Cantor dynamics of renormalizable groups

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Abstract. A group Γ is said to be "finitely non-co-Hopfian," or "renormalizable," if there exists a self-embedding $\varphi\colon \Gamma \to \Gamma$ whose image is a proper subgroup of finite index. Such a proper self-embedding is called a "renormalization for Γ." In this work, we associate a dynamical system to a renormalization φ of Γ. The discriminant invariant $\mathcal{D}_\varphi$ of the associated Cantor dynamical system is a profinite group which is a measure of the asymmetries of the dynamical system. If $\mathcal{D}_\varphi$ is a finite group for some renormalization, we show that Γ/C_φ is virtually nilpotent, where C_φ is the kernel of the action map. We introduce the notion of a (virtually) renormalizable Cantor action, and show that the action associated to a renormalizable group is virtually renormalizable. We study the properties of virtually renormalizable Cantor actions, and show that virtual renormalizability is an invariant of continuous orbit equivalence. Moreover, the discriminant invariant of a renormalizable Cantor action is an invariant of continuous orbit equivalence. Finally, the notion of a renormalizable Cantor action is related to the notion of a self-replicating group of automorphisms of a rooted tree.

1. Introduction

A countable group Γ is *co-Hopfian* if every monomorphism $\varphi\colon \Gamma \to \Gamma$ is an isomorphism [5], and is said to be *non-co-Hopfian* otherwise. If there exists a self-embedding φ whose image is a proper subgroup of finite index, then Γ is said to be *finitely non-co-Hopfian* [57]. A proper self-embedding $\varphi\colon \Gamma \to \Gamma$ with finite index is called a *renormalization* of Γ, in analogy with the case for $\Gamma = \mathbb{Z}^n$. If Γ admits a renormalization, then it is said to be *renormalizable*.

The free abelian group $\mathbb{Z}^n$ is renormalizable, as are many finitely generated nilpotent groups. There are also many examples of renormalizable groups which are not nilpotent, as described for example in [19, 25, 30, 31, 29, 49, 50, 57]. On the other hand, the free group $\mathbb{Z}^{\star n} = \mathbb{Z} \star \cdots \star \mathbb{Z}$ for $n \geq 2$ is non-co-Hopfian, but is not renormalizable. The classification of non-co-Hopfian groups in general appears to be a difficult problem.

There is a related concept of a *scale-invariant* group, introduced by Benjamini (see [54, Section 9.2]). A *scale* for Γ is a descending chain of finite index subgroups $\mathcal{S} = \{\Gamma_\ell \mid \ell \geq 1\}$ whose intersection is a finite group, and such that for each ℓ, there exists an isomorphism $\phi_\ell \colon \Gamma \to \Gamma_\ell$. Benjamini asked if a scale-invariant group must be virtually nilpotent? Nekrashevych and Pete [49, Theorem 1.1] gave examples of scale-invariant groups which are not virtually nilpotent. In the same work, the authors defined the notion of a *strongly scale-invariant* group, as a renormalizable group Γ such that the collection of subgroups $\{\Gamma_\ell = \varphi^\ell(\Gamma) \mid \ell \geq 0\}$ is a scale for Γ. Then [49, Question 1.1] asks if a strongly scale-invariant group must be virtually nilpotent?

In this work, we obtain a partial answer to this question. The key idea is to study the properties of an infinite profinite group $\widehat{\Gamma}_\varphi$ naturally associated to a renormalization φ. The group $\widehat{\Gamma}_\varphi$ is a proper quotient of the full profinite completion of Γ. The key observation is given in Proposition 5.2, which shows that $\varphi \colon \Gamma \to \Gamma$ induces an *open* embedding $\hat{\varphi} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$. Then Theorem 1.5 states that if both the intersections $\bigcap_{\ell > 0} \varphi^\ell(\Gamma)$ and $\bigcap_{\ell > 0} \hat{\varphi}^\ell(\widehat{\Gamma}_\varphi)$ are finite groups, then Γ is virtually nilpotent. In other words, we answer in the affirmative the question of Nekrashevych and Pete above under a stronger assumption, that both Γ and the infinite profinite group $\widehat{\Gamma}_\varphi$ admit a scale.

More precisely, given a renormalization $\varphi \colon \Gamma \to \Gamma$, let $C(\mathcal{G}_\varphi)$ be the largest normal subgroup of $K(\mathcal{G}_\varphi) = \bigcap_{\ell > 0} \varphi^\ell(\Gamma)$. In Definition 2.5 we define a closed subgroup $\mathcal{D}_\varphi \subset \widehat{\Gamma}_\varphi$ which is naturally associated to the renormalization φ. In Theorem 5.3, we prove that $\mathcal{D}_\varphi = \bigcap_{\ell > 0} \hat{\varphi}^\ell(\widehat{\Gamma}_\varphi)$. In particular, φ induces a scale on the profinite group $\widehat{\Gamma}_\varphi$ if and only if $\mathcal{D}_\varphi$ is finite. We have:

Theorem 1.1. *Let Γ be a finitely generated group, and let $\varphi \colon \Gamma \to \Gamma$ be a renormalization of Γ.*

(1) *If $\mathcal{D}_\varphi$ is the trivial group, then $\Gamma/C(\mathcal{G}_\varphi)$ is nilpotent.*

(2) *If $\mathcal{D}_\varphi$ is a finite group, then $\Gamma/C(\mathcal{G}_\varphi)$ is virtually nilpotent.*

The assumption that Γ is finitely generated is essential, as shown by the examples in Section 9.1.

Our approach to the study of renormalizable groups is based on the study of the Cantor dynamical systems naturally associated to their renormalizations. We now briefly discuss this approach.

An action $\Phi: \Gamma \times \mathfrak{X} \to \mathfrak{X}$, or $(\mathfrak{X}, \Gamma, \Phi)$, is said to be a *Cantor action* if Γ is a finitely generated group, $\mathfrak{X}$ is a Cantor metric space, and the action is minimal. The basic properties of Cantor actions are discussed in Section 2. In Section 3, we associate a minimal equicontinuous Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$ to a renormalization φ of Γ. Furthermore, we show that the renormalization $\varphi: \Gamma \to \Gamma$ induces a *renormalization of the action* $(X_\varphi, \Gamma, \Phi_\varphi)$, as in Definitions 7.1 and 7.2.

Let $\Phi_\varphi: \Gamma \to \mathrm{Homeo}(X_\varphi)$ be the action homomorphism associated to the minimal equicontinuous Cantor dynamical system $(X_\varphi, \Gamma, \Phi_\varphi)$. The closure of the image $\Phi_\varphi(\Gamma)$ is an infinite profinite group, denoted by $\widehat{\Gamma}_\varphi$, called the *Ellis group* of the action in the literature [4, 23, 24]. There is an induced transitive action $\widehat{\Phi}_\varphi$ of $\widehat{\Gamma}_\varphi$ on X_φ, and the *discriminant group* $\mathcal{D}_\varphi \subset \widehat{\Gamma}_\varphi$ is the isotropy subgroup of this action at the canonical fixed-point $x_\varphi \in X_\varphi$ of the contraction $\lambda_\varphi: X_\varphi \to X_\varphi$. The isomorphism class of $\mathcal{D}_\varphi$ depends only on the conjugacy class of the action, and has other invariance properties [21, 22, 37, 39]. If Γ is abelian, the discriminant group $\mathcal{D}_\varphi$ is the trivial group for any renormalization, but for Γ non-abelian it need not be trivial. The regularity properties of the restricted action of $\mathcal{D}_\varphi$ on X_φ, as discussed in Section 4, play a fundamental role in the proof of our results.

The proof of Theorem 1.1 is given in Section 6, and uses Theorem 5.3 which is based on the results in Reid [51], quoted as Theorem 5.1 below, and Theorem 4.4 and Proposition 5.2 in this work.

We give an example in Section 9.3 of a renormalization of the Heisenberg group for which $\mathcal{D}_\varphi$ is an infinite profinite group. Thus, while the assumption that $\mathcal{D}_\varphi$ is finite is sufficient to conclude that $\Gamma/C(\mathcal{G}_\varphi)$ is nilpotent, it is not a necessary condition. On the other hand, there are renormalizations of the Heisenberg group for which $\mathcal{D}_\varphi$ is the trivial group. The known examples of renormalizations suggest that it is an interesting problem to study the collection of all renormalizations for a given group Γ, even for the Heisenberg group.

We will now discuss dynamical properties of the Cantor actions associated to renormalizations, which play the key role in the proof of Theorems 1.1, and of its corollary for strongly scale-invariant groups. While most of these properties and results do not require that Γ be finitely-generated, many of our results do require this assumption, as will be pointed out when appropriate.

A Cantor action $(\mathfrak{X}, G, \Phi)$ is *free* if for any $g \in \Gamma$ which is not the identity, the action $\Phi(g)$ has no fixed points. The action is *topologically free*, as in Definition 2.1, if the set of points fixed by at least one non-trivial element of the group is a meager set. The notion of a *quasi-analytic* Cantor action, as in Definition 4.1, was introduced in the works [22, 37] as a generalization of the notion of a topologically free action. The quasi-analytic property of a Cantor action is a fundamental property of renormalizable groups and actions.

Theorem 1.2. *The Cantor action $\Phi_\varphi: \Gamma \times X_\varphi \to X_\varphi$ associated to a renormalization φ is quasi-analytic. Hence, if the action Φ_φ is effective then it is topologically free.*

In fact, Theorem 1.2 is a consequence of a stronger statement. Given a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, let $\Phi(\Gamma) \subset \mathrm{Homeo}(\mathfrak{X})$ denote the image subgroup. If the action is equicontinuous, then the closure $\overline{\Phi(\Gamma)} \subset \mathrm{Homeo}(\mathfrak{X})$ in the *uniform topology of maps* is a separable profinite group. This is discussed further in Section 2.2. For the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$ associated to a renormalization φ, we denote this closure by $\widehat{\Gamma}_\varphi = \overline{\Phi_\varphi(\Gamma)}$. We prove in Theorem 4.4 that the profinite action $\widehat{\Phi}_\varphi \colon \widehat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$ is quasi-analytic, which implies Theorem 1.2. The quasi-analytic property is used to prove that the monomorphism $\varphi \colon \Gamma \to \Gamma$ induces an open embedding of the closure $\widehat{\Gamma}_\varphi$ into itself as below:

Theorem 1.3. *Let φ be a renormalization of the finitely generated group Γ. Then φ induces an injective homomorphism $\widehat{\varphi} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ whose image is a clopen subgroup of $\widehat{\Gamma}_\varphi$.*

This is proved in Section 5, where we use its conclusion to obtain a structure theorem for the closure $\widehat{\Gamma}_\varphi$, an important consequence of which is the following:

Theorem 1.4. *Let φ be a renormalization of the finitely generated group Γ, and $\widehat{\varphi} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ the induced map given by Theorem 1.3. Then, $\mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}^n(\widehat{\Gamma}_\varphi)$.*

Theorem 1.1 and Theorem 1.4 yield an answer to the profinite version of the Nekrashevych–Pete question:

Theorem 1.5. *Let φ be a renormalization of the finitely generated group Γ. Suppose that*

$$K(\mathcal{G}_\varphi) = \bigcap_{\ell>0} \varphi^\ell(\Gamma) \subset \Gamma, \quad \mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi$$

are both finite groups. Then Γ is virtually nilpotent. If both groups are trivial, then Γ is nilpotent.

As mentioned above, our study of renormalizable groups naturally suggests a related notion, that of a *renormalizable* equicontinuous Cantor action, as introduced in Definition 7.1. It is modeled on the concept of a renormalizable dynamical system, and also that of self-similar groups [32, 47] and percolation theory [54, Section 9.2]. The class of renormalizable Cantor actions includes the class of actions associated to a self-embedding $\varphi \colon \Gamma \to \Gamma$, discussed above. We introduce a variant of this notion, the *virtually renormalizable* actions, in Definition 7.2.

The study of renormalizable Cantor actions is motivated, in part, by the examples of Cantor actions on d-adic trees, where elements of the acting group are defined recursively, in terms of the action of a finite set of generators on a rooted d-adic tree for $d \geq 2$, where there is an embedding $\varphi \colon \Gamma \to \Gamma$ whose image is a subgroup of the stabilizer group of a branch of a tree (see for example [47]). The

image $\varphi(\Gamma) \subset \Gamma$ need not be of finite index in Γ, even though the stabilizer group of a branch always has finite index in Γ. We discuss the relation of the notion of a renormalizable action with the notion of self-replicating groups in more detail in Section 7.2.

An equicontinuous Cantor action can be *quasi-analytic*, *locally quasi-analytic* or *wild*, using the notions and classification introduced in the works [37, 38, 39]. For renormalizable actions, there is the following dichotomy:

Theorem 1.6. *A renormalizable equicontinuous Cantor action* $(\mathfrak{X}, \Gamma, \Phi)$ *is either quasi-analytic, and in this case* Γ *is renormalizable and the action is topologically conjugate to the action given by a renormalization of* Γ, *or the action is wild.*

This result motivates the study of the invariants of renormalizable Cantor actions, both to understand the invariants of the renormalization map, and to discover invariants of these actions which distinguish between the quasi-analytic and wild cases of Theorem 1.6. Our final results in this work considers their invariant properties under continuous orbit equivalence.

Theorem 1.7. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *and* $(\mathfrak{X}', \Gamma', \Phi')$ *be minimal equicontinuous Cantor actions which are continuously orbit equivalent. If* $(\mathfrak{X}, \Gamma, \Phi)$ *is renormalizable and locally quasi-analytic, then* $(\mathfrak{X}', \Gamma', \Phi')$ *is virtually renormalizable.*

As a consequence, we obtain that the isomorphism class of the discriminant group $\mathcal{D}_\varphi$ associated to a renormalization φ is an invariant of continuous orbit equivalence.

Theorem 1.8. *Let* $(X_\varphi, \Gamma, \Phi_\varphi)$ *and* $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ *be Cantor actions associated to renormalizations* $\varphi \colon \Gamma \to \Gamma$ *and* $\varphi' \colon \Gamma' \to \Gamma'$, *respectively. If the actions are continuously orbit equivalent, then their discriminant groups* $\mathcal{D}_\varphi$ *and* $\mathcal{D}'_{\varphi'}$ *are isomorphic.*

Examples and applications of our results are discussed in Section 9.

Section 10 discusses open problems. In particular, the works [2, 37, 39] study wild Cantor actions, and the relations between the discriminant invariant and the wild property for the action. It is an interesting problem to further explore this relation for renormalizable actions, as these include many class of branch groups and related constructions, as in [6, 7, 47, 49, 48, 33].

2. Cantor actions

In this section, we recall some of the properties of Cantor actions. A basic reference is [4].

2.1. Basic concepts. For an action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ and $\gamma \in \Gamma$, let $\gamma x = \Phi(\gamma)(x)$. We also sometimes write $\gamma \cdot x = \Phi(\gamma)(x)$ when necessary for notational clarity.

Let $(\mathfrak{X}, \Gamma, \Phi)$ denote an action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$. The orbit of $x \in \mathfrak{X}$ is the subset $\mathcal{O}(x) = \{\gamma x \mid \gamma \in \Gamma\}$. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x)$ is dense in $\mathfrak{X}$.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ with $d_{\mathfrak{X}}(x, y) < \delta$ and all $\gamma \in \Gamma$, we have $d_{\mathfrak{X}}(\gamma x, \gamma y) < \varepsilon$. The equicontinuous property is independent of the choice of the metric on $\mathfrak{X}$ by Proposition 2.2.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is *effective*, or *faithful*, if the action homomorphism $\Phi \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ has trivial kernel. The action is *free* if for all $x \in \mathfrak{X}$ and $\gamma \in \Gamma$, $\gamma x = x$ implies that $\gamma = e$, the identity of the group. The *isotropy group* of $x \in \mathfrak{X}$ is the subgroup

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}. \tag{1}$$

Let $\mathrm{Fix}(\gamma) = \{x \in \mathfrak{X} \mid \gamma x = x\}$, and introduce the *isotropy set*

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists \gamma \in \Gamma, \gamma \neq \mathrm{id}, \gamma x = x\} = \bigcup_{e \neq \gamma \in \Gamma} \mathrm{Fix}(\gamma). \tag{2}$$

Definition 2.1. [12, 43, 52] $(\mathfrak{X}, \Gamma, \Phi)$ is said to be *topologically free* if $\mathrm{Iso}(\Phi)$ is meager in $\mathfrak{X}$.

Note that if $\mathrm{Iso}(\Phi)$ is meager, then $\mathrm{Iso}(\Phi)$ has empty interior.

