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# On hereditarily self-similar $p$-adic analytic pro-$p$ groups

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**Abstract.** A non-trivial finitely generated pro-$p$ group $G$ is said to be strongly hereditarily self-similar of index $p$ if every non-trivial finitely generated closed subgroup of $G$ admits a faithful self-similar action on a $p$-ary tree. We classify the solvable torsion-free $p$-adic analytic pro-$p$ groups of dimension less than $p$ that are strongly hereditarily self-similar of index $p$. Moreover, we show that a solvable torsion-free $p$-adic analytic pro-$p$ group of dimension less than $p$ is strongly hereditarily self-similar of index $p$ if and only if it is isomorphic to the maximal pro-$p$ Galois group of some field that contains a primitive $p$th root of unity. As a key step for the proof of the above results, we classify the 3-dimensional solvable torsion-free $p$-adic analytic pro-$p$ groups that admit a faithful self-similar action on a $p$-ary tree, completing the classification of the 3-dimensional torsion-free $p$-adic analytic pro-$p$ groups that admit such actions.

*Dedicated to Said Sidki on the occasion of his 80th birthday.*

## 1. Introduction

Groups that admit a faithful self-similar action on some regular rooted $d$-ary tree $T_d$ form an interesting class that contains many important examples such as the Grigorchuk 2-group [8], the Gupta–Sidki $p$-groups [9], the affine groups $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ [3], and groups obtained as iterated monodromy groups of self-coverings of the Riemann sphere by post-critically finite rational maps [16]. Recently there has been an intensive study on the self-similar actions of other important families of groups including abelian groups [4], wreath products of abelian groups [5], finitely generated nilpotent groups [2], arithmetic groups [10], and groups of type $\mathrm{FP}_n$ [14]. Self-similar actions of some classes of finite $p$-groups were studied in [1, 25].

We say that a group $G$ is *self-similar of index $d$* if $G$ admits a faithful self-similar action on $T_d$ that is transitive on the first level; moreover, we say that $G$ is *self-similar* if it is self-similar of some index $d$. In [19] we initiated the study of self-similar actions of $p$-adic analytic pro-$p$ groups. In particular, we classified the 3-dimensional *unsolvable* torsion-free $p$-adic analytic pro-$p$ groups for $p \geqslant 5$, and determined which of them admit a faithful self-similar action on a $p$-ary tree. In the present paper, instead, we focus on the study of self-similar actions of torsion-free *solvable* $p$-adic analytic pro-$p$ groups.

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It is fairly easy to show that every free abelian group $\mathbb{Z}^r$ of finite rank $r \geqslant 1$ is self-similar of any index $d \geqslant 2$ (cf. [16, Section 2.9.2]; see also [17]). Hence, every non-trivial subgroup of $\mathbb{Z}^r$ is self-similar of any index $d \geqslant 2$. Similarly, every non-trivial closed subgroup of a free abelian pro-$p$ group $\mathbb{Z}_p^r$ is self-similar of index $p^k$, for $k \geqslant 1$. Motivated by this phenomenon we make the following definitions. A finitely generated pro-$p$ group $G$ is said to be *hereditarily self-similar of index* $p^k$ if any open subgroup of $G$ is self-similar of index $p^k$. If $G$ and all of its non-trivial finitely generated closed subgroups are self-similar of index $p^k$, then $G$ is said to be *strongly hereditarily self-similar of index* $p^k$.

From [19, Proposition 1.5], it follows that any torsion-free $p$-adic analytic pro-$p$ group of dimension 1 or 2 is strongly hereditarily self-similar of index $p^k$ for all $k \geqslant 1$. Moreover, it is not difficult to see that if $p \geqslant 5$, then any 3-dimensional *solvable* torsion-free $p$-adic analytic pro-$p$ group is strongly hereditarily self-similar of index $p^{2m}$ for all $m \geqslant 1$ (see Proposition 3.4). Observe that the latter class contains a continuum of groups that are pairwise incommensurable (see [23]), in contrast to the discrete case, where there are only countably many pairwise non-isomorphic finitely generated self-similar groups (cf. [16, Section 1.5.3]). On the other hand, it is an interesting problem to understand which pro-$p$ groups have the property of being strongly hereditarily self-similar of *index* $p$, and the main result of this paper is the classification of the solvable torsion-free $p$-adic analytic pro-$p$ groups with this property.

