Groups Geom. Dyn. 15 (2021), 1523–1552
DOI 10.4171/GGD/638

Elementary subgroups of virtually free groups

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Abstract. We give a description of elementary subgroups (in the sense of first-order logic) of finitely generated virtually free groups. In particular, we recover the fact that elementary subgroups of finitely generated free groups are free factors. Moreover, one gives an algorithm that takes as input a finite presentation of a virtually free group G and a finite subset X of G, and decides if the subgroup of G generated by X is $\forall\exists$-elementary. We also prove that every elementary embedding of an equationally noetherian group into itself is an automorphism.

Mathematics Subject Classification (2020). Primary: 20F67; Secondary: 03C68.

Keywords. Geometric group theory, virtually free groups, hyperbolic groups, model theory, elementary embedding.

1. Introduction

A map $\varphi\colon H \to G$ between two groups H and G is *elementary* if the following condition holds: for every first-order formula $\theta(x_1, \ldots, x_k)$ with k free variables in the language of groups, and for every k-tuple $(h_1, \ldots, h_k) \in H^k$, the statement $\theta(h_1, \ldots, h_k)$ is true in H if and only if the statement $\theta(\varphi(h_1), \ldots, \varphi(h_k))$ is true in G. In particular, φ is a morphism and is injective. If one only considers a certain fragment $\mathcal{F}$ of the set of first-order formulas (for instance the set of $\forall\exists$-formulas or $\exists^+$-formulas, see paragraph 2.1 for definitions), one says that the map φ is *$\mathcal{F}$-elementary*. When H is a subgroup of G and φ is simply the inclusion of H into G, one says that H is an *elementary subgroup* of G if φ is elementary, and a *$\mathcal{F}$-elementary subgroup* of G if φ is $\mathcal{F}$-elementary.

It was proved by Sela in [21] and by Kharlampovich and Myasnikov in [11] that any non-abelian free factor of a non-abelian finitely generated free group is elementary. Later, Perin proved that the converse holds: if H is an elementary subgroup of F_n, then H is non-abelian and F_n splits as a free product $F_n = H * H'$ (see [17]). Recently, Perin gave another proof of this result (see [19]). More generally, Sela [22] and Perin [17] described elementary subgroups of torsion-free hyperbolic groups.

Our main theorem provides a characterization of ∀∃-elementary subgroups of virtually free groups. Recall that a group is said to be virtually free if it has a free subgroup of finite index. In what follows, all virtually free groups are assumed to be finitely generated and not virtually cyclic (here, and in the remainder of this paper, virtually cyclic means finite or virtually $\mathbb{Z}$). In [1], we classified virtually free groups up to ∀∃-elementary equivalence, i.e. we gave necessary and sufficient conditions for two virtually free groups to have the same ∀∃-theory. In this context, we introduced Definition 1.1 below. Recall that virtually free groups are hyperbolic, and that a non virtually cyclic subgroup N of a hyperbolic group G normalizes a unique maximal finite subgroup of G, denoted by $E_G(N)$ (see [15, Proposition 1] and Section 2.3 for further details).

Definition 1.1 (*legal large extension*). Let G be a hyperbolic group, and let H be a subgroup of G. Suppose that H is not virtually cyclic. One says that G is a *legal large extension* of H if there exists a finite subgroup C of H such that the following three conditions hold.

(1) The group G admits the following presentation:

$$G = \langle H, t \mid [t, c] = 1,\ \forall c \in C\rangle.$$

In particular, H is a hyperbolic group (see Proposition 3.3 and Remark 3.4).

(2) The normalizer $N_H(C)$ of C is not virtually cyclic.

(3) The finite group $E_H(N_H(C))$ is equal to C.

More generally, one says that G is a *multiple legal large extension* of H if there exists a finite sequence of subgroups $H = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that G_{i+1} is a legal large extension of G_i for every integer $0 \leq i \leq n-1$, with $n \geq 1$. Equivalently (see Section 2.3 for details), G is a multiple legal large extension of H if it admits a presentation of the form

$$G = \langle H, t_1, \ldots, t_n \mid [t_i, c] = 1,\ \forall c \in C_i,\ \forall i \in \{1, \ldots, n\}\rangle,$$

where $C_1, \ldots, C_n$ are finite subgroups of H such that $N_H(C_i)$ is not virtually cyclic and $E_H(N_H(C_i))$ is equal to C_i, for every integer $1 \leq i \leq n$.

In terms of graphs of groups, G is a multiple legal large extension of H if it splits as a finite graph of groups over finite groups, whose underlying graph is a rose and whose central vertex group is H, with additional assumptions on the edge groups.

A prototypical example of a multiple legal large extension is given by the splitting of the free group $G = F_k$ of rank $k \geq 3$ as $F_k = \langle F_2, t_1, \ldots, t_{k-2} \mid \varnothing\rangle$. In this example, H is the free group F_2.

Here is a more enlightening example. Take $H = \mathrm{SL}_2(\mathbb{Z})$. Recall that this group splits as an amalgamated free product $H = A *_C B$ where $A = \langle a \rangle \simeq \mathbb{Z}/4\mathbb{Z}$, $B = \langle b \rangle \simeq \mathbb{Z}/6\mathbb{Z}$ and $C = \langle c \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ with $c = a^2 = b^3$. By using the action of H on the Bass-Serre tree of this splitting, one can prove that H is virtually free (more precisely, one can prove that the derived subgroup of H, which is the kernel of the abelianization map $H \twoheadrightarrow \mathbb{Z}/12\mathbb{Z}$, is a free subgroup of index 12) and one can classify the finite subgroups of H, namely the trivial group $\{1\}$, the center $Z(H) = C$ (which is the maximal normal finite subgroup of H), and the conjugates of A, B and $\langle b^2 \rangle \simeq \mathbb{Z}/3\mathbb{Z}$. One can check that $N_H(A) = A$ and $N_H(B) = N_H(\langle b^2 \rangle) = B$. Hence, the only subgroups of H whose normalizer is not virtually cyclic are the trivial group and C. One has $N_H(\{1\}) = N_H(C) = H$, and thus $E_H(N_H(\{1\})) = E_H(N_H(C)) = C$. Therefore, the only possible legal large extension of H is $\langle H, t \mid [t, c] = 1 \rangle$, and the multiple legal large extensions of H are of the form $\langle H, t_1, \ldots, t_n \mid [t_i, c] = 1, \forall i \in \{1, \ldots, n\} \rangle$ for $n \geq 1$.

In [1, Theorem 1.10], we proved the following result.

Theorem 1.2. *Let G be a hyperbolic group, and let H be a subgroup of G. Suppose that H is not virtually cyclic. If G is a multiple legal large extension of H, then H is a $\forall\exists$-elementary subgroup of G.*

Remark 1.3. We conjectured in [1] that H is an elementary subgroup of G. For now, this conjecture is only known to be true when G is torsion-free, in which case $G = H * F_n$ for some $n \geq 1$. See [22].

Remark 1.4. In fact, the following stronger result holds: for every $\exists\forall\exists$-formula $\theta(x_1, \ldots, x_k)$ with k free variables, and for every k-tuple $(h_1, \ldots, h_k) \in H^k$, if the statement $\theta(h_1, \ldots, h_k)$ is true in H, then $\theta(\varphi(h_1), \ldots, \varphi(h_k))$ is true in G.

We shall prove that the converse of Theorem 1.2 holds, provided that G is a virtually free group.

Theorem 1.5. *Let G be a virtually free group, and let H be a $\forall\exists$-elementary proper subgroup of G (in particular, H is not virtually cyclic). Then G is a multiple legal large extension of H.*

Remark 1.6. In particular, Theorem 1.5 recovers the result proved by Perin in [17]: an elementary subgroup of a free group is a free factor.

Putting together Theorem 1.5 and Theorem 1.2, we get the following result.

Theorem 1.7. *Let G be a virtually free group, and let H be a subgroup of G. The following two assertions are equivalent:*

(1) *H is a $\forall\exists$-elementary proper subgroup of G;*

(2) *H is not virtually cyclic and G is a multiple legal large extension of H.*

In addition, we give an algorithm that decides whether or not a finitely generated subgroup of a virtually free group is a $\forall\exists$-elementary subgroup.

Theorem 1.8. *There is an algorithm that, given a finite presentation of a virtually free group G and a finite subset $X \subset G$, outputs 'Yes' if the subgroup of G generated by X is $\forall\exists$-elementary, and 'No' otherwise.*

Remark 1.9. Note that any $\forall\exists$-elementary subgroup of a virtually free group is finitely generated, as a consequence of Theorem 1.5.

Recall that every virtually free group G splits as a finite graph of finite groups (which is not unique), called a Stallings splitting of G (we refer the reader to [12] and Section 2.5 for details). The following result is an immediate consequence of Theorem 1.5.

Corollary 1.10. *Let G be a virtually free group. If the underlying graph of some* (*or, equivalently, any*) *Stallings splitting of G is a tree, then G has no proper elementary subgroup.*

For instance, the virtually free group $\mathrm{SL}_2(\mathbb{Z})$, which is isomorphic to the product $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$, has no proper elementary subgroup.

Last, let us mention another interesting consequence of Theorem 1.5: if an endomorphism φ of a virtually free group G is $\forall\exists$-elementary, then φ is an automorphism. Indeed, note that $\varphi(G)$ is a $\forall\exists$-elementary subgroup of G, and let us prove that $\varphi(G) = G$. Assume towards a contradiction that $\varphi(G)$ is a proper subgroup of G. It follows from Theorem 1.5 that G is a multiple legal large extension of $\varphi(G)$. Hence, there exists an integer $n \geq 1$ such that the abelianizations of G and $\varphi(G)$ satisfy $G^{\mathrm{ab}} = \varphi(G)^{\mathrm{ab}} \times \mathbb{Z}^n$. But $\varphi(G)$ is isomorphic to G since φ is injective, hence $G^{\mathrm{ab}} \simeq G^{\mathrm{ab}} \times \mathbb{Z}^n$, which contradicts the fact that n is non-zero and G is finitely generated. Thus, one has $\varphi(G) = G$ and φ is an automorphism.

In fact, the same result holds for torsion-free hyperbolic groups: if $\varphi\colon G \to G$ is $\forall\exists$-elementary, then G is a hyperbolic tower in the sense of Sela over $\varphi(G) \simeq G$ (see [17]). By definition of a hyperbolic tower, $\varphi(G)$ is a quotient of G. Since torsion-free hyperbolic groups are Hopfian by [20], $\varphi(G) = G$ and φ is an automorphism.

We shall prove the following result, which generalizes the previous observation. Recall that a group is said to be *equationally noetherian* if every infinite system of equations Σ in finitely many variables is equivalent to a finite subsystem of Σ.