The notion of topologically free Cantor actions was introduced by Boyle in his thesis [11], and later used in the works by Boyle and Tomiyama [12] for the study of classification of Cantor actions, by Renault [52] for the study of the C^*-algebras associated to Cantor actions, and by Li [43] in his study of rigidity properties of equicontinuous Cantor actions.

Now assume that $\mathfrak{X}$ is a Cantor space. Let $\mathrm{CO}(\mathfrak{X})$ denote the collection of all clopen (closed and open) subsets of $\mathfrak{X}$, which forms a basis for the topology of $\mathfrak{X}$. For $\phi \in \mathrm{Homeo}(\mathfrak{X})$ and $U \in \mathrm{CO}(\mathfrak{X})$, the image $\phi(U) \in \mathrm{CO}(\mathfrak{X})$. The following result is folklore, and a proof is given in [38, Proposition 3.1].

Proposition 2.2. *A Cantor action* $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ *is equicontinuous if and only if the orbit of every* $U \in \mathrm{CO}(\mathfrak{X})$ *is finite for the induced action* $\Phi_* \colon \Gamma \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$.

Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action. We say that $U \subset \mathfrak{X}$ is *adapted* to the action if U is a non-empty clopen subset, and for any $\gamma \in \Gamma$, if $\Phi(\gamma)(U) \cap U \neq \emptyset$ then $\Phi(\gamma)(U) = U$. The proof of Proposition 3.1 in [38] shows that given $x \in \mathfrak{X}$ and a clopen set W with $x \in W$, there is an adapted clopen set U with $x \in U \subset W$.

The key property of an adapted set U is that the set of "return times" to U,

$$\Gamma_U = \{\gamma \in \Gamma \mid \Phi(\gamma)(U) = U\} \tag{3}$$

is a subgroup of Γ, called the *stabilizer* of U. Then for $\gamma, \gamma' \in \Gamma$ with $\Phi(\gamma)(U) \cap \Phi(\gamma')(U) \neq \emptyset$ we have $\Phi(\gamma^{-1}) \circ \Phi(\gamma')(U) = U$, hence $\gamma^{-1}\gamma' \in \Gamma_U$. Thus, as the action is assumed to be minimal, the translates $\{\Phi(\gamma)(U) \mid \gamma \in \Gamma\}$ form a finite clopen partition of $\mathfrak{X}$, and are in 1-1 correspondence with the elements in the quotient space $X_U = \Gamma/\Gamma_U$. Then Γ acts by permutations of the finite set X_U and so the stabilizer group $\Gamma_U \subset \Gamma$ has finite index.

The action of $\gamma \in \Gamma$ on X_U is trivial precisely when γ is a stabilizer of each coset $\kappa \cdot \Gamma_U$, so $\gamma \in C_U$ where $C_U = \bigcap_{\kappa \in \Gamma} \kappa \Gamma_U \kappa^{-1} \subset \Gamma_U$ is the largest normal subgroup of Γ contained in Γ_U. The action of the finite group $Q_U \equiv \Gamma/C_U$ on X_U by permutations is a finite approximation of the action of Γ on $\mathfrak{X}$, and the isotropy group of the identity coset $e \cdot \Gamma_U$ is $D_U \equiv \Gamma_U/C_U \subset Q_U$.

Definition 2.3. Let $(\mathfrak{X}, \Gamma, \Phi)$ be an equicontinuous Cantor action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is an *adapted neighborhood basis* at $x \in \mathfrak{X}$ for the action Φ, if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 0$, each U_ℓ is adapted to the action Φ, and the intersection $\bigcap_{\ell > 0} U_\ell = \{x\}$.

Given $x \in \mathfrak{X}$ and $\varepsilon > 0$, Proposition 2.2 implies there exists an adapted clopen set $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$ and $\mathrm{diam}(U) < \varepsilon$. Thus, one can choose a descending chain $\mathcal{U}$ of adapted sets in $\mathrm{CO}(\mathfrak{X})$ whose intersection is x, which shows the following:

Proposition 2.4. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action. Given $x \in \mathfrak{X}$, there exists an adapted neighborhood basis $\mathcal{U}$ at x for the action Φ.*

2.2. The dynamical profinite model. Given an equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, let $\Phi(\Gamma) \subset \mathrm{Homeo}(\mathfrak{X})$ denote the image subgroup. Then the closure $\overline{\Phi(\Gamma)} \subset \mathrm{Homeo}(\mathfrak{X})$ in the *uniform topology of maps* is a separable profinite group. This group is identified with the *Ellis group* for the action, as defined in [4, 23, 24]; see also [21, Section 2]. Each element $\hat{g} \in \overline{\Phi(\Gamma)}$ is the uniform limit of a sequence of maps $\{\Phi(g_i) \mid i \geq 1\} \subset \Phi(\Gamma)$. We sometimes denote the limit $\hat{g}$ by (g_i).

For example, if Γ is an abelian group, then $\overline{\Phi(\Gamma)}$ is a compact totally disconnected abelian group, which can be thought of as the group of asymptotic motions of the system. When Γ is non-abelian, the closure $\overline{\Phi(\Gamma)}$ can have much more subtle algebraic properties.

Let $\hat{\Phi} \colon \overline{\Phi(\Gamma)} \times \mathfrak{X} \to \mathfrak{X}$ denote the induced action of $\overline{\Phi(\Gamma)}$ on $\mathfrak{X}$. For $\hat{g} \in \overline{\Phi(\Gamma)}$ we write its action on $\mathfrak{X}$ by $\hat{g}x = \hat{\Phi}(\hat{g})(x)$. For a minimal equicontinuous action

$\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$, the group $\overline{\Phi(\Gamma)}$ acts transitively on $\mathfrak{X}$. Given $x \in \mathfrak{X}$, introduce the isotropy group at x,

$$\overline{\Phi(\Gamma)}_x = \{\hat{g} \in \overline{\Phi(\Gamma)} \mid \hat{g}\, x = x\} \subset \mathrm{Homeo}(\mathfrak{X}), \qquad (4)$$

which is a closed subgroup of $\overline{\Phi(\Gamma)}$, and thus is either finite, or is an infinite profinite group.

Definition 2.5. The group $\overline{\Phi(\Gamma)}_x$ is called the *discriminant* of the action $(\mathfrak{X}, \Gamma, \Phi)$.

There is a natural identification $\mathfrak{X} \cong \overline{\Phi(\Gamma)}/\overline{\Phi(\Gamma)}_x$ of left $\overline{\Phi(\Gamma)}$-spaces, and thus the conjugacy class of $\overline{\Phi(\Gamma)}_x$ in $\overline{\Phi(\Gamma)}$ is independent of the choice of x. If $\overline{\Phi(\Gamma)}_x$ is the trivial group, then $\mathfrak{X}$ is identified with the profinite group $\overline{\Phi(\Gamma)}$, and the action $\hat{\Phi}$ is free. Note that there exists examples of free minimal equicontinuous Cantor actions $(\mathfrak{X}, \Gamma, \Phi)$ for which the action $\hat{\Phi}$ is not free, and in fact $\overline{\Phi(\Gamma)}_x$ is an infinite profinite group. The first such examples were constructed by Fokkink and Oversteegen in [26, Section 8], and further examples are constructed in [22, Section 10].

2.3. Equivalence of Cantor actions.
We recall three notions of equivalence of Cantor actions.

The first and strongest is that of isomorphism of Cantor actions, which is a generalization of the usual notion of conjugacy of topological actions. For $\Gamma = \mathbb{Z}$, isomorphism corresponds to the notion of "flip conjugacy" introduced in the work of Boyle and Tomiyama [12].

Definition 2.6. Two Cantor actions $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are said to be *isomorphic* if there is a homeomorphism $h \colon \mathfrak{X} \to \mathfrak{X}'$ and a group isomorphism $\Theta \colon \Gamma \to \Gamma'$ so that

$$\Phi(\gamma) = h^{-1} \circ \Phi'(\Theta(\gamma)) \circ h \in \mathrm{Homeo}(\mathfrak{X}') \quad \text{for all } \gamma \in \Gamma. \qquad (5)$$

Return equivalence is a form of "virtual isomorphism" for minimal equicontinuous Cantor actions, and is weaker than the notion of isomorphism. This equivalence is natural when considering the Cantor systems arising from geometric constructions, as in the works [37, 38, 39].

Throughout this work, by a small abuse of notation, for a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ and adapted subset $U \subset \mathfrak{X}$, we use Φ_U to denote both the restricted action $\Phi_U \colon \Gamma_U \times U \to U$ and the induced quotient action $\Phi_U \colon H_U \times U \to U$ where $H_U = \Phi(\Gamma_U) \subset \mathrm{Homeo}(U)$.

Definition 2.7. Minimal equicontinuous Cantor actions $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are *return equivalent* if there exists an adapted set $U \subset \mathfrak{X}$ for the action Φ and an adapted set $V \subset \mathfrak{X}'$ for the action Φ', such that the restricted actions (U, H_U, Φ_U) and (V, H_V', Φ_V') are isomorphic.

Continuous orbit equivalence for Cantor actions was introduced in [11, 12], and plays a fundamental role in various approaches to the classification of these actions [52]. Consider the equivalence relation on $\mathfrak{X}$ defined by an action $(\mathfrak{X}, \Gamma, \Phi)$,

$$\mathcal{R}(\mathfrak{X}, \Gamma, \Phi) \equiv \{(x, \gamma x)) \mid x \in \mathfrak{X}, \gamma \in \Gamma\} \subset \mathfrak{X} \times \mathfrak{X}. \tag{6}$$

Given actions $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$, we say they are *orbit equivalent* if there exist a bijection $h: \mathfrak{X} \to \mathfrak{X}'$ which maps $\mathcal{R}(\mathfrak{X}, \Gamma, \Phi)$ onto $\mathcal{R}(\mathfrak{X}', \Gamma', \Phi')$, and similarly for the inverse map h^{-1}.

Definition 2.8. Let $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ be Cantor actions. A *continuous orbit equivalence* between the actions is a homeomorphism $h: \mathfrak{X} \to \mathfrak{X}'$ which is an orbit equivalence, and there exist continuous functions $\alpha: \Gamma \times \mathfrak{X} \to \Gamma'$ and $\beta: \Gamma' \times \mathfrak{X}' \to \Gamma$ such that

(1) for each $x \in \mathfrak{X}$ and $\gamma \in \Gamma$, there exists $\alpha(\gamma, x) \in \Gamma'$ and an open set $x \in U_x \subset \mathfrak{X}$ such that $\Psi(\alpha(\gamma, x)) \circ h|U_x = h \circ \Phi(\gamma)|U_x$;

(2) for each $y \in \mathfrak{X}'$ and $\gamma' \in \Gamma'$, there exists $\beta(\gamma', y) \in \Gamma$ and an open set $y \in V_y \subset \mathfrak{X}'$ such that $\Phi(\beta(\gamma', y)) \circ h|V_y = h \circ \Psi(\gamma')|V_y$.

Note that the maps α and β are not assumed to be cocycles over the respective actions.

3. Renormalizable groups

In this section, we construct the minimal equicontinuous Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$ associated to a renormalization $\varphi: \Gamma \to \Gamma$, and give some of the basic properties of this action.

Set $\Gamma_0 = \Gamma$, and for $\ell \geq 1$, recursively define subgroups $\Gamma_\ell \subset \Gamma$, where $\Gamma_\ell = \varphi(\Gamma_{\ell-1}) \equiv \varphi^\ell(\Gamma)$.

Let $\mathcal{G}_\varphi \equiv \{\Gamma_\ell \mid \ell \geq 0\}$ denote the descending group chain, where each Γ_ℓ has finite index in Γ. Denote the intersection of the group chain by $K(\mathcal{G}_\varphi) \equiv \bigcap_{\ell>0} \Gamma_\ell$. If $K(\mathcal{G}_\varphi)$ is a finite group, then the group Γ is said to be *strongly scale-invariant*, in the terminology of Nekrashevych and Pete [49].

Let $X_\ell = \Gamma/\Gamma_\ell$ be the finite coset space. Note that X_ℓ is not necessarily a group, as the subgroup Γ_ℓ is not assumed to be normal in Γ. Note that Γ acts transitively on the left on X_ℓ, and the inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a natural Γ-invariant quotient map $p_{\ell+1}: X_{\ell+1} \to X_\ell$. The inverse limit space

$$X_\varphi \equiv \varprojlim\{p_{\ell+1}: X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \tag{7}$$

with the Tychonoff topology is a Cantor space. The actions of Γ on the factors X_ℓ induce a minimal equicontinuous action on X_φ, denoted by $\Phi_\varphi: \Gamma \times X_\varphi \to X_\varphi$ or by $(X_\varphi, \Gamma, \Phi_\varphi)$.

Let $\widehat{\Gamma}_\varphi \subset \mathrm{Homeo}(X_\varphi)$ denote the closure of the image $\Phi_\varphi(\Gamma) \subset \mathrm{Homeo}(X_\varphi)$, as introduced in Section 2.2, and let $\widehat{\Phi}_\varphi \colon \widehat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$ denote the induced action.

The embedding φ induces a mapping $\lambda_\varphi \colon X_\varphi \to X_\varphi$, which is defined as the shift map on sequences as follows. A point $\hat{x} \in X_\varphi$ is defined by an equivalence class of sequences $\hat{x} = (g_0, g_1, g_2, \ldots)$ with each $g_\ell \in \Gamma$ satisfying the relations $g_\ell = g_{\ell+1} \bmod \Gamma_\ell$ for all $\ell \geq 0$. Then $\lambda_\varphi(\hat{x}) = (e, \varphi(g_0), \varphi(g_1), \varphi(g_2), \ldots)$ is well defined, and is a contraction on X_φ. Let $x_\varphi \in X_\varphi$ be the unique fixed point for λ_φ. Then $x_\varphi = (e, e, e, \ldots)$ where $e \in \Gamma$ is the identity.

Following Definition 2.5, we obtain a fundamental notion associated to a renormalization of Γ.

Definition 3.1. The *discriminant group* of $(X_\varphi, \Gamma, \Phi_\varphi)$ is $\mathcal{D}_\varphi \equiv (\widehat{\Gamma}_\varphi)_{x_\varphi} \subset \widehat{\Gamma}_\varphi$.

For $k \geq 0$, define

$$
\begin{aligned}
U_k &= \{(g_0, g_1, g_2, \ldots) \in X_\varphi \mid g_i = e \text{ for } 0 \leq i \leq k\} \\
&\cong \varprojlim\{p_{\ell+1} \colon \Gamma_k/\Gamma_{\ell+1} \longrightarrow \Gamma_k/\Gamma_\ell \mid \ell \geq k\},
\end{aligned}
\tag{8}
$$

which is a clopen subset of X_φ adapted to the action Φ_φ, with stabilizer subgroup $\Gamma_{U_k} = \Gamma_k$. The clopen sets $\{U_k \mid k \geq 0\}$ form an adapted neighborhood basis at x_φ, and so we have $x_\varphi = \bigcap_{k>0} U_k$.

Observe that for all $\ell \geq 0$, the contraction mapping $\lambda_\varphi \colon X_\varphi \to X_\varphi$ defined above restricts to a homeomorphism onto $\lambda_\varphi \colon U_\ell \to U_{\ell+1}$.

As the orbit of x_φ is dense in X_φ, for any non-empty open subset $U \subset X$ there exists $g \in \Gamma$ so that $\Phi_\varphi(g)(x_\varphi) \in U$. It follows that there also exists $k > 0$ such that $\Phi_\varphi(g)(U_k) \subset U$.

3.1. The geometric (tree) model. Given a group chain $\mathcal{G}_\varphi$ as above, the associated dynamical system $\Phi_\varphi \colon \Gamma \times X_\varphi \to X_\varphi$ can be represented as an action of a subgroup of the automorphism group of a regular rooted tree as we discuss now. The construction we discuss is applicable to any chain $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset$ of finite index subgroups of the group $\Gamma = \Gamma_0$, without a requirement that the subgroups in the chain are isomorphic to Γ.

A *tree* T consists of a set of vertices $V = \bigsqcup_{\ell \geq 0} V_\ell$, where V_ℓ is a finite vertex set at level ℓ, and of edges joining vertices in $V_{\ell+1}$ and V_ℓ, for all $\ell \geq 0$, defined as follows. For $\ell \geq 0$, identify the vertex set V_ℓ with the coset space $X_\ell = \Gamma/\Gamma_\ell$. Join $v_\ell \in V_\ell$ and $v_{\ell+1} \in V_{\ell+1}$ by an edge if and only if $v_{\ell+1} \subset v_\ell$ as cosets. Let $d = |\Gamma : \varphi(\Gamma)|$ be the index, then for $\ell \geq 0$ the cardinality of V_ℓ is d^ℓ. Such a tree is called a d-ary, or a regular tree.

