Groups Geom. Dyn. 15 (2021), 1363–1398
DOI 10.4171/GGD/633

Cohomology of hyperfinite Borel actions

Sergey Bezuglyi and Shrey Sanadhya

Abstract. We study cocycles of countable groups Γ of Borel automorphisms of a standard Borel space $(X, \mathcal{B})$ taking values in a locally compact second countable group G. We prove that for a hyperfinite group Γ the subgroup of coboundaries is dense in the group of cocycles. We describe all Borel cocycles of the 2-odometer and show that any such cocycle is cohomologous to a cocycle with values in a countable dense subgroup H of G. We also provide a Borel version of Gottschalk–Hedlund theorem.

Mathematics Subject Classification (2020). 37A40, 37A99, 37B05, 37B99, 54H05.

Keywords. Borel automorphism, cocycle, coboundary, hyperfinite countable Borel equivalence relation, odometer.

1. Introduction

Let Γ be a countable group of Borel automorphisms of a standard Borel space $(X, \mathcal{B})$ and G be an abelian locally compact second countable (l.c.s.c.) group. A Borel map $\alpha\colon \Gamma \times X \to G$ is called a *cocycle* if it satisfies the so called *cocycle identity* for all (γ, x):

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 x) + \alpha(\gamma_2, x), \quad \alpha(\mathbf{1}, x) = 0, \tag{1.1}$$

where $\mathbf{1}$ is the identity map and $0 \in G$. A cocycle $\alpha(\gamma, x)$ is called a *coboundary* if there exists a Borel function $f\colon X \to G$ such that $\alpha(\gamma, x) = f(\gamma x) - f(x)$. Two cocycles, α and β, are *cohomologous* if $\alpha - \beta$ is a coboundary. The set $Z^1(\Gamma \times X, G)$ of all cocycles is an abelian group, and the coboundaries form a subgroup $B^1(\Gamma \times X, G)$ of $Z^1(\Gamma \times X, G)$.

Cocycles play an important role in ergodic theory. They are studied up to null sets: if Γ is a countable group of non-singular automorphisms of a standard measure space $(X, \mathcal{B}, \mu)$, then relation (1.1) must be true μ-a.e. Cocycles are widely used in the theory of orbit equivalence of dynamical systems and in various constructions (e.g., skew product) helping to classify dynamical systems and clarify the properties of automorphism groups of a measure space. They are also one of the central tools in the representation theory, theory of groupoids, classification of ergodic actions of amenable and non-amenable groups, etc. Understanding the structure of cocycles for a *hyperfinite automorphism group* (it is a group which is orbit equivalent to a single transformation) led to a more detailed classification than orbit equivalence [5, 12, 14, 15, 18]. We give here several principal references which include some pioneering works of Moore [26], Ramsay [30, 11], Schmidt [31, 32], and Zimmer [37]. (A more detailed list of papers devoted to cocycles is too long to mention all crucial contributions to the theory of cocycles.)

It is well known that there are impressive parallels between ergodic theory and Borel dynamics, though the fact that, for a Borel dynamical system, there is no prescribed measure on the underlying space makes these two theories essentially different. In this paper, we prove several results about cocycles in the context of Borel dynamics. They are motivated by the existing counterparts in the framework of ergodic theory. There are many important problems in dynamics involving Borel cocycles that deserve to be studied. For example, it would be interesting to find out whether the notion of a ratio set makes sense for Borel cocycles taking values in l.c.s.c. (abelian) groups. Another application of cocycles might be related to the study of a Borel version of Mackey range. These concepts are extremely important for the classification of automorphism groups in ergodic theory. Remark that these and other results in ergodic theory remain true for nonabelian groups, in general. In the case of Borel cocycles, even abelian case is not well understood. We hope to contribute to the formulated problems in further works. In this paper, we focused on cocycles with values in abelian groups. It is worth mentioning that various properties of Borel cocycles were considered in the papers [1, 7, 9, 11, 24, 25], and some others.

We fix the main setting for the paper: Γ is a hyperfinite free countable group of Borel automorphisms on a standard Borel space $(X, \mathcal{B})$ and $\alpha \in Z^1(\Gamma \times X, G)$ is a cocycle of Γ with values in an abelian l.c.s.c. group G. In this setting, the following results are proved: (i) we introduce a topology on the space of Borel functions (which is an analogue of the convergence in measure topology) and prove that the set of coboundaries is dense in the set of all cocycles; (ii) using an exact formula that describes cocycles over an odometer, we prove that every cocycle is cohomologous to a cocycle with values in a dense countable subgroup; (iii) we give a criterion (a version of Gottschalk–Hedlund theorem) for a cocycle with values in G to be a coboundary.

The study of Borel cocycles is mostly motivated by the theory of orbit equivalence of groups of Borel automorphisms. The property of orbit equivalence for groups of Borel automorphisms is equivalent to isomorphism of the corresponding equivalence relations generated by orbits. The notion of a countable Borel equivalence relation (CBER) has been extensively studied in the descriptive set theory and Borel dynamics. This concept has many applications in other adjacent areas. We refer to [2, 10, 19, 20, 21, 23, 27, 35, 36], where the reader can find connections of orbit equivalence theory with the descriptive set theory and further references.

Our main results about cocycles are of the following nature. Firstly, it is not hard to see that orbit equivalent groups of Borel automorphisms have isomorphic groups of cocycles and coboundaries and therefore the cohomology groups. This means that the study of cocycles is naturally reduced to the case when cocycles are considered on some "model" CBERs. In this connection, two types of dynamical systems are of crucial importance: odometers and shifts. The classification of hyperfinite CBERs up to isomorphism was a significant achievement due to Dougherty, Jackson, and Kechris [10]. They proved that the complete invariant of isomorphism of hyperfinite CBERs is the cardinality of the set of invariant measures. Odometers represent CBERs with a unique probability invariant measure. They are also the main ingredient for the constructions of CBERs with finite (or countable) set of probability ergodic invariant measures. In Section 5, we give an explicit formula for cocycles of the 2-odometer. Our proof follows the approach used in [13, 14] for measurable dynamical systems.

Another key result about cocycles in ergodic theory states that coboundaries of a non-singular group of automorphisms $\Gamma \subset \mathrm{Aut}(X, \mathcal{B}, \mu)$ are dense in the group of all cocycles if Γ is hyperfinite, see, e.g., [29, 32] for a proof. Here the set $Z^1(\Gamma \times X, G)$ is endowed with the topology of convergence in measure. In Borel dynamics we do not have a prescribed measure on $(X, \mathcal{B})$. Hence, to define an analogue of the topology of convergence in measure, we have to work with all Borel probability measures. (Our approach is similar to that used in [3] where an analogue of the uniform topology on $\mathrm{Aut}(X, \mathcal{B})$ was defined). In Sections 3 and 4, we consider topological properties of $Z^1(\Gamma \times X, G)$ and prove that coboundaries are dense in $Z^1(\Gamma \times X, G)$ if Γ is hyperfinite.

Gottschalk and Hedlund (see [16, Theorem 4.11]) provided a criterion for determining when a bounded cocycle of a minimal homeomorphism of compact space is a coboundary. It was extended to minimal homeomorphisms of non-compact topological space in [6]. It is a well-know fact that every Borel automorphism admits a continuous model, i.e., it is Borel isomorphic to a homeomorphism of a Polish space. Using this model we extend the Gottschalk–Hedlund theorem to bounded Borel cocycles (taking value in an abelian l.c.s.c. group) of minimal homeomorphisms of Polish space (see Theorem 6.1).

The outline of the paper is as follows. In Section 2, we provide basic definitions and preliminary results about groups of Borel automorphisms and cocycles. In Section 3, we define a topology on the space of G-valued functions and discuss properties of this topology. We show that the group of cocycles of a hyperfinite group of automorphisms is a separable Hausdorff topological group. In Section 4, we prove the main result stating that for a hyperfinite Borel action the subgroup of coboundaries is dense in the group of cocycles with respect to the topology defined in Section 3. We study cocycles of the 2-odometer in Section 5. In Section 6, we prove the Borel version of Gottschalk–Hedlund theorem.

Notation and Terminology. Here are a few remarks about the exposition of our results in this paper. Firstly, we prefer to use the terminology which is traditional for dynamical systems in ergodic theory. This means that our principal objects are countable groups of Borel automorphisms not equivalence relations. But we also use the language of CBERs when it is convenient. Secondly, we are aware that some results can be reproved for cocycles with values in non-abelian l.c.s.c. groups, for example, those in Section 5. Meantime, we will work with abelian groups in this section for consistency. The case of non-abelian groups deserves a separate study.

Throughout the paper, we use the following notations.

- $(X, \mathcal{B})$ is a standard Borel space with the σ-algebra of Borel sets $\mathcal{B} = \mathcal{B}(X)$.
- A one-to-one Borel map T of the space $(X, \mathcal{B})$ onto itself is called a *Borel automorphism* of X. In this paper the term "automorphism" means a Borel automorphism of $(X, \mathcal{B})$.
- $\mathrm{Aut}(X, \mathcal{B})$ is the group of all Borel automorphisms of X with the identity map $\mathbb{I} \in \mathrm{Aut}(X, \mathcal{B})$.
- A countable subgroup Γ of $\mathrm{Aut}(X, \mathcal{B})$ is called a *group of Borel automorphisms*. The full group generated by Γ is denoted by $[\Gamma]$.
- $\mathcal{M}_1(X)$ is the set of all Borel probability measures on $(X, \mathcal{B})$.
- $E(S, T) = \{x \in X \mid Tx \neq Sx\} \cup \{x \in X \mid T^{-1}x \neq S^{-1}x\}$ where $S, T \in \mathrm{Aut}(X, \mathcal{B})$.

2. Preliminaries

In this section we provide the basic definitions from Borel dynamics and descriptive set theory.

2.1. Automorphisms of standard Borel space.

Let X denote a separable completely metrizable space (also known as a *Polish space*), and let $\mathcal{B}$ be the σ-algebra generated by the open sets in X. Then the pair $(X, \mathcal{B})$ is called a *standard Borel space*.

A countable subgroup Γ of $\mathrm{Aut}(X, \mathcal{B})$ is called a *Borel automorphism group*. In this paper we focus only on countable Borel automorphism groups. Let G be a countable group with identity e. A *Borel action* of the group G on $(X, \mathcal{B})$ is a group homomorphism

$$\rho: G \longrightarrow \mathrm{Aut}(X, \mathcal{B}), \quad g \longmapsto \rho_g.$$

In other words, for each $g \in G$, $\rho_g: X \to X$ is a Borel automorphism such that (i) $\rho_{gh}(x) = \rho_g(\rho_h(x))$ for every $h \in G$ and (ii) $\rho_e(x) = x$ for every $x \in X$.

Clearly, $\rho(G) = \{\rho_g : g \in G\} \subset \mathrm{Aut}(X, \mathcal{B})$ is a countable subgroup. If, for some $x \in X$, the relation $\rho_g(x) = x$ implies $g = e$, then ρ is called a *free action* of G. In this case, the group homomorphism ρ is injective. We note that every Borel automorphism $T \in \mathrm{Aut}(X, \mathcal{B})$ defines a Borel action of the group $\mathbb{Z}$ by the formula $\mathbb{Z} \ni n \mapsto T^n \in \mathrm{Aut}(X, \mathcal{B})$.

Countable Borel equivalence relation (CBER). An equivalence relation E on $(X, \mathcal{B})$ is called *Borel* if it is a Borel subset of the product space $E \subset X \times X$, where $X \times X$ is equipped with the Borel σ-algebra $\mathcal{B} \times \mathcal{B}$. It is called *countable* if every equivalence class $[x]_E := \{y \in X : (x, y) \in E\}$ is countable for all $x \in X$. If C is a Borel set, then $[C]_E$ denotes the saturation of C with respect to the equivalence relation E, i.e., $[C]_E$ contains the entire class $[x]_E$ for every $x \in C$.

For a countable subgroup Γ of $\mathrm{Aut}(X, \mathcal{B})$, we denote

$$E_X(\Gamma) = \{(x, y) \in X \times X : x = \gamma y \text{ for some } \gamma \in \Gamma\}.$$

Then $E_X(\Gamma)$ is called the *orbit equivalence relation* generated by the group Γ on X. Clearly, $E_X(\Gamma)$ is a CBER. An equivalence relation E is called *aperiodic* if every E-class $[x]_E$ is countably infinite. In contrast, finite E-classes will be called *periodic*. Similarly, a Borel automorphism P is called *periodic at a point* x if there exists $k \in \mathbb{N}$ such that $P^k x = x$. The least such k is called the *period of P at x*.

It turns out that all CBER's are generated by group automorphisms.

Theorem 2.1 (Feldman and Moore [11]). *Let E be a countable Borel equivalence relation on a standard Borel space $(X, \mathcal{B})$. Then there is a countable group Γ of Borel automorphisms of $(X, \mathcal{B})$ such that $E = E_X(\Gamma)$.*

A Borel set B is a *complete section* for an equivalence relation E on $(X, \mathcal{B})$ if it intersects every E-class, i.e., $[B]_E = X$. If a complete section intersects each E-class exactly once then it is called a *Borel transversal*. An equivalence relation E which admits a Borel transversal is called *smooth*. Equivalently, one can say that an equivalence relation E on a standard Borel space $(X, \mathcal{B})$ is *smooth* if there is a Borel function $f : X \to Y$, where Y is a standard Borel space, such that $(x, y) \in E \iff f(x) = f(y)$. We remark that in this paper we will deal only with *non-smooth* CBERs. See [22] for a survey of the state of the art in the theory of countable Borel equivalence relations.

Definition 2.2. Let Γ be a countable automorphism group acting on $(X, \mathcal{B})$. We will denote by $\mathcal{C}_\Gamma$ the collection of Borel subsets C such that C and $X \setminus C$ both are complete sections for $E_X(\Gamma)$.

Full group of automorphisms. For a countable subgroup Γ of $\mathrm{Aut}(X, \mathcal{B})$, we denote by Γx the orbit $\{\gamma x: \gamma \in \Gamma\}$ of x with respect to Γ. We say that Γ is a *free* group of automorphisms if $\gamma x \neq x$ for every $\gamma \neq e$ and $x \in X$.