Theorem 1.11. *Let G be a finitely generated group. Suppose that G is equationally noetherian, or finitely presented and Hopfian. Then, every $\exists^+$-endomorphism of G is an automorphism.*

Remark 1.12. Note that $\forall\exists$-elementary morphisms are *a fortiori* $\exists^+$-elementary. Note also that, contrary to $\forall\exists$-elementary morphisms, $\exists^+$-elementary morphisms are not injective in general.

As a consequence, by Proposition 2 in [14], a finitely generated group G satisfying the hypotheses of Theorem 1.11 above is (strongly) defined by types, and even by $\exists^+$-types, meaning that G is characterized among finitely generated groups, up to isomorphism, by the set $\mathrm{tp}_{\exists^+}(G)$ of all $\exists^+$-types of tuples of elements of G. In particular, Theorem 1.11 answers positively Problem 4 posed in [14] and recovers several results proved in [14].

2. Preliminaries

2.1. First-order logic. For detailed background on first-order logic, we refer the reader to [13]. The language of groups uses the following symbols: the quantifiers $\forall$ and $\exists$, the logical connectors $\wedge$, $\vee$, $\Rightarrow$, the equality and inequality relations $=$ and $\neq$, the symbols 1 (standing for the identity element), $^{-1}$ (standing for the inverse), $\cdot$ (standing for the group multiplication), parentheses (and), and variables $x, y, g, z \ldots$, which are to be interpreted as elements of a group. The *terms* are words in the variables, their inverses, and the identity element (for instance, $x \cdot y \cdot x^{-1} \cdot y^{-1}$ is a term). For convenience, we omit group multiplication. A *first-order formula* is made from terms iteratively: one can first make *atomic formulas* by comparing two terms by means of the symbols $=$ and $\neq$ (for instance, $xyx^{-1}y^{-1} = 1$ is an atomic formula), then one can use logical connectors and quantifiers to make new formulas from old formulas, for instance $\exists x((x \neq 1) \wedge (\forall y(xyx^{-1}y^{-1} = 1)))$. We sometimes drop parentheses when there is no ambiguity. A variable is *free* if it is not bound by any quantifier $\forall$ or $\exists$. A *sentence* is a formula without free variables. Given a formula $\varphi(x_1, \ldots, x_n)$, a group G and a tuple $(g_1, \ldots, g_n) \in G^n$, one says that G satisfies $\varphi(g_1, \ldots, g_n)$ if this statement is true in the usual sense when the variables are interpreted as elements of G.

If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are first-order formulas in the language of groups with free variables $x_1, \ldots, x_n$, we say that $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are *logically equivalent* if for every group G and every n-tuple $(g_1, \ldots, g_n) \in G^n$, the statement $\varphi(g_1, \ldots, g_n)$ is true in G if and only if the statement $\psi(g_1, \ldots, g_n)$ is true in G. Every formula in the language of groups is logically equivalent to a formula in *prenex normal form* (PNF), that is a formula written as a string of quantifiers and bound variables, followed by a quantifier-free part; moreover, one can assume without loss of generality that the quantifier-free part is a disjunction of conjunctions of equations and inequations. Hence, every formula is logically equivalent to a formula of the following form:

$$Q_1 x_1 \ldots Q_n x_n \bigvee_{i=1}^{p} \bigwedge_{j=1}^{q_i} w_{i,j}(x_1, \ldots, x_n, x_{n+1} \ldots, x_m) \varepsilon_i 1,$$

where each Q_i is a quantifier $\forall$ or $\exists$, each ε_i denotes $=$ or $\neq$, and $w_{i,j}$ is a reduced word in the variables $x_1, \ldots, x_m$ and their inverses. The variables $x_{n+1}, \ldots, x_m$ are free.

An *existential formula* (or $\exists$-*formula*) is a formula in PNF in which the symbol $\forall$ does not appear. An *existential positive formula* (or $\exists^+$-*formula*) is a formula in PNF in which the symbols $\forall$ and $\neq$ do not appear. A $\forall\exists$-*formula* is a formula of the form $\varphi(\boldsymbol{x})$: $\forall \boldsymbol{y} \exists \boldsymbol{z}\ \theta(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ where $\theta(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a disjunction of conjunctions of equations and inequations in the variables of the tuples $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. We define $\exists\forall\exists$-formulas in the same way, and so on.

The *existential theory* of a group G, denoted by $\mathrm{Th}_{\exists}(G)$, is the set of first-order sentences that are logically equivalent to an existential formula satisfied by G. We define similarly the *existential positive theory* $\mathrm{Th}_{\exists}^{+}(G)$ of G, and the $\forall\exists$-*theory* $\mathrm{Th}_{\forall\exists}(G)$ of G.

2.2. Hyperbolic groups. In this section, we collect some basic facts about hyperbolic groups and their boundaries that will be useful in the proof of our main result.

Roughly speaking, a *hyperbolic space* is a geodesic metric space (X, d) where all geodesic triangles are thin (see for instance [10, Definition 2.1]). We say that two geodesic rays in X starting at a base-point are equivalent if they remain close to each other, and we define the boundary $\partial_\infty X$ as the set of equivalence classes of geodesic rays starting at a base-point (see [10, Definition 2.7] for a precise definition). The boundary is equipped with the compact-open topology: two rays are "close at infinity" if they stay Hausdorff-close for a long time (see [10, Definition 2.12]).

Recall that a finitely generated group G is *hyperbolic* if for some (equivalently, any) finite generating set S of G, the Cayley graph $\Gamma(G, S)$ equipped with the word metric d_S is hyperbolic. Finitely generated free groups are typical examples of hyperbolic groups. Since hyperbolicity is preserved under quasi-isometry, and since finitely generated groups are quasi-isometric to their finite-index subgroups, finitely generated virtually free groups are hyperbolic.

We define the *boundary* of G by $\partial_\infty G = \partial_\infty \Gamma(G, S)$. Up to homeomorphism, the boundary $\partial_\infty G$ does not depend on the choice of the finite generating set S, since a change of finite generating sets induces a quasi-isometry between the corresponding Cayley graphs and since two quasi-isometric proper hyperbolic spaces have homeomorphic boundaries (see [10, Propostion 2.20 and Definition 2.21]). The group G naturally acts by isometries on its Cayley graph $\Gamma(G, S)$, and this action extends to an action of G on its boundary $\partial_\infty G$ by homeomorphisms (see [10, Propostion 2.20]). This action turns out to be an extremely useful tool for studying the group G. If an element $g \in G$ has infinite order, it fixes exactly two distinct points of $\partial_\infty G$ denoted by g^+ and g^- (see [10, Propostion 4.2]). The stabilizer of the pair $\{g^+, g^-\}$ is the unique maximal virtually cyclic subgroup of G containing g. We denote this subgroup by $M_G(g)$. If h and g are two elements of infinite order, either $M_G(h) = M_G(g)$ or $M_G(h) \cap M_G(g)$ is finite; in the latter case, the subgroup $\langle h, g\rangle$ is not virtually cyclic (i.e. contains a free subgroup of rank 2).

Let K_G denote the maximum order of an element of G of finite order. An element $g \in G$ has infinite order if and only if $g^{K_G!}$ is non-trivial. The following lemma will be useful.

Lemma 2.1. *Let g be an element of G of infinite order.*

(1) *An element $h \in G$ belongs to $M_G(g)$ if and only if the commutator*

$$[g^{K_G!}, hg^{K_G!}h^{-1}]$$

is trivial.

(2) *Let $h \in G$ be an element of infinite order. The following assertions are equivalent*:

  (a) *h belongs to $M_G(g)$,*

  (b) *$M_G(h) = M_G(g)$,*

  (c) *the commutator $[g^{K_G!}, h^{K_G!}]$ is trivial.*

Hence, the subgroup $\langle g, h\rangle$ is virtually cyclic if and only if $[g^{K_G!}, h^{K_G!}] = 1$.

Proof. We only prove the first point, the proof of the second point is similar. If h belongs to $M_G(g)$, then hgh^{-1} belongs to $M_G(g)$. Therefore, $g^{K_G!}$ and $(hgh^{-1})^{K_G!}$ commute, since $M_G(g)$ has a cyclic subgroup of index $\leq K_G$. Conversely, if $g^{K_G!}$ and $hg^{K_G!}h^{-1}$ commute, then $hg^{K_G!}h^{-1}$ fixes the pair of points $\{g^+, g^-\}$. Thus, h fixes $\{g^+, g^-\}$ as well. It follows that h belongs to $M_G(g)$. □

If N is a subgroup of G that is not virtually cyclic, then there exists a unique maximal finite subgroup $E_G(N)$ of G normalized by N. The following fact was proved by Ol'shanskiĭ.

Proposition 2.2 ([15] Proposition 1). *The finite subgroup $E_G(N)$ admits the following description*:

$$E_G(N) = \bigcap_{g \in N^0} M_G(g)$$

where N^0 denotes the set of elements of N of infinite order.

We will need the following easy lemma.

Lemma 2.3. *Let H be a hyperbolic group and let C be a finite subgroup of H. Define $G = \langle H, t \mid [t, c] = 1,\ \forall c \in C\rangle$. Let N be a subgroup of H and suppose that N is not virtually cyclic. Then $E_H(N) = E_G(N)$.*

Remark 2.4. In the next section, we will use this lemma with $N = N_H(C)$.

Proof. First, we prove that for every element $h \in N^0$, the subgroups $M_H(h)$ and $M_G(h)$ coincide. Note that the inclusion $M_H(h) \subset M_G(h)$ is obvious. Conversely, let g be an element of $M_G(h)$. Assume towards a contradiction that g does not belong to H. Then it can be written in normal form: $g = h_0 t^{\varepsilon_1} h_1 t^{\varepsilon_2} h_2 \cdots t^{\varepsilon_n} h_n$ for some integer $n \geq 1$, with $h_i \in H$, $\varepsilon_i \in \{\pm 1\}$ and $h_i \notin C$ if $\varepsilon_i = -\varepsilon_{i+1}$. By Lemma 2.1, one has $[h^{K_G!}, g h^{K_G!} g^{-1}] = 1$. By replacing g with its normal form in this equality, one gets:

$$(h^{K_G!} h_0) t^{\varepsilon_1} \cdots t^{\varepsilon_n} (h_n h^{K_G!} h_n^{-1}) t^{-\varepsilon_n} \cdots t^{-\varepsilon_1} (h_0^{-1} h^{-K_G!} h_0) t^{\varepsilon_1} \cdots$$
$$t^{\varepsilon_n} (h_n h^{-K_G!} h_n^{-1}) t^{-\varepsilon_n} \cdots h_0^{-1} = 1.$$

It follows from Britton's lemma for HNN extensions that this equality is possible only if $h_n h^{K_G!} h_n^{-1}$ belongs to C or $h_0^{-1} h^{-K_G!} h_0$ belongs to C. This is not possible because h has infinite order (as an element of N^0) and C is finite by assumption. This is a contradiction, and thus g belongs to H. Hence, $M_H(h)$ coincides with $M_G(h)$. By Proposition 2.2, one has $E_H(N) = E_G(N)$. □

2.3. Multiple legal large extensions. In Definition 1.1, we first defined a multiple legal large extension as a "tower" of legal large extensions, then we reformulated this definition by means of a presentation by generators and relations, and we claimed that these two points of view are equivalent. The purpose of this section is to prove this claim.