An infinite path in T is a sequence of vertices $(v_\ell)_{\ell \geq 0}$ such that $v_{\ell+1}$ and v_ℓ are joined by an edge, for $\ell \geq 0$. The boundary ∂T of T is the collection of all infinite paths in T, and so it is the subspace

$$
\partial T = \Big\{(v_\ell)_{\ell \geq 0} \subset \prod_{\ell \geq 0} V_\ell \,\Big|\, v_{\ell+1} \text{ and } v_\ell \text{ are joined by an edge}\Big\}.
$$

The space ∂T is a Cantor set with the relative topology from the product topology on $\prod_{\ell \geq 0} V_\ell$. It is immediate that the identification of the vertex sets V_ℓ with the coset spaces X_ℓ induces an identification of ∂T with the inverse limit space X_φ defined by (7), with points in X_φ corresponding to infinite paths in ∂T.

The action of Γ on the coset spaces $X_\ell = V_\ell$, $\ell \geq 0$, is by permutations. Since the action of Γ preserves the containment of cosets, the action preserves the connectedness of the tree T, that is, the vertices $v_\ell \in V_\ell$ and $v_{\ell+1} \in V_{\ell+1}$ are joined by an edge if and only if for any $g \in G$ the images $g \cdot v_\ell \in V_\ell$ and $g \cdot v_{\ell+1} \in V_{\ell+1}$ are joined by an edge. Thus every $g \in \Gamma$ defines an automorphism of the tree T, and we can consider Γ as a subgroup of the group of tree automorphisms $Aut(T)$.

The study of actions of subgroups of automorphism groups of regular trees is an active topic in geometric group theory, see for instance [47]. When studying the dynamical properties of an action $(X_\varphi, \Gamma, \Phi_\varphi)$, sometimes it is useful to represent it as an action on the boundary of a regular tree. However, our results in this paper rely heavily on the properties of the profinite completion $\mathfrak{G}$ of Γ, and the combinatorial character of the methods used to study group actions on rooted trees makes their use in the study of profinite completions rather cumbersome. In this paper, we rely mostly on the algebraic methods we describe further in Sections 3.2 and 3.3, while explaining the implications of our results for the actions of automorphisms of regular rooted trees in Section 7.2.

3.2. The algebraic profinite model. We next introduce an alternate profinite model for the minimal equicontinuous action $(X_\varphi, \Gamma, \Phi_\varphi)$. For each $\ell \geq 1$, let C_ℓ denote the largest normal subgroup (the *core*) of the stabilizer group Γ_ℓ, so

$$C_\ell = \bigcap_{g \in \Gamma} g \Gamma_\ell g^{-1} \subset \Gamma_\ell. \tag{9}$$

As Γ_ℓ has finite index in Γ, the same holds for C_ℓ. Observe that for all $\ell \geq 1$, we have $C_{\ell+1} \subset C_\ell$. Introduce the quotient group $Q_\ell = \Gamma/C_\ell$ with identity element $e_\ell \in Q_\ell$. There are natural quotient maps $q_{\ell+1}\colon Q_{\ell+1} \to Q_\ell$, and we can form the inverse limit group

$$\widehat{\Gamma}_\infty = \varprojlim \{q_{\ell+1}\colon Q_{\ell+1} \longrightarrow Q_\ell \mid \ell \geq 0\}. \tag{10}$$

Theorem 3.2 ([21, Theorem 4.4]). *There is a natural isomorphism $\hat{\tau}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\infty$ which identifies the discriminant group $\mathcal{D}_\varphi$ with the inverse limit group*

$$\mathcal{D}_\infty = \varprojlim \{q_{\ell+1}\colon \Gamma_{\ell+1}/C_{\ell+1} \longrightarrow \Gamma_\ell/C_\ell \mid \ell \geq 0\} \subset \widehat{\Gamma}_\infty. \tag{11}$$

There is an interpretation of the group $\mathcal{D}_\infty$ as an *asymptotic defect* of the Γ-action on X_∞. Suppose that Γ_ℓ is a normal subgroup, so that the quotient Γ/Γ_ℓ is a group. Then Γ/Γ_ℓ acts transitively and freely on X_ℓ. For example, if Γ is

abelian then this is always true. In general, for the normal core $C_\ell \subset \Gamma_\ell$, the finite group $Q_\ell = \Gamma/C_\ell$ acts transitively on X_ℓ and the finite subgroup $D_\ell = \Gamma_\ell/C_\ell$ is the "defect" for the action of Q_ℓ on X_ℓ being a free action. Then $\mathcal{D}_\infty$ is the inverse limit of these finite defects, and provides a measure of the deviation of the action $\hat{\Phi}_\infty$ of $\hat{\Gamma}_\infty$ on X_∞ from being a free action.

Associated to the group chain $\mathcal{G}_\varphi$, there are two subgroups,

$$K(\mathcal{G}_\varphi) = \bigcap_{\ell > 0} \Gamma_\ell, \quad C(\mathcal{G}_\varphi) = \bigcap_{g \in \Gamma} g K(\mathcal{G}_\varphi) g^{-1}. \tag{12}$$

where $C(\mathcal{G}_\varphi)$ is the largest normal subgroup of Γ contained in $K(\mathcal{G}_\varphi)$. Note that for any $g \in C(\mathcal{G}_\varphi)$, the action of $\Phi_\varphi(g)$ on X_φ is trivial.

3.3. The universal profinite model.

We now introduce a model for the action $(X_\varphi, \Gamma, \Phi_\varphi)$ in terms of the profinite completion $\mathfrak{G}$ of Γ. Recall that $\mathfrak{G}$ is the inverse limit of the finite quotient groups Γ/N, for the set $\mathcal{N} = \{N \mid N \subset \Gamma \text{ is a normal subgroup of finite index}\}$ which is partially ordered by inclusion. That is, $\mathfrak{G} = \varprojlim\{\Gamma/N \mid N \in \mathcal{N}\}$.

There is a homomorphism $\psi\colon \Gamma \to \mathfrak{G}$ with dense image, and the kernel of ψ is the group $N(\psi)$ given by the intersection of all normal subgroups of finite index in Γ. Thus, $N(\psi)$ is trivial exactly when the group Γ is residually finite.

By the universal property of the profinite completion, the map $\Phi_\varphi\colon \Gamma \to \hat{\Gamma}_\varphi \subset \text{Homeo}(X_\varphi)$ induces a surjective map $\Pi_\varphi\colon \mathfrak{G} \to \hat{\Gamma}_\varphi$ of profinite groups, and let $\mathfrak{N}(\Pi_\varphi) \subset \mathfrak{G}$ denote its kernel.

Let $\mathfrak{D}_\varphi \equiv \mathfrak{G}_{x_\varphi} \subset \mathfrak{G}$ denote the isotropy subgroup at x_φ of the action $\hat{\Phi}_\mathfrak{G}$ of $\mathfrak{G}$. Then $\mathfrak{N}(\Pi_\varphi) \subset \mathfrak{D}_\varphi$.

We use the universal property of $\mathfrak{G}$ to show a basic fact required for our study of renormalizations.

Proposition 3.3. *The renormalization $\varphi\colon \Gamma \to \Gamma$ induces an open embedding $\hat{\varphi}_\mathfrak{G}\colon \mathfrak{G} \to \mathfrak{G}$.*

Proof. Let $\Gamma_1 = \varphi(\Gamma) \subset \Gamma$ denote the image of φ. Then the partially ordered set

$$\mathcal{N}_1 = \{N_1 \mid N_1 \subset \Gamma_1 \text{ is a subgroup of finite index, normal in } \Gamma_1\}$$

yields the universal profinite completion $\mathfrak{G}_1$ of Γ_1 and φ induces an isomorphism $\hat{\varphi}_1\colon \mathfrak{G} \to \mathfrak{G}_1$.

Next, consider the partially ordered set $\mathcal{N}_1' = \{N \cap \Gamma_1 \mid N \in \mathcal{N}\}$, where $\mathcal{N}$ is the collection of finite-index subgroups which are normal in Γ. Let $\mathfrak{G}_1'$ denote the profinite completion of Γ_1 with respect to $\mathcal{N}_1'$. It is immediate that $\mathfrak{G}_1'$ is the closure of $\psi(\Gamma_1) \subset \mathfrak{G}$ in $\mathfrak{G}$.

Note that if $N' \in \mathcal{N}'$, then N' is also normal in Γ_1 so $N' \in \mathcal{N}_1$. Conversely, if $N \in \mathcal{N}_1$ then its normal core $CN_1 = \bigcap_{g \in \Gamma} g^{-1} N_1 g \subset N_1$ is a normal subgroup

of Γ which has finite index in Γ_1 so $CN_1 \in \mathcal{N}'_1$. Thus, the two partially ordered sets $\mathcal{N}_1$ and $\mathcal{N}'_1$ are cofinal in Γ_1, hence the identity map induces a homeomorphism $\widehat{\mathrm{Id}} \colon \mathfrak{G}_1 \cong \mathfrak{G}'_1$. It follows that the composition $\hat{\varphi}_{\mathfrak{G}} \equiv \widehat{\mathrm{Id}} \circ \hat{\varphi}_1 \colon \mathfrak{G} \to \mathfrak{G}_1 \cong \mathfrak{G}'_1$ is an isomorphism onto the subgroup $\mathfrak{G}'_1 \subset \mathfrak{G}$ of finite index, which is thus open. $\square$

For each $\ell \geq 1$, let $\mathfrak{G}_\ell = \hat{\varphi}_{\mathfrak{G}}^\ell(\mathfrak{G})$ which is an open subgroup of $\mathfrak{G}$ of finite index, with $\mathfrak{G}/\mathfrak{G}_\ell \cong \Gamma/\Gamma_\ell$.

By an argument analogous to the proof of Proposition 3.3, $\mathfrak{G}_\ell$ is identified with the closure of Γ_ℓ in $\mathfrak{G}$, and thus $\mathfrak{G}_\ell = \{g \in \mathfrak{G} \mid \widehat{\Phi}_{\mathfrak{G}}(g)(U_\ell) = U_\ell\}$. That is, U_ℓ is an adapted set for the action $\widehat{\Phi}_{\mathfrak{G}} \colon \mathfrak{G} \times X_\varphi \to X_\varphi$ with stabilizer $\mathfrak{G}_{U_\ell} = \mathfrak{G}_\ell$. It follows that the isotropy group $\mathfrak{D}_\varphi$ at $x_\varphi \in X_\varphi$ of the action $\hat{\varphi}_{\mathfrak{G}}$ is given by $\bigcap_{\ell \geq 0} \mathfrak{G}_\ell$, and so $\Pi_\varphi(\mathfrak{D}_\varphi) = \mathcal{D}_\varphi \subset \widehat{\Gamma}_\varphi$.

Corollary 3.4. *The embedding $\hat{\varphi}_{\mathfrak{G}} \colon \mathfrak{G} \to \mathfrak{G}$ restricts to an isomorphism*

$$\hat{\varphi}_{\mathfrak{G}} \colon \mathfrak{D}_\varphi \longrightarrow \mathfrak{D}_\varphi.$$

Proof. We have $\hat{\varphi}_{\mathfrak{G}}(\mathfrak{D}_\varphi) = \bigcap_{\ell \geq 0} \hat{\varphi}_{\mathfrak{G}}(\mathfrak{G}_\ell) = \bigcap_{\ell \geq 0} \mathfrak{G}_{\ell+1} = \mathfrak{D}_\varphi$, so the restriction $\hat{\varphi}_{\mathfrak{G}} \colon \mathfrak{D}_\varphi \to \mathfrak{D}_\varphi$ is onto. As $\hat{\varphi}_{\mathfrak{G}}$ is an embedding, its restriction to $\mathfrak{D}_\varphi$ is injective. $\square$

Remark 3.5. We want to obtain a version of Proposition 3.3 for the completion $\widehat{\Gamma}_\varphi$ of Γ, in place of the universal profinite completion $\mathfrak{G}$. That is, we will show that $\varphi \colon \Gamma \hookrightarrow \Gamma$ induces an open embedding $\hat{\varphi} \colon \widehat{\Gamma}_\varphi \hookrightarrow \widehat{\Gamma}_\varphi$ (see Proposition 5.2 in the next section).

Unfortunately, the argument in Proposition 3.3 does not directly generalize to the case of the closure of the action $\widehat{\Gamma}_\varphi$. Indeed, in the above proof, the key idea is that the system $\mathcal{N}'_1$ is cofinal in $\mathcal{N}_1$, and this follows because $\mathcal{N}'_1$ contains all normal cores (in Γ) of members of $\mathcal{N}_1$.

On the other hand, the group $\widehat{\Gamma}_\varphi$ is defined as the closure of the action group $\Phi_\varphi(\Gamma) \subset \mathrm{Homeo}(X_\varphi)$, and the map φ induces a natural isomorphism with the closure of the image $\varphi(\Gamma) = \Gamma_1 = \Gamma_{U_1}$ in $\mathrm{Homeo}(U_1)$. In order to obtain an embedding of $\widehat{\Gamma}_\varphi$ into itself, we must relate the closure of Γ_1 in $\mathrm{Homeo}(U_1)$ with that in $\mathrm{Homeo}(X_\varphi)$. Thus, the above algebraic argument using normal cores needs to be replaced by a dynamical argument. The key point is that one needs to show that if $g \in \widehat{\Gamma}_\varphi$ acts trivially on U_1, then it acts trivially on X_φ. This dynamical regularity, i.e. that the action of $g \in \widehat{\Gamma}_\varphi$ is determined by its behavior on any open set, is the goal of the next two sections.

4. Regularity of Cantor actions

In this section we recall the notion of *quasi-analytic* actions, and the localized version of this property. This is a type of regularity property for Cantor actions,

introduced in the works by Álvarez López, Candel, and Moreira Galicia [1, 3], inspired by work of Haefliger [35]. We then consider this property for the action $\hat{\varphi}_{\mathfrak{G}} \colon \mathfrak{G} \times X_\varphi \to X_\varphi$ of the profinite completion $\mathfrak{G}$. In the following definitions, H denotes a topological group which need not be countable.

Definition 4.1. An action $\Phi \colon H \times \mathfrak{X} \to \mathfrak{X}$, where H is a topological group and $\mathfrak{X}$ is a Cantor space, is said to be *quasi-analytic* (*or QA*) if for each clopen set $U \subset \mathfrak{X}$, if the action of $g \in H$ satisfies $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on all of $\mathfrak{X}$.

Note that if an action is not quasi-analytic, then there is some $g \in H$ whose action $\Phi(g)$ on $\mathfrak{X}$ is non-trivial, yet there is a clopen subset U such that the restriction of the action $\Phi(g)$ to U is the identity, and thus the isotropy group for the action at a point $x \in U$ is non-trivial. So, for example, if the space $\mathfrak{X}$ is homeomorphic to a profinite group $\mathfrak{H}$ for which the action Φ is defined by group multiplication, so that the action is induced by a group homomorphism $\Phi \colon H \to \mathfrak{H}$, then the action is quasi-analytic. A topologically free action, as in Definition 2.1, is quasi-analytic. Conversely, the Baire Category Theorem implies that an *effective* quasi-analytic action of a *countable* group is topologically free [52, Section 3].

A local formulation of the QA condition for actions was introduced in the works [22, 37], and has proved very useful for the study of the dynamical properties of equicontinuous Cantor actions.

Definition 4.2. An action $\Phi \colon H \times \mathfrak{X} \to \mathfrak{X}$, where H is a topological group and $\mathfrak{X}$ a Cantor metric space with metric $d_{\mathfrak{X}}$, is *locally quasi-analytic* (or *LQA*), if there exists $\varepsilon > 0$ such that for any non-empty open set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$, and for any non-empty open subset $V \subset U$, if the action of $g \in H$ satisfies $\Phi(g)(V) = V$ and the restriction $\Phi(g)|V$ is the identity map on V, then $\Phi(g)$ acts as the identity on all of U.

Examples of minimal equicontinuous Cantor actions which are locally quasi-analytic, but not quasi-analytic, are elementary to construct; some examples are given in [22, 37].

If $(\mathfrak{X}, H, \Phi)$ is an equicontinuous Cantor action which is not quasi-analytic, then the isotropy group defined in (4) is non-trivial. On the other hand, there are actions with non-trivial isotropy group that are quasi-analytic (see Section 9.3 below, and the examples in [22]). Finally, we define:

Definition 4.3. An equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is said to be *stable* if the associated profinite action $\hat{\Phi} \colon \overline{\Phi(\Gamma)} \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic. The action is said to be *wild* otherwise.

There are many examples of wild Cantor actions. For example, the actions of branch groups on the boundaries of their associated trees are always wild [33]. The work [2] gives the construction of wild Cantor actions exhibiting a variety of characteristic properties, using algebraic methods.