The set

$$[\Gamma] = \{R \in \mathrm{Aut}(X, \mathcal{B}): Rx \in \Gamma x \text{ for all } x \in X\}$$

is called the *full group of automorphisms* generated by Γ. The full group generated by a single automorphism $T \in \mathrm{Aut}(X, \mathcal{B})$ is denoted by $[T]$.

Let $\Gamma \subset \mathrm{Aut}(X, \mathcal{B})$ be a freely acting group of automorphisms of a standard Borel space $(X, \mathcal{B})$. Then, for every $R \in [\Gamma]$, there exists a Borel function $\gamma_R: X \to \Gamma$ such that $Rx = \gamma_R(x)x,\ x \in X$. It follows that every $R \in [\Gamma]$ defines a countable partition of X into Borel sets $A_\gamma = \{x \in X: \gamma_R(x) = \gamma\},\ \gamma \in \Gamma$. Conversely, if $\{A_\gamma\}$ is a Borel partition of X, such that $\{\gamma A_\gamma\}$ also constitutes a Borel partition of X, then the map $Rx = \gamma x$, $x \in A_\gamma$, defines an element of $[\Gamma]$. In case when Γ is generated by a single automorphism T, the same construction holds, and each R from $[T]$ is represented in terms of piecewise constant function $x \mapsto n_R(x)$.

A countable subgroup Γ of $\mathrm{Aut}(X, \mathcal{B})$ is called *hyperfinite* if $\Gamma x = \bigcup_{i=1}^{\infty} \Gamma_i x$ for every $x \in X$, where each Γ_i is a finite subgroup of $[\Gamma]$ and $\Gamma_i \subset \Gamma_{i+1}$ for all i. Equivalently, a countable Borel equivalence relation E is called *hyperfinite* if $E = \bigcup_n E_n$ where $E_n \subset E_{n+1}$ for all n, where each E_n is a finite Borel subequivalence relation of E.

Let Γ_i be a countable subgroup of $\mathrm{Aut}(X_i, \mathcal{B}_i)$, $i = 1, 2$. The groups Γ_1 and Γ_2 are called *orbit equivalent* (denoted *o.e.*) if there exists a Borel isomorphism $\varphi: (X_1, \mathcal{B}_1) \to (X_2, \mathcal{B}_2)$ such that $\varphi \Gamma_1 x = \Gamma_2 \varphi x$, for all $x \in X_1$. Equivalently,

$$\varphi[\Gamma_1]\varphi^{-1} = [\Gamma_2].$$

If $E_X(\Gamma)$ is the equivalence relation generated by a free action of Γ, then the orbit equivalence of Γ_1 and Γ_2 is equivalent to the isomorphism of $E_{X_1}(\Gamma_1)$ and $E_{X_2}(\Gamma_2)$.

We refer readers to [10] for the classification of hyperfinite aperiodic CBER with respect to orbit equivalence.

Theorem 2.3 (Slaman and Steel [33] and Weiss [36]). *Suppose E is a CBER. The following facts are equivalent*:

1. *E is hyperfinite*;
2. *E is generated by a Borel $\mathbb{Z}$-action.*

Below we recall the definition of an *odometer* (known also as an *adding machine*). There are many papers devoted to odometers and their generalizations. We refer the interested reader to [17, 28] for detailed discussion.

Definition 2.4. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of integers such that $p_n \geq 2$ for each n. Let $\Omega = \prod_{n=0}^{\infty}\{0, \dots, p_n - 1\}$ be equipped with product discrete topology. Then Ω is a Cantor set. We define $S\colon \Omega \to \Omega$ as follows: $S(p_0 - 1, p_1 - 1, \dots) = (0, 0, \dots)$, and for any other $x \in \Omega$, find the least k such that $x_k \neq p_k - 1$ and put $S(x) = (0, 0, \dots, 0, x_k + 1, x_{k+1}, x_{k+2}, \dots)$. A Borel automorphism T is called an *odometer* if it is Borel isomorphic to some S. An odometer S is called the 2*-adic odometer*, if $p_n = 2$ for each $n \in \mathbb{N}_0$. In Section 5 we will work with the 2-adic odometer. For brevity, we will call it the 2-odometer.

2.2. Cocycles of Borel automorphism group. As above, let Γ be a countable subgroup of $\mathrm{Aut}(X, \mathcal{B})$ acting freely, and let G denote a locally compact second countable abelian group with identity 0 (we will use the additive group operation). We remark that the assumption that G is an abelian group is made for convenience and can be dropped in the following definitions.

Definition 2.5. A Borel function $a\colon \Gamma \times X \to G$ is called a *cocycle* over Γ if for any elements $\gamma_1, \gamma_2 \in \Gamma$ and all $x \in X$

$$a(\gamma_1\gamma_2, x) = a(\gamma_1, \gamma_2 x) + a(\gamma_2, x) \tag{2.1}$$

and

$$a(\mathbf{1}, x) = 0, \tag{2.2}$$

where $\mathbf{1}$ denotes the identity map. The set of all cocycles of Γ is denoted by $Z^1(\Gamma \times X, G)$.

A cocycle $a\colon \Gamma \times X \to G$ is called a *coboundary* if there exists a Borel function $c\colon X \to G$ such that

$$a(\gamma, x) = c(\gamma x) - c(x), \quad \text{for all } \gamma \in \Gamma, x \in X. \tag{2.3}$$

The set of all coboundaries of Γ is denoted by $B^1(\Gamma \times X, G)$.

Cocycles a_1, $a_2\colon \Gamma \times X \to G$ are called *cohomologous* ($a_1 \sim a_2$) if their difference is a coboundary, i.e., if there exists a Borel function $c\colon X \to G$, such that

$$a_1(\gamma, x) = c(\gamma x) + a_2(\gamma, x) - c(x). \tag{2.4}$$

Sometimes it is useful to define cocycles over an equivalence relation as described below.

Definition 2.6. Let E be a CBER. A Borel function $u\colon E \to G$ is an *orbital cocycle* over E if for every $(x, y), (y, z), (x, z) \in E$

$$u(x, z) = u(x, y) + u(y, z). \tag{2.5}$$

An orbital cocycle is a *coboundary* if there exists a Borel function $c\colon X \to G$ such that for $(x, y) \in E$

$$u(x, y) = c(x) - c(y). \tag{2.6}$$

As before, two orbital cocycles are *cohomologous* if their difference is a coboundary.

Remark 2.7. Let Γ be a freely acting countable group of automorphisms. Given any cocycle $a \in Z^1(\Gamma \times X, G)$, define a function $u_a \colon E_X(\Gamma) \to G$ by the following rule: for any pair $(y, x) \in E_X(\Gamma)$ determine unique $\gamma \in \Gamma$ such that $y = \gamma x$ and then set

$$u_a(y, x) = a(\gamma, x). \tag{2.7}$$

Since Γ is free, u_a is well defined. It is clear that u_a satisfies (2.5), hence it is an orbital cocycle. Moreover, u_a is a coboundary if and only if a is a coboundary. Conversely, every orbital cocycle of $E_X(\Gamma)$ defines a cocycle of Γ.

Remark 2.8. Let T be an automorphism of $(X, \mathcal{B})$ which determines an action of the group $\mathbb{Z}$. Every Borel function $f \colon X \to G$ with values in the group G defines a cocycle $a \colon \mathbb{Z} \times X \to G$ by the formula

$$a(j, x) = \begin{cases} f(x) + f(Tx) + \cdots + f(T^{j-1}x) & \text{if } j \geq 1, \\ 0 & \text{if } j = 0, \\ -f(T^{-1}x) - f(T^{-2}x) - \cdots - f(T^j x) & \text{if } j \leq -1, \end{cases} \tag{2.8}$$

Conversely, if $a \colon \mathbb{Z} \times X \to G$ is a cocycle of the group $\{T^n, n \in \mathbb{Z}\}$, then it is completely determined by the function $f(x) = a(1, x)$. Moreover, the properties of the cocycle $a(j, x)$ are represented in terms of the function f.

Remark 2.9. If $a \colon \Gamma \times X \to G$ is a cocycle of a freely acting countable group of automorphisms Γ, then it can be extended to a cocycle $\hat{a}$ over the full group $[\Gamma]$. Indeed, for $R \in [\Gamma]$ take the uniquely determined function $x \mapsto \gamma(x)$ such that $Rx = \gamma(x)x$. Then we set

$$\hat{a}(R, x) = a(\gamma(x), x), \quad x \in X.$$

It can be easily seen that $\hat{a}$ coincides with a on elements of the group Γ, and $\hat{a}$ satisfies the cocycle identity (2.1) and (2.2).

2.3. Topologies on the group Aut($X, \mathcal{B}$). We will need the notion of convergence of a sequence of Borel automorphisms. Recall that several topologies on $\mathrm{Aut}(X, \mathcal{B})$ were defined and studied in [3]. We will work with the so-called uniform topology τ whose origin lies in ergodic theory (see Section 1 for the definition of $\mathcal{M}_1(X)$ and $E(S, T)$).

Definition 2.10. The *uniform topology* τ, on $\mathrm{Aut}(X, \mathcal{B})$ is defined by the base of neighborhood $\mathcal{V} = \{V(T; \mu_1, \dots, \mu_n; \epsilon)\}$ where, $T \in \mathrm{Aut}(X, \mathcal{B})$, $\mu_1, \dots, \mu_n \in \mathcal{M}_1(X)$, $\epsilon > 0$, and

$$V(T; \mu_1, \dots, \mu_n; \epsilon) = \{S \in \mathrm{Aut}(X, \mathcal{B}) \mid \mu_i(E(S, T)) < \epsilon,\ i = 1, \dots, n\}. \tag{2.9}$$

Remark 2.11. It can be seen that $(\mathrm{Aut}(X, \mathcal{B}), \tau)$ is a Hausdorff, topological group. It is also relevant to mention that topology τ coincides with the topology τ', which is defined by the base of neighborhood $\mathcal{V}' = \{V'(T; \mu_1, \dots, \mu_n; \epsilon)\}$ where, $T \in \mathrm{Aut}(X, \mathcal{B})$, $\mu_1, \dots, \mu_n \in \mathcal{M}_1(X)$, $\epsilon > 0$, and

$$\begin{aligned} &V'(T; \mu_1, \dots, \mu_n; \epsilon) \\ &\quad = \{S \in \mathrm{Aut}(X, \mathcal{B}) \mid \sup_{F \in \mathcal{B}} \mu_i(TF \,\Delta\, SF) < \epsilon,\ i = 1, \dots, n\}. \end{aligned} \tag{2.10}$$

3. Topologies on the space of cocycles

For a standard Borel space $(X, \mathcal{B})$ and an abelian l.c.s.c. group G, we denote by $\mathcal{F}(X, G)$ the set of Borel functions $f: X \to G$. Clearly, this set is an abelian group under pointwise addition of functions. We will write simply $\mathcal{F}$ when X and G are understood. Since G is metrizable, we will denote by $|\cdot|$ a translation invariant metric on G compatible with the topology on G.

In this section we will define and study topologies on $\mathcal{F}(X, G)$ which are analogous to the topology of convergence in measure. For a countable group of Borel automorphisms $\Gamma \subset \mathrm{Aut}(X, \mathcal{B})$, we will consider the subgroups of cocycles and coboundaries in $\mathcal{F}(X, G)$. Our goal is to show that, for a hyperfinite group Γ, coboundaries form a dense subgroup in the group of all cocycles.

Remark 3.1. Let Γ be a hyperfinite countable subgroup of $\mathrm{Aut}(X, \mathcal{B})$. Without loss of generality, we can assume that Γ acts freely. Then Γ is orbit equivalent to a Borel $\mathbb{Z}$-action, i.e., there exists an automorphism $T \in \mathrm{Aut}(X, \mathcal{B})$, such that the orbits $\Gamma(x)$ coincide with those of the group $\{T^n x, n \in \mathbb{Z}\}$. For any two orbit equivalent automorphism groups, their groups of cohomology are isomorphic (see Proposition 3.8 below). This means that, studying cocycles of Γ, it suffices to work with cocycles of the group $\{T^n: n \in \mathbb{Z}\}$. The benefit of this fact is that we can explicitly write down the formula for $\mathbb{Z}$- cocycles as in (2.8). Hence (as was mentioned above), every cocycle $a: \mathbb{Z} \times X \to G$ of $\{T^n, n \in \mathbb{Z}\}$ is represented by a Borel function from X to G.

In the following definition, we discuss several topologies on $\mathcal{F}(X, G)$ which are analogous to the topology defined by convergence of measure.

Definition 3.2. The topologies τ_1, τ_2, τ_3, and τ_4 on $\mathcal{F}(X, G)$ are defined by their bases of neighborhoods $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{W}$ and $\mathcal{W}'$, respectively, where

$$\begin{aligned} \mathcal{U} &= \{U(f; \mu_1, \dots, \mu_n; \epsilon, \delta)\}, \\ \mathcal{U}' &= \{U'(f; \mu_1, \dots, \mu_n; \epsilon)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{W} &= \{W(f; \mu_1, \dots, \mu_n; \epsilon)\}, \\ \mathcal{W}' &= \{W'(f; \mu_1, \dots, \mu_n; \epsilon)\}, \end{aligned}$$

and

$$\begin{aligned} &U(f;\mu_1,\dots,\mu_n;\epsilon,\delta)\\ &\quad := \{g\in\mathcal{F}\colon \mu_i(\{x\colon |f(x)-g(x)|>\epsilon\})<\delta \text{ for all } i=1,\dots,n\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &U'(f;\mu_1,\dots,\mu_n;\epsilon)\\ &\quad := \{g\in\mathcal{F}\colon \mu_i(\{x\colon |f(x)-g(x)|>\epsilon\})<\epsilon \text{ for all } i=1,\dots,n\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &W(f;\mu_1,\dots,\mu_n;\epsilon)\\ &\quad := \left\{g\in\mathcal{F}\colon \int_X \min(|f(x)-g(x)|,1)\,d\mu_i<\epsilon \text{ for all } i=1,\dots,n\right\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &W'(f;\mu_1,\dots,\mu_n;\epsilon)\\ &\quad := \left\{g\in\mathcal{F}\colon \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,d\mu_i<\epsilon \text{ for all } i=1,\dots,n\right\}. \end{aligned} \tag{3.4}$$

In the above definitions, we take $f\in\mathcal{F}(X,G)$, $\mu_1,\dots,\mu_n\in\mathcal{M}_1(X)$, $\epsilon,\delta>0$, and $n\in\mathbb{N}$.

