Groups Geom. Dyn. 15 (2021), 1421–1448
DOI 10.4171/GGD/635

Isolated circular orders of PSL(2, $\mathbb{Z}$)

Shigenori Matsumoto[1]

Abstract. We give a bijection between the isolated circular orders of the group $G = \mathrm{PSL}(2,\mathbb{Z}) \approx (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ and the equivalence classes of Markov systems associated to G. As applications, we present examples of isolated circular orders of the group G.

Mathematics Subject Classification (2010). Primary: 20F65; Secondary: 20F05.

Keywords. Circular orders, dynamical realization, isolated circular orders.

1. Introduction

Throughout this paper, G always stands for the group $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$. Our purpose is to give a bijection between the isolated circular orders of the group G and the symbolic dynamics associated with G, called *Markov systems.* We also use this bijection to construct examples of isolated circular orders of G. The paper is divided into four parts. In Part I, we introduce the necessary prerequisites, we define the Markov systems of G, and we state the main theorem. Part II is devoted to the proof of one half of the main theorem and Part III of the other half. In Part IV, some examples of isolated circular orders are given. In [4], it is proved that the space $\mathrm{LO}(B_3)$ of the left orders (left invariant total orders) of the braid group B_3 of three strings and the space $\mathrm{CO}(G)$ of the circular orders of G are homeomorphic. This provides examples of isolated left orders of B_3.

Part I

We introduce the necessary prerequisites in Sections 2–4, we define the Markov systems associated with G, and we state the main result (Theorem 1) in Section 5.

2. Circular orders

In this section, we provide some preliminary facts about circular orders. Let H be an arbitrary countable group.

[1] The author is partially supported by Grant-in-Aid for Scientific Research (C) No. 25400096.

Definition 2.1. A map $c\colon H^3 \to \{0, 1, -1\}$ is called a *circular order of* H if it satisfies the following conditions:

(1) $c(g_1, g_2, g_3) = 0$ if and only if $g_i = g_j$ for some $i \neq j$;

(2) for any $g_1, g_2, g_3, g_4 \in H$,

$$c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0;$$

(3) for any $g_1, g_2, g_3, g_4 \in H$,

$$c(g_4 g_1, g_4 g_2, g_4 g_3) = c(g_1, g_2, g_3).$$

Definition 2.2. Given a finite set F of H, a *configuration of* F is an equivalence class of injections $\iota\colon F \to S^1$, where two injections ι and ι' are *equivalent* if there is an orientation-preserving homeomorphism h of S^1 such that $\iota' = h\iota$.

Given a circular order c of H, the configuration of the set $\{g_1, g_2, g_3\}$ of three points is determined by the rule that (g_1, g_2, g_3) is ordered anticlockwise if $c(g_1, g_2, g_3) = 1$, and clockwise if $c(g_1, g_2, g_3) = -1$. By condition (2) of Definition 2.1, this is independent of the enumeration of the set. But (2) says more: an easy induction shows the following.

Proposition 2.3. *A circular order of* H *determines the configuration of any finite subset* F *of* H.

Denote by $\mathrm{CO}(H)$ the set of all the circular orders. $\mathrm{CO}(H)$ is a closed subset of the set of maps from H^3 to $\{0, \pm 1\}$, and therefore it is equipped with a totally disconnected compact metrizable topology. A circular order is *isolated* if it is an isolated point of $\mathrm{CO}(H)$. If $c \in \mathrm{CO}(H)$ is isolated, then there is a finite subset S of H such that any circular order which gives the same configuration of S as c is c, and *vice versa*. Such a set S is called a *determining set* of c.

For an automorphism σ and $c \in \mathrm{CO}(H)$, we define $\sigma_* c \in \mathrm{CO}(H)$ by

$$(\sigma_* c)(g_1, g_2, g_3) = c(\sigma^{-1} g_1, \sigma^{-1} g_2, \sigma^{-1} g_3).$$

The order $\sigma_* c$ is called an *automorphic image* of c. An automorphic image of an isolated circular order is isolated. We also say that c and $\sigma_* c$ belong to the same *automorphism class*.

Given $c \in \mathrm{CO}(H)$, we define an action of H on S^1 as follows. Fix an enumeration of H, $H = \{g_i : i \in \mathbb{N}\}$ such that $g_1 = e$, and a base point $x_0 \in S^1$. Define an embedding $\iota\colon H \to S^1$ inductively as follows. First, set $\iota(g_1) = x_0$ and $\iota(g_2) = x_0 + 1/2$. If ι is defined on $\{g_1, \dots, g_n\}$, then there is a connected component of $S^1 \setminus \{\iota(g_1), \dots, \iota(g_n)\}$ where the point g_{n+1} is embedded, by virtue of Proposition 2.3. Let $\iota(g_{n+1})$ be the midpoint of that interval. The left translation of H yields an action of H on $\iota(H)$ which is shown to extend to a continuous action on $\mathrm{Cl}(\iota(H))$. Extend it further to an action on S^1 by requiring that the action on the gaps[1] is linear. The action so obtained is called the *dynamical realization of c based at x_0*.

Definition 2.4. An action ϕ of the group H on S^1 is *tight* at $x_0 \in S^1$ if it satisfies the following two conditions:

- ϕ free at x_0, i.e., the stabilizer of x_0 is trivial;
- if J is a gap of the orbit closure $\mathrm{Cl}(\phi(H)x_0)$, then ∂J is contained in the orbit $\phi(H)x_0$.

We have three lemmas whose proofs are easy and omitted. (Lemma 2.5 is a consequence of the midpoint construction. In fact, it is used in the construction of the dynamical realization, where we extend the H-action from the orbit of x_0 to the orbit closure.)

Lemma 2.5. *The dynamical realization is **tight** at the base point x_0.*

Lemma 2.6. *Two dynamical realizations obtained via different enumerations of H are mutually conjugate by an orientation- and base-point-preserving homeomorphism of S^1.*

Lemma 2.7. *An action ϕ of H on S^1 which is tight at x_0 is topologically conjugate to the dynamical realization of a circular order c based at x_0 by an orientation- and base-point-preserving homeomorphism.*

Henceforth, any action ϕ as in the last lemma is referred to as a *dynamical realization* of c.

3. Preliminaries on G

We study some properties of the group

$$G = \langle \alpha, \beta \colon \alpha^2 = \beta^3 = e \rangle.$$

As is well known, G is isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$, by an isomorphism ϕ such that

$$\phi(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \phi(\beta) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

See Figure 1 for the action of $\mathrm{PSL}(2, \mathbb{Z})$ on the Poincaré upper half plane $\mathbb{H}$.

[1] A *gap* of a closed subset K of S^1 means a connected component of $S^1 \setminus K$.

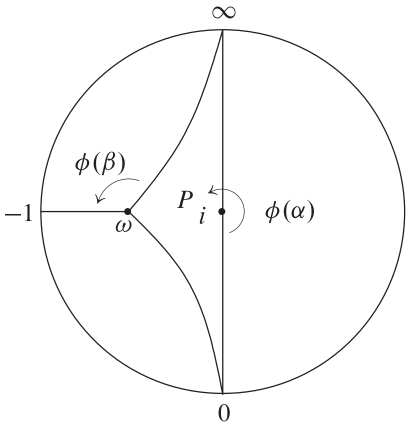

Figure 1. The open disk bounded by the circle is the Poincaré upper half plane $\mathbb{H}$. The element $\phi(\alpha)$ is the $1/2$-rotation around i, and $\phi(\beta)$ the $1/3$-rotation around $\omega = (-1+\sqrt{-3})/2$. The region P bounded by the ideal triangle $\triangle 0\,\infty\,\omega$ is a fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$.

Proposition 3.1. (1) *Any element of $G \setminus \{e\}$ is of order* 2, 3, *or* ∞. *Any element of order* 2 *is conjugate to α, and any element of order* 3 *is conjugate either to β or to β^{-1}.*

(2) *Any torsion free subgroup of G is isomorphic to a free group, either finitely generated or not.*

(3) *The commutator subgroup is a free group freely generated by $\alpha\beta\alpha\beta^{-1}$ and $\alpha\beta^{-1}\alpha\beta$.*

(4) *Any automorphism of G is the conjugation by an element of* $\mathrm{PGL}(2,\mathbb{Z})$ *when we identify G with* $\mathrm{PSL}(2,\mathbb{Z})$ *by ϕ. In other words, the outer automorphism group of G is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, generated by the involution σ_0 defined by $\sigma_0(\alpha) = \alpha$ and $\sigma_0(\beta) = \beta^{-1}$.*

Proof. (1) and (2) are well known, and can be obtained easily by considering the action on $\mathbb{H}$. (3) can be shown, for example, by an induction on word length of elements in $[G, G]$, using the fact that the total exponents of α (resp. β) in the word is even (resp. a multiple of 3). To show (4), notice that any automorphism σ of G sends β to an element which is conjugate either to β or to β^{-1}. We only need to verify that σ is an inner automorphism in the former case. By composing with an appropriate inner automorphism, if necessary, we may assume that $\sigma(\alpha) = \alpha$. Let x be a fixed point of $\phi(\sigma(\beta))$ in $\mathbb{H}$. Since $\sigma(\beta)$ is a conjugate $\gamma\beta\gamma^{-1}$, x is a translate of ω in Figure 1 by an element $\phi(\gamma) \in \mathrm{PSL}(2,\mathbb{Z})$. All such points x, except ω and $\phi(\alpha)(\omega)$, satisfy $d(x,i) > d(\omega,i)$, where d is the Poincaré distance, in which case $\phi(\alpha)$ and $\phi(\sigma(\beta))$ generate a covolume infinite Fuchsian group. Since σ is an automorphism, we have either $x = \omega$ or $\phi(\alpha)\omega$. Accordingly, $\sigma(\beta) = \beta$ or $\sigma(\beta) = \alpha\beta\alpha$, In either case, σ is an inner automorphism. □

Question. Does the equality $(\sigma_0)_*(c) = -c$ hold? Is that true at least for isolated orders c? This is true for all the examples constructed in Sections 13 and 14.

4. Isolated circular orders of *G*

Let c be an isolated circular order of G and ρ a dynamical realization of c based at $x_0 \in S^1$.