Here is our main technical result for the profinite actions associated to renormalizations.

Theorem 4.4. *Let Γ be a finitely generated group and $\varphi\colon \Gamma \to \Gamma$ a renormalization of Γ. Then the profinite action $\hat{\Phi}_{\mathfrak{G}}\colon \mathfrak{G} \times X_{\varphi} \to X_{\varphi}$ is quasi-analytic.*

Proof. Let $g \in \mathfrak{G}$ be such that $\hat{\Phi}_{\mathfrak{G}}(g)$ acts non-trivially on X_{φ}. Suppose there exists a non-empty open set $U \subset X_{\varphi}$ such that $\hat{\Phi}_{\mathfrak{G}}(g)$ acts on U as the identity.

The orbit of every point of X_{φ} is dense in X_{φ} under the action of Γ, so there exists $h \in \Gamma$ such that $\Phi_{\varphi}(h)(x_{\varphi}) \in U$. Set $g' = h^{-1}gh$ so that $\hat{\Phi}_{\mathfrak{G}}(g')$ fixes the open set $U' = \Phi_{\varphi}(h^{-1})(U)$. In particular, $\hat{\Phi}_{\mathfrak{G}}(g')$ fixes x_{φ} and hence $g' \in \mathfrak{D}_{\varphi}$. Thus, we can assume without loss of generality that $\hat{\Phi}_{\mathfrak{G}}(g)$ acts as the identity on U and $x_{\varphi} \in U$, so that $g \in \mathfrak{D}_{\varphi}$.

The nested clopen sets $\mathcal{U} = \{U_{\ell} \mid \ell \geq 0\}$ form a neighborhood basis at x_{φ} so there exists some $k_0 > 0$ such that $U_k \subset U$ for all $k \geq k_0$. Thus, for all $k \geq k_0$ we have $\lambda_{\varphi}^k(X_{\varphi}) \subset U$.

Recall that the restriction $\hat{\varphi}_{\mathfrak{G}}\colon \mathfrak{D}_{\varphi} \to \mathfrak{D}_{\varphi}$ is an automorphism by Corollary 3.4. Thus for $g \in \mathfrak{D}_{\varphi}$ there is a well-defined element $g_{\ell} = \hat{\varphi}_{\mathfrak{G}}^{\ell}(g) \in \mathfrak{D}_{\varphi}$ for all $\ell \in \mathbb{Z}$.

Lemma 4.5. *Let $g \in \mathfrak{D}_{\varphi}$, and suppose g acts trivially on U_{k_0}, for some $k_0 \geq 0$. Then for all $\ell \geq k_0$, the action of $g_{-\ell} = \hat{\varphi}_{\mathfrak{G}}^{-\ell}(g) \in \mathfrak{D}_{\varphi}$ on X_{φ} is trivial.*

Proof. For $x \in X_{\varphi}$ and $\ell \geq 0$, set $x_{\ell} = \lambda_{\varphi}^{\ell}(x)$. Choose $g_x \in \mathfrak{G}$ such that $x = g_x \mathfrak{D}_{\varphi}$ via the identification $X_{\varphi} \cong \mathfrak{G}/\mathfrak{D}_{\varphi}$; that is, x is represented in X_{φ} by the coset $g_x \mathfrak{D}_{\varphi}$. Recall that under this identification, for $h \in \mathfrak{G}$ the action of $\hat{\Phi}_{\mathfrak{G}}(h)$ on X_{φ} becomes left multiplication by h. That is, $\hat{\Phi}_{\mathfrak{G}}(h)(x) = h \cdot g_x \mathfrak{D}_{\varphi} = hg_x \cdot \mathfrak{D}_{\varphi}$. Then for $\ell \geq k_0$ we have that

$$x_{\ell} = \lambda_{\varphi}^{\ell}(x) = \lambda_{\varphi}^{\ell}(g_x \mathfrak{D}_{\varphi}) = \hat{\varphi}_{\mathfrak{G}}^{\ell}(g_x)\mathfrak{D}_{\varphi} \in U_{k_0}. \tag{13}$$

Thus, for $\ell \geq k_0$ we have

$$x_{\ell} = \hat{\Phi}_{\mathfrak{G}}(g)(x_{\ell}) = \hat{\Phi}_{\mathfrak{G}}(g)(\hat{\varphi}_{\mathfrak{G}}^{\ell}(g_x)\mathfrak{D}_{\varphi}) = g\hat{\varphi}_{\mathfrak{G}}^{\ell}(g_x)\mathfrak{D}_{\varphi}. \tag{14}$$

So for $g \in \mathfrak{D}_{\varphi}$ and $\ell \geq k_0$, using that $\hat{\varphi}_{\mathfrak{G}}\colon \mathfrak{G} \to \mathfrak{G}$ is a homomorphism, we have

$$\begin{aligned}
x_{\ell} &= g\hat{\varphi}_{\mathfrak{G}}^{\ell}(g_x)\mathfrak{D}_{\varphi} \\
&= \hat{\varphi}_{\mathfrak{G}}^{\ell}(\hat{\varphi}_{\mathfrak{G}}^{-\ell}(g))\hat{\varphi}_{\mathfrak{G}}^{\ell}(g_x)\mathfrak{D}_{\varphi} \\
&= \hat{\varphi}_{\mathfrak{G}}^{\ell}(\hat{\varphi}_{\mathfrak{G}}^{-\ell}(g)g_x)\mathfrak{D}_{\varphi} \\
&= \hat{\varphi}_{\mathfrak{G}}^{\ell}(g_{-\ell}g_x)\mathfrak{D}_{\varphi}.
\end{aligned} \tag{15}$$

Thus for $g \in \mathfrak{D}_\varphi$, $x \in X_\varphi$ and $\ell \geq k_0$,

$$
\begin{aligned}
x = \lambda_\varphi^{-\ell}(x_\ell) &= \lambda_\varphi^{-\ell}(\hat{\varphi}_{\mathfrak{G}}^\ell(g_{-\ell}g_x)\mathfrak{D}_\varphi) \\
&= \hat{\varphi}_{\mathfrak{G}}^{-\ell}(\hat{\varphi}_{\mathfrak{G}}^\ell(g_{-\ell}g_x))\mathfrak{D}_\varphi \\
&= g_{-\ell}g_x\mathfrak{D}_\varphi = \hat{\Phi}_{\mathfrak{G}}(g_{-\ell})(x).
\end{aligned} \tag{16}
$$

That is, $\hat{\Phi}_{\mathfrak{G}}(g_{-\ell})(x) = x$ for all $x \in X_\varphi$ and $\ell \geq k_0$, as was to be shown.   △

Note that for $g \in \mathfrak{G}$ the equicontinuous action of $\hat{\varphi}_{\mathfrak{G}}(g)$ on X_φ is approximated by the action on the finite quotient spaces X_ℓ for $\ell > 0$. Thus, the assumption that $\hat{\varphi}_{\mathfrak{G}}(g)$ acts non-trivially on X_φ implies there exists some $m_0 > 0$ such that the induced action of $\hat{\varphi}_{\mathfrak{G}}(g)$ on $X_{m_0} = \Gamma/\Gamma_{m_0}$ is non-trivial for some $m_0 > 0$. Denote this action by $\hat{\Phi}_{m_0}(g) \in \mathrm{Perm}(X_{m_0})$, where $\mathrm{Perm}(X_{m_0})$ is the group of permutations of the finite set X_{m_0}, hence $\mathrm{Perm}(X_{m_0})$ is a finite group.

The second key observation required for the proof of Theorem 4.4 is the following "periodicity" of the restricted action $\hat{\Phi}_{m_0} : \mathfrak{D}_\varphi \to \mathrm{Perm}(X_{m_0})$, which allows us to promote properties of the action of $g_{-\ell}$ on small scales U_k (for k large) to global properties of the action of g.

Lemma 4.6. *Let $g \in \mathfrak{D}_\varphi$. Then for every $m_0 \geq 1$, there exists $N_0 \geq 1$ such that for all $\ell \geq 1$ we have $\hat{\Phi}_{m_0}(g_{-N_0\ell}) = \hat{\Phi}_{m_0}(g)$.*

Proof. Let $m_0 \geq 1$. We note the following two elementary properties of $\mathrm{Hom}(\mathfrak{G}, \mathrm{Perm}(X_{m_0}))$:

(1) for any $\sigma \in \mathrm{Hom}(\mathfrak{G}, \mathrm{Perm}(X_{m_0}))$ and $\ell \geq 0$, we have

$$
\sigma \circ \hat{\varphi}_{\mathfrak{G}}^\ell \in \mathrm{Hom}(\mathfrak{G}, \mathrm{Perm}(X_{m_0})).
$$

(2) $\mathrm{Hom}(\mathfrak{G}, \mathrm{Perm}(X_{m_0}))$ is a finite set, as Γ is finitely generated and $\mathrm{Perm}(X_{m_0})$ is finite. Indeed, every element in $\mathrm{Hom}(\mathfrak{G}, \mathrm{Perm}(X_{m_0}))$ is a group homomorphism, and so it is determined by its values on the generating set of a dense subgroup Γ of $\mathfrak{G}$, which is finite.

Now consider $\hat{\Phi}_{m_0} \in \mathrm{Hom}(\mathfrak{G}, \mathrm{Perm}(X_{m_0}))$. By properties (1) and (2) above, there exist $0 \leq N_1 < N_2$ such that $\hat{\Phi}_{m_0} \circ \hat{\varphi}_{\mathfrak{G}}^{N_2} = \hat{\Phi}_{m_0} \circ \hat{\varphi}_{\mathfrak{G}}^{N_1} \in \mathrm{Hom}(\mathfrak{G}, \mathrm{Perm}(X_{m_0}))$. In particular, their restrictions satisfy $\hat{\Phi}_{m_0} \circ \hat{\varphi}_{\mathfrak{G}}^{N_2} = \hat{\Phi}_{m_0} \circ \hat{\varphi}_{\mathfrak{G}}^{N_1} : \mathfrak{D}_\varphi \to \mathrm{Perm}(X_{m_0})$.

Now recall from Corollary 3.4 that $\hat{\varphi}_{\mathfrak{G}}$ restricts to an automorphism on $\mathfrak{D}_\varphi$, so that $\hat{\varphi}_{\mathfrak{G}}^{-N_1} : \mathfrak{D}_\varphi \to \mathfrak{D}_\varphi$ is well defined. It follows that

$$
\begin{aligned}
\hat{\Phi}_{m_0} \circ \hat{\varphi}_{\mathfrak{G}}^{N_2-N_1} &= \hat{\Phi}_{m_0} \circ \hat{\varphi}_{\mathfrak{G}}^{N_2} \circ \hat{\varphi}_{\mathfrak{G}}^{-N_1} \\
&= \hat{\Phi}_{m_0} \circ \hat{\varphi}_{\mathfrak{G}}^{N_1} \circ \hat{\varphi}_{\mathfrak{G}}^{-N_1} \\
&= \hat{\Phi}_{m_0} : \mathfrak{D}_\varphi \longrightarrow \mathrm{Perm}(X_{m_0}).
\end{aligned}
$$

Therefore $N_0 = N_2 - N_1$ satisfies the conclusion of the lemma.   △

We can now complete the proof that the action $\hat{\Phi}_{\mathfrak{G}}\colon \mathfrak{G} \times X_\varphi \to X_\varphi$ is quasi-analytic. If not, then there exists $g \in \mathfrak{D}_\varphi$ such that $\hat{\Phi}_{\mathfrak{G}}(g)$ acts non-trivially on X_φ, and a non-empty open set $U \subset X_\varphi$ with $x_\varphi \in U$ such that $\hat{\varphi}_{\mathfrak{G}}(g)$ acts on U as the identity.

Let $m_0 > 0$ be such that the induced action of $\hat{\varphi}_{\mathfrak{G}}(g)$ on $X_{m_0} = \Gamma/\Gamma_{m_0}$ is non-trivial. Then, there exists some $k_0 > 0$ such that $U_k \subset U$ for all $k \geq k_0$. We assume that $k_0 \geq m_0$. By Lemma 4.5, for all $\ell \geq k_0$ we have $\hat{\Phi}_{m_0}(g_{-\ell})(x) = x$ for all $x \in X_{m_0}$. By Lemma 4.6, there exists $N_0 > 0$ so that $\hat{\Phi}_{m_0}(g_{-N_0\ell}) = \hat{\Phi}_{m_0}(g)$ for all $\ell \geq 0$. However, for $N_0\ell > k_0$ we obtain a contradiction, as $\hat{\Phi}_{m_0}(g)$ is assumed to act non-trivially on X_{m_0} while $\hat{\Phi}_{m_0}(g_{-N_0\ell})$ acts trivially on X_{m_0}. Thus, the action of $\hat{\Phi}_{\mathfrak{G}}$ must be quasi-analytic. $\square$

Finally, note that Theorem 4.4 shows that the profinite action $(X_\varphi, \mathfrak{G}, \hat{\Phi}_{\mathfrak{G}})$ is quasi-analytic, so the same holds for the action $(X_\varphi, \Gamma, \Phi_\varphi)$ obtained by restricting the action to the image $\Phi_{\mathfrak{G}}(\Gamma) \subset \mathfrak{G}$. Then the Baire Category Theorem implies (see [38, Proposition 2.2] for example) that if the action Φ_φ is effective, that is if $\Phi_\varphi\colon \Gamma \to \mathrm{Homeo}(X_\varphi)$ is injective, then the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is topologically free, as asserted in Theorem 1.2.

Corollary 4.7. *Let Γ be a finitely generated group and $\varphi\colon \Gamma \to \Gamma$ a renormalization of Γ. Then the profinite action $\hat{\Phi}_\varphi\colon \hat{\Gamma}_\varphi \times X_\varphi \to X_\varphi$ is quasi-analytic, and the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is stable.*

Proof. Let $U \subset X_\varphi$ be a clopen set and $\hat{g} \in \hat{\Gamma}_\varphi$ such that $\hat{\Phi}_\varphi(\hat{g})$ restricts to the identity on U. Recall that $\Pi_\varphi\colon \mathfrak{G} \to \hat{\Gamma}_\varphi$ is onto, so there exists $g \in \mathfrak{G}$ such that $\Pi_\varphi(g) = \hat{g}$. Then the action $\hat{\Phi}_{\mathfrak{G}}(g)$ restricts to the identity on U, so $\hat{\Phi}_{\mathfrak{G}}(g)$ acts as the identity on X_φ by Theorem 4.4. $\square$

5. Open embeddings

In this section, given a renormalization $\varphi\colon \Gamma \to \Gamma$ with associated Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$, we obtain a structure theory for the profinite group $\hat{\Gamma}_\varphi$ that is the key to the proof of Theorem 1.1. Recall that Proposition 3.3 showed that the induced map $\hat{\varphi}_{\mathfrak{G}}\colon \mathfrak{G} \to \mathfrak{G}$ of the profinite completion $\mathfrak{G}$ of Γ is an open embedding. We thus obtain by [57, Theorem 3.10], which is a reformulation of the results of Reid in [51], the following structure theorem:

Theorem 5.1. *There exist closed subgroups $C_\varphi \subset \mathfrak{G}$ and $Q_\varphi \subset \mathfrak{G}$ so that*

(1) $\mathfrak{G} \cong C_\varphi \rtimes Q_\varphi$, *where C_φ is a pro-nilpotent normal subgroup of $\mathfrak{G}$;*

(2) C_φ *is $\hat{\varphi}_{\mathfrak{G}}$-invariant, and $\hat{\varphi}_{\mathfrak{G}}$ restricts to an open contracting embedding on C_φ;*

(3) Q_φ *is $\hat{\varphi}_{\mathfrak{G}}$-invariant, and $\hat{\varphi}_{\mathfrak{G}}$ restricts to an automorphism of Q_φ.*

Moreover, let $\hat{e} \in \mathfrak{G}$ be the identity element, then we have

$$C_\varphi = \{g \in \mathfrak{G} \mid \lim_{n\to\infty} \hat{\varphi}_\mathfrak{G}^n(g) = \hat{e}\}, \quad Q_\varphi = \bigcap_{n>0} \hat{\varphi}_\mathfrak{G}^n(\mathfrak{G}). \qquad (17)$$

Next, we show that $\hat{\varphi}_\mathfrak{G}$ induces an open embedding $\hat{\varphi}: \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ as promised in Remark 3.5.