Theorem 3.3. *All the topologies τ_1, τ_2, τ_3, and τ_4 from Definition 2.10 coincide on the group $\mathcal{F}(X,G)$.*

Proof. For the entire proof, we assume that $i\in\{1,2,\dots,n\}$. Also note that the notation $\tau_j\subset\tau_k$, for topologies τ_j,τ_k, $j,k\in\{1,2,3,4\}$, $j\neq k$, means that τ_k is stronger than τ_j. Because our topologies are determined in terms of the bases of neighborhoods, it suffices to check that the base for τ_k contains that for τ_j. For example, $\tau_1\subset\tau_2$, implies that for every $f\in\mathcal{F}(X,G)$ and a base element $U(f;\mu_1,\dots,\mu_n;\epsilon,\delta)$ of τ_1 containing f, there exists a base element $U'(f;\mu_1,\dots,\mu_n;\kappa)$ of τ_2 such that

$$U'(f;\mu_1,\dots,\mu_n;\kappa)\subset U(f;\mu_1,\dots,\mu_n;\epsilon,\delta).$$

(1) τ_1 *coincides with* τ_2 *on* $\mathcal{F}(X,G)$. Clearly, for $\delta=\epsilon$, we have $\tau_2\subset\tau_1$. To prove the converse, we will show, as mentioned above, that for a base element $U(f;\mu_1,\dots,\mu_n;\epsilon,\delta)\in\mathcal{U}$, there exists a base element $U'(f;\mu_1,\dots,\mu_n;\kappa)\in\mathcal{U}'$ such that

$$U'(f;\mu_1,\dots,\mu_n;\kappa)\subset U(f;\mu_1,\dots,\mu_n;\epsilon,\delta).$$

If $0<\epsilon<\delta$, take $\kappa=\epsilon$, and we are done, since for $\epsilon<\delta$,

$$U'(f;\mu_1,\dots,\mu_n;\epsilon)\subset U(f;\mu_1,\dots,\mu_n;\epsilon,\delta).$$

Now assume that $0<\delta<\epsilon$. Then take $\kappa=\delta$ and show that

$$U'(f;\mu_1,\dots,\mu_n;\delta)\subset U(f;\mu_1,\dots,\mu_n;\epsilon,\delta).$$

To see this, take any function $g \in U'(f;\mu_1,\dots,\mu_n;\delta)$ and note that $0<\delta<\epsilon$ implies

$$\{x\colon |f(x)-g(x)|>\epsilon\} \subset \{x\colon |f(x)-g(x)|>\delta\}.$$

Thus for all i, we have

$$\mu_i(\{x\colon |f(x)-g(x)|>\epsilon\}) \le \mu_i(\{x\colon |f(x)-g(x)|>\delta\}) < \delta.$$

Hence $g \in U(f;\mu_1,\dots,\mu_n;\epsilon,\delta)$ as needed.

(2) τ_1 *coincides with* τ_3 *on* $\mathcal{F}(X,G)$. First we show that $\tau_3 \subset \tau_1$. We need to verify that, for any neighborhood $W(f;\mu_1,\dots,\mu_n;\epsilon) \in \mathcal{W}$, there exists a neighborhood $U(f;\mu_1,\dots,\mu_n;\epsilon',\delta) \in \mathcal{U}$ such that

$$U(f;\mu_1,\dots,\mu_n;\epsilon',\delta) \subset W(f;\mu_1,\dots,\mu_n;\epsilon).$$

To see this, let $\epsilon' = \epsilon/4$, and consider $g \in U(f;\mu_1,\dots,\mu_n;\epsilon',\delta)$, where $\delta>0$ will be chosen later. Then, for all i, we have

$$\mu_i(\{x\colon |f(x)-g(x)|>\epsilon/4\}) < \delta. \tag{3.5}$$

We will prove that

$$\int_X \min(|f(x)-g(x)|,1)\, d\mu_i < \epsilon. \tag{3.6}$$

Choose a Borel set B such that

$$\min(|f(x)-g(x)|,1) = \begin{cases} |f(x)-g(x)| & \text{if } x \in B,\\ 1 & \text{if } x \in X\setminus B.\end{cases}$$

Define $Q=\{x\in B\colon |f(x)-g(x)|>\epsilon/4\}$. Then, for all i,

$$\int_B |f-g|d\mu_i = \int_Q |f-g|d\mu_i + \int_{B\setminus Q} |f-g|d\mu_i.$$

Choose $\delta>0$, sufficiently small such that the condition $\mu_i(Q)<\delta$ implies

$$\int_Q |f-g|d\mu_i < \epsilon/4.$$

For $x \in B\setminus Q$, we have $|f-g| \le \epsilon/4$. Since every μ_i is a probability measure, we obtain

$$\int_{B\setminus Q} |f-g|d\mu_i < (\epsilon/4)\mu_i(B\setminus Q) < \epsilon/4. \tag{3.7}$$

Thus, for all i, we see that

$$\int_B \min(|f(x) - g(x)|, 1) d\mu_i < \epsilon/2. \tag{3.8}$$

Using (3.5) and choosing $\epsilon/4 < 1$ and $\delta < \epsilon/2$ we get $\mu_i(X \setminus B) < \epsilon/2$. Therefore, for all i, the following inequality holds

$$\int_{X\setminus B} \min(|f(x) - g(x)|, 1) d\mu_i < \epsilon/2. \tag{3.9}$$

Relations (3.8) and (3.9) imply (3.6). This completes the proof of $\tau_3 \subset \tau_1$.

Now we prove that $\tau_1 \subset \tau_3$. We show that, for a base element

$$U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta) \in \mathcal{U},$$

there exists a base element $W(f; \mu_1, \ldots, \mu_n; \kappa) \in \mathcal{W}$ such that

$$W(f; \mu_1, \ldots, \mu_n; \kappa) \subset U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta).$$

For this, let $\kappa = \epsilon\delta$ and let $g \in W(f; \mu_1, \ldots, \mu_n; \kappa)$. Then for all i, we get

$$\int_X \min(|f - g|, 1) d\mu_i < \epsilon\delta. \tag{3.10}$$

Assume, toward a contradiction, that $g \notin U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta)$, i.e.,

$$\mu_i(\{x: |f(x) - g(x)| > \epsilon\}) \geq \delta. \tag{3.11}$$

Denote $P = \{x: |f(x) - g(x)| > \epsilon\}$; then for all i,

$$\int_X |f - g| d\mu_i \geq \int_P |f - g| d\mu_i \geq \epsilon\delta.$$

which contradicts (3.10). Hence, we conclude that $g \in U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta)$ as needed.

(3) τ_1 *coincides with* τ_4 *on* $\mathcal{F}(X, G)$. To see this, let $K = \{x: |f(x) - g(x)| > \epsilon\}$ and note that the equality

$$\{x: |f(x) - g(x)| > \epsilon\} = \left\{x: \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} > \frac{\epsilon}{1 + \epsilon}\right\} := K \tag{3.12}$$

holds. We first show that $\tau_1 \subset \tau_4$. Let $U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta) \in \mathcal{U}$ be a neighborhood from τ_1. Show that there exists a neighborhood $W'(f; \mu_1, \ldots, \mu_n; \kappa) \in \mathcal{W}'$ such that $W'(f; \mu_1, \ldots, \mu_n; \kappa) \subset U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta)$.

Let $\kappa = \frac{\epsilon\delta}{1+\epsilon}$, and let $g \in W'(f;\mu_1,\dots,\mu_n;\kappa)$. Then,

$$\int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu_i < \frac{\epsilon\delta}{1+\epsilon} \quad \text{for all } i.$$

Relation (3.12) implies

$$\frac{\epsilon}{1+\epsilon}\chi_K < \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\chi_K < \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}.$$

Hence,

$$\mu_i(K) < \frac{1+\epsilon}{\epsilon}\int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu_i < \delta.$$

which implies that $g \in U(f;\mu_1,\dots,\mu_n;\epsilon,\delta)$.

It remains to prove that $\tau_4 \subset \tau_1$. Show that for a neighborhood

$$W'(f;\mu_1,\dots,\mu_n;\epsilon) \in \mathcal{W},$$

there exists a basis element $U(f;\mu_1,\dots,\mu_n;\epsilon,\delta) \in \mathcal{U}$ such that

$$U(f;\mu_1,\dots,\mu_n;\epsilon,\delta) \subset W'(f;\mu_1,\dots,\mu_n;\epsilon).$$

Take a function $g \in U(f;\mu_1,\dots,\mu_n;\epsilon,\delta)$, then

$$\mu_i(K) = \mu_i(\{x\colon |f(x)-g(x)| > \epsilon\}) < \delta.$$

Choose δ such that $\delta < \dfrac{\epsilon^2}{1+\epsilon}$; then we obtain for each measure μ_i

$$\begin{aligned}
&\int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu_i \\
&\quad \le \int_K \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu_i + \int_{K^c} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu_i \\
&\quad \le \mu_i(K) + \frac{\epsilon}{1+\epsilon}\mu_i(K^c) \\
&\quad < \delta + \frac{\epsilon}{1+\epsilon} < \epsilon.
\end{aligned}$$

Thus, $g \in W'(f;\mu_1,\dots,\mu_n;\epsilon)$ as needed. □

Remark 3.4. Since the topologies τ_1, τ_2, τ_3 and τ_4, on $\mathcal{F}(X,G)$ coincide, we will will use the notation $\mathcal{T}$ to denote them.

Theorem 3.5. *$\mathcal{F}(X, G)$ is separable Hausdorff topological group with respect to the topology $\mathcal{T}$.*

Proof. We denote by $\mathcal{A} = \{B_i\}_{i\in\mathbb{N}}$ the countable base for the space X which generates $\mathcal{B}$. Recall that G is an abelian l.c.s.c. group with identity 0. Let G_0 be a countable dense subgroup of G. Denote by $\alpha_i \chi_{B_i}$ a function $X \to G$ which takes the value $\alpha_i \in G$ on the set B_i and is 0 everywhere else. Note that we refrain from using the term "characteristic function" as G is an additive abelian group with identity 0 but the notion of multiplicative identity is not defined.

Consider the set $\mathcal{S}(X, G_0)$ of all finite linear combinations of such constant functions with values in G_0, i.e., they can be described as piecewise constant functions that take values from G_0 on sets from the family $\mathcal{A}$ and are zero everywhere else. We will call elements of $\mathcal{S}(X, G_0)$ simple functions.

For notational purpose, we will denote such a function as follows:

$$f(x) = \sum_{l=1}^{p} \alpha_l \chi_{B_l}(x),$$

where $\alpha_l \in G_0$ and $B_l \in \mathcal{A}$ for $l = 1, 2, \ldots, p$.

We first observe that the set $\mathcal{S}(X, G_0)$ is a countable subset of $\mathcal{F}(X, G)$. In what follows we will show that $\mathcal{S}(X, G_0)$ is dense in $\mathcal{F}(X, G)$ with respect to the topology $\mathcal{T}$.

For $f \in \mathcal{F}(X, G)$, consider a neighborhood of f

$$\begin{aligned} &U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta) \\ &\quad = \{g \in \mathcal{F}\colon \mu_i(\{x\colon |f(x) - g(x)| > \epsilon\}) < \delta \text{ for all } i = 1, 2, \ldots, n\}, \end{aligned}$$

where $\mu_1, \ldots, \mu_n \in \mathcal{M}_1(X)$. To prove the result, it suffices to find an element from the set $\mathcal{S}(X, G_0)$ in $U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta)$.

Since $f \in \mathcal{F}(X, A)$ is a Borel function, there exists a sequence $\{s_j\}_{j\in\mathbb{N}}$ of simple function taking value in G_0 which converges pointwise to f. Again using the same notation as above we denote s_j as follows

$$s_j = \sum_{k=1}^{m} \alpha_{k,j} \chi_{E_{k,j}}, \quad j \in \mathbb{N}.$$

where $\alpha_{k,j} \in G_0$ for all $k = 1, 2, \ldots, m$, and

$$E_{k,j} = \{x \in X\colon s_j(x) = c_{k,j} \text{ for all } k\}.$$

For the measure μ_1, we use Egoroff's theorem and find a Borel set $F_1 \in X$ such that $s_j \to f$ uniformly on F_1, and $\mu_1(X \setminus F_1) < \frac{\delta}{n}$. (Note that this convergence is uniform in the usual sense: for $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$, such that for all $j > N_1$ and for all $x \in F$, $|f(x) - s_j(x)| < \epsilon$). Similarly, there exists a Borel set $F_2 \subset F_1$ such that the sequence (s_j) converges uniformly to f on F_2, and $\mu_2(F_1 \setminus F_2) < \frac{\delta}{n}$. Repeating this process n times we obtain a Borel set $F \subset X$ such that the convergence $s_j \to f$ is uniform on F, and, for $i = 1, 2, \ldots, n$, we have $\mu_i(X \setminus F) < \delta$. Hence for any $\epsilon > 0$ one can find some $N \in \mathbb{N}$ such that $|f(x) - s_t(x)| < \epsilon$ for $t > N$ and $x \in F$. In other words, $\mu_i(\{x : |f(x) - s_t(x)| > \epsilon\}) < \delta\}, i = 1, 2, \ldots, n$.