Proposition 4.1. (1) *There is a unique minimal set $\mathcal{M}$ of the action ρ homeomorphic to a Cantor set. Moreover $x_0 \notin \mathcal{M}$.*

(2) *Let I be a gap of $\mathcal{M}$ which contains x_0. Then all the gaps of $\mathcal{M}$ are a translate of I by the action ρ. The stabilizer*[2] *G_I of I is infinite cyclic.*

Proof. It is known by [3] Corollary 1.3 that a dynamical realization ρ of an isolated circular order cannot be minimal (for any countable group). Assume, for sake of contradiction, that ρ admits a finite minimal set of cardinality n. Considering the action of G on the minimal set, one obtains a surjective homomorphism $\xi: G \to \mathbb{Z}/n\mathbb{Z}$. Since $G/[G,G]$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$, n is either 1, 2, 3, or 6. The circular order c of G induces a left order λ on $\mathrm{Ker}(\xi)$ by considering the action of $\rho|_{\mathrm{Ker}(\xi)}$ on the gap containing x_0. In particular, $\mathrm{Ker}(\xi)$ must be torsion free. This eliminates the cases $n = 1, 2, 3$. To eliminate the case $n = 6$, we claim that the induced left order λ on $\mathrm{Ker}(\xi)$ must be isolated. Any left order λ' on $\mathrm{Ker}(\xi)$ together with the cyclic order on $\mathbb{Z}/6\mathbb{Z}$ defines a cyclic order c' of G by the lexicographic construction. Moreover, the map $\lambda' \mapsto c'$ is injective and continuous. Since c is isolated, λ must be isolated, showing the claim. But if $n = 6$, then $\mathrm{Ker}(\xi) = [G,G]$ is a free group on 2 generators, and does not admit an isolated left order [2]. This shows that the minimal set cannot be a finite set. It must be a Cantor set. The uniqueness of a Cantor minimal set $\mathcal{M}$ is easy and well known.

If $x_0 \in \mathcal{M}$, the two boundary points of a gap of $\mathcal{M} = \mathrm{Cl}(\rho(G)x_0)$ cannot be from one orbit of the action, contradicting the tightness of the action ρ at x_0. This shows that $x_0 \notin \mathcal{M}$.

The same argument shows that there are no gaps of $\mathcal{M}$ which are not a translate of I. Such gaps might be gaps of $\mathrm{Cl}(\rho(G)x_0)$.

Finally, let us show that G_I is infinite cyclic. First, G_I is nontrivial. Assume, for sake of contradiction, that it is trivial and consider an interval delimited by x_0 and a point $y \in \mathcal{M} \subset \mathrm{Cl}(\rho(G)x_0)$. Then y cannot be a translate of x_0, again contradicting the tightness. If G_I is not infinite cyclic, G_I is a free group on more

[2] $G_I = \{g \in G : \rho(g)I = I\}$.

than one generator, which does not admit an isolated left order ([2]). As before, the lexicographic construction gives a contradiction. □

The subgroup G_I in the last proposition is called the *linear subgroup* of the isolated circular order of c, and is denoted by L_c. For an automorphism σ of G, we have $L_{\sigma_* c} = \sigma(L_c)$. For a fixed isolated circular order c, consider the minimal word length of a generator of $\sigma(L_c)$ where σ ranges over all the automorphisms of G. This length is an even number, say $2k$.

Definition 4.2. The number k is called the *degree* of the isolated circular order c, and is denoted by $\deg(c)$.

Clearly the degree is an automorphism-class function and it is an odd number (see Section 14).

5. Markov systems

Definition 5.1. A *Markov system* $\mathbb{M} = (a, b, [a], [b], [b^{-1}])$ consists of an orientation-preserving involution a, a homeomorphism b of S^1 of period 3, and three subsets $[a], [b], [b^{-1}]$ of S^1 which satisfy conditions (A)–(E) below.

(A) The sets $[a]$, $[b]$, and $[b^{-1}]$, are disjoint, each set consisting of k closed intervals for some $k \in \mathbb{N}$. The number k is called *multiplicity* of the system $\mathbb{M}$.

A connected component of $[a]$ (resp. $[b]$ and $[b^{-1}]$) is called an *a-interval* (resp. b- and *b^{-1}-interval*). Denote $X = [a] \cup [b] \cup [b^{-1}]$.

(B) Two a-intervals are not adjacent in X. Likewise for b- and b^{-1}-intervals.

A gap of X between an a-interval and a $b^{\pm 1}$-interval is called a *principal gap*. A gap between a b-interval and a b^{-1}-interval is called a *complementary gap*. A maximal interval which consists of $b^{\pm 1}$-intervals and complementary gaps is called a *b-block*. Any b-interval is contained in a unique b-block. The a-intervals and the b-blocks are alternating in S^1 and there are just k b-blocks. The union of all b-blocks is denoted by $[[b]]$. Let us continue the conditions, which are reminiscent of the action of α and β on (the first letter of) the words representing elements of G.

(C) $a[a] = [[b]]$.

This implies $a[[b]] = [a]$ and hence $a[b^{\pm 1}] \subset [a]$.

(D) $b[a] = [b], \ b[b] = [b^{-1}]$.

This implies $b[b^{-1}]=[a]$, since $b^3=\mathrm{id}$, and $b^{-1}[a]=[b^{-1}]$, $b^{-1}[b^{-1}]=[b]$, and $b^{-1}[b]=[a]$.

A principal gap J is always mapped to a principal gap by a, and exactly one of b and b^{-1} maps J to a principal gap, the other to a complementary gap. Therefore, the principal gaps consist of several cycles. Our last condition is the following.

(E) The principal gaps are formed of one cycle.

Definition 5.2. Given two Markov systems $\mathbb{M}=(a,b,[a],[b],[b^{-1}])$ and $\mathbb{M}'=(a',b',[a'],[b'],[(b')^{-1}])$, $\mathbb{M}'$ is a *semiconjugate image of* $\mathbb{M}$ if there is a monotone continuous map $f\colon S^1\to S^1$ of degree one such that $fa=a'f$, $fb=b'f$, $f[a]=[a']$, $f[b]=[b']$, and $f[b^{-1}]=[(b')^{-1}]$. $\mathbb{M}$ and $\mathbb{M}'$ are *equivalent* if they are related by the equivalence relation generated by the semiconjugate image.

Our main result is the following.

Theorem 1. *There is a bijection between the isolated circular orders of G and the equivalence classes of the Markov systems.*

We would like to emphasize that this is not a true classification theorem of isolated circular orders: it is too difficult to classify Markov systems. What we can do so far is to present examples of Markov systems, as in Part IV.

Part II

We construct the Markov partitions associated to isolated circular orders, thereby showing one half of Theorem 1. The result of Part II is summarized as Theorem 6.1.

6. Further properties of isolated circular orders of *G*

This section is devoted to the preparation of the basic facts needed for the proof of the following theorem. Let ϕ be a dynamical realization of an arbitrary isolated circular order of G, based at x_0, and $\mathcal{M}$ be the minimal set of the action ϕ.

Theorem 6.1. *There is a Markov partition $(a,b,[a],[b],[b^{-1}])$ such that $a=\phi(\alpha)$ and $b=\phi(\beta)$.*

We denote $\mathbb{G}=\phi(G)$. The group $\mathbb{G}$ is isomorphic to G (since ϕ is free at x_0) and is generated by a and b.

Definition 6.2. A word of letters $a, b^{\pm 1}$ is *admissible* if it is reduced and contains no consecutive identical letters.

Any element $g \in \mathbb{G} \setminus \{\mathrm{id}\}$ is expressed uniquely by an admissible word denoted by $W(g)$, whose length is denoted by $\|g\|$. The first letter of $W(g)$ is called the *prefix* of g and is denoted by $\mathrm{pre}(g)$. We put $\mathrm{pre}(\mathrm{id}) = \emptyset$ for completeness. If $W(g) = t_1 t_2 \cdots t_r$ and $W(g') = t_i t_{i+1} \cdots t_r$ for some $2 \le i \le r$, then g' is called a *larva* of g. The group $\mathbb{G}$ acts freely at the base point x_0 and all the above terminologies about $\mathbb{G}$ are carried over to the orbit $\mathbb{G}x_0$. If $x = gx_0$ and $x' = g'x_0$ for some $g, g' \in \mathbb{G} \setminus \{\mathrm{id}\}$, the word $W(x)$ is, by definition, the word $W(g)$; the length $\|x\|$ is $\|g\|$; the prefix $\mathrm{pre}(x)$ is $\mathrm{pre}(g)$; and x' is a *larva* of x if g' is a larva of g.

Choose a Riemannian metric on S^1 in such a way that the involution a is an isometry. The distance of two points $x, y \in S^1$ is denoted by $|x - y|$. The length of an interval J is denoted by $|J|$. As a corollary of Proposition 4.1, we get the following.

Corollary 6.3. *There is $\epsilon_1 > 0$ such that if a closed interval J satisfies $|J| < \epsilon_1$ and $x_0 \in J$, then $J \cap \mathbb{G}x_0 = \{x_0\}$.*

Proof. Choose ϵ_1 smaller than the distance of x_0 to the neighbouring points in $\mathbb{G}x_0$. □

Lemma 6.4. *There are finitely many gaps $I_1, \dots, I_r$ of the minimal set $\mathcal{M}$ with the following properties.*

(1) *For any gap $J \ne I_i$, the prefixes of all the points of $\mathbb{G}x_0 \cap J$ are the same.*
(2) *For $J = I_i$, there is an enumeration of points of $\mathbb{G}x_0 \cap J$,*

$$\mathbb{G}x_0 \cap J = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2 \dots\},$$

in the anti-clockwise order of S^1 such that $\mathrm{pre}(x_n)$ is the same for all $n < 0$ and that $\mathrm{pre}(x_n)$ is the same for all $n > 0$.

Proof. Denote by I the gap of $\mathcal{M}$ containing x_0 as before and by G_I the stabilizer of I by the action ϕ. Given an arbitrary gap J of $\mathcal{M}$, choose $g \in \mathbb{G}$ such that $J = gI$ and that $\|g\|$ is the smallest among such g. Let h be a generator of $\phi(G_I)$. Then the word $W(g)$ cannot have $W(h)$ or $W(h^{-1})$ as larvae. This implies that if $\|g\| \ge \|h\|$, then $W(g)$ is not completely cancelled out in the word $W(gh^n)$. In particular, $\mathrm{pre}(gh^n)$ is the same for any $n \in \mathbb{Z}$, showing (1). In the remaining case $\|g\| < \|h\|$, $\mathrm{pre}(gh^n)$ is the same for any $n > 0$ and $\mathrm{pre}(gh^{-n})$ is the same for any $n > 0$, finishing the proof of (2). □

Corollary 6.5. *There exists $\epsilon_2 > 0$ such that if a closed interval J satisfies $|J| < \epsilon_2$ and if there are points $x_1, x_2 \in J \cap \mathbb{G}x_0$ such that $\mathrm{pre}(x_1) \ne \mathrm{pre}(x_2)$, then $\mathrm{Int}(J) \cap \mathcal{M} \ne \emptyset$.*

Proof. Choose $\epsilon_2 > 0$ smaller than the distance of x_0 and $x_{\pm 1}$ for each of the intervals $I_1, \dots, I_r$ in Lemma 6.4. □

For $t \in \{a, b^{\pm 1}\}$, let $\mathbb{G}_t$ be the set of those elements $g \in \mathbb{G} \setminus \{\mathrm{id}\}$ such that the last letter of $W(g)$ is t.