Proposition 5.2. *Let φ be a renormalization of the finitely-generated group Γ. Then φ induces an injective homomorphism $\hat{\varphi}: \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ whose image is a clopen subgroup of $\widehat{\Gamma}_\varphi$.*

Proof. Let $\mathfrak{N}(\Pi_\varphi) = \ker\{\Pi_\varphi: \mathfrak{G} \to \widehat{\Gamma}_\varphi\} \subset \mathfrak{G}$ be the kernel of the homomorphism Π_φ. We claim that $\hat{\varphi}_\mathfrak{G}: \mathfrak{G} \to \mathfrak{G}$ descends to a homomorphism

$$\hat{\varphi}: \widehat{\Gamma}_\varphi \cong \mathfrak{G}/\mathfrak{N}(\Pi_\varphi) \longrightarrow \widehat{\Gamma}_\varphi \cong \mathfrak{G}/\mathfrak{N}(\Pi_\varphi). \qquad (18)$$

Recall that Theorem 1.2 implies that for $g \in \mathfrak{G}$, the action of $\hat{\varphi}_\mathfrak{G}(g)$ on X_φ is locally determined. In particular, the action $\hat{\varphi}_\mathfrak{G}(g)$ is determined by its restriction to the clopen subset $U_1 = \lambda_\varphi(X_\varphi)$.

For $g \in \mathfrak{N}(\Pi_\varphi)$, observe that $\hat{\varphi}_\mathfrak{G}(g)$ acts as the identity on the clopen subset $U_1 = \lambda_\varphi(X_\varphi)$. As the action $\widehat{\Phi}_\mathfrak{G}$ is quasi-analytic, this implies that $\hat{\varphi}_\mathfrak{G}(g)$ acts as the identity on X_φ, and thus $\hat{\varphi}_\mathfrak{G}(g) \in \mathfrak{N}(\Pi_\varphi)$. That is, $\hat{\varphi}_\mathfrak{G}(\mathfrak{N}(\Pi_\varphi)) \subset \mathfrak{N}(\Pi_\varphi) \subset \mathfrak{G}$, and thus we have the composition of homomorphisms

$$\hat{\varphi}: \widehat{\Gamma}_\varphi = \mathfrak{G}/\mathfrak{N}(\Pi_\varphi) \longrightarrow \hat{\varphi}(\mathfrak{G})/\hat{\varphi}(\mathfrak{N}(\Pi_\varphi)) \longrightarrow \mathfrak{G}/\hat{\varphi}(\mathfrak{N}(\Pi_\varphi) \to \mathfrak{G}/\mathfrak{N}(\Pi_\varphi) = \widehat{\Gamma}_\varphi$$

which defines the map (18). We claim that $\hat{\varphi}$ is injective. If not, let $\gamma \in \widehat{\Gamma}_\varphi$ such that $\hat{\varphi}(\gamma) = \mathrm{Id}$. That is, $\hat{\varphi}(\gamma) \in \widehat{\Gamma}_\varphi$ acts as the identity on X_φ. In particular, $\hat{\varphi}(\gamma)$ acts as the identity on $\lambda_\varphi(X_\varphi)$, so for $x \in X_\varphi$,

$$\lambda_\varphi(x) = \hat{\varphi}(\gamma) \cdot \lambda_\varphi(x) = \lambda_\varphi(\gamma x).$$

As λ_φ is an injection, we have $\gamma x = x$ for all $x \in X_\varphi$, and thus $\gamma \in \mathrm{Homeo}(X_\varphi)$ is the identity, as was to be shown. $\qquad \square$

We use the conclusions of Theorem 5.1 and Proposition 5.2 to obtain:

Theorem 5.3. *Let $\varphi: \Gamma \to \Gamma$ be a renormalization for the finitely generated group Γ, with associated Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$. Let $\hat{\varphi}: \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ be the embedding induced from φ. Then there exists a closed **pro-nilpotent normal** subgroup $\widehat{N}_\varphi \subset \widehat{\Gamma}_\varphi$ so that*

(1) *$\widehat{\Gamma}_\varphi \cong \widehat{N}_\varphi \rtimes \mathcal{D}_\varphi$ is a semi-direct product;*

(2) *$\widehat{N}_\varphi$ is $\hat{\varphi}$-invariant, and $\hat{\varphi}$ restricts to an open contracting embedding of $\widehat{N}_\varphi$;*

(3) *$\mathcal{D}_\varphi$ is $\hat{\varphi}$-invariant, and $\hat{\varphi}$ restricts to an automorphism of $\mathcal{D}_\varphi$.*

Moreover, let $\hat{e} \in \widehat{\Gamma}_\varphi$ be the identity element, then we have

$$\widehat{N}_\varphi = \{g \in \widehat{\Gamma}_\varphi \mid \lim_{n \to \infty} \hat{\varphi}^n(g) = \hat{e}\}, \quad \mathcal{D}_\varphi = \bigcap_{n > 0} \hat{\varphi}^n(\widehat{\Gamma}_\varphi). \tag{19}$$

Proof. Recall that by Theorem 5.1 the embedding $\hat{\varphi}_{\mathfrak{G}} \colon \mathfrak{G} \to \mathfrak{G}$ induces an isomorphism $\mathfrak{G} \cong C_\varphi \rtimes Q_\varphi$, where C_φ and Q_φ are characterized by the formulae in (17). First, we show:

Lemma 5.4. $\mathcal{D}_\varphi = \Pi_{\mathfrak{G}}(Q_\varphi) \subset \widehat{\Gamma}_\varphi$.

Proof. Recall that the clopen neighborhoods U_ℓ of x_φ are defined by (8), and for each $\ell \geq 0$ we have $U_\ell = \lambda_\varphi^\ell(X_\varphi)$. For each $\ell \geq 0$, define the clopen subset $\widehat{U}_\ell = \{\gamma \in \widehat{\Gamma}_\varphi \mid \widehat{\Phi}_\varphi(\gamma)(U_\ell) = U_\ell\} \subset \widehat{\Gamma}_\varphi$.

Also, recall that $\mathcal{D}_\varphi = \{\gamma \in \widehat{\Gamma}_\varphi \mid \gamma \cdot x_\varphi = x_\varphi\}$. As $x_\varphi = \bigcap_{\ell \geq 0} U_\ell$, we then have $\mathcal{D}_\varphi = \bigcap_{\ell \geq 0} \widehat{U}_\ell$, and so $\widehat{U}_\ell = \hat{\varphi}^\ell(\widehat{\Gamma}_\varphi)$ where $\hat{\varphi} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ was defined in Proposition 5.2.

Recall that $\mathfrak{G}_\ell = \hat{\varphi}_{\mathfrak{G}}^\ell(\mathfrak{G}) \subset \mathfrak{G}$, and thus $\widehat{U}_\ell = \Pi_{\mathfrak{G}}(\mathfrak{G}_\ell)$. Then,

$$\Pi_{\mathfrak{G}}(Q_\varphi) = \Pi_{\mathfrak{G}}\Big\{\bigcap_{\ell \geq 0} \mathfrak{G}_\ell\Big\} = \bigcap_{\ell \geq 0} \Pi_{\mathfrak{G}}(\mathfrak{G}_\ell) = \bigcap_{\ell \geq 0} \widehat{U}_\ell = \mathcal{D}_\varphi, \tag{20}$$

as was to be shown. △

Next, set $\widehat{N}_\varphi = \Pi_{\mathfrak{G}}(C_\varphi) \subset \widehat{\Gamma}_\varphi$ which is a pro-nilpotent closed subgroup. Then by an argument exactly analogous to the proof of Lemma 5.4, we have

$$\widehat{N}_\varphi = \{\gamma \in \widehat{\Gamma}_\varphi \mid \lim_{\ell \to \infty} \hat{\varphi}^\ell(\gamma) = \hat{e} \in \widehat{\Gamma}_\varphi\}. \tag{21}$$

The proof of Theorem 5.3 now follows. □

Note that the identities (19) in Theorem 5.3 identify the images of the groups C_φ and Q_φ in $\mathrm{Homeo}(X_\varphi)$ in terms of the dynamical properties of the action $\hat{\varphi}$ on $\widehat{\Gamma}_\varphi$.

The conclusions of Theorem 5.3 are illustrated in various examples of renormalizable groups and self-embeddings in Section 9, and also by the examples in the works [49, 56, 57]. Moreover, the conclusion that φ induces an automorphism of the discriminant group $\mathcal{D}_\varphi$ has applications to the constructions of examples of Cantor actions using the *Lenstra method* as given in [37, Section 8.2].

6. Finite discriminant

We next consider the consequences of Theorem 5.3, for $\mathcal{D}_\varphi$ a finite group and prove Theorem 1.1.

Proof of Theorem 1.1. We first assume that the discriminant group $\mathcal{D}_\varphi$ is trivial, and show that the quotient group $\Gamma/C(\mathcal{G}_\varphi)$ is nilpotent, where $C(\mathcal{G}_\varphi)$ is the normal core of the intersection $K(\mathcal{G}_\varphi) = \bigcap_{\ell \geq 0} \Gamma_\ell \subset \Gamma$, as defined in (12). Recall that $C(\mathcal{G}_\varphi) \subset \Gamma$ is identified with the kernel of the homomorphism $\Phi_\varphi \colon \Gamma \to \widehat{\Gamma}_\varphi \subset$ Homeo(X_∞), and that $\hat{\varphi} \colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ was defined in Proposition 5.2.

Note that φ restricts to an isomorphism of $K(\mathcal{G}_\varphi)$ by its definition, and so φ also maps $C(\mathcal{G}_\varphi)$ isomorphically to itself, and thus induces an embedding $\varphi' \colon \Gamma/C(\mathcal{G}_\varphi) \to \Gamma/C(\mathcal{G}_\varphi)$. Then without loss of generality, we can replace Γ with $\Gamma/C(\mathcal{G}_\varphi)$, so can assume that $\Phi_\varphi \colon \Gamma \to \widehat{\Gamma}_\varphi$ is an embedding, and identify Γ with its image $\Phi_\varphi(\Gamma)$. As we assume that $\mathcal{D}_\varphi$ is trivial, by Theorem 5.3 we have $\widehat{\Gamma}_\varphi \cong \widehat{N}_\varphi$ where $\widehat{N}_\varphi$ is a closed pro-nilpotent normal group.

Section 3 of the work [57] gives an overview of some of the structure theory of pro-nilpotent groups, and we recall those aspects as required for the proof of Theorem 1.1. First, $\widehat{N}_\varphi$ admits a splitting by [28, Theorem B] as $\widehat{N}_\varphi \cong \widehat{N}_\infty \times \widehat{N}_{\mathrm{tor}}$ where $\widehat{N}_\infty$ is a torsion-free nilpotent group and $\widehat{N}_{\mathrm{tor}}$ is a torsion group with bounded exponent, by results of [40]. We now claim:

Lemma 6.1. *If $\mathcal{D}_\varphi$ is trivial, then $\widehat{N}_{\mathrm{tor}}$ is the trivial group.*

Proof. Let $\pi_{\mathrm{tor}} \colon \widehat{N}_\varphi \to \widehat{N}_{\mathrm{tor}}$ be the projection, then the image $\pi_{\mathrm{tor}}(\Gamma) \subset \widehat{N}_{\mathrm{tor}}$ is dense.

The abelianization $\widehat{A}_{\mathrm{tor}}$ of $\widehat{N}_{\mathrm{tor}}$ is an abelian group of bounded exponent, which is trivial if and only if $\widehat{N}_{\mathrm{tor}}$ is trivial. By Prüfer's First Theorem (see §24 of [41]), $\widehat{A}_{\mathrm{tor}}$ is a direct sum of (possibly infinitely many) cyclic groups. As Γ is finitely generated, the image of Γ in $\widehat{A}_{\mathrm{tor}}$ is finite rank and dense, and therefore the abelianization $\widehat{A}_{\mathrm{tor}}$ has finite rank. Thus, $\widehat{A}_{\mathrm{tor}}$ is a direct sum of finitely many cyclic groups, hence is a finite group.

Note that the contraction mapping $\hat{\varphi} \colon \widehat{N}_\varphi \to \widehat{N}_\varphi$ induces a contraction mapping $\hat{\varphi}_{\mathrm{tor}} \colon \widehat{N}_{\mathrm{tor}} \to \widehat{N}_{\mathrm{tor}}$.

The second part of Theorem B in Glöckner and Willis [28] proves the existence of a Jordan-Hölder series for bounded exponent contraction groups with each composition factor a *simple* contraction group. Here we say a contraction group with contraction α is *simple* if it has no nontrivial, proper closed normal α-invariant subgroup. Further, the simple contraction groups are classified as shifts on $F^{\mathbb{N}}$ where F is a finite simple group. By considering the first composition factor, we see that $\widehat{N}_{\mathrm{tor}}$ has a quotient of the form $F^{\mathbb{N}}$ where F is a finite simple group. Since $\widehat{N}_{\mathrm{tor}}$ is solvable of bounded exponent [51], we conclude that F is abelian. In particular $\widehat{N}_{\mathrm{tor}}$ has an infinite abelian quotient, which contradicts the fact that $\widehat{A}_{\mathrm{tor}}$ is a finite group, as shown previously. It follows that $\widehat{N}_{\mathrm{tor}}$ must be the trivial group $\triangle$

Since by assumption $\mathcal{D}_\varphi$ is trivial, then $\Gamma \subset \widehat{N}_\infty$. Now observe that by Lemma 6.1, $\widehat{N}_\infty$ is a torsion-free nilpotent group, thus Γ is nilpotent. This concludes the proof of Theorem 1.1 in the case where $\mathcal{D}_\varphi$ is trivial.

Next, assume that $\mathcal{D}_\varphi$ is a finite group. By Theorem 5.3, we have $\mathcal{D}_\varphi = \Pi_\mathfrak{G}(Q_\varphi) \subset \widehat{\Gamma}_\varphi$ and its intersection with $\widehat{N}_\varphi = \Pi_\mathfrak{G}(C_\varphi)$ is the trivial subgroup. As $\mathcal{D}_\varphi$ is a finite group, it follows that $\widehat{N}_\varphi$ is a clopen subset of $\widehat{\Gamma}_\varphi$, and so $\Lambda_\varphi \equiv \Gamma \cap \widehat{N}_\varphi$ is a dense subgroup of $\widehat{N}_\varphi$

The restriction of $\hat{\varphi}$ defines a contraction mapping $\hat{\varphi} \colon \widehat{N}_\varphi \to \widehat{N}_\varphi$. We can thus apply the above arguments for the case when $\mathcal{D}_\varphi$ is trivial to the action of Λ_φ on $\widehat{N}_\varphi$ to conclude that Λ_φ is nilpotent. As Λ_φ has finite index in Γ, this completes the proof of Theorem 1.1. □

Proof of Theorem 1.5. Assume that both $\mathcal{D}_\varphi$ and the subgroup $K(\mathcal{G}_\varphi)$ in (12) are finite groups. Thus its core $C(\mathcal{G}_\varphi) \subset K(\mathcal{G}_\varphi)$ is also finite. Recall that in the above proof of Theorem 1.1, we replaced Γ with the quotient $\Gamma/C(\mathcal{G}_\varphi)$, and concluded that $\Gamma/C(\mathcal{G}_\varphi)$ contains a nilpotent subgroup of finite index. In the case where both groups $\mathcal{D}_\varphi$ and $K(\mathcal{G}_\varphi)$ are trivial, then the claim of the corollary follows directly from Theorem 1.1 and Theorem 5.3. In the case where both groups are finite, we have that $C(\mathcal{G}_\varphi)$ is a finite normal subgroup of Γ and $\Gamma/C(\mathcal{G}_\varphi)$ contains a nilpotent subgroup of finite index, which implies that Γ contains a nilpotent subgroup of finite index. This completes the proof. □

Remark 6.2. We clarify the relation between the groups $K(\mathcal{G}_\varphi)$ and $\mathcal{D}_\varphi$ in the hypothesis of Theorem 1.5. The group $K(\mathcal{G}_\varphi)$ contains every element of Γ which fixes the base point x_φ, while the group $\mathcal{D}_\varphi$ contains every element in $\widehat{\Gamma}_\varphi$ which fixes x_φ. The relationship between $K(\mathcal{G}_\varphi)$ and $\mathcal{D}_\varphi$ is provided by an embedding $\Gamma \to \widehat{\Gamma}_\varphi \subset \mathrm{Homeo}(X_\varphi)$. It follows that if $\mathcal{D}_\varphi$ is a finite group, then the quotient group $K(\mathcal{G}_\varphi)/C(\mathcal{G}_\varphi)$ must be finite, where $C(\mathcal{G}_\varphi)$ is the normal core of $K(\mathcal{G}_\varphi)$. Thus if $\mathcal{D}_\varphi$ is finite, $K(\mathcal{G}_\varphi)$ may still be infinite if its normal core $C(\mathcal{G}_\varphi)$ is infinite. Thus both assumptions in (12) in Theorem 1.5 are necessary.

7. Renormalizable Cantor actions

In this section, we introduce the notions of (*virtually*) *renormalizable* Cantor actions, and study their regularity properties and invariants, yielding a proof of Theorem 1.6.