This implies that, for $t > N$, the functions $s_t = \sum_{k=1}^{m} \alpha_{k,t} \chi_{E_{k,t}}$ belong to $U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta)$. Since this is true for any $\delta > 0$, choose N such that for $t \in N$ we have $s_t \in U\big(f; \mu_1, \ldots, \mu_n; \epsilon, \frac{\delta}{q}\big)$, where q is a positive integer to be chosen later. It follows that

$$\mu_i(\{x : |f(x) - s_t(x)| > \epsilon\}) < \frac{\delta}{q}, \quad i = 1, 2, \ldots, n.$$

In other words, we obtain that, for $k = 1, 2, \ldots, m$,

$$\mu_i(\{x \in E_{k,t} : |f(x) - \alpha_{k,t}| > \epsilon\}) < \frac{\delta}{q}, \quad i = 1, 2, \ldots, n. \tag{3.13}$$

where $\alpha_{k,t} \in G_0$, $k = 1, 2, \ldots, m$ and $t > N$.

Since each $E_{k,t}$ is a Borel set, it can approximated by an open set, i.e., there exists an open set $O^1_{k,t}, \ldots, O^n_{k,t}$ such that

$$\mu_1(O^i_{k,t} \,\Delta\, E_{k,t}) < \frac{\delta}{2q}, \quad i = 1, \ldots, n.$$

Define $O_{k,t} = \bigcap_{i=1}^{n} O^i_{k,t}$, then, for every $i = 1, 2, \ldots n$, one has

$$\mu_i(O_{k,t} \,\Delta\, E_{k,t}) < \frac{\delta}{2q}.$$

Each open set $O_{k,t}$ is a countable union of base elements i.e., $O_{k,t} = \bigcup_{i \in \mathbb{N}} B_i$, where $B_i \in \mathcal{A}$. Thus there exists a finite number, $r(k,t) \in \mathbb{N}$ such that for every $i = 1, 2, \ldots, n$,

$$\mu_i\Big(\Big(\bigcup_{l=1}^{r(k,t)} B_l\Big) \,\Delta\, O_{k,t}\Big) < \frac{\delta}{2q}. \tag{3.14}$$

Let us denote by $I_{k,t}$ the index set $I_{k,t} = \{1, 2, \ldots, r(k,t)\}$. Thus, (3.14) implies that

$$\mu_i\Big(\Big(\bigcup_{l \in I_{k,t}} B_l\Big) \,\Delta\, E_{k,t}\Big) < \frac{\delta}{q}.$$

Since

$$\Big\{x \in \Big(\bigcup_{l \in I_{k,t}} B_l\Big) \,\Delta\, E_{k,t} : |f(x) - \alpha_{k,t}| > \epsilon\Big\} \subset \Big(\bigcup_{l \in I_{k,t}} B_l\Big) \,\Delta\, E_{k,t},$$

we have

$$\mu_i\Big(\Big\{x \in \Big(\bigcup_{l\in I_{k,t}} B_l\Big) \,\Delta\, E_{k,t} \colon |f(x) - \alpha_{k,t}| > \epsilon\Big\}\Big) < \frac{\delta}{q}, \quad i = 1, 2, \ldots, n. \tag{3.15}$$

Now take $q = 2m$ where m is as in the definition of s_j above, then by (3.13) and (3.15), we obtain

$$\mu_i\Big(\Big\{x \in \bigcup_{l\in I_{k,t}} B_l \colon |f(x) - \alpha_{k,t}| > \epsilon\Big\}\Big) < \frac{\delta}{m} \quad \text{for all } i = 1, 2, \ldots, n. \tag{3.16}$$

Note that (3.16) is true for all $m = 1, 2, \ldots, k$. Let $I_t = \bigcup_{k=1}^{m} I_{k,t}$, then

$$\mu_i\Big(\Big\{x \in \bigcup_{l\in I_k} B_l \colon |f(x) - \alpha_{k,t}| > \epsilon\Big\}\Big) < \delta \quad \text{for all } i = 1, 2, \ldots, n. \tag{3.17}$$

Define the sequence of functions s'_t, for $t \in \mathbb{N}$, as follows

$$s'_t(x) = \begin{cases} \alpha_{k,t} & \text{if } x \in B_l, l \in I_{k,t}, \\ 0 & \text{if } x \notin B_l, l \in I_{k,t}. \end{cases}$$

Then, by (3.17), we have

$$\mu_i(\{x \colon |f(x) - s'_t| > \epsilon\}) < \delta \quad \text{for all } i = 1, 2, \ldots, n. \tag{3.18}$$

Relation (3.18) implies that $s'_t \in U(f; \mu_1, \ldots, \mu_n; \epsilon, \delta)$ for $t > N$. Therefore $\mathcal{S}(X, G_0)$ is dense in $\mathcal{F}(X, G)$, and $\mathcal{F}(X, G)$ is a separable space.

To prove the second part of the theorem, we will show that $\mathcal{F}(X, G)$ is a topological group with respect to the topology $\mathcal{T}$. We will do it for the topology τ_3 (see Definition 3.2) because it is easier to work this topology. Note the following facts:

(i) $W(f; \mu_1, \ldots, \mu_n; \epsilon) = -W(-f; \mu_1, \ldots, \mu_n; \epsilon)$;

(ii) $W(f; \mu_1, \ldots, \mu_n; \epsilon/2) + W(g; \mu_1, \ldots, \mu_n; \epsilon/2) \subset W(f + g; \mu_1, \ldots, \mu_n; \epsilon)$.

Both (i) and (ii) are clear by the definition of

$$W(f; \mu_1, \ldots, \mu_n; \epsilon) \quad \text{and} \quad W(g; \mu_1, \ldots, \mu_n; \epsilon).$$

It follows from (i) that the map $f \mapsto -f$ is continuous and (ii) implies that the map $(f, g) \mapsto f + g$ is also continuous.

To see that $\mathcal{F}(X, G)$ is Hausdorff in the topology $\mathcal{T}$, consider $f, g \in \mathcal{F}(X, G)$ such that $f \neq g$. Then there exists $x \in X$,such that $f(x) \neq g(x)$. We work with topology τ_1 and put $\mu_1 = \delta_x$ (the Dirac measure at x). Note that, for $\delta < 1$, the open set U (defined below) contains f but does not contain g:

$$U = \{h \in \mathcal{F} \colon \delta_x(\{y \colon |f(y) - h(y)| > \epsilon\}) < \delta\}.$$

For $\delta < 1$, we get

$$U = \{h \in \mathcal{F}: \delta_x(\{y: |f(y) - h(y)| > \epsilon\}) = 0\}.$$

Therefore $x \notin \{y: |f(y) - h(y)| > \epsilon\}$ and $g \notin U$. □

Proposition 3.6. *Let* Γ *be a hyperfinite countable subgroup of* $\mathrm{Aut}(X, \mathcal{B})$*. The group* $Z^1(\Gamma \times X, G)$ *is closed in* $\mathcal{F}(\Gamma \times X, G)$*, and it is a separable topological group.*

To prove Proposition 3.6, we will show that if $\{a_n\} \subset Z^1(\Gamma \times X, G)$ is a sequence of cocycles such that $a_n \to a$ in τ_1, then $a \in Z^1(\Gamma \times X, G)$. For this, we will prove the following lemma.

Lemma 3.7. *Let* $\{a_n\}$ *be a sequence of cocycles from* $Z^1(\Gamma \times X, G)$*. Then* $a_n \to a$ *in the topology* τ_1 *if and only if for every* $x \in X$ *there exists* $n(x) \in \mathbb{N}$ *such that* $a_n(x) = a(x)$ *for all* $n > n(x)$*.*

Proof. As mentioned in Remark 2.8 the group Γ is orbit equivalent to a group generated by a single automorphism $\{T^n: n \in \mathbb{Z}\}$. It gives us the possibility to represent cocycles a_n as functions on X with values in the group G.

Assume now that $a_n \to a$ in τ_1. Then, for every $\epsilon, \delta > 0$ there exists $n(x) \in \mathbb{N}$ such that $a_n \in U(a; \mu_1, \dots, \mu_p; \epsilon, \delta)$ for $n > n(x)$ (here $\mu_1, \dots, \mu_p \in \mathcal{M}_1(X)$ as usual). Fix $x \in X$ and take $\mu_1 = \delta_x$ (the Dirac measure at x). Thus we have $\delta_x(\{y: |a_n(y) - a(y)| > \epsilon\}) < \delta$. For $\delta < 1$ we get $\delta_x(\{y: |a_n(y) - a(y)| > \epsilon\}) = 0$. Hence, we have $x \notin \{y: |a_n(y) - a(y)| > \epsilon\}$ for all $n > n(x)$. We conclude that $a_n(x) = a(x)$.

Conversely, suppose that, for every $x \in X$, there exists $n(x) \in \mathbb{N}$ such that $a_n(x) = a(x)$ for all $n > n(x)$. Define

$$X_n = \{x \in X: a_m(x) = a(x) \text{ for all } m \geq n\}, \quad n \in \mathbb{N}.$$

Note that $X_n \subset X_{n+1}$, and $\bigcup_{n=1}^{\infty} X_n = X$. For every $\mu \in \mathcal{M}_1(X)$, we see that $\mu(X_n) \to 1$ as $n \to \infty$. Take a neighborhood $U(a; \mu_1, \dots, \mu_p; \epsilon, \delta)$ and find $n_0 \in \mathbb{N}$ such that $\mu_i(X_n) > 1 - \delta$ for $n > n_0$, $i = 1, 2, \dots, p$. Note that, for all $n \in \mathbb{N}$,

$$\{x \in X: |a_n(x) - a(x)| > \epsilon\} \subset X \setminus X_n.$$

Thus $\mu_i(\{x \in X: |a_n(x) - a(x)| > \epsilon\}) < \mu_i(X \setminus X_n) < \delta$. Hence, for $n > n_0$, we deduce that $\mu_i(\{x \in X: |a_n(x) - a(x)| > \epsilon\}) < \delta$ as needed. □

Proof of Proposition 3.4. We switch back to considering a_n and a as functions from $\Gamma \times X$ to G. Since $a_n \in Z^1(\Gamma \times X, G)$, $a_n(\gamma_1\gamma_2, x) = a_n(\gamma_1, \gamma_2 x) + a_n(\gamma_2, x)$, for all $\gamma_1, \gamma_2 \in \Gamma$.

For a fixed $x \in X$, let $n_0 = \max\{n(x), n(\gamma_2 x)\}$, then for $n > n_0$, we have

$$\begin{aligned} a_n(\gamma_1\gamma_2, x) &= a(\gamma_1\gamma_2, x), \\ a_n(\gamma_1, \gamma_2 x) &= a(\gamma_1, \gamma_2 x), \\ a_n(\gamma_2, x) &= a(\gamma_2, x). \end{aligned}$$

Hence, $a(\gamma_1\gamma_2, x) = a(\gamma_1, \gamma_2 x) + a(\gamma_2, x)$ for all $\gamma_1, \gamma_2 \in \Gamma$. Since we can do this for every $x \in X$, $a \in Z^1(\Gamma \times X, G)$. □

Proposition 3.8. *Let $\Gamma_i \in \mathrm{Aut}(X_i, \mathcal{B}_i)$, $i = 1, 2$, be two orbit equivalent countable Borel automorphism groups. Then there exists a topological group isomorphism $\tilde{\varphi}\colon Z^1(\Gamma_1 \times X_1, A) \to Z^1(\Gamma_2 \times X_2, A)$ which carries coboundaries to coboundaries.*

Proof. Since Γ_1 and Γ_2 are orbit equivalent, there exists a Borel map $\varphi\colon X_1 \to X_2$, such that $\varphi[\Gamma_1] = [\Gamma_2]\varphi$. Define $\tilde{\varphi}\colon Z^1(\Gamma_1 \times X_1, A) \to Z^1(\Gamma_2 \times X_2, A)$ as

$$\tilde{\varphi} \circ a_1(\gamma_2, x_2) = a_1(\varphi^{-1}\gamma_2\varphi, \varphi^{-1}x_2)$$

for $a_1 \in Z^1(\Gamma_1 \times X_1, G)$ and $(\gamma_2, x_2) \in \Gamma_2 \times X_2$. Then, $\tilde{\varphi}$ is an isomorphism by definition. If a_1 is a coboundary, $a_1(\gamma_1, x_1) = c(\gamma_1 x_1) - c(x_1)$, where $c\colon X \to G$ is a Borel map.

$$\tilde{\varphi} \circ a_1(\gamma_2, x_2) = a_1(\varphi^{-1}\gamma_2\varphi, \varphi^{-1}x_2) = c(\varphi^{-1}\gamma_2\varphi(\varphi^{-1}x_2)) - c(\varphi^{-1}x_2)$$

is also a coboundary. □

Corollary 3.9. *For a Borel automorphism group Γ of $(X, \mathcal{B})$ the first cohomology group $H^1(\Gamma \times X, G) = Z^1(\Gamma \times X, G)/B^1(\Gamma \times X, G)$ is an invariant of orbit equivalence.*

Remark 3.10. In general, $B^1(\Gamma \times X, G)$ is not closed in the topology described above. Hence $H^1(\Gamma \times X, G)$ should be considered as an abstract group that does not inherit the topological or Borel structure.

Remark 3.11. Let $\mathrm{Ctbl}(X)$ be defined as the subset of $\mathrm{Aut}(X, \mathcal{B})$ consisting of all automorphisms with countable support, that is

$$T \in \mathrm{Ctbl}(X) \iff E(S, \mathbf{I}) \text{ is at most countable.}$$

One can show that $\mathrm{Ctbl}(X)$ is a normal subgroup which is closed with respect to the uniform topology, see (2.9) in Definition 2.10. Therefore $\widehat{\mathrm{Aut}}(X, \mathcal{B}) = \mathrm{Aut}(X, \mathcal{B})/\mathrm{Ctbl}(X)$ is a simple Hausdorff topological group with respect to the quotient topology [3]. Considering elements from $\widehat{\mathrm{Aut}}(X, \mathcal{B})$, we identify Borel automorphisms which differ on at most a countable set. Topological properties of the group $\widehat{\mathrm{Aut}}(X, \mathcal{B})$ are studied in [4]. It was shown that the quotient topology on $\widehat{\mathrm{Aut}}(X, \mathcal{B})$ is in fact generated by neighborhoods $V(T; \mu_1, \dots, \mu_n; \varepsilon)$ where the measures $\mu_1, \dots, \mu_n$ are taken from $M_1^c(X)$, the set of all non-atomic Borel probability measures on a standard Borel space $(X, \mathcal{B})$.