Lemma 6.6. *We have an inclusion* $\mathcal{M} \subset \mathrm{Cl}(\mathbb{G}_t x_0) \setminus \mathbb{G}_t x_0$ *for each* $t \in \{a, b^{\pm 1}\}$.

Proof. It suffices to show that the closed set $X = \mathrm{Cl}(\mathbb{G}_t x_0) \setminus \mathbb{G}_t x_0$ is invariant by $\mathbb{G}$, since $\mathcal{M}$ is the unique minimal set. Given $x \in X$, there is a sequence $\{g_n\}$ in $\mathbb{G}_t$ such that $g_n x_0 \to x$ and $\|g_n\| \to \infty$. For any $f \in \mathbb{G}$, we have $fg_n \in \mathbb{G}_t$ if n is sufficiently large, showing that $fx = \lim_{n\to\infty} fg_n x_0 \in X$. □

Definition 6.7. An action $\psi: G \to \mathrm{Homeo}_+(S^1)$ is ϵ*-close* to the dynamical realization ϕ if $\|\psi(\alpha) - \phi(\alpha)\|_0 < \epsilon$ and $\|\psi(\beta^{\pm 1}) - \phi(\beta^{\pm 1})\|_0 < \epsilon$, where $\psi(g) - \phi(g)$ is a map from the abelian group S^1 to S^1, and $\|\cdot\|_0$ denotes the supremum norm.

Lemma 6.8. *There is* $\epsilon_3 > 0$ *with the following property: if* $\psi: G \to \mathrm{Homeo}_+(S^1)$ *is* ϵ_3*-close to* ϕ*, then* ψ *is free at* x_0 *and the circular order of* G *determined by the orbit* $\psi(G)x_0 \subset S^1$ *is the same as* c.

Proof. Let $S \subset G$ be a finite-determining set of c. One can choose $\epsilon_3 > 0$ so that if ψ is ϵ_3-near to ϕ, then the circular order of $\psi(S)x_0$ is the same as $\phi(S)x_0$ and additionally $\psi(\alpha)$ and $\psi(\beta)$ is not the identity. Assume the isotropy group H of ψ at x_0 is nontrivial. Then H is torsion free, since any torsion element γ of G is conjugate to α or $\beta^{\pm 1}$ and $\psi(\gamma)$ is fixed point free by the additional condition of ϵ_3. Therefore, H is a free group (Proposition 3.1) and admits a left order λ and its reciprocal $-\lambda$. On the other hand, the quotient G/H admits a left G-invariant circular order c' determined by the orbit $\psi(G)x_0$. Now, $\pm\lambda$ and c' determines two distinct circular orders by the lexicographic constructions. This is contrary to the definition of the determining set S. We have shown that ψ is free at x_0. The rest of the assertion follows again by the definition of the determining set. □

7. Continuity of $W(x)$

We show that the assignment $\mathbb{G}x_0 \ni x \mapsto W(x)$ is continuous in some weak sense. The argument here follows closely the proof of [3, Proposition 4.7], but we need an elaboration since the group G is not a free group treated in [3].

Two letters from the set $\{a, b^{\pm 1}\}$ are *congruent* if either they are the same or one is b and the other is b^{-1}. Choose ϵ so that $0 < \epsilon < \min\{\epsilon_1, \epsilon_2, \epsilon_3, 1/2\}$, where $\epsilon_1, \epsilon_2, \epsilon_3$ are the constants defined in the last section.

Proposition 7.1. *If* $x, y \in \mathbb{G}x_0$ *satisfy* $|x - y| < \epsilon$, *then* $\mathrm{pre}(x)$ *and* $\mathrm{pre}(y)$ *are congruent.*

This section is devoted to the proof of the above proposition by *reductio ad absurdum.* We assume the following conditions and we shall deduce a contradiction at the end of the section.

There is a closed interval J with the following properties:

♯1. $|J| < \epsilon$;

♯2. $\partial J = \{x_1, x_2\} \subset \mathbb{G}x_0$ and $\mathrm{pre}(x_1)$ and $\mathrm{pre}(x_2)$ are not congruent.

By Corollary 6.3, (♯1) and (♯2) imply

♯3. $x_0 \notin J$.

An interval which satisfies (♯1)–(♯3) is called a ♯*-interval.*

Lemma 7.2. *There is a ♯-interval* J *such that no larva of a point in* ∂J *is contained in* J.

Proof. Let $J = [x, y]$ be a ♯-interval that minimizes $N(J) = \|x\| + \|y\|$. Then, x has no larva congruent to x and contained in J. Likewise for y. If there are no larvae of x and y contained in J at all, we are done. So, assume one of them, say x, has a larva (necessarily not congruent to x) contained in J. Choose the larva z of x in J which is the nearest to x. Then the interval $J' = [x, z]$ is a ♯-interval. Moreover, there are no larvae of x other than z contained in J'.

Now, since z is a larva of x, both x and z belong to the same $\mathbb{G}_t x_0$ for some $t \in \{a, b^{\pm}\}$. Choose $s \in \{a, b^{\pm 1}\} \setminus \{t\}$. By Corollary 6.5 and Lemma 6.6, $\mathbb{G}_s x_0 \cap J'$ is nonempty since J' is a ♯-interval. Choose a point u of minimal length from $\mathbb{G}_s x_0 \cap J'$. If $\mathrm{pre}(u)$ is congruent to $\mathrm{pre}(x)$, then choose the interval $[u, z]$; otherwise, choose $[x, u]$. □

Finally, let us show that Lemma 7.2 leads to a contradiction. Let J in Lemma 7.2 be such that $J = [x, y]$ and $\mathrm{pre}(x) = a$ and $\mathrm{pre}(y) = b^{\pm 1}$. Choose an open interval $U \supset J$ such that $|U| < \epsilon$ and that all the larvae of both x and y, as well as x_0, are not contained in U. Let h be a homeomorphism of S^1 supported on U such that h sends x to the opposite side of y in U, and define an action ψ of G by setting $\psi(\alpha) = h\phi(\alpha)h^{-1}$ and $\psi(\beta) = \phi(\beta)$. Notice that $\psi(\alpha) = \phi(\alpha)$ except on $U \sqcup \phi(\alpha)U$. (U and $\phi(\alpha)U$ are disjoint since $\epsilon < 1/2$ and $\phi(\alpha)$ is an isometry.) By Lemma 6.8, the cyclic order of G obtained by the orbit $\psi(G)x_0$ must be the same as c, i.e, the one obtained from $\mathbb{G}x_0$. However, by induction on the length of f and g,

$$(\psi(f)x_0, \psi(g)x_0, x_0) = (hx, y, x_0) \quad \text{and} \quad (\phi(f)x_0, \phi(g)x_0, x_0) = (x, y, x_0).$$

A contradiction. This completes the proof of Proposition 7.1.

8. Continuity of $W(x)$-continued

Let $\mathcal{W}_\infty$ be the set of infinite words of letters $a, b^{\pm 1}$, and $\emptyset$ with the following properties:

(1) there are no consecutive appearances of a, b, and b^{-1};

(2) $b^{\mp 1}$ does not follow $b^{\pm 1}$;

(3) all the letters after $\emptyset$ are $\emptyset$.

With abuse of notation, we denote by $W(g) \in \mathcal{W}_\infty$ a finite word followed by a sequence of $\emptyset$. Thus, for example, $W(\mathrm{id}) = \emptyset\,\emptyset\,\emptyset \cdots$, and $W(ab) = ab\,\emptyset\,\emptyset \cdots$. For $n > 0$, the initial subword of length n of $W(g)$ is denoted by $W_n(g)$. We also define $W(x)$ and $W_n(x)$ for a point $x = gx_0 \in \mathbb{G}x_0$ by $W(x) = W(g)$ and $W_n(x) = W_n(g)$, respectively. As a consequence of Proposition 7.1, we get:

Proposition 8.1. *For any $n > 0$, there exists $\epsilon(n) > 0$ such that if two points $x, y \in \mathbb{G}x_0$ satisfy $|x - y| < \epsilon(n)$, then $W_n(x) = W_n(y)$.*

Proof. For any $\eta > 0$, define $\delta(\eta) \in (0, \eta)$ so that if $|x - y| < \delta(\eta)$, then $|b^{\pm 1}x - b^{\pm 1}y| < \eta$. (Recall that a is an isometry.) Then for $\epsilon_1 = \delta(\epsilon)$, $|x - y| < \epsilon_1$ implies $W_1(x) = W_1(y)$. For if not, we may assume $W_1(x) = b$ and $W_1(y) = b^{-1}$, since $\epsilon_1 < \epsilon$. But then $W_1(bx) = b^{-1}$ and $W_1(by) = a$ or $\emptyset$, contrary to the assumption $|bx - by| < \epsilon$.

For general n, define

$$\epsilon(n) = \overbrace{\delta(\delta(\cdots\delta))}^{n}(\epsilon).$$

An induction on n shows the proposition. ☐

9. Construction of Markov systems

Given $x \in \mathcal{M}$, choose a point $x_i \in \mathbb{G}x_0$ from the $\epsilon(i)/2$-neighbourhood of x. Then $W_i(x_i)$ is independent of the choice of x_i. Moreover, $W_i(x_i) = W_i(x_j)$ if $i < j$. Thus, the sequence $\{W_i(x_i)\}$ stabilizes.

Definition 9.1. For any $x \in \mathcal{M}$, define a word $W(x) \in \mathcal{W}_\infty$ as the limit of $W_i(x_i)$. Also, define $W_n(x)$ to be the initial subword of length n of $W(x)$.

Notice that $W(x)$ is a word of letters a and $b^{\pm 1}$: $\emptyset$ never shows up.

Definition 9.2. For an admissible word w of length n of letters $a, b^{\pm 1}$, we define the subset $[w]$ of S^1 to be the union of the points $x \in \mathcal{M}$ such that $W_n(x) = w$ and the gaps (x, y) of $\mathcal{M}$ such that $W_n(x) = W_n(y) = w$.

Lemma 9.3. (1) *For any finite admissible word w, $[w]$ is a finite union of closed disjoint intervals.*

(2) *If $v \neq w$ are reduced words of the same length, then $[v] \cap [w] = \emptyset$.*

(3) $a[ab^{\pm 1}] = [b^{\pm 1}]$.

(4) $b[a] = [b]$, $b[b] = [b^{-1}]$, *and* $b[b^{-1}] = [a]$.