Lemma 2.5. *Let G be a hyperbolic group, and let H be a subgroup of G. The following two assertions are equivalent.*

(1) *There exists a finite sequence of subgroups $H = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that G_{i+1} is a legal large extension of G_i for every integer $0 \leq i \leq n-1$, with $n \geq 1$.*

(2) *The group G admits a presentation of the form*

$$G = \langle H, t_1, \ldots, t_n \mid [t_i, c] = 1,\ \forall c \in C_i,\ \forall i \in \{1, \ldots, n\} \rangle,$$

where $C_1, \ldots, C_n$ are finite subgroups of H such that $N_H(C_i)$ is not virtually cyclic and $E_H(N_H(C_i))$ is equal to C_i, for every integer $1 \leq i \leq n$.

Proof. We first prove the implication (2) ⇒ (*1*). Define $G_0 = H$, and for every $1 \leq i \leq n$ define G_i as the subgroup of G generated by H and $t_1, \ldots, t_i$. Note that G_{i+1} can be written as $\langle G_i, t_{i+1} \mid [t_{i+1}, c] = 1,\ \forall c \in C_{i+1} \rangle$, for every $0 \leq i \leq n-1$. Observe that $N_{G_i}(C_{i+1})$ is not virtually cyclic since it contains $N_H(C_{i+1})$, which is not virtually cyclic by assumption. We claim that the equality

$E_{G_i}(N_{G_i}(C_{i+1})) = C_{i+1}$ holds. Note that the inclusion $E_{G_i}(N_{G_i}(C_{i+1})) \supset C_{i+1}$ is obvious since C_{i+1} is normalized by its normalizer. Let us prove that the converse inclusion holds. First, note that $E_{G_i}(N_{G_i}(C_{i+1}))$ is contained in $E_{G_i}(N_H(C_{i+1}))$, because $N_H(C_{i+1})$ is contained in $N_{G_i}(C_{i+1})$. By Lemma 2.3 (applied i times), one has $E_{G_i}(N_H(C_{i+1})) = E_H(N_H(C_{i+1}))$. By assumption, $E_H(N_H(C_{i+1}))$ is equal to C_{i+1}, and thus one has $E_{G_i}(N_{G_i}(C_{i+1})) \subset C_{i+1}$, which concludes the proof of the implication (*2*) $\Rightarrow$ (*1*).

Now, let us prove that (*2*) follows from (*1*). We will only prove this implication for $n = 2$, and the general case can be proved in exactly the same way.

First, note that every finite subgroup C_2 of $G_1 = \langle H, t_1 \mid [t_1, c] = 1,\ \forall c \in C_1 \rangle$ is conjugate to a subgroup of H. Indeed, as a finite group, C_2 fixes a point in the Bass-Serre tree T of the splitting $\langle H, t \mid [t_1, c] = 1,\ \forall c \in C_1 \rangle$ of G_1. Since all vertices of T are translates of the unique vertex v_H of T fixed by H, there exists an element $g \in G_1$ such that C_2 fixes gv_H, i.e. such that $g^{-1}C_2 g$ is a subgroup of H. As a consequence, a legal large extension $G_2 = \langle G_1, t_2 \mid [t_2, c] = 1,\ \forall c \in C_2 \rangle$ of G_1 can be written without loss of generality as $G_2 = \langle H, t_1, t_2 \mid [t_i, c] = 1,\ \forall c \in C_i,\ \forall i \in \{1, 2\} \rangle$, up to replacing C_2 with $g^{-1}C_2 g$ and t_2 with $g^{-1}t_2 g$.

It remains to prove that $N_H(C_2)$ is not virtually cyclic, and that $E_H(N_H(C_2)) = C_2$. Note that $N_{G_1}(C_2)$ is not virtually cyclic, and that $E_{G_1}(N_{G_1}(C_2)) = C_2$ since G_2 is a legal large extension of G_1. We distinguish two cases.

First case: suppose that C_2 is not conjugate to a subgroup of C_1. We will prove that $N_H(C_2) = N_{G_1}(C_2)$ (and thus $N_H(C_2)$ is not virtually cyclic). The inclusion $N_H(C_2) \subset N_{G_1}(C_2)$ is obvious since $N_H(C_2) = H \cap N_{G_1}(C_2)$; let us prove that the converse inclusion holds. Let g be an element of $N_{G_1}(C_2)$, and prove that g fixes v_H, i.e. that g belongs to H. Note that C_2 is contained both in H and in gHg^{-1}. Assume towards a contradiction that $gv_H \neq v_H$, then C_2 is contained in the stabilizer of the path in T between v_H and gv_H; since all edge groups of T are conjugates of C_1, the group C_2 is contained in a conjugate of C_1. This is a contradiction. Now, prove that $E_H(N_H(C_2)) = C_2$. Since $N_H(C_2) = N_{G_1}(C_2)$, one has $E_H(N_H(C_2)) = E_H(N_{G_1}(C_2))$. Moreover, $E_H(N_{G_1}(C_2))$ is obviously contained in $E_{G_1}(N_{G_1}(C_2)) = C_2$. Hence, one has $E_H(N_H(C_2)) \subset C_2$. The converse inclusion holds by definition of $E_H(N_H(C_2))$.

Second case: suppose that C_2 is conjugate to a subgroup of C_1. One can suppose without loss of generality that C_2 is contained in C_1. First, note that $N_H(C_2)$ is not virtually cyclic; indeed, it contains the centralizer $Z_H(C_1)$ of C_1 in H, which is not virtually cyclic as a finite-index subgroup of $N_H(C_1)$. It remains to prove that $E_H(N_H(C_2)) = C_2$. Let $n \geq 1$ be an integer such that, for every $h \in N_H(C_1)$, the element h^n centralizes C_1 (and thus centralizes C_2). Note that $M_H(h) = M_H(h^n)$ for h of infinite order (see Lemma 2.1). By Proposition 2.2, the following holds:

$$C_1 = E_H(N_H(C_1)) = \bigcap_{h \in N_H(C_1)^0} M_H(h^n) \supset \bigcap_{h \in N_H(C_2)^0} M_H(h) = E_H(N_H(C_2)).$$

Hence, $E_H(N_H(C_2))$ is contained in C_1. It follows that $E_H(N_H(C_2))$ is normalized both by $N_H(C_2)$ (by definition) and by t_1 (as a subgroup of C_1). Therefore, $E_H(N_H(C_2))$ is normalized by $N_{G_1}(C_2)$, which is generated by $N_H(C_2)$ and t_1. As a consequence, $E_H(N_H(C_2))$ is contained in $E_{G_1}(N_{G_1}(C_2)) = C_2$. Conversely, $C_2 \subset E_H(N_H(C_2))$ by definition, and thus $E_H(N_H(C_2)) = C_2$. □

2.4. Properties relative to a subgroup. Let G be a finitely generated group, and let H be a subgroup of G.

Definition 2.6. An *action* of the pair (G, H) on a simplicial tree T is an action of G on T such that H fixes a point of T. We always assume that the action is *minimal*, which means that there is no proper subtree of T invariant under the action of G. The quotient graph of groups T/G (or sometimes the tree T itself), which is finite since the action is minimal, is called a *splitting of* (G, H), or a *splitting of* G *relative to* H. The action is said to be *trivial* if G fixes a point of T.

Definition 2.7. We say that G is *one-ended relative to* H if G does not split as an amalgamated product $A *_C B$ or as an HNN extension $A*_C$ such that C is finite and H is contained in a conjugate of A or B. In other words, G is one-ended relative to H if any action of the pair (G, H) on a simplicial tree with finite edge stabilizers is trivial.

Definition 2.8. The group G is said to be *co-Hopfian relative to* H if every monomorphism $\varphi: G \hookrightarrow G$ that coincides with the identity on H is an automorphism of G.

The following result was first proved by Sela in [20] for torsion-free one-ended hyperbolic groups, with H trivial.

Theorem 2.9 (see [2] Theorem 2.31). *Let G be a hyperbolic group, let H be a subgroup of G. Assume that G is one-ended relative to H. Then G is co-Hopfian relative to H.*

Remark 2.10. In [2], this result is stated and proved under the assumption that H is finitely generated. However, Lemma 3.2 below shows that this hypothesis is not necessary.

2.5. Relative Stallings splittings. A splitting of a group G is said to be *reduced* if the following holds: if $e = [v, w]$ is an edge in the Bass-Serre tree of the splitting such that $G_e = G_v = G_w$, then v and w are in the same G-orbit.

Let G be a finitely generated group. Under the hypothesis that there exists a constant K such that every finite subgroup of G has order at most K, Linnell proved in [12] that G splits as a finite graph of groups with finite edge groups and all of whose vertex groups are finite or one-ended. Such a splitting is called a *Stallings splitting* of G. It is not unique in general, but the conjugacy classes of one-ended vertex groups do not depend on the splitting (in other words, the G-orbits of one-ended vertex groups are the same in all Bass-Serre trees of Stallings splittings of G). In addition, the conjugacy classes of finite vertex groups do not depend on a given reduced Stallings splittings of G (in other words, the G-orbits of finite vertex groups are the same in all Bass-Serre trees of reduced Stallings splittings of G). A one-ended subgroup of G that appears as a vertex group of a Stallings splitting is called a *one-ended factor* of G.

Recall that there is a uniform bound on the order of a finite subgroup of a hyperbolic group, and thus the aforementioned result of Linnell applies to hyperbolic groups. Moreover, G is virtually free if and only if all vertex groups in some (equivalently, any) Stallings splitting of G are finite. For instance, as discussed in the introduction, the group $\mathrm{SL}_2(\mathbb{Z})$ splits as a graph of groups with exactly two vertex groups of order 4 and 6 respectively, and one edge group of order 2.

Given a subgroup H of G, Linnell's result can be generalized as follows: the pair (G, H) splits as a finite graph of groups with finite edge groups such that each vertex group is finite or one-ended relative to a conjugate of H. Such a splitting is called a *Stallings splitting of G relative to H*. It is not unique in general. However, if H is infinite, then there exists a unique vertex group containing H and this group does not depend on the splitting. This vertex group is called the *one-ended factor of G relative to H*. Moreover, the G-orbits of one-ended vertex groups that do not contain a conjugate of H are the same in all Bass-Serre trees of Stallings splittings of G relative to H, and the G-orbits of finite vertex groups are the same in all Bass-Serre trees of reduced Stallings splittings of G relative to H. For further details, we refer the reader to [9, Section 3.3].

Note that if G is infinite and if the finite subgroups of G are of order at most K, then every $\exists$-elementary subgroup of G is infinite since it satisfies the sentence $\exists x\ (x^{K!} \neq 1)$. As a consequence, in the context of Theorem 1.5, the one-ended factor of G relative to H is well-defined.