Using a similar approach, we identify two functions f and g if they differ on at most a countable set. In other words, we define the quotient set $\widehat{\mathcal{F}}$ with elements $\hat{g} = \{g \circ T : T \in \mathrm{Ctbl}(X)\}$ where $g \in \mathcal{F}(X, \mathcal{B})$. Then one can show that the quotient topology $\hat{\tau}$ on $\widehat{\mathcal{F}}$ is defined by neighborhoods $V(f; \mu_1, \dots, \mu_k; \epsilon)$, where $\mu_1, \dots, \mu_k \in M_1^c(X)$.

Based on Remark 3.11, we can obtain the following result. The proof is left for the reader because we do not use this result in the paper.

Proposition 3.12. *Let $\hat{\tau}$ be the topology on $\widehat{\mathcal{F}}(X, G)$ defined as in Remark 3.11 by atomless measures from $M_1^c(X)$. Then, for $\hat{f}_n$ and $\hat{f}$ from $\widehat{\mathcal{F}}$,*

$$\hat{f}_n \xrightarrow{\hat{\tau}} \hat{f}$$

if and only if $(\hat{f}_n)$ converges to $\hat{f}$ uniformly.

4. Density of coboundaries for hyperfinite Borel actions

In this section we prove following result.

Theorem 4.1. *Let $\Gamma \subset \mathrm{Aut}(X, \mathcal{B})$ be a hyperfinite Borel automorphism group. Then $B^1(\Gamma \times X, G)$ is dense in $Z^1(\Gamma \times X, G)$ with respect to the topology $\mathcal{T}$ where G is a l.c.s.c. group.*

Since Γ is hyperfinite, it is orbit equivalent to a Borel $\mathbb{Z}$-action. By Corollary 3.9, the first cohomology group is an invariant of orbit equivalence. Hence, without loss of generality, it suffices to prove the statement for a single Borel automorphism $T \in \mathrm{Aut}(X, \mathcal{B})$. To prove the theorem, we will use the Kakutani tower construction for an aperiodic Borel automorphism which gives the possibility to use periodic automorphisms to approximate T. This construction is described in [27, Chapter 7] and [3]. We include it here for convenience of the reader.

Recall that a Borel set $A \subset X$ is called a *complete section* (or simply a *T-section*) for an automorphism $T \in \mathrm{Aut}(X, \mathcal{B})$ if every T-orbit meets A at least once. If there exists a complete Borel section A such that A meets every T-orbit exactly once, then T is called *smooth*. In this case, $X = \bigcup_{i \in \mathbb{Z}} T^i A$ and all the sets $T^i A$ are disjoint. A measurable set W is said to be *wandering* with respect to $T \in \mathrm{Aut}(X, \mathcal{B})$ if the sets $T^n W$, $n \in \mathbb{Z}$, are pairwise disjoint. The σ-ideal generated by all T-wandering sets in $\mathcal{B}$ is denoted by $\mathcal{W}(T)$. By the Poincaré recurrence lemma, one can state that given $T \in \mathrm{Aut}(X, \mathcal{B})$ and $A \in \mathcal{B}$ there exists $N \in \mathcal{W}(T)$ such that for each $x \in A \setminus N$ the points $T^n x$ return to A for infinitely many positive n and also for infinitely many negative n. The points from the set $A \setminus N$ are called *recurrent*.

Remark 4.2. Assume that all points from a given set A are recurrent for a Borel automorphism T. Then for $x \in A$, let $n(x) = n_A(x)$ be the smallest positive integer such that $T^{n(x)}x \in A$ and $T^i x \notin A$, $0 < i < n(x)$. Let $C_k = \{x \in A \mid n_A(x) = k\}$, $k \in \mathbb{N}$, then $T^k C_k \subset A$ and $\{T^i C_k \mid i = 0, \dots, k-1\}$ are pairwise disjoint. Note that some C_k's may be empty. Since $T^n x \in A$ for infinitely many positive and negative n, we obtain

$$\bigcup_{n\geq 0} T^n A = \bigcup_{n\in\mathbb{Z}} T^n A = X$$

and

$$X = \bigcup_{n\geq 0} T^n A = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{k-1} T^i C_k.$$

This union decomposes X into T-towers $\xi_k = \{T^i C_k \mid i = 0, \dots, k-1\}$, $k \in \mathbb{N}$, where C_k is the base and $T^{k-1}C_k$ is the top of ξ_k. Depending on T, the set of these towers $\{\xi_k\}$ can be, in general, countable.

Lemma 4.3. *Let $T \in \mathrm{Aut}(X, \mathcal{B})$ be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$. Then there exists a sequence (A_n) of Borel sets such that*

(i) $X = A_0 \supset A_1 \supset A_2 \supset \cdots$;

(ii) $\bigcap_n A_n = \emptyset$;

(iii) *A_n and $X \setminus A_n$ are complete T-sections, $n \in \mathbb{N}$;*

(iv) *every point in A_n is recurrent, $n \in \mathbb{N}$.*

Proof. See [2, Lemma 4.5.3], where (i)–(iii) have been proved in more general settings of countable Borel equivalence relations. It is shown in [27, Chapter 7] that one can refine the choice of (A_n) to get (iv). □

Definition 4.4. A sequence of Borel sets satisfying conditions (i)–(vi) of Lemma 4.3 is called a *vanishing sequence of markers*.

Proposition 4.5. *Let $T \in \mathrm{Aut}(X, \mathcal{B})$ be an aperiodic Borel automorphism of a standard Borel space $(X, \mathcal{B})$. Then there exists a sequence of periodic automorphisms (P_n) of $(X, \mathcal{B})$ converging to T in the uniform topology (see Definition 2.10). Moreover, the periodic automorphisms P_n can all be taken from $[T]$.*

Proof. This propositions was proved in [3, Section 2]. We give the proof here as it will be used in Lemma 4.6.

Let (A_n) be a vanishing sequence of markers for T. Then, as we have seen above, A_n generates a decomposition of X into T-towers

$$\xi_k(n) = \{T^i C_k(n) \mid i = 0, \dots, k-1\}$$

and $\bigcup_k C_k(n) = A_n$. Define

$$P_n x = \begin{cases} Tx & \text{if } x \notin B_n = \bigcup_{k=1}^{\infty} T^{k-1} C_k(n), \\ T^{-k+1} x & \text{if } x \in T^{k-1} C_k(n) \text{ for some } k. \end{cases} \tag{4.1}$$

Then P_n belongs to $[T]$, and the period of P_n on $\xi_k(n)$ is k. Note that P_n equals T everywhere on X except the set B_n which is the union of the tops of the towers.

It follows from Lemma 4.3 that (A_n) is a decreasing sequence of Borel subsets such that $\bigcap_n A_n = \emptyset$. This means that for any $x \in X$ there exists $n(x)$ such that $x \notin A_n$, $n \geq n(x)$. Moreover, if for some $x \in X$, $P_n x = Tx$, then $P_{n+k} x = Tx$ for all k. These facts prove that, for every x, the sequence $(P_n x)$ is eventually stabilized and it is and equal to Tx. Hence, P_n converges to T in the topology τ. □

Lemma 4.6 is well known in the theory of dynamical systems. We include it here for convenience of the reader.

Lemma 4.6 (folklore). (1) *Let P be a periodic automorphism of a standard Borel space $(X, \mathcal{B})$. Then any cocycle of P is a coboundary.*

(2) *The same result holds for a smooth automorphism of a standard Borel space $(X, \mathcal{B})$.*

Proof. (1) Let $a \in Z^1(P \times X, G)$, be a cocycle for P taking value in l.c.s.c. abelian group G with identity 0. Denote by C_k the base of P-tower ξ_k where P has period k. Then X is the disjoint union of ξ_k. We define a Borel function $f : X \to G$ by setting $f(x) = f_k(x), x \in \xi_k, k \in \mathbb{N}$, where

$$f_k(x) = \begin{cases} a(P^j, P^{-j} x) & \text{if } x \in P^j C_k \text{ for } 1 \leq j \leq k-1, \\ 0 & \text{if } x \in C_k. \end{cases}$$

It suffices to check that a is a coboundary on every tower ξ_k. For every $x \in X$, there exist k and $j \in \{0, \dots, k-1\}$ such that $x \in P^j C_k$. Let $n \in \mathbb{N}$, then $P^n x \in P^m C_k$ where $n = m - j + ik$. Therefore, we have

$$f_k(P^n x) - f_k(x) = a(P^m, P^{-j} x) - a(P^j, P^{-j} x).$$

Since $0 = a(P^j P^{-j}, x) = a(P^j, P^{-j}x) + a(P^{-j}, x)$, we obtain

$$f_k(P^n x) - f_k(x) = a(P^m, P^{-j}x) + a(P^{-j}, x) = a(P^m P^{-j}, x) = a(P^n, x).$$

Hence, a is a coboundary.

Statement (2) is proved analogously. □

Let $T \in \mathrm{Aut}(X, \mathcal{B})$ and f be a Borel function on X. By $a(f)$ we denote the cocycle generated by f:

$$a(f)(j,x) = \begin{cases} f(x) + f(Tx) + \cdots + f(T^{j-1}x) & \text{if } j \geq 1, \\ 0 & \text{if } j = 0, \\ -f(T^{-1}x) - f(T^{-2}x) - \cdots - f(T^j x) & \text{if } j \leq -1. \end{cases} \tag{4.2}$$

Lemma 4.7. *Suppose a sequence of Borel functions (f_i) converges to f in the topology $\mathcal{T}$. Then the sequence of cocycles $a(f_i)$ converges to $a(f)$, i.e., for every $j \in \mathbb{Z}$,*

$$a(f_i)(j,x) \xrightarrow{\mathcal{T}} a(f)(j,x), \quad i \to \infty.$$

Proof. To prove the lemma, we need to show that for any positive ϵ and δ and for any finite set of Borel probability measures $\mu_1, \ldots, \mu_n$ there exists $N \in N$ such that

$$\mu_l(\{x : |a_i(j,x) - a(j,x)| > \epsilon\}) < \delta, \quad l = 1, \ldots, n. \tag{4.3}$$

Fix a natural number j (the case of negative j is considered similarly). Take a finite set of Borel probability measures $\mu_1, \ldots, \mu_n$. Define

$$\{\nu_1, \ldots, \nu_s\} = \{\mu_i \circ T^k : i = 1, \ldots, n,\ k = 0, 1, \ldots, j-1\}$$

(here $s = ij$). It follows from the condition of the lemma that for any positive ϵ_1 and δ_1 there exists $N = N(\epsilon_1, \delta_1) \in \mathbb{N}$ such that for all $i > N$

$$\nu_l(\{x : |f_i - f| > \epsilon_1\}) < \delta_1, \quad l = 1, \ldots, s. \tag{4.4}$$

For convenience, we introduce the following sets

$$A_k(i, \epsilon_1) = \{x : |f_i \circ T^k - f \circ T^k| > \epsilon_1\}, \quad k = 0, \ldots, j-1,$$

and

$$C(i, \epsilon) = \{x : |a_i(j,x) - a(j,x)| > \epsilon\}.$$

Denote

$$S(i, \epsilon) = \Big\{x : \sum_{k=0}^{j-1} |f_i(T^k x) - f(T^k x)| > \epsilon\Big\}.$$

Since

$$|a_i(j,x) - a(j,x)| \leq \sum_{k=0}^{j-1} |f_i(T^k x) - f(T^k x)|,$$

we see that $C(i,\epsilon) \subset S(i,\epsilon)$. Take $\epsilon_1 = \frac{\epsilon}{j}$ and $\delta_1 = \frac{\delta}{j}$; then it follows from the above definitions that

$$\bigcup_{k=0}^{j-1} A_k\Big(i, \frac{\epsilon}{j}\Big) \supset S(i,\epsilon).$$

We need to prove that $\mu_l(C(i,\epsilon)) < \delta$ for all sufficiently large i and $l = 1,\dots,n$. Indeed, it follows from (4.4) that, for $i > N(\epsilon_1,\delta_1)$,

$$\begin{aligned}\mu_l(C(i,\epsilon)) \leq \mu_l(S(i,\epsilon)) &\leq \sum_{k=0}^{j-1} \mu_l\Big(A_k\Big(i,\frac{\epsilon}{j}\Big)\Big)\\ &= \sum_{k=0}^{j-1} \mu_l \circ T^k\Big(A_0\Big(i,\frac{\epsilon}{j}\Big)\Big) < j\frac{\delta}{j} = \delta.\end{aligned}$$

This proves the lemma. □

Proposition 4.8. *Let $a: \mathbb{Z} \times X \to G$ be a cocycle of an aperiodic $T \in \mathrm{Aut}(X,\mathcal{B})$. Then there exists a sequence of coboundaries (a_n) of T such that (a_n) converges to a in the topology $\mathcal{T}$ (see Remark 3.4 and Definition 3.2).*

Proof. It is obvious that, for any cocycle $a: \mathbb{Z} \times X \to G$ of $T \in \mathrm{Aut}(X,\mathcal{B})$, there is a Borel function f such that $a = a(f)$, i.e.,

$$a(j,x) = \begin{cases} f(x) + f(Tx) + \cdots + f(T^{j-1}x) & \text{if } j \geq 1,\\ 0 & \text{if } j = 0,\\ -f(T^{-1}x) - f(T^{-2}x) - \cdots - f(T^j x) & \text{if } j \leq -1.\end{cases} \tag{4.5}$$

In the proof, we will use the notation introduced in this section above. By Proposition 4.5, for every $T \in \mathrm{Aut}(X,\mathcal{B})$, there exists a sequence of periodic automorphisms (P_i) of $(X,\mathcal{B})$ converging to T in the topology τ (see Definition 2.10). It can be easily seen that P_i and P_{i+1} agree (that is $P_i x = P_{i+1}x$ everywhere except on top of the T-towers $\xi_k(i)$ built over A_i where (A_i) is a vanishing sequence of markers. Let D_i denote the union of the top levels of T-towers $\xi_k(i)$. Since $D_i \supseteq D_{i+1}$ and $\bigcap_i A_i = \emptyset$, we see that $\bigcap_i D_i = \emptyset$. Therefore, for every x, there exists a smallest number $n(x)$ such that, for all $i \geq n(x)$, $P_i x$ are all the same and equal to Tx.