(5) *The cardinalities of the components of $[a]$, $[b]$, and $[b^{-1}]$ are the same.*

Proof. (1) is a consequence of Proposition 8.1. (2)–(4) are clear from the definitions. (5) follows from (4). □

Proof of Theorem 6.1. Define $\mathbb{M} = (a, b, [a], [b, [b^{-1}]])$. It is obvious that $\mathbb{M}$ satisfies conditions (A)–(D). (E) also follows basically from the transitivity of gaps by the dynamical realization (Proposition 4.1 (2)). □

Part III

We define two properties (*) and (**), and show that any Markov system is equivalent to one satisfying both (*) and (**). Next, to any Markov partition satisfying both (*) and (**) we assign an isolated circular order, thereby showing the other half of Theorem 1. The result is summarized as Theorem 12.1.

10. Fundamental properties of Markov systems and modifications

Let $\mathbb{M} = (a, b, [a], [b], [b^{-1}])$ be an arbitrary Markov system. Recall that the map a sends a principal gap J to a principal gap, and either one of b or b^{-1} sends J to a principal gap, the other to a complementary gap. By condition (E), the principal gaps I_i, I_i' and the complementary gaps I_i'' ($i \in \mathbb{Z}/k\mathbb{Z}$) are dynamically related as in (10.1) below, where b_i is either b or b^{-1}. The gaps I_i''' are gaps between components of $[ab]$ and $[ab^{-1}]$ to be defined later.

$$\begin{array}{ccccccccccccc} I_1 & \xrightarrow{b_1} & I_1' & \xrightarrow{a} & I_2 & \xrightarrow{b_2} & I_2' & \xrightarrow{a} & \cdots & \xrightarrow{b_k} & I_k' & \xrightarrow{a} & I_{k+1} = I_1 \\ {\scriptstyle b_1}\nwarrow & & \swarrow{\scriptstyle b_1} & & {\scriptstyle b_2}\nwarrow & & \swarrow{\scriptstyle b_2} & & & & & & \\ & I_1'' & & & & I_2'' & & & & & & & \\ & {\scriptstyle a}\downarrow & & & & {\scriptstyle a}\downarrow & & & & & & & \\ & I_1''' & & & & I_2''' & & & & & & & \end{array} \qquad (10.1)$$

In this section, we study fundamental properties of Markov systems. Besides, we show that any Markov system can be modified in its equivalence class to another one with good properties. First of all, consider the following property concerning diagram (10.1).

(*) The map[3] $f_1 = ab_k \cdots ab_1$ which leaves I_1 invariant admits no fixed point in the open interval I_1, and for any $z \in I_1$, $\lim_{n\to\infty} f_1^n(z) \to x$, where x is a point in $\partial J \cap [a]$.

Our first modification result is the following.

Lemma 10.1. *Any Markov system is equivalent to a Markov system satisfying (*).*

Proof. In the admissible word $f_1 = ab_k \cdots ab_1$, the last map a (first in the word) is a transposition of I'_k and I_1. If we change the map a by the conjugation by a homeomorphism supported in I_1 and leave b unchanged, then the new maps still satisfy all the requirement for Markov systems. Since the modification is free, one gets (*). □

We introduce basic terminologies and notations.

Definition 10.2. For subsets P and Q consisting of finite disjoint closed intervals of S^1, the inclusion $P \subset Q$ is called *precise* if any boundary point of Q is contained in P.

The inclusion $[b] \cup [b^{-1}] \subset [[b]]$ is precise. The composite of precise inclusions is precise.

Definition 10.3. For the Markov system $\mathbb{M} = (a, b, [a], [b], [b^{-1}])$, denote by $\mathbb{G} = \mathbb{G}(\mathbb{M})$ the subgroup of $\mathrm{Homeo}_+(S^1)$ generated by a and b.

By *word*, we always mean a word of letters $a, b^{\pm 1}$. Any map of $\mathbb{G} \setminus \{\mathrm{id}\}$ is expressed uniquely as an admissible word, as can be shown by (4) and (7) of the next lemma.

Definition 10.4. For a admissible word $w = vt$, where t is the last letter of w, we define $[w] = v[t]$.

Lemma 10.5. (1) *For an admissible word wv, we have $w[v] = [wv]$.*

(2) *If wv, $v^{-1}u$ and wu are admissible, then $wv[v^{-1}u] = [wu]$.*

(3) *If wa is admissible, then $[wa] = [w]$.*

(4) *If wv is admissible, then $[wv] \subset [w]$.*

[3] In $f_1 = ab_k \cdots ab_1$, b_1 is the first map.

(5) *For an admissible word w, $[w]$ consists of k disjoint closed intervals.*

(6) *The inclusion $[wb] \cup [wb^{-1}] \subset [w]$ is precise for an admissible word w which ends at a.*

(7) *If w and w' are distinct admissible words of the same length, then $[w]$ and $[w']$ are disjoint.*

Proof. (1) is obtained by an easy induction on the length of v.

For (2),

$$wv[v^{-1}u] = wvv^{-1}[u] = w[u] = [wu].$$

For (3), if $w = vb^{\pm 1}$, then

$$[wa] = vb^{\pm 1}[a] = v[b^{\pm 1}] = [vb^{\pm 1}] = [w].$$

For (4), notice that if $w = ua$ and $v = b^{\pm 1}$, then

$$[wv] = ua[b^{\pm 1}] \subset u[a] = [w].$$

Together with (3), this implies $[wv] \subset [w]$ if $\|v\| = 1$. The general case follows by an induction on $\|v\|$.

For (5), if $w = vt$, where t is the last letter of v, $[vt] = v[t]$, $[t]$ consists of disjoint k intervals and v is a homeomorphism.

For (6), since the inclusion $[b] \cup [b^{-1}] \subset [[b]]$ is precise, if we write $w = va$, then the inclusion

$$[wb] \cup [wb^{-1}] = w([b] \cup [b^{-1}]) \subset w[[b]] = va[[b]] = v[a] = [w]$$

is precise.

(7) follows from an induction of word length. □

For an infinite admissible word $\underline{w} = t_1 t_2 \cdots$, define $[\underline{w}] = \bigcap_i [t_1 \cdots t_i]$. It is a nonempty set consisting of k closed intervals, some possibly degenerate to points. We have $v[\underline{w}] = [v\underline{w}]$ for any finite admissible word v and any infinite admissible word $\underline{w}$.

Definition 10.6. Define $X_\infty = \bigcap_{f \in \mathbb{G}} f^{-1} X$ for $X = [a] \cup [b] \cup [b^{-1}]$.

The set X_∞ is closed and $\mathbb{G}$-invariant. The following lemma says that it is nonempty.

Lemma 10.7. *We have $X_\infty = \bigcup_{\underline{w}} [\underline{w}]$, where $\underline{w}$ runs over all the infinite admissible words.*

Proof. The inclusion $\supset$ is easy: for any $f \in \mathbb{G}$, we have $f[\underline{w}] = [f\underline{w}] \subset X$, showing $[\underline{w}] \subset f^{-1}X$. Let us show $\subset$. For any $x \in X_\infty$, define $t_1 \in \{a, b^{\pm 1}\}$ by the condition $x \in [t_1]$, then t_2 by $t_1^{-1}(x) \in [t_2]$, t_3 by $t_2^{-1}t_1^{-1}(x) \in [t_3]$ .... The word $\underline{w} = t_1 t_2 \cdots$ we obtained is admissible and $x \in [\underline{w}]$. □

For any $x \in X_\infty$, define $\widehat{W}(x) = \underline{w}$ if $x \in [\underline{w}]$. The inclusion $X_\infty \subset X$ is "precise" in the sense that any boundary point of X is contained in X_∞. This stems from Lemma 10.5 (6) and Lemma 10.7. Therefore a gap of X is a gap of X_∞.

Lemma 10.8. *The gaps of X_∞ form one orbit of the $\mathbb{G}$-action.*

Proof. Any gap of X_∞ other than principal or complementary gaps is a gap contained in $[w]$ between components of $[wb]$ and $[wb^{-1}]$ for some admissible word w ending at a, and hence is the image by w of a complementary gap. □

Lemma 10.9. *The set X_∞ admits no isolated component.*

Proof. The proof is by contradiction. Let C be an isolated component contained in $[\underline{w}]$. Let $\underline{w} = t_1 t_2 t_3 \cdots$ and, for each $m \in \mathbb{N}$, let C_m be the component of $[t_1 \cdots t_m]$ containing C. We claim that the decreasing sequence $C_1 \supset C_2 \supset \cdots \downarrow C$ stabilizes, i.e., that there is m_0 such that $C_{m_0+i} = C_{m_0}$ for any $i \in \mathbb{N}$. Assume not. For any neighbourhood U of C, some C_m is contained in U. But if the sequence does not stabilize, there is a component of $[\underline{w}]$ distinct from C contained in C_m and hence in U. This shows that C is not isolated. The contradiction shows the claim. Now, the interval $t_{m_0-1}^{-1} \cdots t_1^{-1} C$ is at the same time a component of X and of X_∞,

It is no loss of generality to assume that C itself is a component of both X and X_∞. Let $C(1) = C$ and $C(i) = t_{i-1}^{-1} \cdots t_1^{-1} C$ for any $i > 1$. Then $C(i)$ is also a component of both X and X_∞. Since X has only finitely many components, the sequence $\{C(i)\}$ is eventually periodic.

Without loosing generality, one may assume that $C(1)$ is in a periodic cycle: $C(n+1) = C(1)$ and n such smallest. Notice that the intertwining arrows of the cycle are $t_i^{-1}\colon C(i) \to C(i+1)$ and that $C(i)$ is a t_i-interval. In particular,

(1) if $C(i)$ is an a-interval, then $t_i^{-1} = a$.

Let $J(i)$ be the gap of X right to $C(i)$. $J(i)$ is also a gap of X_∞, either principal or complementary. It forms a cycle with the same arrows $t_i^{-1}\colon J(i) \to J(i+1)$ as the arrows $t_i^{-1}\colon C(i) \to C(i+1)$ of the cycle $\{C(i)\}$. The cycle, consisting of principal and complementary gaps, is contained in the first and second lines of diagram (10.1). But, since there is no consecutive b or b^{-1} in the arrows of the cycle, it is either the cycle in the first line or its reciprocal. In particular, $n = 2k$ and all the $J(i)$'s are principal.

Now, any a-interval appears in the cycle $\{C(i)\}$, since its right gap is principal and any principal gap appears in the cycle $\{J(i)\}$. By (1), if $C(i)$ is an a-interval, then it is mapped by a to a $b^{\pm 1}$- interval $C(i+1)$. That is, *any a-interval must be mapped by a to a $b^{\pm 1}$-interval*. But there are k a-intervals and $2k$ $b^{\pm 1}$-intervals, and some a-interval must be mapped by a to a nontrivial b-block containing more than one $b^{\pm 1}$-interval. A contradiction. □

We consider the following property and the second modification.