**Theorem A.** *Let $p$ be a prime, and let $G$ be a solvable torsion-free $p$-adic analytic pro-$p$ group. Suppose that $p > d := \dim(G)$. Then $G$ is strongly hereditarily self-similar of index $p$ if and only if $G$ is isomorphic to one of the following groups:*

(1) *for $d \geqslant 1$, the abelian pro-$p$ group $\mathbb{Z}_p^d$;*
(2) *for $d \geqslant 2$, the metabelian pro-$p$ group $G^d(s) := \mathbb{Z}_p \ltimes \mathbb{Z}_p^{d-1}$, where the canonical generator of $\mathbb{Z}_p$ acts on $\mathbb{Z}_p^{d-1}$ by multiplication by the scalar $1 + p^s$, for some integer $s \geqslant 1$.*

Observe that the "if" part of the theorem holds in greater generality (Proposition 3.7); we also remark that the condition $p > d$ makes it possible to apply Lie methods (see Section 3, in particular, Remark 3.2). It is worth noting that during the last decade the groups listed in Theorem A have appeared in the literature in different contexts (see, for example, [12, 13, 20–22, 24]). The reader will find a more detailed account of the related results at the end of Section 3.

Let $K$ be a field. The absolute Galois group of $K$ is the profinite group $G_K = \mathrm{Gal}(K_s/K)$, where $K_s$ is a separable closure of $K$. The maximal pro-$p$ Galois group of $K$, denoted by $G_K(p)$, is the maximal pro-$p$ quotient of $G_K$. More precisely, $G_K(p) = \mathrm{Gal}(K(p)/K)$, where $K(p)$ is the composite of all finite Galois $p$-extensions of $K$ (inside $K_s$). Describing absolute Galois groups of fields among profinite groups is one of the most important problems in Galois theory. Already describing $G_K(p)$ among pro-$p$ groups is a remarkable challenge. Theorem A and a result of Ware [26] yield the following.

**Theorem B.** *Let $p$ be a prime, and let $G$ be a non-trivial solvable torsion-free $p$-adic analytic pro-$p$ group. Suppose that $p > \dim(G)$. Then $G$ is strongly hereditarily self-similar of index $p$ if and only if $G$ is isomorphic to the maximal pro-$p$ Galois group of some field that contains a primitive $p$th root of unity.*

Similarly to Theorem A, the "if" part holds in greater generality (Proposition 3.8).

The proof of Theorem A is by induction on $d = \dim(G)$. As mentioned above, for $d = 1, 2$ matters are trivial, while for $d = 3$ interesting phenomena start to occur. Indeed, as a basis for the induction, one has to consider the case $d = 3$, and this leads us to the classification result below. This result is interesting on its own right since it completes the classification started by [19, Theorem B] of the 3-dimensional torsion-free $p$-adic analytic pro-$p$ groups that are self-similar of index $p$.

**Theorem C.** *Let $p \geqslant 5$ be a prime and fix $\rho \in \mathbb{Z}_p^*$ a non-square modulo $p$. Let $G$ be a 3-dimensional solvable torsion-free $p$-adic analytic pro-$p$ group. Then the following holds.*

(1) *$G$ is self-similar of index $p^2$.*

(2) *Let $L$ be the $\mathbb{Z}_p$-Lie lattice associated with $G$. Then $G$ is self-similar of index $p$ if and only if $L$ is isomorphic to a Lie lattice presented in the following irredundant list (cf. Remark 2.21; the parameters below take values $s, r, t \in \mathbb{N}$, $c \in \mathbb{Z}_p$, and $\varepsilon \in \{0, 1\}$):*

