \ell_1^{-1})x_2\},$$
$$C_{\sqrt{2}} = \{(\ell_{\sqrt{2}} \# s_{\sqrt{2}} \# \ell_1^{-1})x_2(\ell_{\sqrt{2}} \# t_1\uparrow_f^{\sqrt{2}} \# \ell_{\sqrt{2}}^{-1})\}.$$

Write $C_1 = \{h_1 x_2\}$ and $C_{\sqrt{2}} = \{h_2 x_j\}$. We may conjugate (C_1, C_2) to $(\{x_2\}, \{x_j h'\})$, and write $x_j h' = h'' x_k$.

If $k = 1$, then $\tilde{g}$ is conjugate to $z^2 - 1\colon (\widehat{\mathbb{C}}, \{\infty, -1, 0, 1, \sqrt{2}\}) \circlearrowleft$. Otherwise, $\tilde{g}$ is conjugate to $z^2 - 1\colon (\widehat{\mathbb{C}}, \{\infty, -1, 0, 1, -\sqrt{2}\}) \circlearrowleft$.

6.1.3. Mapping class bisets. We continue the discussion from §6.1.2, and consider the related mapping class bisets. The biset $M(B)$ is of course reduced to $\{B\}$, with $\mathbf{Mod}(G) = 1$.

On the other hand, the biset $M(\widetilde{B})$ is a left-free $\mathbf{Mod}(\widetilde{G})$-biset of degree 2. This can be seen in various ways: analytically, the point $\sqrt{2}$ may be moved to $-\sqrt{2}$, and a basis of $M(\widetilde{B})$ may be chosen as $\{(z^2 - 1, \{\infty, -1, 0, 1, \sqrt{2}\}), (z^2 - 1, \{\infty, -1, 0, 1, -\sqrt{2}\})\}$. More symbolically, the action of exchanging $\sqrt{2}$ with $-\sqrt{2}$ amounts to changing, in the wreath recursion of $\tilde{f}$, the entry "$\gamma_1 = \langle\langle 1, \gamma_{\sqrt{2}} \rangle\rangle$" into "$\gamma_1 = \langle\langle \gamma_{\sqrt{2}}, 1 \rangle\rangle$".

Note that the biset $M(\widetilde{B})$ is nevertheless connected, and $M(\widetilde{B}) = M^*(\widetilde{B})$: indeed the right action by the mapping class that pushes 1 once along the circle $\{|z| = 1\}$ has the effect of exchanging the two left orbits.

Example 6.1. Let us now consider the sets $A = \{\infty, -1, 0, 1\}$ and $D = \{\sqrt{2}\}$, still with the map $f(z) = z^2 - 1\colon (\widehat{\mathbb{C}}, A) \circlearrowleft$ and $\tilde{f} = f\colon (\widehat{\mathbb{C}}, A \sqcup D) \circlearrowleft$. Then the fibres of the map $\mathcal{F}_{D,D}\colon M^*(\tilde{f}) = M(\tilde{f}) \to M(f)$ are not connected (by the second claim of Proposition 2.21).

Indeed the left action of $\mathbf{Mod}(\widehat{\mathbb{C}} \setminus A, D)$ has two orbits, while the right action does not identify these orbits since 1 is not allowed to move around a critical value.

6.2. Belyi maps. The Basilica map $f(z) = z^2 - 1$ is an example of dynamical *Belyi map*, namely a map whose post-critical set consists of 3 points. All such maps are realizable as holomorphic maps, and the three points may be normalized as $\{0, 1, \infty\}$.

In this subsection, we briefly state how the main results of this article simplify considerably. We concentrate on the dynamical situation, namely a map $f\colon (S^2, A) \circlearrowleft$ covered by $\tilde{f}\colon (S^2, A \sqcup D) \circlearrowleft$, with $\#A = 3$.

In that case, $\mathbf{Mod}(S^2, A) = 1$, and we have a short exact sequence

$$1 \longrightarrow \mathbf{knBraid}(S^2, A \sqcup D) \longrightarrow \mathbf{Mod}(S^2, A \sqcup D) \longrightarrow \pi_1(S^2 \setminus A, D) \longrightarrow 1.$$

Assume first that D consists only of $\tilde{f}$-fixed points. Then the extension of bisets decomposition of $M(\tilde{f})$ from Theorem B reduces to the statement that $M(\tilde{f})$ is an inert extension of $B(f)^D$. The case of D consisting of a single fixed point was considered in [5, §9.2].

More concretely: if we are given a wreath recursion $G \to G \wr S\!\downarrow$ for $B(f)$, with $G = \pi_1(S^2 \setminus A, *)$, then a generating set for $\mathbf{Mod}(S^2, A \sqcup D)$ can be chosen to consist of $\#D$ copies of a generating set of G, corresponding to point pushes of D in $S^2 \setminus A$, together with some additional knitting elements. A wreath recursion for $M(\tilde{f})$ will then be of the form $\mathbf{Mod}(S^2, A \sqcup D) \to \mathbf{Mod}(S^2, A \sqcup D) \wr (S^D)\!\downarrow$, consisting of D parallel copies of the wreath recursion of $B(f)$. The wreath recursion associated with the knitting elements is trivial.