7.1. Renormalizable actions. For a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ and an adapted set $U \subset \mathfrak{X}$, note that $H_U = \Phi(\Gamma_U) \subset \mathrm{Homeo}(U)$ acts faithfully on U, so (U, H_U, Φ_U) is always an effective action.

Definition 7.1. A Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is *renormalizable* if it is equicontinuous, and there exists an adapted proper clopen set $U \subset \mathfrak{X}$ such that the actions $(\mathfrak{X}, \Gamma, \Phi)$ and (U, H_U, Φ_U) are isomorphic (as in Definition 2.6) by a homeomorphism $\lambda \colon \mathfrak{X} \to U$ and isomorphism $\Theta \colon \Gamma \to H_U$, and the intersection $\bigcap_{\ell \geq 0} \lambda^\ell(\mathfrak{X})$ is a point.

For example, let $(X_\varphi, \Gamma, \Phi_\varphi)$ be the Cantor action associated to a renormalization φ of Γ. Suppose the action is effective, then by Theorem 1.2 it is topologically free, and thus the map $\Phi_U \colon \Gamma_U \to H_U$ is an injection. Then the action is renormalizable with $\lambda = \lambda_\varphi$ and $\Theta = \Phi_U \circ \varphi \colon \Gamma \to H_U$. In general, for a renormalizable action, there is no requirement that the map $\Phi_U \colon \Gamma_U \to H_U$ is injective, and so H_U need not be identified with a subgroup of Γ.

Definition 7.2. A Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is *virtually renormalizable* if it is equicontinuous, and there exists an adapted set $V \subset \mathfrak{X}$ such that the restricted action (V, H_V, Φ_V) is renormalizable.

The class of virtually renormalizable actions is much more general than the renormalizable actions, as it allows for the case when the action map $\Phi \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ has a non-trivial kernel. In the following, we show some properties of these actions. We first show:

Proposition 7.3. *Suppose that the Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic, then it is quasi-analytic.*

Proof. We assume there is given a homeomorphism $\lambda \colon \mathfrak{X} \to U$ and group isomorphism $\Theta \colon \Gamma \to H_U$ implementing an isomorphism of $(\mathfrak{X}, \Gamma, \Phi)$ with (U, H_U, Φ_U) as in (5).

First, suppose that the map $\Phi_U \colon \Gamma_U \to H_U$ is injective, and hence is an isomorphism, as it is onto by the definition of H_U. Then the composition $\varphi \equiv \Phi_U^{-1} \circ \Theta \colon \Gamma \to \Gamma$ is a proper inclusion with image $\Gamma_U \subset \Gamma$. As U is adapted, Γ_U has finite index in Γ, and thus φ is a renormalization of Γ. It follows from Theorem 1.2 that the action Φ is quasi-analytic.

It thus suffices to show that if $\Phi_U \colon \Gamma_U \to H_U$ has a non-trivial kernel $K_U \subset \Gamma$, then the action Φ is not locally quasi-analytic, which yields a contradiction. We show this using a recursive argument.

Set $U_0 = \mathfrak{X}$, then $U_1 = \lambda(\mathfrak{X})$ is a clopen set by assumption. Also define clopen sets $U_\ell = \lambda^\ell(U_0)$ for $\ell > 1$, so that $U_\ell \subset U_{\ell-1}$. The assumption in Definition 7.1 that the intersection $\bigcap_{\ell \geq 0} \lambda^\ell(\mathfrak{X})$ is a point, denoted by $x_\lambda \in \mathfrak{X}$, implies that $\{U_\ell \mid \ell \geq 0\}$ is an adapted neighborhood basis at x_λ.

Now set $\Gamma_\ell = \Gamma_{U_\ell}$ for $\ell \geq 0$, and let $H_\ell = \Phi_{U_\ell}(\Gamma_\ell) \subset \mathrm{Homeo}(U_\ell)$ for $\ell \geq 0$. Let $\{\Gamma_\ell \mid \ell \geq 0\}$ be the associated group chain. Recall that as the action of H_1 on U_1 is effective, and the actions $(\mathfrak{X}, \Gamma, \Phi)$ and (U_1, H_1, Φ_{U_1}) are isomorphic,

so the action of Γ on $\mathfrak{X}$ is effective. That is, the kernel $K_0 \subset \Gamma$ of Φ is trivial, and $\Phi\colon \Gamma \to H_0$ is an isomorphism onto. To avoid cumbersome notation, we will identify $\Gamma = H_0$ and write the action as $g \cdot x = \Phi(g)(x)$.

Now observe that

$$\Gamma_{\ell+1} = \{g \in \Gamma \mid g \cdot U_{\ell+1} = U_{\ell+1}\} = \{g \in \Gamma_\ell \mid g \cdot U_{\ell+1} = U_{\ell+1}\} = (\Gamma_{U_\ell})_{U_{\ell+1}}$$

since $g \cdot U_{\ell+1} = U_{\ell+1}$ implies $g \cdot U_\ell = U_\ell$, as U_ℓ is an adapted clopen set and $U_{\ell+1} \subset U_\ell$.

We give the first step of the recursive argument. Define

$$K_1 \equiv \ker\{\Phi_{U_1}\colon \Gamma_1 \to H_1 \subset \mathrm{Homeo}(U_1)\} \subset \Gamma_1 \subset \Gamma. \tag{22}$$

By assumption, the subgroup K_1 is non-trivial.

Let $\Phi^1_{U_1}\colon H_1 \times U_1 \to U_1$ denote the action of H_1, and let $(H_1)_{U_2} \subset H_1$ denote the elements of H_1 which map U_2 to itself. Then introduce the subgroup $K'_2 \subset (H_1)_{U_2}$ of elements which restrict to the identity on U_2. Then we have

$$
\begin{aligned}
K'_2 &= \ker\{\Phi^1_{U_2}\colon (H_1)_{U_2} \to \mathrm{Homeo}(U_2)\} \\
&= \ker\{\Phi^1_{U_2}\colon (H_1)_{\lambda(U_1)} \to \mathrm{Homeo}(\lambda(U_1))\} \\
&= \ker\{\Phi^1_{U_2}\colon \Theta(\Gamma)_{\lambda(U_1)} \to \mathrm{Homeo}(\lambda(U_1))\} \\
&= \Theta(\ker\{\Phi_{U_1}\colon \Gamma_{U_1} \to \mathrm{Homeo}(U_1)\}) = \Theta(K_1),
\end{aligned}
\tag{23}
$$

where the last equality follows because $(\mathfrak{X}, \Gamma, \Phi)$ and (U_1, H_1, Φ_{U_1}) are isomorphic.

By assumption K_1 is a non-trivial subgroup, so by (23) we have $K'_2 = \Theta(K_1)$ is also non-trivial. That is, if $g \in K_1 \subset \Gamma_1$ is not the identity, then g acts non-trivially on $U_0 = \mathfrak{X}$ and restricts to the identity on U_1 by the definition (22) of K_1. Thus, $h = \Theta(g) \in H_1$ acts non-trivially on U_1 and restricts to the identity on U_2. Since $H_1 = \Phi_{U_1}(\Gamma_1)$, there exists $g' \in \Gamma_1$ such that $\Phi_{U_1}(g') = h$. We have found $g' \in \Gamma_1$, such that $g' \notin K_1$ and $g' \in K_2$. Therefore, K_1 is a non-trivial proper subgroup of K_2.

Set $K_\ell = \ker\{\Phi_{U_\ell}\colon \Gamma_\ell \to \mathrm{Homeo}(U_\ell)\}$ for $\ell \geq 2$, then by repeating the above arguments in (23), we have $K_\ell \subset K_{\ell+1} \subset \Gamma$ is a proper inclusion for all $\ell \geq 1$. As the diameter of the sets U_ℓ tends to 0 as ℓ increases, given any adapted set $V \subset \mathfrak{X}$ for the action Φ, there exist $\ell > 0$ and $\gamma \in \Gamma$ such that $O = \gamma \cdot U_\ell \subset V$. This implies that the dynamics of Γ_ℓ acting on U_ℓ is conjugate to the restricted action of Γ_V on the adapted clopen set O. Thus, there exists some element $\gamma' \in \Gamma$ such that $\gamma' \cdot O = O$ and the action of $\Phi(\gamma')$ restricted to O is non-trivial, but restricts to the identity on some open set that is a translate of $U_{\ell+1}$ in O, namely $\gamma' = \gamma^{-1} \circ s \circ \gamma$, where $s \in K_{\ell+1}$ and $s \notin K_\ell$. As this holds for all $\ell > 0$, the action Φ is not locally quasi-analytic. $\qquad \square$

We have the following consequence of the above proof of Proposition 7.3.

Proposition 7.4. *Suppose that the Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic, then the action is isomorphic to an action $(X_\varphi, \Gamma, \Phi_\varphi)$ associated to a renormalization $\varphi \colon \Gamma \to \Gamma$, and in particular Γ is renormalizable, and the action $(\mathfrak{X}, \Gamma, \Phi)$ is stable.*

Proof. As in the proof of Proposition 7.3, let $\{U_\ell \mid \ell \geq 0\}$ be an adapted neighborhood basis at x_φ, and let $\{\Gamma_\ell \mid \ell \geq 0\}$ is the associated group chain. The action $(\mathfrak{X}, \Gamma, \Phi)$ is quasi-analytic by Proposition 7.3, so we have isomorphisms $\Gamma_\ell \cong H_\ell$, and in particular the composition $\varphi \equiv \Phi_{U_1}^{-1} \circ \Theta \colon \Gamma \to \Gamma$ is a proper inclusion with image $\Gamma_1 \subset \Gamma$ a subgroup of finite index. It then follows that $\Gamma_\ell = \varphi^\ell(\Gamma)$, and by the results in Section 3 (see also [15, 21, 22]) the Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to the action $(X_\varphi, \Gamma, \Phi_\varphi)$. Then the action $(\mathfrak{X}, \Gamma, \Phi)$ is stable by Corollary 4.7. $\square$

As a consequence of the above, we have the following result, which implies Theorem 1.6.

Theorem 7.5. *Let φ be a renormalization of Γ, then $(X_\varphi, \Gamma, \Phi_\varphi)$ is virtually renormalizable. Conversely, suppose that a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic, then Γ is renormalizable, and there is a renormalization φ such that $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to $(X_\varphi, \Gamma, \Phi_\varphi)$.*

7.2. Renormalizable actions on trees. We will now discuss the relationship between renormalizable actions on Cantor sets and self-similarity properties of groups acting on rooted trees. Recall that, given a group chain $\{\Gamma_\ell\}_{\ell \geq 0}$ consisting of finite index subgroups of $\Gamma = \Gamma_0$, the tree model is a natural action of Γ on a rooted tree (see Section 3.1). We start by briefly recalling this construction. As in the rest of the paper, the action of $\Gamma \subset \text{Aut}(T)$ on the boundary of T is assumed minimal in this section.

Recall that $V_\ell = X_\ell = \Gamma/\Gamma_\ell$, and $V = \bigsqcup_{\ell \geq 0} V_\ell$ is the vertex set of a tree T. The boundary ∂T of T consists of all infinite connected paths $(v_\ell)_{\ell \geq 0} \in \partial T$. Let $\hat{e} = (e_\ell)_{\ell \geq 0} = (e \cdot \Gamma_\ell)$ be the path passing through the coset of the identity $e \in \Gamma$ at each level V_ℓ. Then, as in (8), for $k \geq 0$,

$$\begin{aligned} U_k &= \{(w_\ell)_{\ell \geq 0} \in \partial T \mid w_\ell = e \cdot \Gamma_\ell, 0 \leq \ell \leq k\} \\ &= \{(w_\ell)_{\ell \geq 0} \in \partial T \mid w_k = e \cdot \Gamma_k\} \end{aligned} \tag{24}$$

is a clopen neighborhood of $\hat{e}$ and Γ_k is the stabilizer subgroup of U_k. Since the action of Γ on ∂T is minimal, the induced action of Γ on each vertex level V_ℓ is transitive, for $\ell \geq 0$.

We now discuss Definition 7.1 of a renormalizable action as applied to actions on rooted trees, described in the previous paragraph. Suppose the action $(\partial T, \Gamma)$

is renormalizable with $U = U_1$. The set U_1 contains all infinite paths in ∂T which pass through the vertex $e_1 = e \cdot \Gamma_1$, and every element in Γ_1 fixes e_1. For each $g \in \Gamma_1 \subset \mathrm{Aut}(T)$, denote by $g_1 = g|U_1$ the restriction. If the action $(\partial T, \Gamma)$ is quasi-analytic, then there is precisely one element $g \in \Gamma$ which restricts to g_1, and so the map $\Phi_{U_1}: \Gamma_1 \to H_{U_1}$ is invertible. As discussed at the beginning of this section, in this case there is an injective homomorphism $\varphi = \Phi_{U_1}^{-1} \circ \Theta: \Gamma \to \Gamma_1 \subset \Gamma$. In particular, the conditions that $(\partial T, \Gamma)$ is quasi-analytic and minimal implies that there are no elements in Γ whose support is contained entirely in U_1, and it follows that the class of groups which admit renormalizations does not contain weakly branch groups (see [7, 32] for more details about weakly branch groups).

If the action $(\partial T, \Gamma)$ is renormalizable but not quasi-analytic, then the elements in H_{U_1} can be extended from U_1 to ∂T in multiple ways, and Γ may have elements with support contained entirely in U_1. Such renormalizable actions are wild. Some actions of branch groups belong to this class, for instance, the action of the Grigorchuk group, as we show below.

We will show that renormalizability of the action of Γ on ∂T is closely related to the *self-replicating* property of the action of Γ on T. Given a vertex $v_\ell \in V$, let $v_\ell T$ be a subtree of T with root v_ℓ. Fix an isomorphism of rooted trees $p_{v_\ell}: T \to v_\ell T$ and let $\partial p_{v_\ell}: \partial T \to \partial(v_\ell T)$ be the induced homeomorphism of boundaries. Note that the inclusion $v_\ell T \hookrightarrow T$ also induces a homeomorphism between $\partial(v_\ell T)$ and the clopen subset U_{v_ℓ} of ∂T that consists of all paths passing through the vertex v_ℓ. If $v_\ell = e_\ell$, then $U_{v_\ell} = U_\ell$ for the set U_ℓ defined in (24).

Denote by Γ_{v_ℓ} the subgroup of elements $g \in \Gamma$ which fix v_ℓ and so preserve U_{v_ℓ}. For $g \in \Gamma_{v_\ell}$ denote by $g_\ell = g|U_{v_\ell}$ the restriction, and consider the pullback $p_\ell^* g_\ell$ to ∂T. We refer to [32] for a precise definition of a self-replicating group, but it implies that for any $v_\ell \in V_\ell$ and any $\ell \geq 0$, the morphisms

$$\tilde{p}_\ell: \Gamma_{v_\ell} \longrightarrow \mathrm{Aut}(T), \quad g \longmapsto p_\ell^* g_\ell = p_\ell^*(g|U_{v_\ell}) \tag{25}$$

have image in $\Gamma \subset Aut(T)$ and are surjective onto Γ. We are now in the position to establish the connection between self-replicating groups and renormalizable actions.

Proposition 7.6. *Let Γ be a self-replicating group acting on the boundary ∂T of a regular tree T. Then the action $(\partial T, G)$ is renormalizable in the sense of Definition* 7.1.

Proof. In the notation of above, fix the vertex $v_\ell = e \cdot \Gamma_\ell$, so $U_{v_\ell} = U_\ell$ and $\Gamma_{v_\ell} = \Gamma_\ell$. The group $H_{U_\ell} \subset \mathrm{Homeo}(U_\ell)$ is a quotient group of Γ_ℓ. The pullback map $p_\ell^*: H_{U_\ell} \to \Gamma$ is clearly an injective group homomorphism, as H_{U_ℓ} is a group of homeomorphisms of a subtree. The map p_ℓ^* is surjective by the definition of a self-replicating group and (25). It follows that the action of Γ is renormalizable with maps $\lambda = p_\ell: \partial T \to \partial(v_\ell T)$ and $\Theta = (p_\ell^*)^{-1}: \Gamma \to H_{U_\ell}$. $\qquad\square$

An example of a group whose action is renormalizable and not quasi-analytic is the Grigorchuk group, which is known to be self-replicating [32]. We refer to [32] for other examples of self-replicating groups, acting on trees, and to [8] for the overview of the relation between the notions of self-similar groups and other notions of renormalizability, for instance that of tilings.