(a) $\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = 0,\ [x_0, x_2] = 0\rangle$;

(b) *for* $s \geqslant 1$, $\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^s x_1,\ [x_0, x_2] = p^s x_2\rangle$;

(c) *for* $s, r \geqslant 1$ *and* $v_p(c) = 1$,

$$\begin{aligned}\big\langle x_0, x_1, x_2 \mid [x_1, x_2] &= 0,\\ [x_0, x_1] &= p^s x_1 + p^{s+r} c x_2,\\ [x_0, x_2] &= p^{s+r} x_1 + p^s x_2\big\rangle;\end{aligned}$$

(d) $\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^{s+1} \rho^\varepsilon x_2,\ [x_0, x_2] = p^s x_1\rangle$;

(e) *for* $s \geqslant 1$, $\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^s x_2,\ [x_0, x_2] = p^s x_1\rangle$;

(f) *for* $r \geqslant 1$ *and* $v_p(c) = 1$,

$$\big\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^{s+r} x_1 + p^s c x_2,\ [x_0, x_2] = p^s x_1\big\rangle;$$

(g) *for* $s \geqslant 1$ *and* $v_p(1 + 4c) = 1$,

$$\big\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^s x_1 + p^s c x_2,\ [x_0, x_2] = p^s x_1\big\rangle.$$

In dimension 3, Theorem C and [19, Theorem B] yield the following stronger version of Theorem A.

**Theorem D.** *Let $p \geqslant 5$ be a prime, and let $G$ be a 3-dimensional torsion-free $p$-adic analytic pro-$p$ group. Then the following are equivalent.*

(1) *$G$ is hereditarily self-similar of index $p$.*

(2) *$G$ is strongly hereditarily self-similar of index $p$.*

(3) *$G$ is isomorphic to $\mathbb{Z}_p^3$ or to $G^3(s)$ for some integer $s \geqslant 1$.*

We believe that one can drop the assumption of solvability in Theorem A even in higher dimension.

**Conjecture E.** *Let $p$ be a prime, and let $G$ be a torsion-free $p$-adic analytic pro-$p$ group of dimension $d$. Suppose that $p > d$. Then $G$ is strongly hereditarily self-similar of index $p$ if and only if $G$ is isomorphic to $\mathbb{Z}_p^d$ for $d \geqslant 1$ or to $G^d(s)$ for $d \geqslant 2$ and some integer $s \geqslant 1$.*

**Main strategy and outline of the paper.** For the proof of the main results we use Lie methods. More precisely, we use the language of virtual endomorphisms (see, for instance, [19, Proposition 1.3]) to translate self-similarity problems on $p$-adic analytic groups to problems on $\mathbb{Z}_p$-Lie lattices (Proposition 3.1). Recall from [19] that a $\mathbb{Z}_p$-Lie lattice $L$ is said to be self-similar of index $p^k$ if there exists a homomorphism of algebras $\varphi : M \to L$, where $M \subseteq L$ is a subalgebra of index $p^k$ and $\varphi$ is simple, which means that there are no non-zero ideals of $L$ that are $\varphi$-invariant.

In Section 2, we prove results on Lie lattices, and for the main ones mentioned here we assume that $p \geqslant 3$. The first main result of that section is *Theorem* 2.22, where we classify the 3-dimensional solvable $\mathbb{Z}_p$-Lie lattices that are self-similar of index $p$, complementing the analogue result for unsolvable lattices proven in [19, Theorem 2.32]. In Definition 2.32, we introduce the notion of (strongly) hereditarily self-similar Lie lattice. Thanks to the classification result, we are able to prove *Proposition* 2.41, which is a classification of the 3-dimensional $\mathbb{Z}_p$-Lie lattices that are (strongly) hereditarily self-similar of index $p$. This result is particularly relevant since it is used as the basis of the induction (which is on dimension) for the proof of the second main result on Lie lattices, *Theorem* 2.34, which provides a classification of the solvable $\mathbb{Z}_p$-Lie lattices that are strongly hereditarily self-similar of index $p$. At the beginning of Section 2 the reader will find a more detailed account of its structure.