Next, we define $K_j := \{x \in X : n(x) = j\}$, $j \in \mathbb{N}$. Note that $K_j \subset K_{j+1}$ and $\bigcup_j K_j = X$. Fix a finite set of probability measures $\mu_1, \mu_2, \dots \mu_n \in \mathcal{M}_1(X)$ and take $\epsilon > 0$. Then there exists $j \in \mathbb{N}$, such that $\mu_l(K_j) > 1-\epsilon$ for $l = 1,2,\dots,n$.

We recall that the periodic automorphisms P_i are taken from the full group $[T]$ and therefore the cocycle $a \in Z^1(\Gamma \times X, T)$ can be extended to P_i. This observation allows us to define

$$f_n(x) := a(P_n, x) \quad \text{for all } x \in X.$$

By Lemma 4.6, every cocycle of P_n is a coboundary. Hence there exists a sequence of Borel functions $g_n \colon X \to G$ such that $f_n(x) = g_n(x) - g_n(P_n x)$. Moreover, recall that $P_n x = Tx$ for every $x \in K_n$. As a result, for every $x \in K_n$ we have $f_n(x) = a(P_n, x) = a(T, x) = f(x)$. We further define a sequence of Borel functions $F_n \colon X \to G$ as follows:

$$F_n(x) = g_n(x) - g_n(Tx) \quad \text{for all } x \in X.$$

By definition, the function F_n is a T-coboundary for every n.

It remains to show that $F_n \xrightarrow{\mathcal{T}} f$ (see Definition 3.2). For this, we prove that for every $\epsilon, \delta > 0$ there exists $n \in \mathbb{N}$ such that

$$\mu_l(\{x \colon |F_n(x) - f(x)| > \epsilon\}) < \delta \quad \text{for all } l = 1, 2, \ldots, n. \tag{4.6}$$

Note that if $x \in K_n$, then $f_n(x) - f(x)$ and

$$|F_n(x) - f(x)| = |g_n(P_n x) - g_n(Tx)| = 0.$$

Hence,

$$\mu_l(\{x \colon |F_n(x) - f(x)| > \epsilon\}) \subset X \setminus K_n \quad \text{for all } l = 1, 2, \ldots, n.$$

For every $\delta > 0$, we can find N such that for all $n \geq N$, $\mu_l(X \setminus K_n) < \delta$ for $l = 1, 2, \ldots, n$, and then (4.6) follows.

To finish the proof, we define the sequence of T-coboundaries (a_n) by functions F_n as in (4.5). It follows from Lemma 4.7 that the converges of (F_n) to the function f in the topology $\mathcal{T}$ implies that $a_n(F_n)$ converges to $a(f)$ in $\mathcal{T}$. It completes the proof. □

Proof of Theorem 4.1. In light of Theorem 2.3, Proposition 4.8 implies Theorem 4.1. □

5. Cocycle over odometer action

The goal of this section is to describe explicitly cocycles defined by 2-odometers. In fact, the results of this section can be used for arbitrary uniquely ergodic Borel automorphisms since they are Borel isomorphic to the 2-odometer. We will use the following definition of the 2-odometer which is equivalent to Definition 2.4.

Consider the space $(X = \{0,1\}^{\mathbb{N}}, \mathcal{B})$, where $\mathcal{B}$ is the Borel sigma-algebra generated by cylinder sets. Let $\Gamma \subset \mathrm{Aut}(X, \mathcal{B})$ be the group of Borel automorphisms generated by automorphisms $\langle \delta_1, \dots, \delta_n, \dots \rangle$ where δ_n acts on $x = (x_i) \in X$ by the formula

$$(\delta_n x)_i = \begin{cases} x_i & \text{if } i \neq n, \\ x_i + 1 \pmod 2 & \text{if } i = n. \end{cases} \tag{5.1}$$

We see that every δ_n is periodic, $\delta_n^2 = \mathbf{1}$, and any two generators δ_n, δ_k commute. Obviously, the orbit equivalence relation $E_X(\Gamma)$ is *hyperfinite* and preserves the product measure $\mu = \bigotimes_i \mu_i$ where $\mu_i(\{0\}) = \mu_i(\{1\}) = 1/2$. The group Γ is orbit equivalent to the 2-odometer acting on $(\{0,1\}^{\mathbb{N}}, \mathcal{B})$.

Cocycles over odometers have been extensively studied in ergodic theory. We refer, in particular, to the papers [13, 14] where the authors proved several important results. Firstly, it was shown that every cocycle is cohomologous to a cocycle that takes values in a countable subgroup H of G, and, secondly, cocycles with dense range are unique in the following sense: let α and β be two cocycles with values in G such that the skew products $\Gamma(\alpha)$ and $\Gamma(\beta)$ are ergodic, then there exists an automorphism R in the normalizer $N[\Gamma]$ such that α and $\beta \circ R$ are cohomologous (see Section 1).

We use a similar approach to prove the first result in the setting of Borel dynamics. We do not know whether the second result holds. We remark that for consistency with other parts of this paper our proof is given for an abelian group G though the same proof works for non-abelian groups.

We reprove the following statement that was implicitly formulated in [13].

Proposition 5.1. *Let the group $\Gamma = \langle \delta_1, \dots \delta_n, \dots \rangle$ of Borel automorphisms of $\{0,1\}^{\mathbb{N}}$ be defined as in (5.1). Then for every cocycle $c\colon \Gamma \times X \to G$, there exists a sequence of Borel functions $(f_n\colon X \to G)_{n\in\mathbb{N}}$ such that*

$$\begin{aligned} c(\delta_n, x) = {} & x_1 f_1(\delta_n x) + \dots + x_{n-1} f_{n-1}(\delta_n x) \\ & + (-1)^{x_n} f_n(x) - x_{n-1} f_{n-1}(x) - \dots - x_1 f_1(x), \end{aligned} \tag{5.2}$$

where the function f_n is invariant with respect to $\delta_1, \delta_2, \dots, \delta_n$, $n \in \mathbb{N}$.

Conversely, let $(f_n\colon X \to G)_{n\in\mathbb{N}}$, be a sequence of Borel maps such that each f_n is invariant with respect to $\delta_1, \delta_2, \dots, \delta_n$. Then $(f_n)_{n\in\mathbb{N}}$ generates a cocycle c according to (5.2).

Proof. Since the transformations δ_i, $i \in \mathbb{N}$, are pairwise commuting, relation (5.2) can be extended to all $\gamma = \delta_{i_1} \cdots \delta_{i_k} \in \Gamma$. First we show that if there exist a sequence of functions (f_n) with the invariance property as described above, then (5.2) defines a cocycle of Γ. To do this, we show that

$$c(\delta_n \delta_k, x) = c(\delta_k \delta_n, x) \quad \text{and} \quad c(\delta_n^2, x) = 0 \quad \text{for all } n, k \in \mathbb{N}, x \in X.$$

In other words, we need to prove that the definition of c by (5.2) gives the same result for two ways to compute $c(\delta_n\delta_k, x)$.

By the cocycle identity, we have $c(\delta_n\delta_k, x) = c(\delta_n, \delta_k x) + c(\delta_k, x)$. For definiteness, we can assume that $n > k$. In what follows, we will use the obvious property $(\delta_k x)_i = x_i$ if $i \neq k$ and $(\delta_k x)_k = x_k + 1 \pmod 2$. Then

$$\begin{aligned} c(\delta_n, \delta_k x) = x_1 f_1(\delta_n\delta_k x) + \cdots + (\delta_k x)_k f_k(\delta_n\delta_k x) + \cdots + x_{n-1} f_{n-1}(\delta_n\delta_k x) \\ + (-1)^{(\delta_k x)_n} f_n(\delta_k x) - x_{n-1} f_{n-1}(\delta_k x) - \cdots \\ - (\delta_k x_k) f_k(\delta_k x) - \cdots - x_1 f_1(\delta_k x). \end{aligned}$$

Using the fact that, for each $i \in \mathbb{N}$, the function f_i is invariant with respect to $\delta_1, \delta_2, \dots, \delta_i$, we get

$$\begin{aligned} c(\delta_n, \delta_k x) = x_1 f_1(\delta_n\delta_k x) + \cdots + (\delta_k x)_k f_k(\delta_n x) + \cdots \\ + x_{n-1} f_{n-1}(\delta_n x) + (-1)^{(\delta_k x)_n} f_n(x) \\ - x_{n-1} f_{n-1}(x) - \cdots - (\delta_k x)_k f_k(x) \\ - x_{k-1} f_{k-1}(\delta_k x) - \cdots - x_1 f_1(\delta_k x). \end{aligned}$$

Similarly, we have by (5.2)

$$\begin{aligned} c(\delta_k, x) = x_1 f_1(\delta_k x) + \cdots + x_{k-1} f_{k-1}(\delta_k x) + (-1)^{x_k} f_k(x) \\ - x_{k-1} f_{k-1}(x) - \cdots - x_1 f_1(x). \end{aligned}$$

After taking the sum and simplifying, we obtain that

$$\begin{aligned} c(\delta_n\delta_k, x) = x_1 f_1(\delta_n\delta_k x) + \cdots + (\delta_k x)_k f_k(\delta_n x) + \cdots + x_{n-1} f_{n-1}(\delta_n x) \\ + (-1)^{(\delta_k x)_n} f_n(x) - x_{n-1} f_{n-1}(x) - \cdots - (\delta_k x)_k f_k(x) \\ + (-1)^{x_k} f_k(x) - x_{k-1} f_{k-1}(x) - \cdots - x_1 f_1(x). \end{aligned} \tag{5.3}$$

Next, we represent $c(\delta_k\delta_n, x)$ as $c(\delta_k, \delta_n x) + c(\delta_n, x)$ and compute noticing that $(\delta_n x)_k = x_k$:

$$\begin{aligned} c(\delta_k, \delta_n x) = x_1 f_1(\delta_k\delta_n x) + \cdots + (-1)^{x_k} f_k(\delta_n x) \\ - x_{k-1} f_{k-1}(\delta_n x) - \cdots - x_1 f_1(\delta_n x) \end{aligned}$$

and

$$\begin{aligned} c(\delta_n, x) = x_1 f_1(\delta_n x) + \cdots + x_{k-1} f_{k-1}(\delta_n x) + x_k f_k(\delta_n x) + \cdots \\ + (-1)^{x_n} f_n(x) - x_{n-1} f_{n-1}(x) - \cdots \\ - x_{k+1} f_{k+1}(x) - x_k f_k(x) - x_{k-1} f_{k-1}(x) - \cdots - x_1 f_1(x). \end{aligned}$$

Thus, we get

$$\begin{aligned} c(\delta_k, \delta_n x) + c(\delta_n, x) = x_1 f_1(\delta_k\delta_n x) + \cdots + (-1)^{x_k} f_k(\delta_n x) + x_k f_k(\delta_n x) + \cdots \\ + (-1)^{x_n} f_n(x) - x_{n-1} f_{n-1}(x) - \cdots \\ - x_k f_k(x) - x_{k-1} f_{k-1}(x) - \cdots - x_1 f_1(x). \end{aligned} \tag{5.4}$$

One can easily see (by considering all possible values for x_k) that the following relations hold:

$$(\delta_k x)_k f_k(\delta_n x) = (-1)^{(\delta_n x)_k} f_k(\delta_n x) + x_k f_k(\delta_n x)$$

and

$$-x_k f_k(x) = (\delta_k x)_k f_k(x) + (-1)^{x_k} f_k(x).$$

Comparing (5.3) and (5.4), we conclude that

$$c(\delta_n, \delta_k x) + c(\delta_k, x) = c(\delta_k, \delta_n x) + c(\delta_n x)$$

for all distinct integers n, k.

To see that, for every $n \in \mathbb{N}$, the cocycle c has the property $c(\delta_n^2, x) = 0$, we observe

$$\begin{aligned} c(\delta_n, \delta_n x) + c(\delta_n, x) = {} & (\delta_n x)_1 f_1(\delta_n^2 x) + \cdots + (-1)^{(\delta_n x)_n} f_n(\delta_n x) - \cdots \\ & - (\delta_n x)_{n-1} f_{n-1}(\delta_n x) - \cdots - (\delta_n x)_1 f_1(\delta_n x) \\ & + x_1 f_1(\delta_n x) + \cdots + (-1)^{x_n} f_n(x) - \cdots - x_1 f_1(x). \end{aligned}$$

Because $\delta_n^2 = \mathbf{1}$ and f_n is δ_n-invariant, we see that

$$c(\delta_n^2, x) = (-1)^{(\delta_n x)_n} f_n(x) + (-1)^{x_n} f_n(x) = 0.$$

This proves that relation (5.2) defines a cocycle of the group Γ.

Conversely, if a cocycle c is given, then the functions f_n are determined as follows: set $f_n'(x) = c(\delta_n, x)$ for x from the cylinder set $A_n(0, \dots, 0)$ generated by the first n zeros. Then f_n' is extended on X by invariance with respect to the subgroup $\langle \delta_1, \dots \delta_n \rangle$ to obtain the function f_n. □

Let α and β be two cocycles of Γ, which are determined as in Proposition 5.1 by sequences of Borel functions $f_n \colon X \to G$ and $\overline{f_n} \colon X \to G$, respectively. Define two new sequences of functions $\psi_n \colon X \to G$ and $\overline{\psi_n} \colon X \to G$ as follows:

$$\psi_n(x) = -x_n f_n(x) - x_{n-1} f_{n-1}(x) - \cdots - x_1 f_1(x), \tag{5.5}$$

$$\overline{\psi_n}(x) = -x_n \overline{f_n}(x) - x_{n-1} \overline{f_{n-1}}(x) - \cdots - x_1 \overline{f_1}(x). \tag{5.6}$$

We denote by $\{W_i\}_{i=1}^{\infty}$ a system of neighborhoods of $0 \in G$ with the following properties:

(i) W_i is compact for every i;

(ii) W_i is symmetric for every i (i.e., $W_i = -W_i$);

(iii) $W_{i+1} + W_{i+1} \subset W_i, i \in \mathbb{N}$.