2.6. The JSJ decomposition and the modular group. Let us denote by $\mathcal{Z}$ the class of groups that are either finite or virtually cyclic with infinite center. Let G be a hyperbolic group, and let H be a subgroup of G. Suppose that G is one-ended relative to H. In [9], Guirardel and Levitt construct a splitting of G relative to H called the canonical JSJ splitting of G over $\mathcal{Z}$ relative to H. In what follows, we refer to this decomposition as the *$\mathcal{Z}$-JSJ splitting of G relative to H*. This tree T enjoys particularly nice properties and is a powerful tool for studying the pair (G, H). Before giving a description of T, let us recall briefly some basic facts about hyperbolic 2-dimensional orbifolds.

A compact connected 2-dimensional orbifold with boundary $\mathcal{O}$ is said to be *hyperbolic* if it is equipped with a hyperbolic metric with totally geodesic boundary. It is the quotient of a closed convex subset $C \subset \mathbb{H}^2$ by a proper discontinuous group of isometries $G_{\mathcal{O}} \subset \mathrm{Isom}(\mathbb{H}^2)$. We denote by $p: C \to \mathcal{O}$ the quotient map. By definition, the orbifold fundamental group $\pi_1(\mathcal{O})$ of $\mathcal{O}$ is $G_{\mathcal{O}}$. We may also view $\mathcal{O}$ as the quotient of a compact orientable hyperbolic surface with geodesic boundary by a finite group of isometries. A point of $\mathcal{O}$ is *singular* if its preimages in C have non-trivial stabilizer. A *mirror* is the image by p of a component of the fixed point set of an orientation-reversing element of $G_{\mathcal{O}}$ in C. Singular points not contained in mirrors are *conical points*; the stabilizer of the preimage in $\mathbb{H}^2$ of a conical point is a finite cyclic group consisting of orientation-preserving maps (rotations). The orbifold $\mathcal{O}$ is said to be *conical* if it has no mirror. For instance, $\langle a, b, c, d, x, y \mid [a, b][c, d]x^7 y = 1\rangle$ is the orbifold fundamental group of the orbifold whose underlying surface is orientable of genus 2 with one boundary component, and with one conical point of order 7.

Definition 2.11. A group G is called a *finite-by-orbifold group* if it is an extension

$$1 \longrightarrow F \longrightarrow G \longrightarrow \pi_1(\mathcal{O}) \longrightarrow 1$$

where $\mathcal{O}$ is a compact connected hyperbolic conical 2-orbifold, possibly with (totally geodesic) boundary, and F is an arbitrary finite group called the *fiber*. We call an *extended boundary subgroup* of G the preimage in G of a boundary subgroup of the orbifold fundamental group $\pi_1(\mathcal{O})$ (for an indifferent choice of regular base point). We define in the same way *extended conical subgroups*.

Definition 2.12. A vertex v of a graph of groups is said to be *quadratically hanging* (denoted by *QH*) if its stabilizer G_v is a finite-by-orbifold group $1 \to F \to G \to \pi_1(\mathcal{O}) \to 1$ such that $\mathcal{O}$ has non-empty boundary, and such that any incident edge group is finite or contained in an extended boundary subgroup of G. We also say that G_v is QH.

Definition 2.13. Let G be a one-ended hyperbolic group, and let H be a finitely generated subgroup of G. Let T be the $\mathcal{Z}$-JSJ decomposition of G relative to H. A vertex group G_v of T is said to be *rigid* if it is elliptic in every splitting of G over $\mathcal{Z}$ relative to H.

The following proposition is crucial (see Section 6 of [9], Theorem 6.5 and the paragraph below Remark 9.29). We keep the same notations as in the previous definition.

Proposition 2.14. *If G_v is not rigid, i.e. if it fails to be elliptic in some splitting of G over $\mathcal{Z}$ relative to H, then G_v is quadratically hanging.*

Proposition 2.15 below summarizes the properties of the $\mathbb{Z}$-JSJ splitting relative to H that are useful in the proof of Theorem 1.5.

Proposition 2.15. *Let G be a hyperbolic group, and let H be a subgroup of G. Suppose that G is one-ended relative to H. Let T be its $\mathbb{Z}$-JSJ decomposition relative to H.*

- *The tree T is bipartite: every edge joins a vertex carrying a maximal virtually cyclic group to a vertex carrying a non virtually cyclic group.*
- *The action of G on T is acylindrical in the following strong sense: if an element $g \in G$ of infinite order fixes a segment of length ≥ 2 in T, then this segment has length exactly 2 and its midpoint has virtually cyclic stabilizer.*
- *Let v be a vertex of T, and let e, e' be two distinct edges incident to v. If G_v is not virtually cyclic, then the group $\langle G_e, G_{e'} \rangle$ is not virtually cyclic.*
- *If v is a QH vertex of T, every edge group G_e of an edge e incident to v coincides with an extended boundary subgroup of G_v. Moreover, given any extended boundary subgroup B of G_v, there exists a unique incident edge e such that $G_e = B$.*
- *The subgroup H is contained in a rigid vertex group.*

Remark 2.16. The rigid vertex group containing H may be QH. For instance, let $G = \langle a, b, c, d \mid [a, b] = [c, d] \rangle$ be the fundamental group of the closed orientable surface of genus 2. Let H be the subgroup of G generated by a, b, and consider the splitting $H *_{[a,b]=[c,d]} \langle c, d \rangle$ of G. This splitting is the $\mathbb{Z}$-JSJ splitting of G relative to H, and the vertex group containing H (namely H itself) is both QH and rigid relative to H.

Definition 2.17. Let G be a hyperbolic group and let H be a subgroup of G. Suppose that G is one-ended relative to H. We denote by $\mathrm{Aut}_H(G)$ the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms whose restriction to H is the conjugacy by an element of G. The *modular group* $\mathrm{Mod}_H(G)$ of G relative to H is the subgroup of $\mathrm{Aut}_H(G)$ consisting of all automorphisms σ satisfying the following conditions:

- the restriction of σ to each rigid or virtually cyclic vertex group of the $\mathbb{Z}$-JSJ splitting of G relative to H coincides with the conjugacy by an element of G;
- the restriction of σ to each finite subgroup of G coincides with the conjugacy by an element of G;
- σ acts trivially on the underlying graph of the $\mathbb{Z}$-JSJ splitting relative to H.

Remark 2.18. In [18], the relative modular group fixes pointwise the subgroup H, whereas we allow conjugation. As a consequence, our modular group is bigger than the one defined in [18], and thus all results of [18] claiming the existence of a modular automorphism are true *a fortiori* with our definition of $\mathrm{Mod}_H(G)$.

We will need the following result.

Theorem 2.19. *Let G be a hyperbolic group, let H be an infinite subgroup of G and let U be the one-ended factor of G relative to H. There exist a finite subset $F \subset U \setminus \{1\}$ and a finitely generated subgroup $H' \subset H$ such that, for every non-injective homomorphism $\varphi\colon U \to G$ that coincides with the identity on H' up to conjugation, there exists an automorphism $\sigma \in \mathrm{Mod}_H(U)$ such that* $\ker(\varphi \circ \sigma) \cap F \neq \varnothing$.

Proof. This result is stated and proved in [2] under the assumption that H is finitely generated (see Theorem 2.32), in which case one can take $H' = H$. We only give a brief sketch of how the proof can be adapted if H is not assumed to be finitely generated. In [2], the assumption that H is finitely generated is only used in the proof of Proposition 2.27 in order to ensure that the group H fixes a point in a certain real tree T with virtually cyclic arc stabilizers (namely the tree obtained by rescaling the metric of a Cayley graph of G by a given sequence of positive real numbers going to infinity). Let $\{h_1, h_2, \ldots\}$ be a generating set for H, and let H_n be the subgroup of H generated by $\{h_1, \ldots, h_n\}$. If H is not finitely generated, then there exists an integer n_0 such that, for all $n \geq n_0$, the subgroup H_n is not virtually cyclic. It follows that all H_n fix the same point of T for $n \geq n_0$, which proves that H is elliptic in T. Hence, one can just take $H' = H_{n_0}$. □

2.7. Related homomorphisms and Preretractions. We denote by $\mathrm{ad}(g)$ the inner automorphism $h \mapsto ghg^{-1}$.

Definition 2.20 (related homomorphisms). Let G be a hyperbolic group and let H be a subgroup of G. Assume that G is one-ended relative to H. Let G' be a group. Let Λ be the $\mathcal{Z}$-JSJ splitting of G relative to H. Let φ and φ' be two homomorphisms from G to G'. We say that φ and φ' are *$\mathcal{Z}$-JSJ-related* or *Λ-related* if the following two conditions hold:

- for every vertex v of Λ such that G_v is rigid or virtually cyclic, there exists an element $g_v \in G'$ such that

$$\varphi'_{|G_v} = \mathrm{ad}(g_v) \circ \varphi_{|G_v};$$

- for every finite subgroup F of G, there exists an element $g \in G'$ such that

$$\varphi'_{|F} = \mathrm{ad}(g) \circ \varphi_{|F}.$$

Remark 2.21. Note that in [18, Definition 5.15], an additional assumption is made about the QH vertex groups. This technical assumption is not required in our paper.

Definition 2.22 (preretraction). Let G be a hyperbolic group, and let H be a subgroup of G. Assume that G is one-ended relative to H. Let Λ be the $\mathbb{Z}$-JSJ splitting of G relative to H. A *$\mathbb{Z}$-JSJ-preretraction* or *Λ-preretraction* of G is an endomorphism of G that is Λ-related to the identity map. More generally, if G is a subgroup of a group G', a preretraction from G to G' is a homomorphism Λ-related to the inclusion of G into G'. Note that a Λ-preretraction coincides with a conjugacy on H, since H is contained in a rigid vertex group of Λ.

The following easy lemma shows that being Λ-related can be expressed in first-order logic. This lemma is stated and proved in [2] (see Lemma 2.22) under the assumption that H is finitely generated, but this hypothesis is not used in the proof.

Lemma 2.23. *Let G be a hyperbolic group and let H be a subgroup of G. Assume that G is one-ended relative to H. Let G' be a group. Let Λ be the $\mathbb{Z}$-JSJ splitting of G relative to H. Let $\{g_1, \ldots, g_n\}$ be a generating set of G. There exists an existential formula $\theta(x_1, \ldots, x_{2n})$ with $2n$ free variables such that, for every $\varphi, \varphi' \in \mathrm{Hom}(G, G')$, φ and φ' are Λ-related if and only if G' satisfies $\theta(\varphi(g_1), \ldots, \varphi(g_n), \varphi'(g_1), \ldots, \varphi'(g_n))$.*

The proof of the following lemma is identical to that of Proposition 7.2 in [3].