In Section 3, we prove the main theorems of the paper and provide additional results on hereditarily self-similar groups. We observe that Theorem C follows from Theorem 2.22, Theorem D follows from Proposition 2.41, and Theorem A follows from Theorem 2.34. In Section 4, we state some open problems that we consider challenging and that we believe will stimulate future research on the subject.

**Notation.** Throughout the paper, $p$ denotes a prime number and $\equiv_p$ denotes equivalence modulo $p$. For $p \geqslant 3$ we fix $\rho \in \mathbb{Z}_p^*$ a non-square modulo $p$. We denote the $p$-adic valuation by $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$. The set $\mathbb{N}$ of natural numbers is assumed to contain 0. For the lower central series $\gamma_n(G)$ and the derived series $\delta_n(G)$ of a group (or Lie algebra) $G$

we use the conventions $\gamma_0(G) = G$ and $\delta_0(G) = G$. By a $\mathbb{Z}_p$-lattice we mean a finitely generated free $\mathbb{Z}_p$-module. Let $L$ be a $\mathbb{Z}_p$-lattice. When $M \subseteq L$ is a submodule, we denote the isolator of $M$ in $L$ by $\mathrm{iso}_L(M) := \{x \in L : \exists k \in \mathbb{N}\ p^k x \in M\}$. We denote by $\langle x_1, \dots, x_n \rangle$ the *submodule* of $L$ generated by $x_1, \dots, x_n \in L$. When $L$ has the structure of a Lie algebra, we denote its center by $Z(L)$.

## 2. Results on Lie lattices

In this section, which is self-contained, we prove results about self-similarity of $\mathbb{Z}_p$-Lie lattices. The main results, mentioned in Section 1, are proved under the assumption that $p \geqslant 3$. On the other hand, most of the auxiliary results are valid and proved for any $p$, and we believe that they constitute a large part of the work needed to generalize the main results to $p = 2$. The structure of the section is as follows. In Section 2.1, we prove some basic results on $\mathbb{Z}_p$-Lie lattices that admit an abelian ideal of codimension 1; these results are used both for the study of 3-dimensional lattices and of lattices in higher dimension. After two preparatory technical sections (Sections 2.2 and 2.3), in Section 2.4, we prove one of the main theorems on Lie lattices (Theorem 2.22). After another preparatory section (Section 2.5), in Section 2.6, we prove the other two main results (Proposition 2.41 and Theorem 2.34). Apart from the main results, a few statements are worth to be mentioned here, for instance, Propositions 2.6, 2.13, and 2.36. The most difficult technical results are the proofs of non-self-similarity of Propositions 2.20 and 2.25.

We will be dealing with several families of $\mathbb{Z}_p$-Lie lattices, which we list in the definition below. For $p \geqslant 3$, families from (0) to (5) are needed for the classification of 3-dimensional solvable $\mathbb{Z}_p$-Lie lattices (see Remark 2.21). Family (6) generalizes families (0) and (1), while family (7) generalizes families (4) and (5).

**Definition 2.1.** We define eight families of 3-dimensional solvable $\mathbb{Z}_p$-Lie lattices through presentations.

(0) $L_0 = \langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = 0,\ [x_0, x_2] = 0 \rangle$.

(1) For $s \in \mathbb{N}$, $L_1(s) = \langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^s x_1,\ [x_0, x_2] = p^s x_2 \rangle$.

(2) For $s, r \in \mathbb{N}$ and $c \in \mathbb{Z}_p$,

$$\begin{aligned} L_2(s, r, c) = \big\langle x_0, x_1, x_2 \mid [x_1, x_2] &= 0, \\ [x_0, x_1] &= p^s x_1 + p^{s+r} c x_2, \\ [x_0, x_2] &= p^{s+r} x_1 + p^s x_2 \big\rangle. \end{aligned}$$

(3) For $s \in \mathbb{N}$, $L_3(s) = \langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = 0,\ [x_0, x_2] = p^s x_1 \rangle$.