Proposition 5.2. *Let α and β be two cocycles of the group Γ with values in a l.c.s.c. group G. Let (f_n) and $(\overline{f_n})$ be the sequences of functions determined by α and β, respectively, according to Proposition 5.1. Assume that, for all $x \in X$ and $n \in \mathbb{N}$,*

$$f_n(x) - \overline{f_n}(x) \in W_n,$$

where the neighborhoods (W_n) satisfy conditions (i)–(iii). *Then the cocycles α and β are cohomologous.*

Proof. Define a sequence of functions

$$g_n(x) := -\psi_n(x) + \overline{\psi_n}(x), \quad n \in \mathbb{N},$$

where ψ_n and $\overline{\psi_n}$ are as in (5.5) and (5.6). Thus for all $n, k \in \mathbb{N}$, we have

$$g_{n+k}(x) - g_n(x) = -\psi_{n+k}(x) + \overline{\psi_{n+k}}(x) + \psi_n(x) - \overline{\psi_n}(x).$$

It follows from (5.5) and (5.6) that

$$-\psi_{n+k}(x) = x_{n+k} f_{n+k}(x) + \cdots + x_{n+1} f_{n+1}(x) + \psi_n(x),$$

and a similar formula holds for $\overline{\psi_{n+k}}(x)$. Hence,

$$\begin{aligned} g_{n+k}(x) - g_n(x) = x_{n+k} f_{n+k}(x) - x_{n+k} \overline{f_{n+k}}(x) + \cdots \\ + x_{n+1} f_{n+1}(x) - x_{n+1} \overline{f_{n+1}}(x). \end{aligned}$$

It follows from the condition of Proposition that $f_{n+i}(x) - \overline{f_{n+i}}(x) \in W_{n+i}$ for all $i, n \in \mathbb{N}$. Hence,

$$x_{n+i} f_{n+i}(x) - x_{n+i} \overline{f_{n+i}}(x) \in W_{n+i} \quad \text{for all } i, n \in \mathbb{N}$$

By the choice of W_i, we obtain

$$\begin{aligned} g_{n+k}(x) - g_n(x) &\in W_{n+k} + W_{n+k-1} + \cdots + W_{n+1} \\ &\subset W_{n+k-1} + W_{n+k-1} + \cdots + W_{n+1} \\ &\subset W_{n+k-2} + \cdots + W_{n+1} \\ &\vdots \\ &\subset W_n. \end{aligned}$$

Using the Cauchy criterion, there exists a Borel function $g\colon X \to G$ such that, g_n converges uniformly to g on X.

Without loss of generality, we can assume that $n \geq k$. Since $(\delta_n x)_i = x_i$ where $i = 1, \ldots, k-1$, and

$$-\overline{\psi_{k-1}}(\delta_k(x)) = x_1 \overline{f_1}(\delta_k x) + \cdots + x_{k-1} \overline{f_1}(\delta_k x),$$

$$\overline{\psi_{k-1}}(x) = -x_{k-1} \overline{f_{k-1}}(x) + \cdots + -x_1 \overline{f_1}(x),$$

we can compute $\hat{\beta}(\delta_k, x) = g_n(\delta_k x) + \beta(\delta_k, x) - g_n(x)$ as follows:

$$\begin{aligned}\hat{\beta}(\delta_k, x) = & -\psi_n(\delta_k x) + \overline{\psi_n}(\delta_k x) + x_1 \overline{f_1}(\delta_k x) + \cdots \\ & + x_{k-1} \overline{f_1}(\delta_k x) + (-1)^{x_k} \overline{f_k}(x) \\ & - x_{k-1} \overline{f_{k-1}}(x) - \cdots - x_1 \overline{f_1}(x) - (-\psi_n(x) + \overline{\psi_n}(x)) \\ = & -\psi_n(\delta_k x) + \overline{\psi_n}(\delta_k x) - \overline{\psi_{k-1}}(\delta_k(x)) + (-1)^{x_k} \overline{f_k}(x) + \overline{\psi_{k-1}}(x) \\ & + \psi_n(x) - \overline{\psi_n}(x) \\ = & -\psi_n(\delta_k x) - x_n \overline{f_n}(\delta_k x) - x_{n-1} \overline{f_{n-1}}(\delta_k x) - \cdots - (\delta_k x)_k \overline{f_k}(\delta_k x) \\ & + (-1)^{x_k} \overline{f_k}(x) + x_n \overline{f_n}(x) + \cdots + x_k \overline{f_k}(x) + \psi_n(x).\end{aligned}$$

Since $n \geq k$, the function f_n is invariant with respect to $\delta_1, \ldots, \delta_k$, we have

$$\begin{aligned}\hat{\beta}(\delta_k, x) = & -\psi_n(\delta_k x) - x_n \overline{f_n}(x) - x_{n-1} \overline{f_{n-1}}(x) - \cdots \\ & - (\delta_k x)_k \overline{f_k}(x) + (-1)^{x_k} \overline{f_k}(x) \\ & + x_k \overline{f_k}(x) + \cdots + x_n \overline{f_n}(x) + \psi_n(x).\end{aligned}$$

After simplifying, we obtain that

$$\hat{\beta}(\delta_k, x) = -\psi_n(\delta_k x) - (\delta_k x)_k \overline{f_k}(x) + (-1)^{x_k} \overline{f_k}(x) + x_k \overline{f_k}(x) + \psi_n(x).$$

It remains to show that

$$(\delta_k x)_k \overline{f_k}(x) + (-1)^{x_k} \overline{f_k}(x) + x_k \overline{f_k}(x) = 0. \tag{5.7}$$

Indeed, if $x_k = 0$, then $x_k \overline{f_k}(x) = 0$, and $(\delta_k x)_k = 1$ implies that

$$-(\delta_k x)_k \overline{f_k}(x) = -\overline{f_k}(x).$$

If $x_k = 1$, then $-(\delta_k x)_k \overline{f_k}(x) = 0$ and $(-1)^{x_k} \overline{f_k}(x) = -\overline{f_k}(x)$. Thus, in both cases we get

$$g_n(\delta_k x) + \beta(\delta_k, x) - g_n(x) = -\psi_n(\delta_k x) + \psi_n(x).$$

On the other hand,

$$\begin{aligned}-\psi_n(\delta_k x) + \psi_n(x) = & \, x_n f_n(\delta_k x) + \cdots + (\delta_k x_k) f_k(\delta_k x) + \cdots \\ & + x_1 f_1(\delta_k x) - x_n f_n(x) - \cdots - x_k f_k(x) - \cdots - x_1 f_1(x).\end{aligned}$$

By invariance of f_n with respect to of $\delta_1, \ldots, \delta_k$, we can write down the above equality as

$$\begin{aligned}-\psi_n(\delta_k x) + \psi_n(x) & = x_n f_n(x) + \cdots + (\delta_k x)_k f_k(x) + x_{k-1} f_{k-1}(\delta_k x) + \cdots \\ & \quad + x_1 f_1(\delta_k x) - x_n f_n(x) - \cdots - x_k f_k(x) - \cdots - x_1 f_1(x) \\ & = -\psi_{k-1}(\delta_k x) + (-1)^{x_k} f_k(x) + \psi_{k-1}(x) \\ & = \alpha(\delta_k, x).\end{aligned} \tag{5.8}$$

The first equality in (5.8) is due to relation (5.7), applied to the function f_k, and the second equality is, in fact, a short form of the definition of α.

Thus, we proved that, for every $n \geq k$ and all $x \in X$,

$$g_n(\delta_k x) + \beta(\delta_k, x) - g_n(x) = \alpha(\delta_k, x).$$

Since $g_n(x) \to g(x)$ as $n \to \infty$, we conclude that

$$g(\delta_k x) + \beta(\delta_k, x) - g(x) = \alpha(\delta_k, x).$$

Because the group Γ is generated by $\delta_k, k \in \mathbb{N}$,, we see that the cocycles α and β are cohomologous. □

Theorem 5.3. *Let Γ be a free group of Borel automorphisms which is orbit equivalent to the 2-odometer. Let α be a Γ-cocycle with values in a l.c.s.c. group G and H a dense countable subgroup of G. Then the cocycle $\alpha\colon \Gamma \times X \to G$ is cohomologous to a cocycle β with values the subgroup H.*

Proof. Without loss of generality, we can consider cocycles of the 2-odometer. By Proposition 5.1 the cocycle α is determined by the functions $f_n\colon X \to G$, $n \in \mathbb{N}$. Take a sequence of symmetric neighborhoods of 0 in G which satisfies the properties (i)–(iii) (see above). Approximate each function $f_n(x)$ by a function $\overline{f_n}(x)$ with values in H so that $f_n(x) - \overline{f_n}(x) \in W_n$ for each $x \in X$, and additionally, $\overline{f_n}(\delta_j x) = \overline{f_n}(x)$, for $1 \leq j \leq n$. Clearly it can be done because the functions f_n have this property.

Hence, we satisfy the conditions of Proposition 5.2. Construct the Γ-cocycle β which is determined by the sequence of functions $\overline{f_n}(x)$, then β is cohomologous to α. □

6. Borel version of Gottschalk–Hedlund theorem

The following is a version of the Gottschalk–Hedlund (G–H) theorem for Borel automorphisms. Our proof is a modification of the proof of Gottschalk–Hedlund theorem given by F. Browder [6].

We will consider homeomorphisms of a Polish space. It is well known that every Borel automorphism admits a continuous model, i.e., it is Borel isomorphic to a homeomorphism of a Polish space, see, e.g., [21]. We say that a homeomorphism $T \in \mathrm{Aut}(X, \mathcal{B})$ acting on a Polish space X is *minimal* if every T-orbit is dense in X, i.e., for every $x \in X$, $\overline{\{T^i x\colon i \in \mathbb{Z}\}} = X$. There exist Polish spaces that admit minimal homeomorphisms (we thank [34] for examples of such spaces).

We note that in Theorem 6.1 we consider *bounded* cocycles of homeomorphism of a Polish space, while the G–H theorem for topological dynamics (see [16]) has no such restriction. This is due to the fact that the underlying space in Theorem 6.1 is a non-compact Polish space. In topological dynamics *continuous* cocycles of homeomorphism of a compact space are studied. Here we study *Borel* cocycles of homeomorphisms of a non-compact Polish space. Hence, we have to limit our discussion to bounded cocycles. We do not know whether the result holds without this assumption.

In the proof of Theorem 6.1 we will use the following fact: every locally compact second countable group G has a left-invariant metric d which is proper, that is every closed d-bounded set in G is compact (see [8, Theorem 2.B.4]).

Theorem 6.1. *Let* $(X, \mathcal{B})$ *be a Polish space and* $T \in \mathrm{Aut}(X, \mathcal{B})$ *is a minimal homeomorphism of* $(X, \mathcal{B})$*. Let* $h\colon X \to G$ *be a bounded Borel map from* X *to a l.c.s.c. abelian group* G*. Then, the function* h *is a coboundary* (*i.e., there exists a bounded Borel function* $f\colon X \to G$ *such that* $f(Tx) - f(x) = h(x)$*,* $x \in X$)*, if and only if there exists* $M > 0$ *such that*

$$\sup_{x\in X}\Big|\sum_{k=-j}^{j} h(T^k x)\Big| \le M,$$

for all $j \ge 0$.

Before we begin to prove Theorem 6.1, we define some maps and prove Lemmas 6.3–6.5. Let $\psi\colon X \times G \to G$, as $\psi(x, g) = g + h(x)$ where $h(x)$ is the Borel map as in the statement of Theorem 6.1. Next, we define the skew product $X \times G \to X \times G$ as $\pi(x, g) = (Tx, \psi(x, g)) = (Tx, g + h(x))$.

Denote by $\mathrm{Orb}_\pi(x, g) = \bigcup_{n\in\mathbb{Z}}\{\pi^n(x, g)\}$ the orbit of (x, g) under π and by $F(x, g) = \overline{\mathrm{Orb}_\pi(x, g)}$ the orbit closure in $X \times G$. Let p_X and p_G denote the natural projections from $X \times G$ to X and G, respectively. We assume that for each point $(x, g) \in X \times G$ the set $p_G(F(x, g))$ is contained in a compact subset of G.

Remark 6.2. We note that the condition that $\sum_{k=-j}^{j} h(T^k x)$ is bounded in G for all $x \in X$ and $j \ge 0$ is equivalent to the fact that the orbit (with respect to π) of any point $(x, g) \in X \times G$ has a bounded and hence a precompact image in G under the projection map p_G of $X \times G$ into G. This in turn implies that $p_G(F(x, g))$ is contained in a compact subset of G.

Consider the family J of subsets F of $X \times G$ such that

$$J = \{F \mid F \text{ is a nonempty closed subset of} X \times G; (x, g) \in F \implies \pi(x, g) \in F; \\ p_G(F) \text{ is contained in a compact subset of } G\}.$$

Obviously, J is nonempty since, for any point $(x_0, g_0) \in X \times G$ the set $F(x_0, g_0)$ is in J.

Lemma 6.3. *If* $F \in J$*, then* $p_X(F) = X$.

Proof. Let $(x_0, g_0) \in F$. Since $\pi^n(x_0, g_0) \in F$, $p_X(\pi^n(x_0, g_0)) \in p_X(F)$. Thus $p_X(F)$ contains the dense set $\{T^k(x_0)\}$. Hence $p_X(F)$ is dense in X.

Next, for $F \subset X \times G$, we have $F \subset X \times \overline{p_G(F)}$ and $\overline{p_G(F)}$ is a compact set in G. Since the projection $p_X(F)$ is a closed map, we obtain that $p_X(F)$ is closed in X. We showed that $p_X(F)$ is dense and closed in X, hence $p_X(F) = X$. □

Lemma 6.4. *The family of sets J has a minimal element under inclusion. Every orbit closure $F(x, g)$, $(x, g) \in X \times G$, contains a minimal element of J.*

Proof. We use Zorn's lemma. Consider a totally ordered (with respect to inclusion) chain $\{F_\alpha\}$ in J. Let $F_0 = \bigcap_\alpha F_\alpha$. Then F_0 is a closed π-invariant set and $p_G(F_0)$ is clearly contained in a compact set of G. To prove that $F_0 \in J$, we show $F_0 \neq \emptyset$.