Lemma 2.24. *Let G be a hyperbolic group. Suppose that G is one-ended relative to a subgroup H. Let Λ be the $\mathbb{Z}$-JSJ splitting of G relative to H. Let φ be a Λ-preretraction of G. If φ sends every QH vertex group of Λ isomorphically to a conjugate of itself, then φ is injective.*

2.8. Centered graph of groups

Definition 2.25 (centered graph of groups). A graph of groups over $\mathbb{Z}$, with at least two vertices, is said to be *centered* if the following conditions hold:

- the underlying graph is bipartite, with a particular QH vertex v such that every vertex different from v is adjacent to v;
- every stabilizer G_e of an edge incident to v coincides with an extended boundary subgroup or with an extended conical subgroup of G_v (see Definition 2.11);
- given any extended boundary subgroup B, there exists a unique edge e incident to v such that G_e is conjugate to B in G_v;
- if an element of infinite order fixes a segment of length ≥ 2 in the Bass-Serre tree of the splitting, then this segment has length exactly 2 and its endpoints are translates of v.

The vertex v is called *the central vertex*.

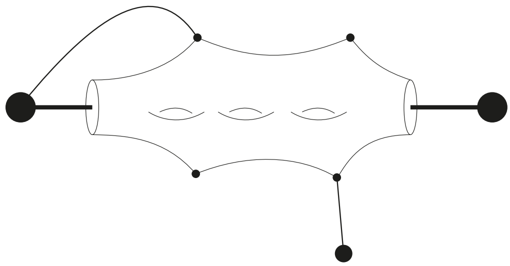

Figure 1. A centered graph of groups. Edges with infinite stabilizer are depicted in bold.

We also need to define relatedness and preretractions in the context of centered graphs of groups.

Definition 2.26 (Δ-related homomorphisms). Let G and G' be two groups. Let H be a subgroup of G. Suppose that G has a centered splitting Δ, with central vertex v. Suppose that H is contained in a non-central vertex of Δ. Let φ and φ' be two homomorphisms from G to G'. We say that φ and φ' are Δ-*related* (relative to H) if the following two conditions hold:

- for every vertex $w \neq v$, there exists an element $g_w \in G'$ such that

$$\varphi'_{|G_w} = \mathrm{ad}(g_w) \circ \varphi_{|G_w};$$

- for every finite subgroup F of G, there exists an element $g \in G'$ such that

$$\varphi'_{|F} = \mathrm{ad}(g) \circ \varphi_{|F}.$$

Definition 2.27 (Δ-preretraction). Let G be a hyperbolic group, let H be a subgroup of G, and let Δ be a centered splitting of G. Let v be the central vertex of Δ. Suppose that H is contained in a non-central vertex of Δ. An endomorphism φ of G is called a Δ-*preretraction* (relative to H) if it is Δ-related to the identity of G in the sense of the previous definition. A Δ-preretraction is said to be *non-degenerate* if it does not send G_v isomorphically to a conjugate of itself.

3. Elementary subgroups of virtually free groups

In this section, we prove Theorem 1.5. Recall that this theorem claims that if G is a virtually free group and H is a proper $\forall\exists$-elementary subgroup of G, then G is a multiple legal large extension of H.

3.1. Elementary subgroups are one-ended factors. As a first step, we will prove the following result.

Proposition 3.1. *Let G be a virtually free group. Let H be a subgroup of G. If H is $\forall\exists$-elementary, then H coincides with the one-ended factor of G relative to H. In other words, H appears as a vertex group in a splitting of G over finite groups.*

The proof of Proposition 3.1, which is inspired from [17], consists in showing that if G is a hyperbolic group and H is strictly contained in the one-ended factor of G relative to H, then there exists a centered splitting Δ of G relative to H, and a non-degenerate Δ-preretraction of G (see Lemmas 3.5 and 3.7 below). However, if G is virtually free, Lemma 3.8 below shows that such a Δ-preretraction cannot exist.

We shall prove Proposition 3.1 after establishing a series of preliminary lemmas. The following result is a generalization of Lemma 4.20 in [17]. Recall that all group actions on trees considered in this paper are assumed to be minimal (see Definition 2.6). As a consequence, trees have no vertex of valence 1. We say that a tree T endowed with an action of a group G is *non-redundant* if there exists no valence 2 vertex v such that both boundary monomorphisms into the vertex group G_v are isomorphisms.

Lemma 3.2. *Let G be a finitely generated group, and let H be a subgroup of G. Suppose that G is one-ended relative to H and that there is a constant C such that every finite subgroup of G has order at most C. Then there exists a finitely generated subgroup H'' of H such that G is one-ended relative to H''.*

Proof. Let $\{h_1, h_2, \ldots\}$ be a generating set for H, possibly infinite. For every integer $n \geq 1$, let H_n be the subgroup of H generated by $\{h_1, \ldots, h_n\}$. By Theorem 1 in [23], there is a maximum number m_n of orbits of edges in a non-redundant splitting of G relative to H_n over finite groups. Let T_n be such a splitting with m_n orbits of edges, and let G_n be the vertex group of T_n containing H_n.

We shall prove that G_{n+1} is contained in G_n for all n sufficiently large. First, note that the sequence of integers $(m_n)_{n\in\mathbb{N}}$ is non-increasing, because T_{n+1} is a splitting of G relative to H_n. In particular, there exists an integer n_0 such that $m_n = m_{n+1}$ for every $n \geq n_0$. We claim that G_{n+1} is elliptic in T_n. Otherwise, there exists a non-trivial splitting $G_{n+1} = A *_C B$ or $G_{n+1} = A *_C$ with C finite and $H_n \subset A$, and one gets a non-redundant splitting of G relative to H_n over finite groups with $m_{n+1} + 1 = m_n + 1$ edges by replacing the vertex group G_{n+1} of the graph of groups T_{n+1}/G with the previous one-edge splitting of G_{n+1}, which contradicts the definition of m_n.

Hence, for $n \geq n_0$, one has $G_n \subset G_{n_0}$. In particular, G_{n_0} contains H_n for every integer n. Thus, G_{n_0} contains H. Since G is assumed to be one-ended relative to H, one has $G = G_{n_0}$ and one can take $H'' = H_{n_0}$. □

We will need the following well-known result in the proof of Lemma 3.5 below.

Proposition 3.3 ([4], Proposition 1.2). *If a hyperbolic group splits over quasi-convex subgroups, then every vertex group is quasi-convex* (*hence hyperbolic*).

Remark 3.4. In a hyperbolic group, virtually cyclic subgroups are quasi-convex. Therefore, the previous result applies to splittings of hyperbolic groups over virtually cyclic subgroups.

Lemma 3.5. *Let G be a hyperbolic group. Let H be a $\forall\exists$-elementary subgroup of G. Let U be the one-ended factor of G relative to H. Let Λ be the $\mathbb{Z}$-JSJ splitting of U relative to H. If H is strictly contained in U, then there exists a non-injective Λ-preretraction $U \to G$.*

Proof. Let H' be the finitely generated subgroup of H given by Theorem 2.19 and let H'' be the finitely generated subgroup of H given by Lemma 3.2 above. Let H_0 be the finitely generated subgroup of H generated by $H' \cup H''$.

Let us prove that every morphism $\varphi: U \to H$ whose restriction to H_0 coincides with the identity is non-injective. First, note that U is one-ended relative to H_0 (since it is one-ended relative to H'' which is contained in H_0), and that U is hyperbolic by Proposition 3.3 above. Therefore, by Theorem 2.9, U is co-Hopfian relative to H_0. Hence, a putative monomorphism $\varphi: U \to H \subset U$ whose restriction to H_0 coincides with the identity is surjective, viewed as an endomorphism of U. But $\varphi(U)$ is contained in H, which shows that $U = \varphi(U)$ is contained in H. This is a contradiction since H is strictly contained in U, by assumption.

Let i denote the inclusion of H into G. We proved in the previous paragraph that every morphism $\varphi: U \to H$ whose restriction to H_0 coincides with the identity is non-injective, and thus $i \circ \varphi: U \to G$ is non-injective. Therefore, by Theorem 2.19, for every morphism $\varphi: U \to H$ whose restriction to H_0 (which contains H') coincides with the identity, there exists an automorphism $\sigma \in \mathrm{Mod}_H(U)$ such that $i \circ \varphi \circ \sigma$ kills an element of the finite set $F \subset U \setminus \{1\}$ given by Theorem 2.19. The morphism $i: H \to G$ being injective, $\varphi \circ \sigma$ kills an element of F. In addition, note that the morphisms $\varphi \circ \sigma$ and φ are Λ-related (see Definition 2.26). Hence, for every morphism $\varphi: U \to H$ whose restriction to H_0 coincides with the identity, there exists a morphism $\varphi': U \to H$ that kills an element of the finite set F, and which is Λ-related to φ. We will see that this statement ($\star$) is expressible by means of a $\forall\exists$-sentence with constants in H.

Let $U = \langle u_1, \ldots, u_n \mid R(u_1, \ldots, u_n) = 1 \rangle$ be a finite presentation of U. Let $\{h_1, \ldots, h_p\}$ be a finite generating set for H_0. For every $1 \leq i \leq p$, the element h_i can be written as a word $w_i(u_1, \ldots, u_n)$. Likewise, one can write $F = \{v_1(u_1, \ldots, u_n), \ldots, v_k(u_1, \ldots, u_n)\}$.

Observe that there is a one-to-one correspondence between the set of homomorphisms $\mathrm{Hom}(U, H)$ and the set of solutions in H^n of the system of equations

$R(x_1,\dots,x_n)=1$. The group H satisfies the following formula, expressing the statement ($\star$):

$$\mu(h_1,\dots,h_p)\colon \forall x_1\dots\forall x_n\Big[\Big(R(x_1,\dots,x_n)=1\wedge\bigwedge_{i=1}^{p}w_i(x_1,\dots,x_n)=h_i\Big)$$
$$\Rightarrow\Big(\exists x_1'\dots\exists x_n'\ R(x_1',\dots,x_n')=1$$
$$\wedge\,\theta(x_1,\dots,x_n,x_1',\dots,x_n')=1\wedge\bigvee_{i=1}^{k}v_i(x_1',\dots,x_n')=1\Big)\Big]$$

where θ is the formula given by Lemma 2.23, expressing that the homomorphisms φ and φ' defined by $\varphi\colon h_i\mapsto x_i$ and $\varphi'\colon h_i\mapsto x_i'$ are Λ-related, where Λ denotes the $\mathbb{Z}$-JSJ splitting of U relative to H. This formula is logically equivalent to the following $\forall\exists$-formula in prenex normal form:

$$\mu^*(h_1,\dots,h_p)\colon \forall x_1\dots\forall x_n\exists x_1'\dots\exists x_n'$$
$$\Big(R(x_1,\dots,x_n)\neq 1\vee\bigvee_{i=1}^{p}w_i(x_1,\dots,x_n)\neq h_i$$
$$\vee\Big(R(x_1',\dots,x_n')=1\wedge\theta(x_1,\dots,x_n,x_1',\dots,x_n')$$
$$=1\wedge\bigvee_{i=1}^{k}v_i(x_1',\dots,x_n')=1\Big)\Big).$$

Since H is a $\forall\exists$-elementary subgroup of G, we know that the group G satisfies $\mu^*(h_1,\dots,h_p)$ as well. For $x_i=u_i$ for $1\leq i\leq p$, the interpretation of $\mu^*(h_1,\dots,h_p)$ in G provides a tuple $(g_1,\dots,g_n)\in G^n$ such that the application $p\colon U\to G$ defined by $u_i\mapsto g_i$ for every $1\leq i\leq p$ is a homomorphism, is Λ-related to the inclusion of U into G (see Definition 2.26), and kills an element of F. As a conclusion, p is a non-injective Λ-preretraction from U to G (see Definition 2.22). □

The following easy lemma is proved in [2, Lemma 4.5].