(4) For $p \geqslant 3$, $s, t \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$,

$$L_4(s, t, \varepsilon) = \big\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^{s+t} \rho^{\varepsilon} x_2,\ [x_0, x_2] = p^s x_1 \big\rangle.$$

(5) For $s, r \in \mathbb{N}$ and $c \in \mathbb{Z}_p$,

$$\begin{aligned} L_5(s,r,c) = \big\langle x_0, x_1, x_2 \mid [x_1, x_2] &= 0, \\ [x_0, x_1] &= p^{s+r} x_1 + p^s c x_2, \\ [x_0, x_2] &= p^s x_1 \big\rangle. \end{aligned}$$

(6) For $a \in \mathbb{Z}_p$, $L_6(a) = \langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = a x_1,\ [x_0, x_2] = a x_2 \rangle$.

(7) For $s \in \mathbb{N}$ and $a, c \in \mathbb{Z}_p$,

$$L_7(s,a,c) = \big\langle x_0, x_1, x_2 \mid [x_1, x_2] = 0,\ [x_0, x_1] = p^s a x_1 + p^s c x_2,\ [x_0, x_2] = p^s x_1 \big\rangle.$$

### 2.1. On a class of metabelian Lie lattices

Given an integer $d \geqslant 1$, we are going to consider $(d+1)$-dimensional $\mathbb{Z}_p$-Lie lattices that admit a $d$-dimensional abelian ideal. Greek indices will take values in $\{0, \dots, d\}$, while Latin indices will take values in $\{1, \dots, d\}$. For matrices in $\mathrm{gl}_{d+1}(\mathbb{Q}_p)$ we use a notation like $\bar{U} = (U_{\alpha\beta})$; moreover, for such a matrix, we denote $U = (U_{ij}) \in \mathrm{gl}_d(\mathbb{Q}_p)$.

Let $L$ be a $(d+1)$-dimensional antisymmetric $\mathbb{Z}_p$-algebra. Observe that $L$ admits a $d$-dimensional abelian ideal if and only if there exists a basis $\boldsymbol{x} = (x_0, \dots, x_d)$ of $L$ and a matrix $A \in \mathrm{gl}_d(\mathbb{Z}_p)$, $A = (A_{ij})$, such that for all $i, j$ we have

$$\begin{aligned} [x_i, x_j] &= 0, \\ [x_0, x_i] &= \sum_l A_{li} x_l. \end{aligned}$$

In this case, $\langle x_1, \dots, x_d \rangle$ is a $d$-dimensional abelian ideal. It is immediate to see that, for such an $L$, the Jacobi identity holds, and that $\delta_2(L) = \{0\}$; in other words, $L$ is a metabelian Lie lattice. When it exists, a basis as above is called a *good basis* of $L$, and $A$ is called the *matrix of* $L$ with respect to the (good) basis $\boldsymbol{x}$. Observe that $A$ is the matrix of the homomorphism of lattices $[x_0, \cdot] : \langle x_1, \dots, x_d \rangle \to \langle x_1, \dots, x_d \rangle$ with respect to the displayed bases.

Let $L$ be a $(d+1)$-dimensional $\mathbb{Z}_p$-Lie lattice that admits a $d$-dimensional abelian ideal, let $\boldsymbol{x}$ be a good basis of $L$, and let $A$ be the corresponding matrix. Observe that $\mathrm{rk}(A) = \dim[L, L]$, so that $\mathrm{rk}(A)$ is an isomorphism invariant of $L$. In particular, $A$ is invertible over $\mathbb{Q}_p$ if and only if $\dim[L, L] = d$, a relevant special case. Let $M \subseteq L$ be a finite-index submodule, let $\boldsymbol{y} = (y_0, \dots, y_d)$ be a basis of $M$, and let $\bar{U} = (U_{\alpha\beta}) \in \mathrm{gl}_{d+1}(\mathbb{Z}_p)$ be the matrix of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$, namely, $y_\beta = \sum_\alpha U_{\alpha\beta} x_\alpha$. Observe that $M \cap \langle x_1, \dots, x_d \rangle = \langle y_1, \dots, y_d \rangle$ if and only if $U_{0i} = 0$ for all $i$; moreover, there exists a basis of $M$ such that $U_{\alpha\beta} = 0$ for all $\alpha < \beta$. We also observe that $\dim[M, M] = \dim[L, L]$.