(**) $\operatorname{Int} X_\infty = \emptyset$.

Lemma 10.10. *Any Markov system is equivalent to a system with properties (*) and (**).*

Proof. This is done by an anti-Denjoy modification: one collapses each component $[\underline{w}]$ of X_∞ to a point, and define new maps a and b of the collapsed S^1. The modification does not collapse $[a]$- or $[b^{\pm 1}]$-intervals to points, thanks to Lemma 10.9, and does not spoil property (**). □

11. Circular order determined by Markov systems

In this section, we shall show that any Markov system with (*) and (**) is a dynamical realization of some circular order, i.e., that the associated action is tight at some point. This is done by showing that the set X_∞ is a minimal set with good properties.

Definition 11.1. For an admissible word v of even length, (v) denotes the infinite admissible word which repeats v.

Lemma 11.2. *The boundary point $x \in \partial I_1 \cap [a]$ satisfies $\widehat{W}(x) = (ab_k \cdots ab_1)$ and the other boundary point y satisfies $\widehat{W}(y) = (b_1^{-1}a \cdots b_k^{-1}a)$.*

Proof. Since x is fixed by f_1, we have an equality of infinite words

$$ab_k \cdots ab_1 \widehat{W}(x) = \widehat{W}(x),$$

where the left-hand side is before reducing. If there is no cancellation in the left-hand side, we get $\widehat{W}(x) = (ab_k \cdots ab_1)$. If $ab_k \cdots ab_1$ is completely canceled out, we get $\widehat{W}(x) = (b_1^{-1}a \cdots b_k^{-1}a)$. Otherwise, there is an intermediate cancellation, the left-hand side begins at a, and the right-hand side begins at b_1^{-1}. We thus obtained either $\widehat{W}(x) = (ab_k \cdots ab_1)$ or $\widehat{W}(x) = (b_1^{-1}a \cdots b_k^{-1}a)$. A like statement holds for $\widehat{W}(y)$. But, since $x \in [a]$, then $\widehat{W}(x) = (ab_k \cdots ab_1)$, and, since I_1 is a principal gap, then $\widehat{W}(y) = (b_1^{-1}a \cdots b_k^{-1}a)$. □

Lemma 11.3. *The stabilizer* $\mathbb{G}_{I_1}$ *of* I_1 *is infinite cyclic generated by* f_1.

Proof. Assume $h \neq \mathrm{id}$ stabilizes I_1. One may assume that h admits neither f_1 nor f_1^{-1} as an initial subword, by replacing h with a shorter word if necessary. Since $hx = x$ for x in the previous lemma, we have an equality of infinite words

$$h(ab_k \cdots ab_1) = (ab_k \cdots ab_1). \tag{11.1}$$

If there is an intermediate cancellation in the left-hand side, then h begins and ends at a. But then, since $hy = y$, we have another equality

$$h(b_1^{-1}a \cdots b_k^{-1}a) = (b_1^{-1}a \cdots b_k^{-1}a), \tag{11.2}$$

where the left-hand side begins at a and the right-hand side at b_1^{-1}, leading to a contradiction.

Consider the case where there is no cancellation at all in the left had side of (11.1). Then either h is f_1 or its initial subword of length 2ℓ, $\ell < k$. In the latter case, the word $(ab_k \cdots ab_1)$ is periodic of period 2ℓ and $2k$, hence of period $2(k,\ell)$. This shows that $h = ab_\ell \cdots ab_1$. But then in the diagram (10.1) the principal gap $I_{\ell+1}$ must be equal to I_1. A contradiction. In the remaining case, where w is cancelled out in the left-hand side of (11.1), there is no cancellation in the left-hand side of (11.2), and one can show that $h = f_1^{-1}$ by a like argument. □

Lemma 11.4. *For any Markov system with* (**), X_∞ *is a minimal set of* $\mathbb{G}$.

Proof. First of all, notice that X_∞ is a Cantor set by Lemma 10.9 and (**). This, together with Lemma 10.8, shows that the orbit of a boundary point of a gap is dense in X_∞. Assume there is a minimal set Y properly contained in X_∞. Then any boundary point of a gap of X_∞ cannot be contained in Y. Therefore, any gap K of Y contains infinitely many gaps J_i of X_∞. Since each J_i belongs to the orbit of I_1, it is left invariant by a map represented in an admissible word as $h_i = v_i f_i v_i^{-1}$, where f_i is a cyclic permutation of f_1. Since all the h_i leaves a boundary point z of K invariant, we get $v_i f_i v_i^{-1}\widehat{W}(z) = \widehat{W}(z)$. By the same argument as the proof of Lemma 11.2, $\widehat{W}(z) = v_i(f_i)$ or $\widehat{W}(z) = v_i(f_i^{-1})$ for any i. This contradicts the uniqueness of $\widehat{W}(z)$: the admissibility of the word $v_i f_i v_i^{-1}$ implies that v_i contains neither f_i nor f_i^{-1} as the terminal subword. □

Definition 11.5. Given a Markov system $\mathbb{M} = (a, b, [a], [b], [[b]])$, define a homomorphism $\phi_{\mathbb{M}}\colon G \to \mathrm{Homeo}_+(S^1)$ by $\phi_{\mathbb{M}}(\alpha) = a$ and $\phi_{\mathbb{M}}(\beta) = b$.

We have $\phi_{\mathbb{M}}(G) = \mathbb{G}$.

Lemma 11.6. *For any Markov system* $\mathbb{M}$ *satisfying* (*) *and* (**), $\phi_{\mathbb{M}}$ *is a dynamical realization of a circular order, denoted by* $c_{\mathbb{M}}$, *based at some point* $x_0 \in I_1$.

Proof. By Lemma 2.7, we only need to show that $\phi_{\mathbb{M}}$ is tight at x_0. Clearly $\phi_{\mathbb{M}}$ is free at x_0 by (*). To show the other condition, notice that X_∞ is a Cantor minimal set and that any gap of X_∞ is a translate of I_1. Now, choose an arbitrary gap J of $\mathrm{Cl}(\phi_{\mathbb{M}}(G)x_0)$ for $x_0 \in I_1$. If one endpoint of J belongs to the orbit $\phi_{\mathbb{M}}(G)x_0$, so does the other endpoint. On the other hand, there is no gap of $\mathrm{Cl}(\phi_{\mathbb{M}}(G)x_0)$ whose endpoints belong to the minimal set X_∞. □

12. Isolation

The following theorem is the goal of Part II.

Theorem 12.1. *For any Markov system* $\mathbb{M}$ *satisfying* (*) *and* (**), *the circular order* $c_{\mathbb{M}}$ *given by Lemma* 11.6 *is isolated. Moreover,* $\deg(c_{\mathbb{M}})$ *is equal to the multiplicity of* $\mathbb{M}$.

The statement about the degree (Definition 4.2) is obvious. The rest of this section is devoted to the proof of the isolation of $c_{\mathbb{M}}$, by a ping-pong argument. Let

$$Y = [ab] \cup [ab^{-1}] \cup [b] \cup [b^{-1}],$$

$$h_1 = abab^{-1}, \quad h_2 = ab^{-1}ab,$$

$$\Omega(h_1) = [ab], \quad \Omega(h_2) = [ab^{-1}],$$

$$\Omega(h_1^{-1}) = [b], \quad \Omega(h_2^{-1}) = [b^{-1}].$$

Notice that the "address" of $\Omega(h_i^{\pm 1})$ is just the first two letters of $h_i^{\pm 1}$, since $[b^{\pm 1}] = [b^{\pm 1}a]$.

Lemma 12.2. *For* $i = 1, 2$, *we have a precise inclusion*

$$h_i(Y - \Omega(h_i^{-1})) \subset \Omega(h_i)$$

and

$$h_i^{-1}(Y - \Omega(h_i)) \subset \Omega(h_i^{-1}).$$

Proof. For example, the first inclusion for $i = 1$ follows from a sequence of precise inclusions,

$$abab^{-1}([a] \cup [b^{-1}]) = aba([b^{-1}] \cup [b]) = ab([ab^{-1}] \cup [ab]) \subset ab([a]) = [ab].$$

The other inclusions are similar. □

The ping-pong property on Y is not sufficient to prove the isolation of $c_{\mathbb{M}}$.

Lemma 12.3. *There are open neighbourhoods $N_i^{\pm}$ of $\Omega(h_i^{\pm 1})$ such that*

(1) *the closures $\mathrm{Cl}(N_i^{\pm})$ are mutually disjoint, and*

(2) $h_i(S^1 - N_i^-) \subset N_i^+$, *or equivalently*, $h_i^{-1}(S^1 - N_i^+) \subset N_i^-$, $i = 1, 2$.

Proof. We assume that the point x in $\partial I_1 \cap [a]$ is the right endpoint of I_1. Thus, the map $f_1 = ab_k \cdots ab_1$ is an increasing homeomorphism of I_1. One can choose two points in all the principal gaps I_i and I_i' in (10.1) which satisfy the relations in Figure 2, where the thin lines indicate the correspondences by the intertwining maps.

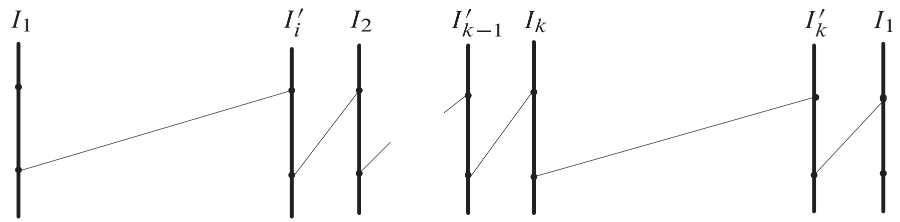

Figure 2. The right side of the intervals are drawn upward.

This is possible simply because the first return map is increasing.

Then choose three points in I_i'' and I_i''' as the images of the previous points. See Figure 3. All these points are called *distinguished points*.

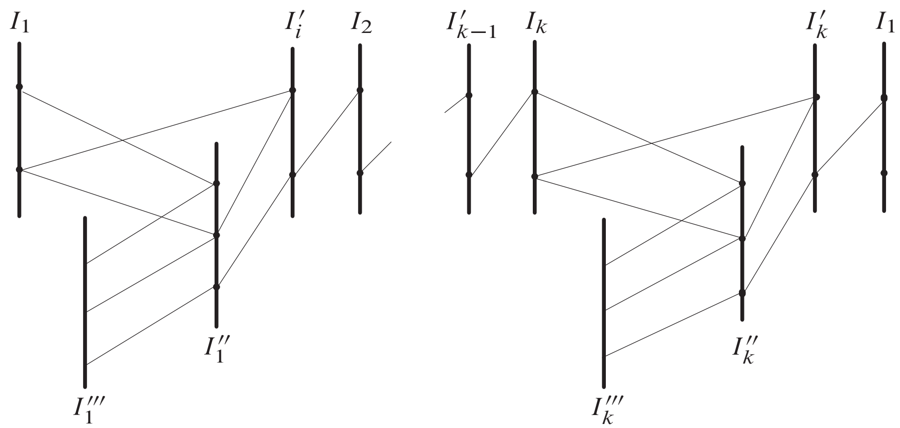

Figure 3

Define the neighbourhood $N_i^{\pm}$ of $\Omega(h_i^{\pm 1})$ by expanding each component until it reaches the first distinguished point. Condition (1) of the lemma is satisfied.