In case D consists of periodic points of period > 1, then the actions of G on the left and the right should be appropriately permuted. If D contains n_i cycles of period i, then abstractly $M(\tilde{f})$ will consist in the direct product of n_i copies of $B(f^i) = B(f)^{\otimes i}$.

Example 6.2. Consider the map $\tilde{f}(z) = f(z) = z^3$ with $A = \{0, 1, \infty\}$ and $D = \{\omega = \exp(2\pi i/3)\}$. The biset $M(f)$ is of course reduced to $\{f\}$, while $M(\tilde{f}) = \mathbf{Mod}(\widehat{\mathbb{C}} \setminus A, D) \cong \pi_1(\widehat{\mathbb{C}} \setminus A)$ with trivial right action. Indeed D is not in the image of f, so pushing ω has no effect.

However, $M^*(\tilde{f})$ has two left $\mathbf{Mod}(\widehat{\mathbb{C}} \setminus A, D)$-orbits by Lemma 2.20. Representatives may be chosen as $\{(z^3, A \sqcup \{\omega\}), (z^3, A \sqcup \{\omega^2\})\}$, or equivalently (if the marked set is to remain $A \sqcup D$) as the maps $\{z^3, z^3 \circ m\}$ for a homeomorphism $m\colon (\widehat{\mathbb{C}}, A) \circlearrowleft$ that pushes ω to ω^2.

6.3. Cyclic bisets. We finally consider the easy case of a cyclic biset, namely the biset ${}_G B_G$ of a monomial map $f(z) = z^d : (\widehat{\mathbb{C}}, \{0, \infty\}) \circlearrowleft$ for some $d \in \mathbb{Z} \setminus \{-1, 0, 1\}$. Then $G \cong \mathbb{Z}$ and B is a left-free right-principal biset. Choosing $b_0 \in B$ we can identify $B = b_0 G$ with $\mathbb{Z}$ with the actions are given by

$$m \cdot b \cdot n = dm + b + n.$$

Observe first that $b \to b + k$ is an automorphism of B for all $k \in \mathbb{Z}$. Therefore, contrarily Proposition A we have $\operatorname{Aut}(B) \cong \mathbb{Z}$.

Suppose that $(G_a, B_a)_{a \in \widetilde{A}}$ is a portrait of bisets in B. Then $\operatorname{Aut}(B)$ acts on portraits by

$$(G_a, B_a)_{a \in \widetilde{A}} + k := (G_a, B_a + k)_{a \in \widetilde{A}}.$$

We denote by $(G_a, B_a)_{a \in \widetilde{A}} / \operatorname{Aut}(B)$ the orbit of this action. We can now adjust Theorem C for cyclic bisets:

Lemma 6.3. *Let $\widetilde{G} \twoheadrightarrow G$ be a forgetful morphism of groups with $G \cong \mathbb{Z}$, and let ${}_G B_G$ be the biset of $z^d : (\widehat{\mathbb{C}}, \{0, \infty\}) \circlearrowleft$.*

There is then a bijection between, on the one hand, conjugacy classes of portraits of bisets $(B_a)_{a \in \widetilde{A}}$ in B considered up to the action of $\operatorname{Aut}(B)$ and, on the other hand, $\widetilde{G}$-$\widetilde{G}$-bisets projecting to B under $\widetilde{G} \twoheadrightarrow G$ considered up to composition with the biset of a knitting element. This bijection maps every minimal portrait of bisets of $\widetilde{B}$ to $(B_a)_{a \in \widetilde{A}} / \operatorname{Aut}(B)$.

Proof. Mark an extra fixed point in $(\widehat{\mathbb{C}}, \{0, \infty\})$ so as to remove automorphisms, and apply Theorem C. □

To illustrate Lemma 6.3, consider $\tilde{f}(z) = z^d : (\widehat{\mathbb{C}}, \{0, \infty\} \cup D) \circlearrowleft$ and let $(G_a, B_a)_{a \in \widetilde{A}}$ be the induced portrait of bisets on $(\widehat{\mathbb{C}}, \{0, \infty\})$. Let us also assume that $d > 1$. For $a \in D$ write $B_a = \{b_a\}$ with $b_a \in B \cong \mathbb{Z}$.

Let us consider a twisted map $\tilde{g} = m \tilde{f} n$. For $a \in D$ let m_a and n_a be the number of times m and n push a around 0. If $(G_a, C_a)_{a \in \widetilde{A}}$ is the induced portrait of bisets, then $C_a = \{d m_a + b_d + n_{f_*(a)}\} = \{c_a\}$ for $a \in A$. For $a \in D$ set

$$x_a := c_a / d + c_{f_*(a)} / d^2 + c_{f_*^2(a)} / d^3 + \cdots \mod \mathbb{Z} \in \mathbb{R}/\mathbb{Z}.$$

Then $(C_a)_{a \in D}$ shadows $(x_a)_{a \in D}$. The map $\tilde{g}$ is unobstructed if and only if all points x_a are pairwise different. If $\tilde{g}$ is unobstructed, then $(x_a)_{d \in D}$ considered up to the action

$$(x_a)_{a \in D} \longrightarrow (x_a + k)_{a \in D}, k \in \mathbb{Z}/(d-1)\mathbb{Z}$$

is a complete conjugacy invariant of $\tilde{g}$.

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