Grigorchuk [32, Proposition 11.6] showed that a countable self-replicating group Γ which acts freely on the boundary ∂T of a tree T is scale-invariant, see the Introduction for the definitions. Nekrashevych and Pete [49] provided examples of finitely generated scale-invariant groups that are not strongly scale-invariant. Our results can be used to strengthen Grigorchuk's result in [32, Proposition 11.6] for free actions of finitely generated self-replicating groups to show that they are strongly scale-invariant.

Proposition 7.7. *Let Γ be a finitely generated self-replicating group, and suppose the action of Γ on the boundary ∂T of a regular tree T is free. Then Γ is strongly scale-invariant.*

Proof. By the argument above the action of a self-replicating group is renormalizable, and if it is free, then it is quasi-analytic. Then by Proposition 7.4 Γ is strongly scale-invariant, that is, there is a renormalization $\varphi \colon \Gamma \to \Gamma$, and the group chain $\{\varphi^\ell(\Gamma) \mid \ell \geq 0\}$ has trivial intersection. □

It is natural to ask if the converse of Proposition 7.6 holds, that is, if an effective renormalizable action is always that of a self-replicating group of homeomorphisms. In Definition 7.1 of a renormalizable action the map $\Theta \colon \Gamma \to H_U$ is allowed to be any group isomorphism, introducing a possibility of a 'twist'. We leave the question whether the converse holds as an open problem.

Problem 7.8. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an effective renormalizable Cantor action, and suppose $(\mathfrak{X}, \Gamma, \Phi)$ is conjugate to an action of Γ on $(\partial T, \Gamma)$, where ∂T is the boundary of a regular tree T. Prove that Γ is a self-replicating group, or find a counterexample to this statement.*

8. Continuous orbit equivalence

We next give the proofs of Theorems 1.7 and 1.8, which consider the properties of renormalizable actions which are preserved by continuous orbit equivalence.

8.1. Proof of Theorem 1.7. Let $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ be minimal equicontinuous Cantor actions which are continuously orbit equivalent, and assume that $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable and locally quasi-analytic. We claim that $(\mathfrak{X}', \Gamma', \Phi')$ is virtually renormalizable.

First note that by Proposition 7.3, the action $(\mathfrak{X}, \Gamma, \Phi)$ is quasi-analytic, and by Proposition 7.4, there exists a proper self-embedding $\varphi \colon \Gamma \to \Gamma$ such that the action $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to the action $(X_\varphi, \Gamma, \Phi_\varphi)$. Thus, the Cantor actions $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are continuously orbit equivalent, where $(X_\varphi, \Gamma, \Phi_\varphi)$ is quasi-analytic by Theorem 1.2, and stable by Corollary 4.7. Then Theorem 6.9 of [39] implies that $(\mathfrak{X}', \Gamma', \Phi')$ is locally quasi-analytic.

The hypotheses of Theorem 1.5 in [38] are then satisfied, so that $(X_\varphi, \Gamma, \Phi_\varphi)$ is return equivalent to $(\mathfrak{X}', \Gamma', \Phi')$. Thus, there exists adapted sets $V \subset X_\varphi$ for the action $(X_\varphi, \Gamma, \Phi_\varphi)$ and $V' \subset \mathfrak{X}'$ for the action $(\mathfrak{X}', \Gamma', \Phi')$, so that the restricted actions (V, H_V, Φ_V) and $(V', H'_{V'}, \Phi'_{V'})$ are isomorphic, where $H_V = \Phi_V(\Gamma_V) \subset \mathrm{Homeo}(V)$ and $H'_{V'} = \Phi'_{V'}(\Gamma'_{V'}) \subset \mathrm{Homeo}(V')$.

Let $x_\varphi \in X_\varphi$ denote the fixed-point for the contraction $\lambda_\varphi \colon X_\varphi \to X_\varphi$. The action $(X_\varphi, \Gamma, \Phi_\varphi)$ is minimal, so by conjugating by an element of Γ, we can assume that $x_\varphi \in V$.

Let $h \colon V \to V'$ be a homeomorphism and $\Theta \colon H_V \to H'_{V'}$ a group isomorphism which realizes the isomorphism between (V, H_V, Φ_V) and $(V', H'_{V'}, \Phi'_{V'})$ as in Definition 2.6.

For the action $(X_\varphi, \Gamma, \Phi_\varphi)$, we have an adapted neighborhood basis $\{U_\ell = \lambda^\ell(X_\varphi) \mid \ell \geq 0\}$ and a group chain $\mathcal{G}_\varphi = \{\Gamma_\ell = \varphi^\ell(\Gamma) \mid \ell \geq 0\}$ as before.

Choose $\ell_0 > 0$ sufficiently large so that $U_{\ell_0} \subset V$ and $h(U_{\ell_0}) \subset V'$. Then set $W = U_{\ell_0}$. Note that $\lambda_\varphi(U_\ell) = U_{\ell+1}$ for all $\ell \geq 0$, so $W_1 = \lambda_\varphi(W) \subset W$. Set $W' = h(W) \subset V'$ and $W'_1 = h(W_1) \subset W'$. Then the restriction of φ to $\Gamma_W = \Gamma_{\ell_0}$ yields a proper self-embedding $\varphi_W \colon \Gamma_W \to \Gamma_W$.

Since the action $(X_\varphi, \Gamma, \Phi_\varphi)$ is quasi-analytic, the map $\Phi_W \colon \Gamma_W \to H_W$ is an isomorphism. Thus, φ_W induces a proper self-embedding $\hat{\varphi}_W \colon H_W \to H_W$. Then set $H_\ell = \hat{\varphi}_W^\ell(H_V)$ for all $\ell \geq 0$. It then follows from the constructions that the Cantor action (W, H_W, Φ_W) is isomorphic with the Cantor action associated to $\hat{\varphi}_W \colon H_W \to H_W$.

Finally, the isomorphism between (V, H_V, Φ_V) and $(V', H'_{V'}, \Phi'_{V'})$ restricts to an isomorphism between (W, H_W, Φ_W) and $(W', H'_{W'}, \Phi'_{W'})$ which then defines a self-embedding of $H'_{W'}$. Thus, the Cantor action $(\mathfrak{X}', \Gamma', \Phi')$ is virtually renormalizable. This completes the proof of Theorem 1.7.

8.2. Proof of Theorem 1.8. Let $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ be Cantor actions associated to renormalizations $\varphi \colon \Gamma \to \Gamma$ and $\varphi' \colon \Gamma' \to \Gamma'$, respectively. Assume that $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ are continuously orbit equivalent. We must show that the discriminant groups $\mathcal{D}_\varphi$ and $\mathcal{D}'_{\varphi'}$ for these actions are isomorphic.

By Corollary 4.7, the actions $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ are quasi-analytic and stable. Then Theorem 1.5 in [38] implies that the actions $(\mathfrak{X}, \Gamma, \Phi)$ and $(X'_{\varphi'}, \Gamma', \Phi'_{\varphi'})$ are return equivalent.

Thus, there exist adapted sets $V \subset \mathfrak{X}$ for the action $(\mathfrak{X}, \Gamma, \Phi)$ and $V' \subset \mathfrak{X}'$ for the action $(\mathfrak{X}', \Gamma', \Phi')$ so that the restricted actions (V, H_V, Φ_V) and

$(V', H'_{V'}, \Phi'_{V'})$ are isomorphic, where recall that $H_V = \Phi_V(\Gamma_V) \subset \mathrm{Homeo}(V)$ and $H'_{V'} = \Phi'_{V'}(\Gamma'_{V'}) \subset \mathrm{Homeo}(V')$. As the actions are quasi-analytic, the maps Φ_V and $\Phi'_{V'}$ are monomorphisms, hence are isomorphisms. Thus, the actions (V, Γ_V, Φ_V) and $(V', \Gamma'_{V'}, \Phi'_{V'})$ are isomorphic, induced by a homeomorphism $h \colon V \to V'$.

Let $\mathcal{D}_V$ denote the discriminant group for the restricted action (V, Γ_V, Φ_V). Then by the arguments in [37, Section 4], there is a surjective map $\rho_{\mathfrak{X}, V} \colon \mathcal{D}_\varphi \to \mathcal{D}_V$ which is an isomorphism when the profinite action $\hat{\Phi}_\varphi \colon G \times X_\varphi \to X_\varphi$ is quasi-analytic. Likewise, for the discriminant $\mathcal{D}'_{V'}$ of the action $(V', \Gamma'_{V'}, \Phi'_{V'})$, there is an isomorphism $\rho_{\mathfrak{X}', V'} \colon \mathcal{D}_{\varphi'} \to \mathcal{D}'_{V'}$.

The isomorphism class of the discriminant group is an invariant for isomorphism of Cantor actions, so we conclude $\mathcal{D}_\varphi \cong \mathcal{D}_V \cong \mathcal{D}'_{V'} \cong \mathcal{D}'_{\varphi'}$ as claimed. This completes the proof of Theorem 1.8.

9. Applications and Examples

In this section, we discuss some of the applications of the results of this paper, then give a selection of examples to illustrate these results.

For a compact manifold M without boundary, an expansive diffeomorphism $\phi \colon M \to M$ gives rise to a renormalization $\varphi \colon \Gamma \to \Gamma$ of the fundamental group $\Gamma = \pi_1(M, x)$. In this case, Shub showed in [55] that the universal covering of M has polynomial growth type, and hence by Gromov [34] the group Γ has a finite-index nilpotent subgroup. There are a variety of constructions of expansive diffeomorphisms on nilmanifolds, and the invariants associated to the renormalization φ of Γ are then invariants of the expansive map ϕ.

The construction of generalized Hirsch foliations in [10, 36] is based on choosing a renormalization $\varphi \colon \Gamma \to \Gamma$ of the fundamental group of a compact manifold M. Thus, invariants of the renormalization yield invariants for this genre of foliated manifolds.

The classification of M-like laminations, where M is a fixed compact manifold, is reduced to the classification of renormalizations in the work [13].

These applications are all based on the constructions of renormalizations for groups with the non-co-Hopfian property. Many finitely generated nilpotent groups are renormalizable, as shown for example in [9, 14, 18, 17, 16, 42]. There is also a variety of examples of renormalizable groups which are not nilpotent, as described for example in [19, 25, 30, 31, 29, 49, 50, 57]. While these works show the existence of a proper self-embedding for a particular class of groups, they do not calculate the groups $\mathcal{D}_\varphi$ and $\hat{N}_\varphi$ which are associated to an embedding φ by Theorem 5.3. In the following, we make these calculations for a selected set of examples of renormalizable groups.

In Section 9.1 we give a basic example of a renormalizable Cantor action, where the group Γ is not finitely generated, and Γ is not virtually nilpotent.

In Section 9.2 we give a basic example of the cross-product construction of renormalizable groups, for which the discriminant is a non-trivial finite group.

In Section 9.3 we calculate the discriminant $\mathcal{D}_\varphi$ and the induced map $\hat{\varphi}\colon \mathcal{D}_\varphi \to \mathcal{D}_\varphi$ for an "untwisted" embedding $\varphi\colon \mathcal{H} \to \mathcal{H}$ of the Heisenberg group $\mathcal{H}$.

In Section 9.4 we give an example of a renormalizable group that arises in the study of arboreal representations of absolute Galois groups of number fields.

9.1. Infinitely generated actions.
The assumption that Γ is finitely generated is essential for the conclusion of Theorem 1.1, as shown by the following example. Let F be a finite nonabelian simple group and set $\Gamma := \bigoplus_{i=0}^\infty F$. Then Γ is a countable group, but not finitely generated.

Observe that Γ admits a renormalization, given by the shift map,

$$\varphi(f_0, f_1, \ldots) = (e, f_0, f_1, \ldots),$$

for $e \in F$ the identity element. The associated Cantor space is the infinite product $X_\varphi = \prod_0^\infty F$. The action Φ_φ of Γ on X_φ is free, so the discriminant $\mathcal{D}_\varphi$ is trivial in this case. However, Γ is not virtually nilpotent.

9.2. Multihedral groups.
This is an elementary example of a group Γ with self-embedding φ and non-trivial finite discriminant group $\mathcal{D}_\varphi \subset \Gamma$.

Let $\Lambda = \mathbb{Z}^k$ be the free abelian group on k generators. Let $H \subset \mathrm{Perm}(k)$ be a non-trivial subgroup of the finite symmetric group $\mathrm{Perm}(k)$ on k symbols, and assume that H is a simple group. Let $\mathrm{Perm}(k) \subset \mathrm{GL}(k, \mathbb{Z})$ be the standard embedding permuting the coordinates.

Let $\Gamma = \mathbb{Z}^k \rtimes H$ be the semi-direct product of these groups. Fix $m > 1$, then define $\varphi\colon \Gamma \to \Gamma$ to be multiplication by m on the $\mathbb{Z}^k$ factor. That is, for $(\vec{v}, g) \in \Gamma$ set $\varphi(\vec{v}, g) = (m \cdot \vec{v}, g)$. Then,

$$\Gamma_\ell = \{(m^\ell \cdot \vec{v}, g) \mid \vec{v} \in \mathbb{Z}^k, g \in H\} = m^\ell \mathbb{Z}^k \rtimes H, \tag{26}$$

$$K(\mathcal{G}_\varphi) = \{(0, 0, g) \mid g \in H\} \cong H, \tag{27}$$

where $\mathcal{G}_\varphi = \{\Gamma_\ell \mid \ell \geq 0\}$. Then we have $X_\varphi \cong \widehat{\mathbb{Z}}_m^k$, the product of k-copies of the inverse limit space $\widehat{\mathbb{Z}}_m = \varprojlim\{\mathbb{Z}/m^{k+1}\mathbb{Z} \to \mathbb{Z}/m^k\mathbb{Z}, k \geq 0\}$. The subgroup H acts on X_φ by permutations of the coordinates, so the adjoint action on X_φ of a non-identity element $g \in H$ is a non-trivial permutation of the coordinate axes, hence is non-trivial. Thus, the normal core $C(\mathcal{G}_\varphi) \subset K(\mathcal{G}_\varphi)$ is trivial, and we have $K(\mathcal{G}_\varphi) \subset \mathcal{D}_\varphi$. Thus, a calculation shows that the normal core $C_\ell \subset \Gamma_\ell$ is the subgroup of (26) where $g = e \in H$ is the identity, so $\Gamma_\ell/C_\ell \cong H$ for all $\ell > 0$. Thus, $\mathcal{D}_\varphi \cong H$. Also, the subgroup $\widehat{N}_\varphi$ is the product of k copies of $\widehat{\mathbb{Z}}_m$, or the m-adic k-torus.

Observe that the map φ restricts to the identity on the subgroup H, while φ acts as multiplication by m on the normal subgroup $\mathbb{Z}^k$. Thus, $\hat{\varphi} \colon \mathcal{D}_\varphi \to \mathcal{D}_\varphi$ in Theorem 5.3, item 3, is the identity map, and $\hat{\varphi} \colon \widehat{N}_\varphi \to \widehat{N}_\varphi$ in Theorem 5.3, item 2, is induced by coordinate-wise multiplication by m on $\mathbb{Z}^k$.

9.3. Nilpotent endomorphisms.
The 3-dimensional Heisenberg group $\mathcal{H}$ is the simplest non-abelian nilpotent group, and we give a self-embedding for which $\mathcal{D}_\varphi$ is an infinite profinite group.

A general construction for self-embeddings of 2-step nilpotent groups is given by Lee and Lee in [42], of which this example is a special case. More generally, group chains in $\mathcal{H}$ were studied in detail by Lightwood, Şahin and Ugarcovici in [44], where they give a complete description for the subgroups of $\mathcal{H}$ and a characterization of which subgroups are normal. This work also gives a discussion of twisted and untwisted subgroups of the Heisenberg group, which can be used to construct further examples of renormalizations.

Group chains in $\mathcal{H}$ whose discriminant invariant is an infinite group were first constructed by Dyer in her thesis [20], and also described in [21, Example 8.1]. In the following, we construct such a group chain realized via a self-embedding of $\mathcal{H}$.