Let us check if (2) is satisfied. First of all, we only consider the right gap J of each component of $\Omega(h_i^{\mp 1})$ and check if $J \setminus N_i^{\mp}$ is mapped by $h_i^{\pm 1}$ into $h_i^{\pm 1}(J) \cap N_i^{\pm}$.

We do not need to consider the left gaps, since the statement for them follows automatically. We can see this by Figure 4. Our strategy is the following. We choose any gap $J \in \{I_i, I_i', I_i'', I_i'''\}$ from diagram (10.1). We consider the left endpoint $\partial_- J$ and calculate to which class ($[ab]$, $[ab^{-1}]$, $[b]$ or $[b^{-1}]$) it belongs. It tells us which one of $\Omega(h_i^{\pm 1})$ the left neighbour of J is, and thus which one of the maps $h_i^{\pm 1}$ we should check. We explain it concretely with an example $J = I_1$. Recall that the left endpoint y of I_1 satisfies $\widehat{W}(y) = (b_1^{-1} a \cdots b_k^{-1} a)$ and it belongs to $[b_1^{-1}] = \Omega(b_1^{-1} a b_1 a)$. Therefore, the map we should check is $a b_1^{-1} a b_1$. For any gap J, we calculate its left endpoint and the corresponding map. The actual proof is divided into four cases.

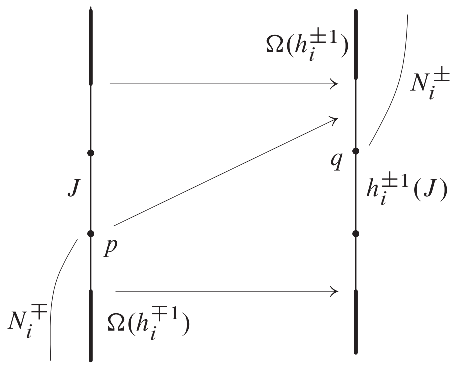

Figure 4. The arrows are $h_i^{\pm 1}$. We shall check if p is mapped above q. If this is true, then the inverse $h_i^{\mp 1}$ maps q below p.

Case 1. $J = I_i$. Since

$$\begin{aligned} \widehat{W}(\partial_- I_i) &= a b_{i-1} \cdots a b_1 \widehat{W}(y) \\ &= a b_{i-1} \cdots a b_1 (b_1^{-1} a \cdots b_k^{-1} a) = b_i^{-1} a b_{i+1}^{-1} \cdots, \end{aligned}$$

we have $\partial_- I_i \in [b_i^{-1}] = \Omega(b_i^{-1} a b_i a)$ and the map in concern is $h = a b_i^{-1} a b_i$.

According as $b_{i+1} = b_i^{-1}$ or $b_{i+1} = b_i$, h is either of the following composites:

$$\begin{array}{ccccccccc} I_i & \xrightarrow{b_i} & I_i' & \xrightarrow{a} & I_{i+1} & \xrightarrow{b_{i+1}} & I_{i+1}' & \xrightarrow{a} & I_{i+2} \\ & & & & & \searrow^{b_{i+1}^{-1}} & & & \\ & & & & & & I_{i+1}'' & \xrightarrow{a} & I_{i+1}''' \end{array}$$

In any case, h satisfies the required property. See Figure 5.

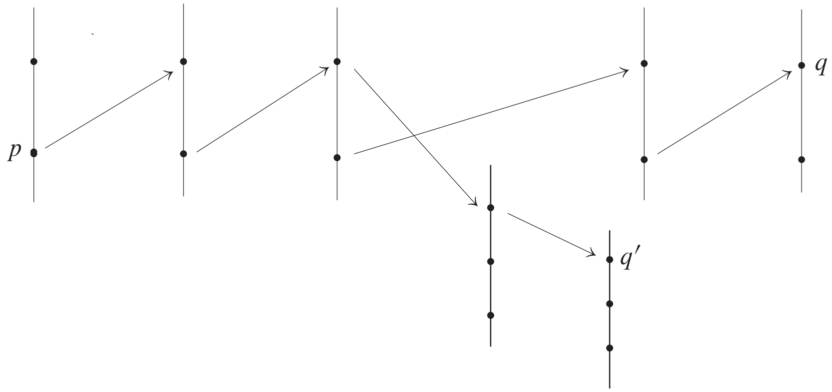

Figure 5. h maps p above q or q'.

Case 2. $\boldsymbol{J = I_i'}$**.** Since $\widehat{W}(\partial_- I_i') = b_i \widehat{W}(\partial_- I_i) = a b_{i+1}^{-1} \cdots$, we have $\partial_- I_i' \in \Omega(a b_{i+1}^{-1} a b_{i+1})$ and the map in concern is $b_{i+1}^{-1} a b_{i+1} a$. This case is analogous to Case 1, and is omitted. The point is that the map $b_{i+1}^{-1} a b_{i+1} a$ defined on I_i' also goes from left to right in (10.1).

Case 3. $\boldsymbol{J = I_i''}$**.** Since $\widehat{W}(\partial_- I_i'') = b_i^{-1} \widehat{W}(\partial_- I_i) = b_i \cdots$, we have $\partial_- I_i'' \in \Omega(b_i a b_i^{-1} a)$ and the map in concern is $a b_i a b_i^{-1}$.

It is either of the following composites:

$$\begin{array}{ccccccccc} & & I_i' & \xrightarrow{a} & I_{i+1} & \xrightarrow{b_{i+1}} & I_{i+1}' & \xrightarrow{a} & I_{i+2} \\ & \nearrow^{b_i^{-1}} & & & & \searrow^{b_{i+1}^{-1}} & & & \\ I_i'' & & & & & & I_{i+1}'' & \xrightarrow{a} & I_{i+1}''' \end{array}$$

See Figure 6. In the bottom composite, the point p is mapped exactly to q'. In this case, we enlarge the neighbourhood of $\Omega(b_i a b_i^{-1} a)$ slightly so as to contain the point p. Then its complement in J is mapped above q'.

Case 4. $\boldsymbol{J = I_i'''}$**.** Since $\widehat{W}(\partial_- I_i''') = a \widehat{W}(\partial_- I_i''') = a b_i \cdots$, we have $\partial_- I_i''' \in \Omega(a b_i a b_i^{-1})$ and the map in concern is $b_i a b_i^{-1} a$. This case is similar to Case 3 and is omitted. □

For the proof of Theorem 12.1, we need [3, Proposition 3.3]. The statement below is adapted for our purpose, and we shall give a complete proof in the appendix.

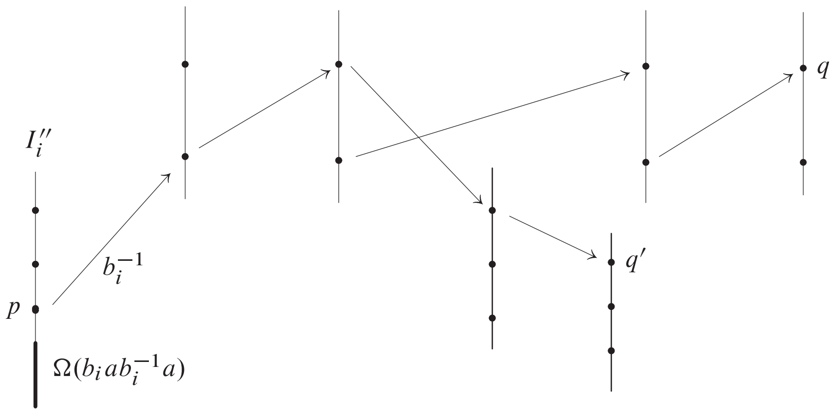

Figure 6. p is mapped above q, but exactly to q'.

Proposition 12.4. *Let ϕ be a dynamical realization of a circular order c based at x_0. Given any neighbourhood U of ϕ in* $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$*, there is a neighbourhood V of c in* $\mathrm{CO}(G)$ *such that any order in V has a dynamical realization based at x_0 contained in U.*

We also need the following proposition whose proof is again contained in the appendix.

Proposition 12.5. *The space* $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$ *is locally pathwise connected. In fact, more can be said: any neighbourhood of any point contains a pathwise-connected neighbourhood.*

Proof of Theorem 12.1. Recall that $\phi_{\mathbb{M}}$ is a dynamical realization of the circular order $c_{\mathbb{M}}$ given by Lemma 11.6 based at a point $x_0 \in I_1$. One can choose x_0 in $\mathrm{Int}(S^1 \setminus \cup N_i^{\pm})$, where $N_i^{\pm}$ is from Lemma 12.3. Choose a pathwise connected neighbourhood U of $\phi_{\mathbb{M}}$ in $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$ such that any action $\psi \in U$ satisfies

$$h_i(\psi)(S^1 \setminus N_i^-) \subset N_i^+$$

(and equivalently $h_i(\psi)^{-1}(S^1 \setminus N_i^+) \subset N_i^-$), where $h_1(\psi) = \psi(\alpha\beta\alpha\beta^{-1})$ and $h_2(\psi) = \psi(\alpha\beta^{-1}\alpha\beta)$.

The ping-pong lemma (Klein's criterion) asserts that the subgroup $[G, G]$ generated by $\alpha\beta\alpha\beta^{-1}$ and $\alpha\beta^{-1}\alpha\beta$ acts freely at x_0 by any $\psi \in U$. We claim that G itself acts freely at x_0 by ψ. In fact, if H is the stabilizer at x_0, then $H \cap [G, G] = \{e\}$. That is, the canonical projection $G \to G/[G, G] \cong \mathbb{Z}/6\mathbb{Z}$ is injective on H, and hence H, if nontrivial, contains an element γ of finite order. But γ is conjugate either to α or to $\beta^{\pm 1}$, and therefore $\psi(\gamma)$ is fixed point free. A contradiction shows the claim.