Let $x_0 \in X$, consider $G_\alpha = F_\alpha \cap p_X^{-1}(x_0)$. By Lemma 6.3, $p_X(F_\alpha) = X$, therefore G_α is a nonempty closed subset for any $x_0 \in X$ and any α. Moreover, $G_\alpha \subset x_0 \times \overline{p_G(F_\alpha)}$. We note that $x_0 \times \overline{p_G(F_\alpha)}$ is compact since it is mapped homeomorphically by p_G to a compact set $\overline{p_G(F_\alpha)}$. Since G_α is compact for each α, $G_0 = \bigcap_\alpha G_\alpha$ is non-empty. Since $G_0 \subset F_0$ we conclude that F_0 is non-empty. □

Let $\xi\colon G \to G$ be a homeomorphism of G such that it commutes with ψ i.e., $\psi(x, \xi g) = \xi\psi(x, g)$ for all $x \in X$ and $g \in G$. Let $S_\xi\colon X \times G \to X \times G$ be a homeomorphism defined by $S_\xi(x, g) = (x, \xi g) = (x, \xi g)$.

Lemma 6.5. *Let F_0 be a minimal element of J and suppose that for a fixed point $x_0 \in X$, the points (x_0, g_0), (x_0, g_1) lie in F_0. Suppose further that there exists a homeomorphism ξ of G onto itself such that it commutes with ψ and $\xi(g_0) = g_1$. Then $S_{\xi_k} F_0 = F_0$.*

Proof. Since ξ commutes with ψ, we get

$$\begin{aligned} S_\xi \pi(x_0, g_0) &= S_\xi(Tx_0, \psi(x_0, g_0)) = (Tx_0, \xi\psi(x_0, g_0)) \\ &= (Tx_0, \psi(x_0, \xi g_0)) = \pi(x_0, \xi g_0) = \pi S_\xi(x_0, g_0). \end{aligned}$$

Thus $S_\xi \pi^n = \pi^n S_\xi$, i.e., $S_\xi(\mathrm{Orb}_\pi(x_0, g_0)) = \mathrm{Orb}_\pi(x_0, \xi g_0)$. Using the fact that S_ξ is a homeomorphism we get $S_\xi F(x_0, g_0) = F(x_0, \xi g_0)$. Since F_0 is a minimal element of J, by assumption it contains both (x_0, g_0) and (x_0, g_1) we get $F_0 = F(x_0, g_0) = F(x_0, g_1)$. But $S_\xi F_0 = S_\xi F(x_0, g_0) = F(x_0, \xi g_0) = F(x_0, g_1) = F_0$. □

Proof of Theorem 6.1. Let $B(0,r)$ denote the ball of radius r centered at $0 \in G$ with respect to a translation invariant metric on G. We first assume that there exists a bounded Borel function $f: X \to G$ such that $f(x) \in B(0,m)$ for some $m > 0$, and $h(x) = f(Tx) - f(x)$ for all $x \in X$. Then, it is clear that

$$\sum_{k=-j}^{j} h(T^k x) = -f(T^{-j}x) + f(T^{(j+1)}x) \in B(0,2m).$$

Hence, $\sum_{k=-j}^{j} h(T^k x)$ is bounded in G for all x.

Conversely, assume that, for all $x \in X$ and for all $j \geq 0$, $\sum_{k=-j}^{j} h(T^k x)$ is bounded in G. Thus, for any point $(x_0, g_0) \in X \times G$, the set $p_G(F(x_0, g_0))$ is contained in a compact set of G (see Remark 6.2). Therefore, we can apply Lemmas 6.3–6.5.

Let F_0 be a minimal closed invariant set in $X \times G$ with respect to π. We will show that, for any $x_0 \in X$, F_0 contains at most one point of the form (x_0, g). To see this, assume that for some $x_0 \in X$, the set $p_X^{-1} x_0 \cap F_0$ contains two distinct points (x_0, g_0) and (x_0, g_1). Let $k = g_1 - g_0$; then the map $\xi_k(g) = g + k$ is a homeomorphism of G onto itself which commutes with ψ, and $\xi_k(g_0) = g_1$. By Lemma 6.5, $S_{\xi_k} F_0 = F_0$ where $S_{\xi_k}(x,g) = (x, g+k)$. Hence, $S_{\xi_k}^i F_0 = F_0$ for any integer i. This contradicts the boundness of $p_G(F_0)$. Thus, F_0 has at most one point (x_0, g_0) for arbitrary $x_0 \in X$. Therefore, we can uniquely define a function $f: X \to G$ by the condition $f(x_0) = g_0$ where $(x_0, g_0) \in F_0$. By Lemma 6.3, the function f is defined at every point of X. Moreover, f can also be considered as a function on X with values in the compact set $\overline{p_G(F_0)}$.

Recall following result: If Y is a topological space, Z a compact space, and $s: Y \to Z$ is a function, then the graph of s is closed if and only if s is continuous.

Since the set F_0 is the graph of f and F_0 is closed, we conclude that f is a continuous function. Finally, for $\pi(x_0, f(x_0)) \in F_0$, we have $(Tx_0, f(x_0) + h(x_0)) \in F_0$. Thus, by definition of f, we get $f(Tx_0) = f(x_0) + h(x_0)$ as needed. □

Acknowledgments. The authors are very thankful to Palle Jorgensen and Paul Muhly for their interest to this work and useful discussions. We thank the referee for the careful reading and valuable suggestions.

References

[1] H. Becker, Cocycles and continuity. *Trans. Amer. Math. Soc.* **365** (2013), no. 2, 671–719. Zbl 1315.03081 MR 2995370

[2] H. Becker and A. S. Kechris, *The descriptive set theory of Polish group actions*. London Mathematical Society Lecture Note Series, 232. Cambridge University Press, Cambridge, 1996. Zbl 0949.54052 MR 1425877

[3] S. Bezuglyi, A. H. Dooley, and J. Kwiatkowski, Topologies on the group of Borel automorphisms of a standard Borel space. *Topol. Methods Nonlinear Anal.* **27** (2006), no. 2, 333–385. Zbl 1136.37002 MR 2237460

[4] S. Bezuglyi and K. Medynets, Smooth automorphisms and path-connectedness in Borel dynamics. *Indag. Math.* (*N.S.*) **15** (2004), no. 4, 453–468. Zbl 1082.37010 MR 2114930

[5] S. I. Bezuglyi and V. Y. Golodets, Weak equivalence and the structures of cocycles of an ergodic automorphism. *Publ. Res. Inst. Math. Sci.* **27** (1991), no. 4, 577–625. Zbl 0743.28008 MR 1140678

[6] F. E. Browder, On the iteration of transformations in noncompact minimal dynamical systems. *Proc. Amer. Math. Soc.* **9** (1958), 773–780. Zbl 0092.12602 MR 96975

[7] J.-P. Conze and A. Raugi, On the ergodic decomposition for a cocycle. *Colloq. Math.* **117** (2009), no. 1, 121–156. Zbl 1177.37014 MR 2539552

[8] Y. Cornulier and P. de la Harpe, *Metric geometry of locally compact groups*. EMS Tracts in Mathematics, 25. European Mathematical Society (EMS), Zürich, 2016. Zbl 1352.22001 MR 3561300

[9] A. I. Danilenko, Quasinormal subrelations of ergodic equivalence relations. *Proc. Amer. Math. Soc.* **126** (1998), no. 11, 3361–3370. Zbl 0917.28019 MR 1610944

[10] R. Dougherty, S. Jackson, and A. S. Kechris, The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.* **341** (1994), no. 1, 193–225. Zbl 0803.28009 MR 1149121

[11] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *Trans. Amer. Math. Soc.* **234** (1977), no. 2, 289–324. Zbl 0369.22009 MR 578656

[12] J. Feldman, C. E. Sutherland, and R. J. Zimmer, Subrelations of ergodic equivalence relations. *Ergodic Theory Dynam. Systems* **9** (1989), no. 2, 239–269. Zbl 0654.22003 MR 1007409

[13] V. Ya. Golodets, A description of the representations of anticommutation relations. *Uspehi Mat. Nauk* **24** (1969), no. 4(148), 3–64. In Russian. Zbl 0186.46305 MR 0264409

[14] V. Ya. Golodets and S. D. Sinelshchikov, Outer conjugacy for actions of continuous amenable groups. *Publ. Res. Inst. Math. Sci.* **23** (1987), no. 5, 737–769. Zbl 0656.46053 MR 934670

[15] V. Ya. Golodets and S. D. Sinelshchikov, Classification and structure of cocycles of amenable ergodic equivalence relations. *J. Funct. Anal.* **121** (1994), no. 2, 455–485. Zbl 0821.28010 MR 1272135

[16] W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*. American Mathematical Society Colloquium Publications, 36. American Mathematical Society, Providence, R.I., 1955. Zbl 0067.15204 MR 0074810

[17] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiĭ, Automata, dynamical systems, and groups. *Tr. Mat. Inst. Steklova* **231** (2000), Din. Sist., Avtom. i Beskon. Gruppy, 134–214. In Russian. English translation, *Proc. Steklov Inst. Math.* **2000,** no. 4(231), 128–203. Zbl 1155.37311 MR 1841755

[18] T. Hamachi, Canonical subrelations of ergodic equivalence relations-subrelations. *J. Operator Theory* **43** (2000), no. 1, 3–34. MR 1740892

[19] G. Hjorth, *Classification and orbit equivalence relations*. Mathematical Surveys and Monographs, 75. American Mathematical Society, Providence, R.I., 2000. Zbl 0942.03056 MR 1725642

[20] S. Jackson, A. S. Kechris, and A. Louveau, Countable Borel equivalence relations. *J. Math. Log.* **2** (2002), no. 1, 1–80. Zbl 1008.03031 MR 1900547

[21] A. S. Kechris, *Classical descriptive set theory*. Graduate Texts in Mathematics, 156. Springer, New York, 1995. Zbl 0819.04002 MR 1321597

[22] A. S. Kechris, The theory of countable Borel equivalence relations. Preprint, 2019. http://www.math.caltech.edu/~kechris/papers/lectures%20on%20CBER01.pdf

[23] A. S. Kechris and B. D. Miller, *Topics in orbit equivalence*. Lecture Notes in Mathematics, 1852. Springer, Berlin, 2004. Zbl 1058.37003 MR 2095154

[24] B. Miller, The existence of measures of a given cocycle. I. Atomless, ergodic σ-finite measures. *Ergodic Theory Dynam. Systems* **28** (2008), no. 5, 1599–1613. Zbl 1167.37007 MR 2449546

[25] B. D. Miller, Coordinatewise decomposition, Borel cohomology, and invariant measures. *Fund. Math.* **191** (2006), no. 1, 81–94. Zbl 1097.03042 MR 2232198

[26] C. C. Moore, Restrictions of unitary representations to subgroups and ergodic theory: Group extensions and group cohomology. In V. Bargmann (ed.), *Group representations in mathematics and physics. Battelle Seattle 1969 Rencontres.* Lecture Notes in Physics, 6. Springer, Berlin etc., 1970, 1–35. Zbl 0223.22020 MR 0279232

[27] M. G. Nadkarni, *Basic ergodic theory*. Third edition. Texts and Readings in Mathematics, 6. Hindustan Book Agency, New Delhi, 2013. Zbl 1270.28013 MR 2963410

[28] V. Nekrashevych, *Self-similar groups*. Mathematical Surveys and Monographs, 117. American Mathematical Society, Providence, R.I., 2005. Zbl 1087.20032 MR 2162164

[29] K. R. Parthasarathy and K. Schmidt, On the cohomology of a hyperfinite action. *Monatsh. Math.* **84** (1977), no. 1, 37–48. Zbl 0384.28017 MR 457680

[30] A. Ramsay, Virtual groups and group actions. *Advances in Math.* **6** (1971), 253–322. Zbl 0216.14902 MR 281876

[31] K. Schmidt, *Cocycles on ergodic transformation groups*. Macmillan Lectures in Mathematics, 1. Macmillan Company of India, Delhi etc., 1977. Zbl 0421.28017 MR 0578731

[32] K. Schmidt, *Algebraic ideas in ergodic theory*. CBMS Regional Conference Series in Mathematics, 76. Published for the Conference Board of the Mathematical Sciences, Washington, D.C., by the American Mathematical Society, Providence, R.I., 1990. Zbl 0719.28006 MR 1074576

[33] T. A. Slaman and J. R. Steel, Definable functions on degrees. In A. S. Kechris, D. A. Martin, and J. R. Steel (eds.), *Cabal Seminar 81–85.* Lecture Notes in Mathematics, 1333. Springer, Berlin, 1988. 37–55. Zbl 0677.03038 MR 960895

[34] v. Snoha, private communication, 2019.

[35] V. S. Varadarajan, Groups of automorphisms of Borel spaces. *Trans. Amer. Math. Soc.* **109** (1963), 191–220. Zbl 0192.14203 MR 159923

[36] B. Weiss, Measurable dynamics. In R. Beals, A. Beck, A. Bellow, and A. Hajian (ed.), *Conference in modern analysis and probability.* (Yale University, 1982.) Contemporary Mathematics, 26. American Mathematical Society, Providence, R.I., 1984, 395–421. Zbl 0599.28023 MR 737417

[37] R. J. Zimmer, *Ergodic theory and semisimple groups*. Monographs in Mathematics, 81. Birkhäuser, Basel, 1984. Zbl 0571.58015 MR 776417

Received February 21, 2020

Sergey Bezuglyi, Department of Mathematics, University of Iowa, Iowa City, 52242 IA, USA

e-mail: sergii-bezuglyi@uiowa.edu

Shrey Sanadhya, Department of Mathematics, University of Iowa, Iowa City, 52242 IA, USA

e-mail: shrey-sanadhya@uiowa.edu