Lemma 3.6. *Let G be a group endowed with a splitting over finite groups. Let T_G denote the Bass-Serre tree associated with this splitting. Let U be a group endowed with a splitting over infinite groups, and let T_U be the associated Bass-Serre tree. If $p\colon U\to G$ is a homomorphism injective on edge groups of T_U, and such that $p(U_v)$ is elliptic in T_G for every vertex v of T_U, then $p(U)$ is elliptic in T_G.*

Lemma 3.7. *Let G be a hyperbolic group. Let H be an infinite subgroup of G. Let U be the one-ended factor of U relative to H (which is unique since H is infinite). Let Λ be the $\mathbb{Z}$-JSJ splitting of G relative to H. Suppose that there exists a non-injective Λ-preretraction $p: U \to G$. Then there exists a centered splitting of G relative to H, called Δ, and a non-degenerate Δ-preretraction of G.*

Proof. First, we will prove that there exists a QH vertex x of Λ such that U_x is not sent isomorphically to a conjugate of itself by p. Assume towards a contradiction that this claim is false, i.e. that each stabilizer U_x of a QH vertex x of Λ is sent isomorphically to a conjugate of itself by p. We claim that $p(U)$ is contained in a conjugate of U. Let Γ be a Stallings splitting of G relative to H. By definition of U, there exists a vertex u of the Bass-Serre tree T of Γ such that $G_u = U$. First, let us check that the hypotheses of Lemma 3.6 are satisfied.

(1) By definition, Γ is a splitting of G over finite groups, and Λ is a splitting of U over infinite groups.

(2) p is injective on edge groups of Λ (as a Λ-preretraction).

(3) if x is a QH vertex of Λ, then $p(U_x)$ is conjugate to U_x by assumption. In particular, $p(U_x)$ is contained in a conjugate of U in G. As a consequence, $p(U_x)$ is elliptic in T (more precisely, it fixes a translate of the vertex u of T such that $G_u = U$). If x is a non-QH vertex of Λ, then $p(U_x)$ is conjugate to U_x by definition of a Λ-preretraction. In particular, $p(U_x)$ is elliptic in T.

By Lemma 3.6, $p(U)$ is elliptic in T. It remains to prove that $p(U)$ is contained in a conjugate of U. Observe that U is not finite-by-(closed orbifold), as a virtually free group. Therefore, there exists at least one non-QH vertex x in Λ. Moreover, since p is inner on non-QH vertices of Λ, there exists an element $g \in G$ such that $p(U_x) = gU_xg^{-1}$. Hence, $p(U_x)$ is contained in both $p(U)$ and gUg^{-1}. Moreover, note that $p(U_x)$ is infinite (since U_x is infinite and $p(U_x)$ is conjugate to U_x), and thus the intersection $p(U) \cap gUg^{-1}$ is infinite. Now, let v be a vertex of T fixed by $p(U)$. If v does not coincide with gu (the unique vertex of T fixed by gUg^{-1}), then the stabilizer of the path joining v to gu is infinite; indeed, this stabilizer is $G_v \cap G_{gu}$, and one has $G_v \supset p(U)$ and $G_{gu} = gG_ug^{-1} = gUg^{-1}$, therefore $G_v \cap G_{gu}$ contains the infinite group $p(U) \cap gUg^{-1}$. This is not possible since edge groups of the Bass-Serre tree T of Γ are finite. As a consequence, the vertex v coincides with gu, which proves that $p(U)$ is contained in gUg^{-1}.

Now, up to composing p by the conjugation by g^{-1}, one can assume that p is an endomorphism of U. By Lemma 2.24, p is injective. This is a contradiction since p is non-injective by hypothesis. Hence, we have proved that there exists a QH vertex x of Λ such that U_x is not sent isomorphically to a conjugate of itself by p.

Then, we refine Γ by replacing the vertex u fixed by U by the $\mathbb{Z}$-JSJ splitting Λ of U relative to H (which is possible since edge groups of Γ adjacent to u are finite, and thus are elliptic in Λ). With a little abuse of notation, we still denote by x the vertex of Γ corresponding to the QH vertex x of Λ. Then, we collapse to a point every connected component of the complement of star(x) in Γ (where star(x) stands for the subgraph of Γ constituted of x and all its incident edges). The resulting graph of groups, denoted by Δ, is non-trivial. One easily sees that Δ is a centered splitting of G, with central vertex x.

The homomorphism $p\colon U \to G$ is well-defined on G_x because $G_x = U_x$ is contained in U. Moreover, p restricts to a conjugation on each stabilizer of an edge e of Δ incident to x. Indeed, either e is an edge coming from Λ, either G_e is a finite subgroup of U; in each case, $p_{|G_e}$ is a conjugation since p is Λ-related to the inclusion of U into G. Now, one can define an endomorphism $\varphi\colon G \to G$ that coincides with p on $G_x = U_x$ and coincides with a conjugation on every vertex group G_y of Γ, with $y \neq x$. By induction on the number of edges of Γ, it is enough to define φ in the case where Γ has only one edge. If $G = U_x *_C B$ with $p_{|C} = \mathrm{ad}(g)$, one defines $\varphi\colon G \to G$ by $\varphi_{|U_x} = p$ and $\varphi_{|B} = \mathrm{ad}(g)$. If $G = U_x *_C = \langle U_x, t \mid tct^{-1} = \alpha(c), \forall c \in C\rangle$ with $p_{|C} = \mathrm{ad}(g_1)$ and $p_{|\alpha(C)} = \mathrm{ad}(g_2)$, one defines $\varphi\colon G \to G$ by $\varphi_{|U_x} = p$ and $\varphi(t) = g_2^{-1} t g_1$.

The endomorphism φ defined above is Δ-related to the identity of G (in the sense of Definition 2.26), and φ does not send G_x isomorphically to a conjugate of itself. Hence, φ is a non-degenerate Δ-preretraction of G (see Definition 2.27). □

The following result is proved in [2] (Lemma 4.4).

Lemma 3.8. *Let G be a virtually free group, and let Δ be a centered splitting of G. Then G has no non-degenerate Δ-preretraction.*

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Let U be the one-ended factor of G relative to H. Assume towards a contradiction that H is strictly contained in U. Then by Proposition 3.5, there exists a non-injective preretraction $U \to G$ (with respect to the $\mathbb{Z}$-JSJ splitting of U relative to H). By Lemma 3.7, there exists a centered splitting Δ of G relative to H such that G has a non-degenerate Δ-preretraction. This contradicts Lemma 3.8. Hence, H is equal to U. □

3.2. Proof of Theorem 1.5. Recall that Theorem 1.5 claims that if H is a $\forall\exists$-elementary proper subgroup of a virtually free group G, then G is a multiple legal large extension of H. Before proving this result, we will define five numbers associated with a hyperbolic group, which are encoded into its $\forall\exists$-theory (see Lemma 3.10 below).

Definition 3.9. Let G be a hyperbolic group. We associate to G the following five integers:

- the number $n_1(G)$ of conjugacy classes of finite subgroups of G,

- the sum $n_2(G)$ of $|\mathrm{Aut}_G(C_k)|$ for $1 \leq k \leq n_1(G)$, where the C_k are representatives of the conjugacy classes of finite subgroups of G, and

$$\mathrm{Aut}_G(C_k) = \{\alpha \in \mathrm{Aut}(C_k) \mid \exists g \in N_G(C_k),\ \mathrm{ad}(g)_{|C} = \alpha\},$$

- the number $n_3(G)$ of conjugacy classes of finite subgroups C of G such that $N_G(C)$ is infinite virtually cyclic,
- the number $n_4(G)$ of conjugacy classes of finite subgroups C of G such that $N_G(C)$ is not virtually cyclic (finite or infinite),
- the number $n_5(G)$ of conjugacy classes of finite subgroups C of G such that $N_G(C)$ is not virtually cyclic (finite or infinite) and $E_G(N_G(C)) \neq C$.

The following lemma shows that these five numbers are preserved under ∀∃-equivalence. Its proof is quite straightforward and is postponed after the proof of Theorem 1.5.

Lemma 3.10. *Let G and G' be two hyperbolic groups. Suppose that* $\mathrm{Th}_{\forall\exists}(G) = \mathrm{Th}_{\forall\exists}(G')$. *Then $n_i(G) = n_i(G')$, for $1 \leq i \leq 5$.*

Theorem 1.5 will be an easy consequence of the following result.

Proposition 3.11. *Let G be a virtually free group. Let H be a proper subgroup of G. Suppose that the following three conditions are satisfied*:

(1) $n_i(H) = n_i(G)$ *for all* $1 \leq i \leq 5$,

(2) *H appears as a vertex group in a splitting of G over finite groups,*

(3) *two finite subgroups of H are conjugate in H if and only if they are conjugate in G.*

Then G is a multiple legal large extension of H (*see Definition* 1.1).

Proof. First, note that the equality $n_4(H) = n_4(G)$ implies that H is non virtually cyclic. Indeed, if H is virtually cyclic, then $n_4(H) = 0$, whereas $n_4(G)$ is greater than 1 since $N_G(\{1\}) = G$ is not virtually cyclic by assumption.

Let T be the Bass-Serre tree of the splitting of G given by the second condition. Up to refining this splitting, one can assume without loss of generality that the vertex groups of T which are not conjugate to H are finite. In other words, T is a Stallings splitting of G relative to H, in which H is a vertex group by assumption. Moreover, up to collapsing some edges, one can assume that T is *reduced*, which means that if $e = [v, w]$ is an edge of T such that $G_e = G_v = G_w$, then v and w are in the same orbit. We denote by Γ the quotient graph of groups T/G.

We will deduce from the third condition that the underlying graph of Γ has only one vertex. Assume towards a contradiction that the Bass-Serre tree T of Γ has at least two orbits of vertices. Hence, there is a vertex v of T which is not in the orbit of the vertex v_H fixed by H. By definition of Γ, the vertex stabilizer G_v is finite. It follows from the equality $n_1(H) = n_1(G)$ that every finite subgroup of G is conjugate to a subgroup of H, and thus there exists an element $g \in G$ such that gG_vg^{-1} is contained in H. Therefore, G_v stabilizes the path of edges in T between the vertices v and $g^{-1}v_H$. It follows that G_v coincides with the stabilizer of an edge incident to v in T, which contradicts the assumption that T is reduced.