Now, let V be a neighbourhood of $c_{\mathbb{M}}$ in $\mathrm{CO}(G)$ such that a dynamical realization ψ of an arbitrary order c' of V belongs to U. Then there is a path ψ_t, $0 \le t \le 1$, joining $\phi_{\mathbb{M}}$ and ψ contained in U. Since all the ψ_t act freely at x_0, the circular orders of G obtained from the orbit $\psi_t(G)x_0$ are all the same, independent of t. (Recall that $\mathrm{CO}(G)$ is totally disconnected.) This shows $c' = c_{\mathbb{M}}$. The proof of Theorem 12.1 is now complete. □

Proof of Theorem 1. We have constructed a map from the set of the isolated circular orders to the set of the equivalence classes of the Markov systems in Part II, and a map in the reverse direction in Part III. It is clear from the constructions that one is the inverse of the other. □

Part IV

We present some examples of isolated circular orders of G.

13. Primary examples

We revisit isolated circular orders of G constructed in [4].

Standard example. Here is a Markov system $\mathbb{M}_1$ with multiplicity 1. Just place intervals $[a], [b], [b^{-1}]$ in the anti-clockwise order on S^1. Clearly, there are an involution a of S^1 which interchanges $[a]$ and $[[b]]$ and a period-3 homeomorphism b which circulates $[a]$, $[b]$, and $[b^{-1}]$. Transitivity of gaps is also clear. Thus we obtain an isolated circular order $c_1 = c_{\mathbb{M}_1}$. If the placement of $[a], [b], [b^{-1}]$ is clockwise, we get another order c'. But this is in the same automorphism class of c_1: $c' = (\sigma_0)_* c_1$ for σ_0 in Proposition 3.1 (4). By the homeomorphism $\mathrm{CO}(G) \approx \mathrm{LO}(B_3)$, c_1 corresponds to the Dubrovina–Dubrovin order [1].

Finite lifts of the standard example. Let ϕ_1 be a dynamical realization of the previous example c_1. We shall construct more examples starting with finite lifts of ϕ_1. For $k > 1$, let $p_k\colon S^1 \to S^1$ be the k-fold covering map. A G-action ϕ_k on S^1 is called a *k-fold lift of* ϕ_1 if it satisfies $p_k\phi_k(g) = \phi_1(g)$ for any $g \in G$. Then the rotation numbers satisfy $k\,\mathrm{rot}(\phi_k(g)) = \mathrm{rot}(\phi_1(g))$. This shows that a lift of an involution is an involution if and only if k is odd. Likewise, a lift of a 3-periodic map is 3-periodic if and only if k is coprime to 3. Therefore, a k-fold lift ϕ_k of ϕ_1 exists if and only if $k \equiv \pm 1$ (6): moreover it is unique if it exists. The map $\phi_k(\beta)$ is 3-periodic and has rotation number $\pm 1/3$ according as $k \equiv \pm 1$ (6), since $(6\ell \pm 1) \cdot (\epsilon/3) \equiv 1/3$ (1) implies $\epsilon = \pm 1$.

We shall show that the lift ϕ_k is associated with a Markov system $\mathbb{M}_k$ for $k = 6\ell + 1$. The case $k = 6\ell - 1$ is similar and is left to the reader. Let $[a]_i$, $[b]_i$, and $[b^{-1}]_i$ ($i \in \mathbb{Z}/k\mathbb{Z}$) be the lifts of $[a]$, $[b]$, and $[b^{-1}]$ of the previous example $\mathbb{M}_1$ by p_k, ordered anticlockwise in S^1 as

$$\dots, \quad [a]_i, \quad [b]_i, \quad [b^{-1}]_i, \quad [a]_{i+1}, \quad \dots. \tag{13.1}$$

The maps a, b of the system $\mathbb{M}_k$ must be $\phi_k(\alpha)$ and $\phi_k(\beta)$. The sequence (13.1) has $3k = 18\ell + 3$ terms, and b, being the lift of $\phi_1(\beta)$, maps each term to the term $6\ell + 1$ right to it. Therefore we have

$$b[a]_i = [b]_{i+2\ell}, \quad b[b]_i = [b^{-1}]_{i+2\ell}, \quad b[b^{-1}]_i = [a]_{i+2\ell+1}. \tag{13.2}$$

The sequence (13.1) is contracted into a sequence

$$\dots, \quad [a]_i, \quad [[b]]_i, \quad [a]_{i+1}, \quad [[b]]_{i+1}, \quad \dots$$

of $12\ell + 2$ terms. The map a sends each term to the term $6\ell + 1$ right to it. That is,

$$a[a]_i = [[b]]_{i+3\ell}, \quad a[[b]]_i = [a]_{i+3\ell+1}. \tag{13.3}$$

In order to check (E), we shall classify all principal gaps into two families J_i's and J_i''s ($i \in \mathbb{Z}/(6\ell + 1)\mathbb{Z}$) by the following ordering:

$$\dots, \quad [a]_i, \quad J_i, \quad [b]_i, \quad [b^{-1}]_i, \quad J_i', [a]_{i+1}, \quad \dots.$$

By (13.2) and (13.3), we get

$$b(J_i') = J_{i+2\ell+1} \tag{13.4}$$

and

$$ab(J_i') = J'_{i+5\ell+1}. \tag{13.5}$$

Now, (13.5) shows that the group generated by ab acts transitively on the family J_i''s, since $(6\ell + 1, 5\ell + 1) = 1$, and (13.4) shows that any J_i from the other family is mapped by b^{-1} to an element of this family. This shows (E).

The circular order defined by $\mathbb{M}_k$ is denoted by c_k. Each of them is from a distinct automorphism class, since $\deg(c_k) = k$.

14. Further example

Notice that the $\deg(c)$ of any isolated circular order c is odd, since the involution a transposes $[a]$ and $[[b]]$. According to our calculation, there are no new examples of isolated circular orders up to degree ≤ 7. But there is one in degree 9, which we shall present below.

This example is not well ordered as the previous one and it is no use to give an index to each component. Any component of $[a]$ is denoted by the same letter $\underline{a}$. Likewise, we use the notations $\underline{b}$ and $\underline{b}^{-1}$. Also a component of $[[b]]$ is denoted by $[\underline{b}]$. Consider the following ordering of 27 intervals in S^1 which is grouped into three:

$$\underline{a}\,\underline{b}^{-1}\underline{a}\,\underline{b}^{-1}\underline{a}\,\underline{b}^{-1}\underline{b}\,\underline{a}\,\underline{b}^{-1}\colon \underline{b}\,\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}^{-1}\underline{b}\underline{a}\colon \underline{b}^{-1}\underline{b}\,\underline{b}^{-1}\underline{b}\,\underline{b}^{-1}\underline{b}\,\underline{a}\,\underline{b}^{-1}\underline{b}. \qquad (14.1)$$

Define a period-3 homeomorphism b of S^1 by permuting the three groups cyclically to the right. Notice that b so defined satisfies $b\underline{a} = \underline{b}$, $b\underline{b} = \underline{b}^{-1}$, and $b\underline{b}^{-1} = \underline{a}$. The sequence (14.1) is contracted to

$$\underline{a}\,[\underline{b}]\,\underline{a}\,[\underline{b}]\,\underline{a}\,[\underline{b}]\,\underline{a}\,[\underline{b}]\,\underline{a}\colon [\underline{b}]\,\underline{a}\,[\underline{b}]\,\underline{a}\,[\underline{b}]\,\underline{a}\,[\underline{b}]\,\underline{a}\,[\underline{b}] \qquad (14.2)$$

Define an involution a by transposing the groups. The maps a and b satisfies (A)–(D). To show (E), indicate the principal gaps by $\left[\begin{smallmatrix} i\\ j\end{smallmatrix}\right]$ and the complementary gaps $\left[\begin{smallmatrix} i\\ *\end{smallmatrix}\right]$ as follows:

$$\underline{a}\begin{bmatrix}1\\1\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}2\\2\end{bmatrix}\underline{a}\begin{bmatrix}3\\3\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}4\\4\end{bmatrix}\underline{a}\begin{bmatrix}5\\5\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}6\\ *\end{bmatrix}\underline{b}\begin{bmatrix}7\\6\end{bmatrix}\underline{a}\begin{bmatrix}8\\7\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}9\\ *\end{bmatrix}$$

$$\underline{b}\begin{bmatrix}1\\8\end{bmatrix}\underline{a}\begin{bmatrix}2\\9\end{bmatrix}\underline{b}\begin{bmatrix}3\\1\end{bmatrix}\underline{a}\begin{bmatrix}4\\2\end{bmatrix}\underline{b}\begin{bmatrix}5\\3\end{bmatrix}\underline{a}\begin{bmatrix}6\\4\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}7\\ *\end{bmatrix}\underline{b}\begin{bmatrix}8\\5\end{bmatrix}\underline{a}\begin{bmatrix}9\\6\end{bmatrix}$$

$$\underline{b}^{-1}\begin{bmatrix}1\\ *\end{bmatrix}\underline{b}\begin{bmatrix}2\\ *\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}3\\ *\end{bmatrix}\underline{b}\begin{bmatrix}4\\ *\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}5\\ *\end{bmatrix}\underline{b}\begin{bmatrix}6\\7\end{bmatrix}\underline{a}\begin{bmatrix}7\\8\end{bmatrix}\underline{b}^{-1}\begin{bmatrix}8\\ *\end{bmatrix}\underline{b}\begin{bmatrix}9\\9\end{bmatrix}$$

Notice that one of maps $b^{\pm 1}$ sends $\left[\begin{smallmatrix} i\\ j\end{smallmatrix}\right]$ to some $\left[\begin{smallmatrix} i\\ k\end{smallmatrix}\right]$ and the other one to $\left[\begin{smallmatrix} i\\ *\end{smallmatrix}\right]$, and that the map a sends $\left[\begin{smallmatrix} i\\ j\end{smallmatrix}\right]$ to some $\left[\begin{smallmatrix} k\\ j\end{smallmatrix}\right]$. We can find a cycle of principal gaps:

$$\begin{bmatrix}1\\1\end{bmatrix}\xrightarrow{a}\begin{bmatrix}3\\1\end{bmatrix}\xrightarrow{b^{-1}}\begin{bmatrix}3\\3\end{bmatrix}\xrightarrow{a}\begin{bmatrix}5\\3\end{bmatrix}\xrightarrow{b^{-1}}\begin{bmatrix}5\\5\end{bmatrix}\xrightarrow{a}\begin{bmatrix}8\\5\end{bmatrix}\xrightarrow{b^{-1}}\begin{bmatrix}8\\7\end{bmatrix}\xrightarrow{a}\begin{bmatrix}6\\7\end{bmatrix}\xrightarrow{b^{-1}}\begin{bmatrix}6\\4\end{bmatrix}\xrightarrow{a}$$

$$\begin{bmatrix}4\\4\end{bmatrix}\xrightarrow{b}\begin{bmatrix}4\\2\end{bmatrix}\xrightarrow{a}\begin{bmatrix}2\\2\end{bmatrix}\xrightarrow{b}\begin{bmatrix}2\\9\end{bmatrix}\xrightarrow{a}\begin{bmatrix}9\\9\end{bmatrix}\xrightarrow{b^{-1}}\begin{bmatrix}9\\6\end{bmatrix}\xrightarrow{a}\begin{bmatrix}7\\6\end{bmatrix}\xrightarrow{b^{-1}}\begin{bmatrix}7\\8\end{bmatrix}\xrightarrow{a}\begin{bmatrix}1\\8\end{bmatrix}\xrightarrow{b^{-1}}$$

Therefore (E) is also satisfied. This yields an isolated circular order c_9 of degree 9. It is not in the automorphism classes of the previous examples. If we arrange the intervals in the clockwise order, we obtain another order c'. But again $c' = (\sigma_0)^* c_9$.