Let $\mathcal{H}$ be represented as $(\mathbb{Z}^3, *)$ with the group operation $*$. We have

$$(x, y, z)*(u, v, w) = (x+u, y+v, z+w+xv), \quad (x, y, z)^{-1} = (-x, -y, -z+xy),$$
$$(28)$$

for $x, u, y, v, z, w \in \mathbb{Z}$. This is equivalent to the upper triangular representation in $\mathrm{GL}(\mathbb{Z}^3)$. In particular, we have

$$(x, y, z) * (u, v, w) * (x, y, z)^{-1} = (u, v, w + xv - yu). \tag{29}$$

For integers $p, q > 0$ define $\varphi \colon \mathcal{H} \to \mathcal{H}$ by a self-embedding by $\varphi(x, y, z) = (px, qy, pqz)$. Then

$$\mathcal{H}_\ell = \varphi^\ell(\mathcal{H}) = \{(p^\ell x, q^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}, \quad \bigcap_{\ell > 0} \mathcal{H}_\ell = \{e\}.$$

Now assume that $p, q > 1$ are distinct prime numbers. Formula (29) implies that the normal core for $\mathcal{H}_\ell$ is given by

$$C_\ell = \mathrm{core}(\mathcal{H}_\ell) = \{((pq)^\ell x, (pq)^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}.$$

Thus, the finite group

$$Q_\ell = \mathcal{H}/C_\ell = \{(x, y, z) \mid x, y, z \in \mathbb{Z}/(pq)^\ell \mathbb{Z}\}. \tag{30}$$

The profinite group $\widehat{\mathcal{H}}_\infty$ is the inverse limit of the quotient groups Q_ℓ so we have

$$\widehat{\mathcal{H}}_\infty = \{(x, y, z) \mid x, y, z \in \widehat{\mathbb{Z}}_{pq}\}$$

with multiplication on each finite quotient induced given by the formula (29). To identify the discriminant subgroup $\mathcal{D}_\infty$ first note

$$\mathcal{H}_\ell / C_\ell = \{(p^\ell x, q^\ell y, 0) \mid x \in \mathbb{Z}/q^\ell\mathbb{Z}, y \in \mathbb{Z}/p^\ell\mathbb{Z}\} \subset Q_\ell, \qquad (31)$$

$$\mathcal{H}_{\ell+1} / C_{\ell+1} = \{(p^{\ell+1} x, q^{\ell+1} y, 0) \mid x \in \mathbb{Z}/q^{\ell+1}\mathbb{Z}, y \in \mathbb{Z}/p^{\ell+1}\mathbb{Z}\}. \qquad (32)$$

The bonding map $q_{\ell+1}\colon \mathcal{H}_{\ell+1}/C_{\ell+1} \to \mathcal{H}_\ell/C_\ell$ from the definition (11) for $\mathcal{D}_\infty$ is induced from the inclusion $\mathcal{H}_{\ell+1} \subset \mathcal{H}_\ell$ modulo quotient by

$$\mathcal{H}_{\ell+1} \cap C_\ell = \{(p^{\ell+1} q^\ell x, p^\ell q^{\ell+1} y, (pq)^{\ell+1} z) \mid x, y, z \in \mathbb{Z}\}.$$

Thus, in terms of the coordinates x, y in (32) the bonding map is given by

$$q_{\ell+1}(x, y, 0) = (x \bmod q^\ell\mathbb{Z}, y \bmod p^\ell\mathbb{Z}, 0).$$

It then follows by formula (11) that

$$\mathcal{D}_\varphi \cong \mathcal{D}_\infty = \varprojlim\{q_{\ell+1}\colon \mathcal{H}_{\ell+1}/C_{\ell+1} \to \mathcal{H}_\ell/C_\ell \mid \ell \geq 0\} \cong \widehat{\mathbb{Z}}_q \times \widehat{\mathbb{Z}}_p. \qquad (33)$$

The induced map $\hat{\varphi}\colon \mathcal{D}_\varphi \to \mathcal{D}_\varphi$ is given by multiplication by p on $\widehat{\mathbb{Z}}_q$ in the first x-coordinate, and multiplication by q on $\widehat{\mathbb{Z}}_p$ in the second y-coordinate, so that $\hat{\varphi}$ acts as an isomorphism on $\mathcal{D}_\varphi$, as asserted in Theorem 5.3.

Finally, consider the subgroup of Q_ℓ in (30) which is complementary to the subgroup $\mathcal{H}_\ell/C_\ell$,

$$N_\ell = \{(q^\ell x, p^\ell y, z) \mid x \in \mathbb{Z}/p^\ell\mathbb{Z}, y \in \mathbb{Z}/q^\ell\mathbb{Z}, z \in \mathbb{Z}/(pq)^\ell\mathbb{Z}\} \subset Q_\ell. \qquad (34)$$

The map φ induces a map on N_ℓ given by multiplication by p in the first x-coordinate, and multiplication by q in the second y-coordinate, so the action is nilpotent on N_ℓ. The inverse limit of the groups N_ℓ is a subgroup of $\widehat{\mathcal{H}}_\infty$ identified with

$$\widehat{N}_\varphi \cong \widehat{\mathcal{H}}_\infty / \mathcal{D}_\infty \cong \{(x, y, z) \mid x \in \widehat{\mathbb{Z}}_p, y \in \widehat{\mathbb{Z}}_q, z \in \widehat{\mathbb{Z}}_{pq}\},$$

and is a pro-nilpotent group as it has the finite nilpotent groups N_ℓ as quotients. Moreover, the induced map $\hat{\varphi}\colon \widehat{N}_\varphi \to \widehat{N}_\varphi$ is a contraction, as asserted in Theorem 5.3.

Note that if we take $p = q$ in the above calculations, so $\varphi\colon \mathcal{H} \to \mathcal{H}$ is the "diagonal expansion" by p on the abelian factor $\mathbb{Z}^2$, then $\mathcal{H}_{2\ell} \subset C_\ell$. So while each quotient $\mathcal{H}_{2\ell}/C_{2\ell}$ is non-trivial, its image under the composition of bonding maps in (11) vanishes in $\mathcal{H}_\ell/C_\ell$, hence $\mathcal{D}_\varphi$ is the trivial group in the inverse limit. Correspondingly, the inverse limit space X_φ has a well-defined group structure.

9.4. Semi-direct product of dyadic integers with its group of units.
This example can be viewed as a more sophisticated version of Example 9.2. It arises,

in particular, as the profinite arithmetic iterated monodromy group associated to a certain post-critically finite quadratic polynomial, as discussed in [46].

Let $\widehat{\Gamma} = \widehat{\mathbb{Z}}_2 \rtimes \widehat{\mathbb{Z}}_2^\times$, where $\widehat{\mathbb{Z}}_2$ is the dyadic integers, and $\widehat{\mathbb{Z}}_2^\times$ is the multiplicative group of dyadic integers. Denote by a the topological generator of the abelian group $\widehat{\mathbb{Z}}_2$, that is, a is identified with $([1]) \in \widehat{\mathbb{Z}}_2$, where $[1]$ is the equivalence class of 1 in $\mathbb{Z}/2^n\mathbb{Z}$, $n \geq 1$.

Recall that $\widehat{\mathbb{Z}}_2^\times$ is the automorphism group of $\widehat{\mathbb{Z}}_2$. The multiplicative units in the 2-adic integers can be computed by computing the units in $\mathbb{Z}/2^n\mathbb{Z}$ for any n, and taking the inverse limit (see [53, Theorem 4.4.7]) so we have $\widehat{\mathbb{Z}}_2^\times \cong \mathbb{Z}/2\mathbb{Z} \times \widehat{\mathbb{Z}}_2$. Here, $\mathbb{Z}/2\mathbb{Z}$ is generated by $([-1]) \in \widehat{\mathbb{Z}}_2^\times$, where $[-1]$ denotes the equivalence class of -1 in $\mathbb{Z}/2^n\mathbb{Z}$ for $n \geq 1$, and the the second factor is generated by $([5]) \in \widehat{\mathbb{Z}}_2^\times$, where $[5]$ is the equivalence class of 5 in $\mathbb{Z}/2^n\mathbb{Z}$ for $n \geq 1$. Denote these generators by b and c respectively. Then let

$$\Gamma \cong \langle a, b, c \mid b^2 = 1, \, bab^{-1} = a^{-1}, \, cac^{-1} = a^5, \, bcb^{-1}c^{-1} = 1\rangle, \quad (35)$$

where b and c commute since they are generators of different factors in a product space.

Define a self-embedding $\varphi: \Gamma \to \Gamma$ by setting $\varphi(a) = a^2$, $\varphi(b) = b$ and $\varphi(c) = c$. That is, we have

$$\Gamma_1 = \varphi(\Gamma) \cong \langle a^2, b, c \mid b^2 = 1, ba^2b^{-1} = a^{-2},$$
$$ca^2c^{-1} = (a^2)^5, bcb^{-1}c^{-1} = 1\rangle,$$

and so we obtain a group chain $\Gamma_\ell = \langle a^{2^\ell}, b, c\rangle$, $\ell \geq 1$. The discriminant group of the action defined by this group chain was computed in [46, Section 7]. In particular, computing the normal cores of the subgroups Γ_ℓ we obtain $C_\ell = \langle a^{2^\ell}, c^{2^{\ell-2}}\rangle \subset \Gamma_\ell$, and it follows that

$$\mathcal{D}_\varphi = \varprojlim\{\Gamma_{\ell+1}/C_{\ell+1} \longrightarrow \Gamma_\ell/C_\ell\} \cong \widehat{\mathbb{Z}}_2^\times.$$

10. Problems

The study of the properties of the dynamical systems of the form $(X_\varphi, \Gamma, \Phi_\varphi)$ suggests the following approach to the classification problem for renormalizable groups and their proper self-embeddings.

Problem 10.1. *Classify the structure of renormalizable groups Γ which satisfy*:

(1) $\mathcal{D}_\varphi$ *is the trivial group*;

(2) $\mathcal{D}_\varphi$ *is a finite group*;

(3) $\mathcal{D}_\varphi$ *is an infinite profinite group*.

Case (1) is discussed further in Section 10.1 below. There are numerous and varied constructions of examples of case (2), where $\mathcal{D}_\varphi$ is a finite group. See Section 9.2 for some typical examples.

The most interesting problems arise for case (3), where $\mathcal{D}_\varphi$ is an infinite profinite group. Corollary 4.7 implies that all of the direct limit group invariants for Cantor actions defined in [39] are bounded for these examples. Thus, the problem is to refine the invariants constructed from the adjoint action of $\mathcal{D}_\varphi$ on the pro-nilpotent normal subgroup $\widehat{N}_\varphi \subset \widehat{\Gamma}_\varphi$ to distinguish these various examples. Note that if the group chain $\mathcal{G}_\varphi$ has trivial intersection, then the intersection $\mathcal{D}_\varphi \cap \Gamma$ is trivial, so the invariants constructed using the adjoint action of $\mathcal{D}_\varphi$ are only "seen" when considering the action of the profinite group $\widehat{\Gamma}_\varphi$.

10.1. Renormalizable nilpotent groups.

Suppose that Γ admits a renormalization $\varphi \colon \Gamma \to \Gamma$, such that each of the subgroups $\Gamma_\ell = \varphi^\ell(\Gamma)$ is a *normal* subgroup of Γ. Then the third author showed in the work [56] that the quotient $\Gamma/C(\mathcal{G}_\varphi)$ must be free abelian. In particular, if the group chain $\mathcal{G}_\varphi = \{\Gamma_\ell \mid \ell \geq 0\}$ has trivial intersection, then Γ is free abelian. Theorem 1.1 is a more general form of this result, where the assumption that $\mathcal{G}_\varphi$ has finite discriminant implies that Γ is virtually nilpotent.

The remarks at the end of Section 9.3 show that $\mathcal{D}_\varphi$ is trivial when $p = q$ for the construction in Section 9.3. In fact, these remarks apply in general to the diagonal action on the nilpotent subgroup of upper triangular integer matrices, where φ is given by multiplication by a constant factor p on the super-diagonal entries; that is, those directly above the diagonal. This suggests that the non-triviality of the discriminant invariant $\mathcal{D}_\varphi$ for an endomorphism of a nilpotent group is a measure of the "asymmetry" of the embedding φ. It is an interesting problem to make this statement more precise for the general nilpotent group.

Problem 10.2. *Let Γ be a finitely generated torsion free nilpotent group, and φ a renormalization such that $\mathcal{G}_\varphi = \{\Gamma_\ell \mid \ell \geq 0\}$ has trivial intersection. Develop the relationship between the properties of the discriminant group $\mathcal{D}_\varphi$, the embedding φ, and the nilpotent structure theory of Γ, as developed for example in [14, 16].*

10.2. Algebraic invariants.

The reduced group C^*-algebra $C_r^*(X_\varphi, \Gamma, \Phi_\varphi)$ obtained from the group action $(X_\varphi, \Gamma, \Phi)$ is a source of invariants for the group Γ and the embedding φ. In the case when $\Gamma = \mathbb{Z}^n$ is free abelian, the work [27] shows that the ordered K-theory of this C^*-algebra is a complete invariant of the action. It is natural to ask whether similar results are possible in more generality:

Problem 10.3. *Let Γ be a finitely generated nilpotent group, and φ a renormalization of Γ. What information about the nilpotent structure constants of Γ and the embedding φ is determined by the K-theory groups $K_*(C_r^*(X_\varphi, \Gamma, \Phi_\varphi))$?*

Note that by Theorem 1.8, the isomorphism class of the discriminant group $\mathcal{D}_\varphi$ is an invariant of the continuous orbit equivalence class of the Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$, and the isomorphism class of $C_r^*(X_\varphi, \Gamma, \Phi_\varphi)$ is also invariant. It seems natural that these two invariants should be closely related.

Problem 10.4. *Let Γ be a renormalizable group. How does the algebraic structure of $C_r^*(X_\varphi, \Gamma, \Phi_\varphi)$ reflect the properties of the profinite group $\mathcal{D}_\varphi$?*

Theorem 5.3 shows that the profinite group $\widehat{\Gamma}_\varphi$ is a semi-direct product with $\mathcal{D}_\varphi$ as a factor. One approach to Problem 10.4 would be to relate the decomposition $\widehat{\Gamma}_\varphi \cong \widehat{N}_\varphi \rtimes \mathcal{D}_\varphi$ in Theorem 5.3 to the algebraic structure of $C_r^*(X_\varphi, \Gamma, \Phi_\varphi)$.

10.3. Realization. Given any pro-finite group $\mathcal{D}$ which is topologically countably generated, it was shown in [37, 39], using the Lenstra method, that there exists a finitely generated group Γ and Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ whose discriminant is isomorphic to $\mathcal{D}$.

Problem 10.5. *What profinite groups can be realized as the discriminant of a Cantor action associated to a renormalizable group Γ?*

10.4. Renormalizable Cantor actions. Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action of wild type; that is, the action $\widehat{\Phi} \colon \overline{\Phi(\Gamma)} \to \mathrm{Homeo}(\mathfrak{X})$ is not locally quasi-analytic. The action is said to be *wild of finite type* if, in addition, for some $x \in \mathfrak{X}$ and every adapted set with $x \in U$, the kernel of the restriction $\ker\{\widehat{\Phi}_U \colon \overline{\Phi(\Gamma)}_x \to \mathrm{Homeo}(U)\}$ is finite. Examples of wild actions constructed by the first two authors in [37] are of finite type. However, the examples in [37] are not renormalizable.

Problem 10.6. *Do there exist renormalizable Cantor actions which are wild of finite type?*

Problem 10.7. *Suppose that $(\mathfrak{X}, \Gamma, \Phi)$ is a renormalizable Cantor action which is wild. What can be said about the algebraic properties of Γ? For example, must Γ have exponential growth type? What can be said about the profinite group $\overline{\Phi(\Gamma)} \subset \mathrm{Homeo}(\mathfrak{X})$ for such actions?*

10.5. Representations of Galois groups. The works of the second author [45, 46] define the discriminant invariants associated to arboreal representations of absolute Galois groups for number fields and function fields. Such a representation is a profinite group, obtained as the inverse limit of finite Galois groups, which act on finite extensions of the ground field, obtained by adjoining the roots of the n-th iteration of the same polynomial, for $n \geq 1$.

The example given in Section 9.4 is an example of an arboreal representation of an absolute Galois group, which is isomorphic to a Cantor action associated to

a renormalization. For many polynomials the associated action is known to be not locally quasi-analytic [46] and, therefore, by Theorem 1.2 it cannot be associated to a renormalization of a group. This suggests the following problem:

Problem 10.8. *For which arboreal representations of absolute Galois groups does there exists a dense finitely generated group Γ and a renormalization $\varphi\colon \Gamma \to \Gamma$, such that the arboreal representation of Γ is return equivalent to a Cantor action associated to $(X_\varphi, \Gamma, \Phi_\varphi)$?*

Although, as discussed above, many arboreal representations are not associated to a finite-index embedding $\varphi\colon \Gamma \to \Gamma$, since they are associated to a structure built using iterations of the same polynomial, it is natural to look for a formalism similar to the non-co-Hopfian setting for the study of these groups. This motivated the definition of renormalizable actions in Section 7, and suggests the following interesting problem:

Problem 10.9. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be an equicontinuous minimal Cantor action, and suppose that $(\mathfrak{X}, \Gamma, \Phi)$ is renormalizable as in Definition 7.1. Develop a structure theory for the group obtained as the closure of the action $(\mathfrak{X}, \Gamma, \Phi)$ in $\mathrm{Homeo}(\mathfrak{X})$, analogous to Theorem 5.3.*

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