Hence, the underlying graph of Γ is a rose, and the central vertex group of Γ is H. Moreover, edge stabilizers of Γ are finite. In other words, there exist pairs of finite subgroups $(C_1, C_1'), \ldots, (C_n, C_n')$ of H, together with automorphisms $\alpha_1 \in \mathrm{Isom}(C_1, C_1'), \ldots, \alpha_n \in \mathrm{Isom}(C_n, C_n')$ such that G has the following presentation:

$$G = \langle H, t_1, \ldots, t_n \mid \mathrm{ad}(t_i)_{|C_i} = \alpha_i, \ \forall i \in [\![1, n]\!]\rangle.$$

By assumption, the integers $n_i(G)$ and $n_i(H)$ are equal, for $1 \leq i \leq 5$. From the equality $n_1(G) = n_1(H)$, one deduces immediately that the finite groups C_i and C_i' are conjugate in H for every integer $i \in \{1, \ldots, n\}$. Therefore, one can assume without loss of generality that $C_i' = C_i$.

Note that for every finite subgroup C of H, the group $\mathrm{Aut}_H(C)$ is contained in $\mathrm{Aut}_G(C)$. Thus, the equality $n_2(G) = n_2(H)$ guarantees that $\mathrm{Aut}_H(C_i)$ is in fact equal to $\mathrm{Aut}_G(C_i)$, for every $1 \leq i \leq n$. Hence, since the automorphism $\mathrm{ad}(t_i)_{|C_i}$ of C_i belongs to $\mathrm{Aut}_G(C_i)$, there exists an element $h_i \in N_H(C_i)$ such that $\mathrm{ad}(h_i)_{|C_i} = \mathrm{ad}(t_i)_{|C_i}$. Up to replacing t_i with $t_ih_i^{-1}$, the group G has the following presentation:

$$G = \langle H, t_1, \ldots, t_n \mid \mathrm{ad}(t_i)_{|C_i} = \mathrm{id}_{C_i}, \ \forall i \in [\![1, n]\!]\rangle.$$

In order to prove that G is a multiple legal large extension of H (see Definition 1.1), it remains to prove that the following two conditions hold, for every integer $1 \leq i \leq n$:

(1) the normalizer $N_H(C_i)$ is non virtually cyclic (finite or infinite),

(2) and the finite group $E_H(N_H(C_i))$ coincides with C_i.

The equalities $n_3(G) = n_3(H)$ and $n_4(G) = n_4(H)$ ensure that $N_H(C_i)$ is not virtually cyclic. Indeed, if $N_H(C_i)$ were finite, then $N_G(C_i)$ would be infinite virtually cyclic and $n_3(G)$ would be at least $n_3(H) + 1$; similarly, if $N_H(C_i)$ were infinite virtually cyclic, then $N_G(C_i)$ would be non virtually cyclic and $n_4(G) \geq n_4(H) + 1$. Hence, the first condition above is satisfied.

Last, it follows from the equality $n_5(G) = n_5(H)$ that the finite group $E_H(N_H(C_i))$ coincides with C_i, otherwise one would have $n_5(G) \geq n_5(H) + 1$, since $E_G(N_G(C_i)) = C_i$. Thus, the second condition above holds. As a conclusion, G is a multiple legal large extension of H in the sense of Definition 1.1. □

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Let G be a virtually free group, and let H be a $\forall\exists$-elementary subgroup of G. By Proposition 3.1, H is a vertex group in a splitting of G over finite groups, which means that the second condition of Proposition 3.11 is satisfied. In addition, H is hyperbolic by Proposition 3.3.

Note that G and H have the same $\forall\exists$-theory. It follows from Lemma 3.10 that $n_i(H)$ is equal to $n_i(G)$ for all $1 \leq i \leq 5$. Hence, the first condition of Proposition 3.11 holds.

It remains to check the third condition of Proposition 3.11, namely that two finite subgroups of H are conjugate in H if and only if they are conjugate in G. First, recall that H and G have the same number of conjugacy classes of finite subgroups, since $n_1(G) = n_1(H)$. Then, the conclusion follows from the following observation: if two finite subgroups $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ of H are not conjugate in H, then they are not conjugate in G. Indeed, H satisfies the following universal formula:

$$\theta(a_1, \ldots, a_m, b_1, \ldots, b_m)\colon \forall x \bigvee_{i=1}^{m} \bigwedge_{j=1}^{m} x a_i x^{-1} \neq b_j.$$

Since H is $\forall\exists$-elementary (in particular $\forall$-elementary), G satisfies this sentence as well. Therefore, A and B are not conjugate in G. □

It remains to prove Lemma 3.10.

Proof of Lemma 3.10. Let us denote by K_G the maximal order of a finite subgroup of G. Since G and G' have the same existential theory, we have $K_G = K_{G'}$. Let $n \geq 1$ be an integer. If $n_1(G) \geq n$, then the following $\exists\forall$-sentence, written in natural language for convenience of the reader and denoted by $\theta_{1,n}$, is satisfied by G: there exist n finite subgroups $C_1, \ldots, C_n$ of G such that, for every $g \in G$ and $1 \leq i \neq j \leq n$, the groups gC_ig^{-1} and C_j are distinct. Since G and G' have the same $\exists\forall$-theory, the sentence $\theta_{1,n}$ is satisfied by G' as well. As a consequence, $n_1(G') \geq n$. It follows that $n_1(G') \geq n_1(G)$. By symmetry, we have $n_1(G) = n_1(G')$.

In the rest of the proof, we give similar sentences $\theta_{2,n}, \ldots, \theta_{5,n}$ such that the following series of equivalences hold: $n_i(G) \geq n \Leftrightarrow G$ satisfies $\theta_{i,n} \Leftrightarrow G'$ satisfies $\theta_{i,n} \Leftrightarrow n_i(G') \geq n$.

One has $n_2(G) \geq n$ if and only if G satisfies the following $\exists\forall$-sentence $\theta_{2,n}$: there exist ℓ finite subgroups $C_1, \ldots, C_\ell$ of G and, for every $1 \leq i \leq \ell$, a finite subset $\{g_{i,j}\}_{1 \leq j \leq n_i}$ of $N_G(C_i)$ such that

- for every $g \in G$ and $1 \leq i \neq j \leq n$, the groups gC_ig^{-1} and C_j are distinct;
- the sum $n_1 + \cdots + n_\ell$ is equal to n;
- for every $1 \leq i \leq \ell$, and for every $1 \leq j \neq k \leq n_i$, the automorphisms $\mathrm{ad}(g_j)_{|C_i}$ and $\mathrm{ad}(g_k)_{|C_i}$ of C_i are distinct.

One has $n_3(G) \geq n$ if and only if G satisfies the following $\exists\forall$-sentence $\theta_{3,n}$: there exist n finite subgroups $C_1, \ldots, C_n$ of G and n elements $g_1 \in N_G(C_1), \ldots, g_n \in N_G(C_n)$ of infinite order (i.e. satisfying $g_i^{K_G!} \neq 1$) such that

- for every $g \in G$ and $1 \leq i \neq j \leq n$, the groups gC_ig^{-1} and C_j are distinct;
- for every $1 \leq i \leq n$ and $g \in N_G(C_i)$, the subgroup $\langle g, g_i \rangle$ of $N_G(C_i)$ is virtually cyclic, i.e. $[g^{K_G!}, g_i^{K_G!}] = 1$ (see Lemma 2.1).

One has $n_4(G) \geq n$ if and only if G satisfies the following $\exists\forall$-sentence $\theta_{4,n}$: there exist n finite subgroups $C_1, \ldots, C_n$ of G and, for every $1 \leq i \leq n$, a couple of elements $(g_{i,1}, g_{i,2})$ normalizing C_i such that

- for every $g \in G$ and $1 \leq i \neq j \leq n$, the groups gC_ig^{-1} and C_j are distinct;
- for every $1 \leq i \leq n$, the subgroup $\langle g_{i,1}, g_{i,2} \rangle$ is not virtually cyclic (i.e. $[g_{i,1}^{K_G!}, g_{i,2}^{K_G!}]$ is non-trivial, by Lemma 2.1).

One has $n_5(G) \geq n$ if and only if G satisfies the following $\exists\forall$-sentence $\theta_{5,n}$: there exist $2n$ finite subgroups $C_1, \ldots, C_n$ and $C_1' \supsetneq C_1, \ldots, C_n' \supsetneq C_n$ of G and, for every $1 \leq i \leq n$, a couple of elements $(g_{i,1}, g_{i,2})$ normalizing C_i, such that

- for every $g \in G$ and $1 \leq i \neq j \leq n$, the groups gC_ig^{-1} and C_j are distinct;
- for every $1 \leq i \leq n$, the subgroup $\langle g_{i,1}, g_{i,2} \rangle$ is not virtually cyclic;
- every element of G that normalizes C_i also normalizes C_i'. □

4. Algorithm

In this section, we shall prove the following theorem.

Theorem 4.1. *There is an algorithm that, given a finite presentation of a virtually free group G and a finite subset $X \subset G$, outputs 'Yes' if the subgroup of G generated by X is $\forall\exists$-elementary, and 'No' otherwise.*

We shall use the following fact.

Lemma 4.2. *A subgroup H of G is $\forall\exists$-elementary if and only if the three conditions of Proposition 3.11 are satisfied.*

Proof. If the conditions of Proposition 3.11 are satisfied, then either $H = G$, or H is a proper subgroup and G is a multiple legal large extension of H, by Proposition 3.11. In both cases, the subgroup H is $\forall\exists$-elementary by Theorem 1.2. Conversely, if H is $\forall\exists$-elementary, then either $H = G$ or H is a proper subgroup of G and G is a multiple legal large extension of H, by Theorem 1.5. □

The proof of Theorem 4.1 consists in showing that the conditions of Proposition 3.11 can be decided by an algorithm.

4.1. Algorithmic tools. First, we collect several algorithms that will be useful in the proof of Theorem 4.1.

4.1.1. Solving equations in hyperbolic groups. The following theorem is the main result of [7].

Theorem 4.3. *There exists an algorithm that takes as input a finite presentation of a hyperbolic group G and a finite system of equations and inequations with constants in G, and decides whether there exists a solution or not.*

4.1.2. Computing a finite presentation of a subgroup given by generators. The following result is well known.

Theorem 4.4. *There is an algorithm that, given a finite presentation of a hyperbolic and locally quasiconvex group G, and a finite subset X of G, produces a finite presentation for the subgroup of G generated by X.*

Recall that a group is said to be *locally quasiconvex* if every finitely generated subgroup is quasiconvex. Marshall Hall Jr. proved that every finitely generated subgroup of a finitely generated free group is a free factor in a finite-index subgroup, which shows in particular that finitely generated free groups are locally quasiconvex. It follows easily that finitely generated virtually free groups are locally quasiconvex. Thus, Theorem 4.4 applies when G is virtually free.