15. Appendix

We shall give proofs of Propositions 12.4 and 12.5. The former holds true for an arbitrary countable group H.

Proposition 12.4. *Let ϕ be a dynamical realization of a circular order c based at x_0. Given any neighbourhood U of ϕ in* $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$*, there is a neighbourhood V of c in* $\mathrm{CO}(G)$ *such that any order in V has a dynamical realization based at x_0 contained in U.*

Proof. Given a finite set F of H and $\epsilon > 0$, let

$$U(F, \epsilon, \phi) = \{\psi \in \mathrm{Hom}(H, \mathrm{Homeo}_+(S^1)) \colon \|\psi(g) - \phi(g)\|_0 < \epsilon \text{ for all } g \in F\}.$$

One may replace the given U in the proposition by a smaller neighbourhood $U(F, \epsilon, \phi)$. One may also assume that F is symmetric: $g \in F$ implies $g^{-1} \in F$. Given a finite subset S of H, define a neighbourhood $V_S(c)$ of c in $\mathrm{CO}(H)$ by

$$V_S(c) = \{c' \in \mathrm{CO}(H) \colon c'|_{(S \cup \{e\})^3} = c|_{(S \cup \{e\})^3}\}.$$

Then any $c' \in V_S(c)$ admits a dynamical realization $\psi_{c',S}$ based at x_0 such that $\psi_{c',S}(g)x_0 = \phi(g)x_0$ for any $g \in S$. Our aim is first to find a good S and then to alter $\psi_{c',S}$ by a x_0-preserving conjugacy to obtain a homomorphism contained in $U(F, \epsilon, \phi)$. Choose $\delta > 0$ so that, whenever $f \in F$ and $|x - y| < \delta$, we have $|\phi(f)x - \phi(f)y| < \epsilon$.

Case 1. *$\phi(H)x_0$ is dense in S^1.* Choose a finite subset S' of H so that $X = \phi(S')x_0$ is $\delta/2$-dense in S' and define $S = FS' \cup S'$. In this case, there is no need for the alteration: we shall show that $\psi_{c',S} \in U(F, \epsilon, \phi)$. In fact, for any $x \in X$, we have $x = \psi_{c',S}(g)x_0 = \phi(g)x_0$ for $g \in S'$. Moreover $\psi_{c',S}(f)x = \phi(f)x$, for any $x \in X$ and $f \in F$, because $\phi(f)x = \phi(fg)x_0$, $\psi_{c',S}(f)x = \psi_{c',S}(fg)x_0$ and $fg \in S$. Choose any $f \in F$ and any $y \in S^1$. Then y belongs to the closure $\mathrm{Cl}(J)$ of some gap J of X. Since X is $\delta/2$-dense, we have $|J| < \delta$, and hence $|\phi(f)J| < \epsilon$. Since both points $\phi(f)y$ and $\psi_{c',S}(f)y$ belong to the same interval $\phi(f)\,\mathrm{Cl}(J) = \psi_{c',S}(f)\,\mathrm{Cl}(J)$, we have $|\phi(f)y - \psi_{c',S}(f)y| < \epsilon$. Since $f \in F$ and $y \in S^1$ are arbitrary, we obtain $\psi_{c',S} \in U(F, \epsilon, \phi)$, as is required.

Case 2. *$\phi(H)x_0$ is not dense in S^1.* Denote by J_0 the gap of $\mathrm{Cl}(\phi(H)x_0)$ whose left endpoint is x_0. Define $h_{\min} \in H$ so that $\phi(h_{\min})x_0$ is the right endpoint of J_0. Since ϕ is tight at x_0, any gap J of $\mathrm{Cl}(\phi(H)x_0)$ is a translate of J_0. Denote by $\mathcal{V}$ the set of the gaps J such that $|J| \geq \delta$. Let

$$S_1 = \{g \in H \colon \phi(g)(J_0) \in \mathcal{V}\}$$

and let $S_2 = S_1 \cup S_1 h_{\min}$. Thus, the endpoints of any interval of $\mathcal{V}$ belongs to $\phi(S_2)x_0$. Add some more elements to S_2 to form a finite subset S_3 such that $\phi(S_3)x_0$ is $\delta/2$-dense in $\mathrm{Cl}(\phi(H)x_0)$. Notice that any gap of $\phi(S_3)x_0$ either belongs to $\mathcal{V}$ or is of length $< \delta$. Finally, let $S = FS_3 \cup S_3$. We have the following property by the same argument as in Case 1.

(1) *For any point* $x \in \phi(S_3)x_0$ *and* $f \in F$, *we have* $\psi_{c',S}(f)x = \phi(f)x$.

Define a one dimensional simplicial complex $\mathcal{K}$ as follows. The vertex set of $\mathcal{K}$ is $\mathcal{V}$. Two vertices J and J' are joined by an edge if there is $f \in F$ such that $\phi(f)J = J'$. Such f is unique since $\phi(H)$ acts simply transitively on the gaps of $\mathrm{Cl}(\phi(H)x_0)$: $\phi(g)J_0 = J_0$ implies $g = e$. We label the directed edge from J to J' by f. For a directed edge $J \xrightarrow{f} J'$ of $\mathcal{K}$, we have $\psi_{c',S}(f)(J) = J'$ by (1). The simple transitivity on the gaps shows the following.

(2) *For a directed cycle*

$$J_1 \xrightarrow{f_1} J_2 \xrightarrow{f_2} \cdots J_n \xrightarrow{f_n} J_{n+1} = J_1,$$

we have

$$f_n \cdots f_2 f_1 = e.$$

Let us consider $h \in \mathrm{Homeo}_+(S^1)$ supported on the union of the closures of the intervals in $\mathcal{V}$ and leaving the endpoints of the intervals fixed. We claim that there is h such that for any directed edge $J \xrightarrow{f} J'$ of $\mathcal{K}$, $h\psi_{c',S}(f)h^{-1} = \phi(f)$ on J. To show this, consider a spanning tree T_ν of each component $\mathcal{K}_\nu$ of $\mathcal{K}$. Define h to be the identity on a prescribed base vertex J_1 of T_ν. For any directed edge $J_1 \xrightarrow{f} J_2$ of T_ν, define h on J_2 so that $h\psi_{c',S}(f)h^{-1} = \phi(f)$ holds on J_1. We continue this process along directed paths in T_ν issuing at J_1 until we define h on all the vertices of $\mathcal{K}_\nu$. Then for any directed edge $J \xrightarrow{f} J'$ of T_ν, $h\psi_{c',S}(f)h^{-1} = \phi(f)$ on J (even if the edge is directed toward the base vertex). Also, for an edge of $\mathcal{K}_\nu$ not in T_ν, we have the same equality thanks to the relation (2). The proof of the claim is over.

Let us define a homomorphism ψ by $\psi(g) = h\psi_{c',S}(g)h^{-1}$ for any $g \in H$. The homomorphism ψ still satisfies (1) with $\psi_{c',S}$ replaced by ψ. Let J be any gap of $\phi(S_3)x_0 = \psi(S_3)x_0$ and f any element of F. If either $|J| < \delta$ or $|\phi(f)J| < \delta$, then $\psi(f)$ is ϵ-near to $\phi(f)$ on J. If not, $J \xrightarrow{f} \phi(f)J$ is an edge of $\mathcal{K}$, and $\psi(f) = \phi(f)$ on J. This shows $\psi \in U(F, \epsilon, \phi)$, as is required. □

Proposition 12.5. *The space* $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$ *is locally pathwise connected. In fact, more can be said: any neighbourhood of any point contains a pathwise-connected neighbourhood.*

Proof. The space in question is homeomorphic to $Q_2 \times Q_3$, where

$$Q_2 = \{a \in \mathrm{Homeo}_+(S^1) \colon a^2 = \mathrm{id}\},$$
$$Q_3 = \{b \in \mathrm{Homeo}_+(S^1) \colon b^3 = \mathrm{id}\}.$$

Choose a base point $x_0 \in S^1$. A map a in Q_2 is specified by a point $a(x_0)$ and an orientation-preserving homeomorphism from $[x_0, a(x_0)]$ to $[a(x_0), x_0]$. So, Q_2 is homeomorphic to $(0,1) \times \mathrm{Homeo}_+(0,1)$. Likewise, Q_3 is homeomorphic to $(0,1) \times \mathrm{Homeo}_+(0,1) \times \mathrm{Homeo}_+(0,1)$. It suffices to show that the space $\mathrm{Homeo}_+(0,1)$ satisfies the claimed property. For any neighbourhood U of a point $f \in \mathrm{Homeo}_+(0,1)$, choose $\epsilon > 0$ so that

$$U(f,\epsilon) = \{g \in \mathrm{Homeo}_+(0,1) : \|g - f\| < \epsilon\}$$

is contained in U. Then for any $g \in U(f,\epsilon)$, the path $\{(1-t)f + tg\}_{0 \le t \le 1}$ is contained in $U(f,\epsilon)$. □

References

[1] T. V. Dubrovina and N. I. Dubrovin, *On braid groups. Sb. Math.* **192** (2001), no. 5–6, 693–703. English translation of *Mat. Sb.* **192** (2001), no. 5, 53–64, in Russian. Zbl 1037.20036 MR 1859702

[2] S. H. McCleary, Free lattice ordered groups represented as o-2 transitive ℓ-permutation groups. *Trans. Amer. Math. Soc.* **290** (1985), no. 1, 69–79. Zbl 0546.06013 MR 0787955

[3] K. Mann and C. Rivas, Group orderings, dynamics and rigidity. *Ann. Inst. Fourier (Grenoble)* **68** (2018), no. 4, 1399–1445. Zbl 07002300 MR 3887426

[4] S. Matsumoto, Dynamics of isolated left orders. *J. Math. Soc. Japan* **72** (2020), no. 1, 185–211. Zbl 07196503 MR 4055461

Received July 1, 2019

Shigenori Matsumoto, Department of Mathematics, College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan

e-mail: matsumo@math.cst.nihon-u.ac.jp