4.1.3. Basic algorithms

Lemma 4.5. *There is an algorithm that takes as input a finite presentation of a hyperbolic group and computes a list of representatives of the conjugacy classes of finite subgroups in this hyperbolic group.*

Proof. There exists an algorithm that computes, given a finite presentation $\langle S \mid R\rangle$ of a hyperbolic group G, a hyperbolicity constant δ of G (see [16]). In addition, it is well-known that the ball of radius 100δ in G contains at least one representative of each conjugacy class of finite subgroups of G (see [5]). Moreover, two finite subgroups C_1 and C_2 of G are conjugate if and only if there exists an element g whose length is bounded by a constant depending only on δ and on the size of the generating set S of G, such that $C_2 = gC_1g^{-1}$ (see [6]). □

Lemma 4.6 ([8], Lemma 2.5). *There is an algorithm that computes a set of generators of the normalizer of any given finite subgroup in a hyperbolic group.*

Lemma 4.7 ([8], Lemma 2.8). *There is an algorithm that decides, given a finite set S in a hyperbolic group, whether $\langle S\rangle$ is finite, virtually cyclic infinite, or non virtually cyclic (finite or infinite).*

Lemma 4.8. *There is an algorithm that takes as input a finite presentation of a hyperbolic group G and a finite subgroup C of G such that $N_G(C)$ is non virtually cyclic (finite or infinite), and decides whether or not $E_G(N_G(C)) = C$.*

Proof. By Lemma 4.5, one can compute some representatives $A_1, \dots, A_k$ of the conjugacy classes of finite subgroups of G. Given an element $g \in G$, let $\theta_g(x)$ be a quantifier-free formula expressing the following fact: there exists an integer $1 \leq i \leq k$ such that the finite set $\{C, g\}$ is contained in xA_ix^{-1}. Note that the group $\langle C, g\rangle$ is finite if and only if the existential sentence $\exists x\ \theta_g(x)$ is true in G.

One can compute a finite generating set S for $N_G(C)$ using Lemma 4.6. By Theorem 4.3 above, one can decide if the following existential sentence with constants in G is satisfied by G: there exist two elements g and g' such that

(1) g does not belong to C;

(2) $\theta_g(g')$ is satisfied by G (hence, the subgroup $C' := \langle C, g\rangle$ is finite);

(3) for every $s \in S$, one has $sC's^{-1} = C'$.

Note that such an element g exists if and only if C is strictly contained in $E_G(N_G(C))$. This concludes the proof of the lemma. □

The following lemma is an immediate corollary of Lemmas 4.5–4.8 above.

Lemma 4.9. *There is an algorithm that takes as input a finite presentation of a hyperbolic group G and computes the five numbers $n_1(G), \dots, n_5(G)$ (see Definition 3.9).*

4.2. Decidability of the first condition of Proposition 3.11

Lemma 4.10. *There is an algorithm that, given a finite presentation of a virtually free group G and a finite subset $X \subset G$ generating a subgroup $H = \langle X\rangle$, outputs 'Yes' if $n_i(H) = n_i(G)$ for all $i \in \{1,2,3,4,5\}$ and 'No' otherwise.*

Proof. By Theorem 4.4, there is an algorithm that takes as input a finite presentation $G = \langle S_G \mid R_G\rangle$ and X, and produces a finite presentation $\langle S_H \mid R_H\rangle$ for H. By Lemma 4.9, one can compute $n_i(G)$ and $n_i(H)$ for every $i \in \{1,2,3,4,5\}$. □

4.3. Decidability of the second condition of Proposition 3.11

Lemma 4.11. *There is an algorithm that, given a finite presentation of a virtually free group G and a finite subset $X \subset G$ generating a subgroup $H = \langle X\rangle$, outputs 'Yes' if H is infinite and coincides with the one-ended factor of G relative to H (well-defined since H is infinite), and 'No' otherwise.*

Proof. By Lemma 4.7, one can decide if H is finite or infinite. By Lemma 8.7 in [8], one can compute a Stallings splitting of G relative to H. Let T be the Bass–Serre tree of this splitting. Let U be the one-ended factor of G relative to H and let u be the vertex of T fixed by U. By Corollary 8.3 in [8], one can decide if there exists an automorphism φ of G such that $\varphi(H) = U$, which is equivalent to deciding if $U = H$. Indeed, if $\varphi(H) = U$, then H fixes the vertex u for the action of G on T twisted by φ. Thus, by definition of U as the one-ended factor relative to H, the pair (U, H) acts trivially on the tree T for the action twisted by φ. Consequently, $\varphi(U)$ fixes u as well. Therefore, one has $\varphi(U) = U = \varphi(H)$, and it follows that $U = H$ since φ is an automorphism of G. □

4.4. Decidability of the third condition of Proposition 3.11

Lemma 4.12. *There is an algorithm that, given a finite presentation of a virtually free group G and a finite subset $X \subset G$ generating a subgroup $H = \langle X \rangle$, decides whether or not every finite subgroup of G is conjugate to a subgroup of H.*

Proof. By Theorem 4.4, there is an algorithm that takes as input a finite presentation $G = \langle S_G \mid R_G \rangle$ and X, and produces a finite presentation $\langle S_H \mid R_H \rangle$ for H. By Lemma 4.5, there is an algorithm that computes two lists $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_n\}$ of representatives of the conjugacy classes of finite subgroups of G and H respectively. Then, for every finite subgroup A_i of G in the first list, deciding if A_i is conjugate in G to B_j for some $j \in \{1, \dots, n\}$ is equivalent to solving the following finite disjunction of systems of equations with constants in G, which can be done using Theorem 4.3:

$$\theta(x) \colon \exists x \; (x A_i x^{-1} = B_1) \vee \cdots \vee (x A_i x^{-1} = B_n).$$

Hence, there is an algorithm that outputs 'Yes' if every finite subgroup of G is conjugate to a subgroup of H, and 'No' otherwise. □

References

[1] S. André, On Tarski's problem for virtually free groups. Preprint, 2019. arXiv:arXiv:1910.08464

[2] S. André, Virtually free groups are almost homogeneous. Preprint, 2018. arXiv:1810.11200

[3] S. André, Hyperbolicity and cubulability are preserved under elementary equivalence. *Geom. Topol.* **24** (2020), no. 3, 1075–1147. Zbl 07256603 MR 4157551

[4] B. Bowditch, Cut points and canonical splittings of hyperbolic groups. *Acta Math.* **180** (1998), no. 2, 145–186. Zbl 0911.57001 MR 1638764

[5] N. Brady, Finite subgroups of hyperbolic groups. *Internat. J. Algebra Comput.* **10** (2000), no. 4, 399–405. Zbl 1010.20030 MR 1776048

[6] M. Bridson and J. Howie, Conjugacy of finite subsets in hyperbolic groups. *Internat. J. Algebra Comput.* **15** (2005), no. 4, 725–756. Zbl 1083.20032 MR 2160576

[7] F. Dahmani and V. Guirardel, Foliations for solving equations in groups: free, virtually free, and hyperbolic groups. *J. Topol.* **3** (2010), no. 2, 343–404. Zbl 1217.20021 MR 2651364

[8] F. Dahmani and V. Guirardel, The isomorphism problem for all hyperbolic groups. *Geom. Funct. Anal.* **21** (2011), no. 2, 223–300. Zbl 1258.20034 MR 2795509

[9] V. Guirardel and G. Levitt, JSJ decompositions of groups. *Astérisque* **395** (2017), vii+165. MR 3758992 Zbl 1391.20002

[10] I. Kapovich and N. Benakli, Boundaries of hyperbolic groups. In S. Cleary, R. Gilman, A. G. Myasnikov and V. Shpilrain (eds.), *Combinatorial and geometric group theory.* (New York, 2000 and in Hoboken, N.J., 2001.) Contemporary Mathematics, 296. American Mathematical Society, Providence, R.I., 2002, 39–93. Zbl 1044.20028 MR 1921706

[11] O. Kharlampovich and A. Myasnikov, Elementary theory of free non-abelian groups. *J. Algebra* **302** (2006), no. 2, 451–552. Zbl 1110.03020 MR 2293770

[12] P. Linnell, On accessibility of groups. *J. Pure Appl. Algebra* **30** (1983), no. 1, 39–46. Zbl 0545.20020 MR 716233

[13] D. Marker, *Model theory*. An introduction. Graduate Texts in Mathematics, 217. Springer, New York, 2002. Zbl 1003.03034 MR 1924282

[14] A. Myasnikov and N. Romanovskii, Characterization of finitely generated groups by types. *Internat. J. Algebra Comput.* **28** (2018), no. 08, 1613–1632. Zbl 07003138 MR 3899227

[15] A. Ol'shanskiĭ, On residualing homomorphisms and G-subgroups of hyperbolic groups. *Internat. J. Algebra Comput.* **3** (1993), no. 4, 365–409. Zbl 0830.20053 MR 1250244

[16] P. Papasoglu, An algorithm detecting hyperbolicity. In G. Baumslag, D. Epstein, R. Gilman, H. Short, and C. Sims (eds.), *Geometric and computational perspectives on infinite groups.* (Minneapolis, MN, and New Brunswick, N.J., March 17–20, 1994.) DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 25. American Mathematical Society, Providence, R.I., 1996, 193–200. Zbl 0857.20017 MR 1364185

[17] C. Perin, Elementary embeddings in torsion-free hyperbolic groups. *Ann. Sci. Éc. Norm. Supér.* (4) **44** (2011), no. 4, 631–681. Zbl 1245.20052 MR 2919979

[18] C. Perin, Erratum to: "Elementary embeddings in torsion-free hyperbolic groups." *Ann. Sci. Éc. Norm. Supér.* (4) **46** (2013), no. 5, 851–856. Zbl 1284.20046 Zbl 3185354

[19] C. Perin, Elementary subgroups of the free group are free factors – a new proof. Preprint, 2019. arXiv:1903.05876

[20] Z. Sela, Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. *Geom. Funct. Anal.* **7** (1997), no. 3, 561–593. Zbl 0884.20025 MR 1466338

[21] Z. Sela, Diophantine geometry over groups. VI. The elementary theory of a free group. *Geom. Funct. Anal.* **16** (2006), no. 3, 707–730. Zbl 1118.20035 MR 2238945

[22] Z. Sela, Diophantine geometry over groups. VII. The elementary theory of a hyperbolic group. *Proc. Lond. Math. Soc.* (3) **99** (2009), no. 1, 217–273. Zbl 1241.20049 MR 2520356

[23] R. Weidmann, On accessibility of finitely generated groups. *Q. J. Math.* **63** (2012), no. 1, 211–225. MR 2889188 Zbl 1280.20032

Received February 19, 2020

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