**Lemma 2.2.** *Let $d, L, \boldsymbol{x}, A, M, \boldsymbol{y}, \bar{U} = (U_{\alpha\beta})$ be as above. Assume that $M \cap \langle x_1, \dots, x_d \rangle = \langle y_1, \dots, y_d \rangle$. Then $U = (U_{ij})$ is invertible over $\mathbb{Q}_p$ (it is a $d \times d$ matrix), and one may define $B \in \mathrm{gl}_d(\mathbb{Q}_p)$ by $B = U_{00} U^{-1} A U$. Then the following holds.*

(1) *$M$ is a subalgebra of $L$ if and only if $B$ has entries in $\mathbb{Z}_p$.*

(2) *Assume that $M$ is a subalgebra of $L$. Then $\boldsymbol{y}$ is a good basis of $M$ and $B$ is the matrix of $M$ with respect to $\boldsymbol{y}$.*

*Proof.* Since $y_j = \sum_i U_{ij} x_i$, it follows that $[y_i, y_j] = 0$. Over $\mathbb{Q}_p$, we have

$$[y_0, y_j] = U_{00} \sum_i U_{ij} [x_0, x_i] = U_{00} \sum_{i,l} U_{ij} A_{li} x_l = U_{00} \sum_{i,l,k} U_{ij} A_{li} U_{kl}^{-1} y_k,$$

so that $[y_0, y_j] = \sum_k B_{kj} y_k$. The lemma follows. ■

Observe that the case $M = L$ is included in the above discussion. In this case, $U$ is invertible over $\mathbb{Z}_p$, and the defining formula of $B$ is the change-of-basis formula for the matrix of $L$ (under lower block-triangular changes of basis).

We now study homomorphisms of algebras.

**Lemma 2.3.** *Let $L, M$ be $(d+1)$-dimensional $\mathbb{Z}_p$-Lie lattices endowed with good bases $\boldsymbol{x}, \boldsymbol{y}$, and let $A, B$ be the respective matrices. Let $\varphi : M \to L$ be a homomorphism of modules, and let $\overline{F} \in \mathrm{gl}_{d+1}(\mathbb{Z}_p)$ be the matrix of $\varphi$ with respect to the given bases; namely, $\varphi(y_\beta) = \sum_\alpha F_{\alpha\beta} x_\alpha$. Then the following holds.*

(1) *The homomorphism $\varphi$ is a homomorphism of algebras if and only if, for all $i, j$:*
   (a) $\sum_l F_{0l} B_{lj} = 0$;
   (b) $F_{0i}(AF)_{kj} - F_{0j}(AF)_{ki} = 0$, *for all $k$;*
   (c) $(FB)_{ij} = F_{00}(AF)_{ij} - F_{0j} \sum_l A_{il} F_{l0}$.

(2) *Assume that $\varphi$ is a homomorphism of algebras and* $\dim[M, M] = d$. *Then $F_{0i} = 0$ for all $i$.*

(3) *Assume that $F_{0i} = 0$ for all $i$. Then $\varphi$ is a homomorphism of algebras if and only if $FB = F_{00} AF$.*

*Proof.* The homomorphism $\varphi$ is a homomorphism of algebras if and only if, for all $i, j$, $[\varphi(y_i), \varphi(y_j)] = 0$ and $\varphi([y_0, y_j]) = [\varphi(y_0), \varphi(y_j)]$. One computes

$$\big[\varphi(y_i), \varphi(y_j)\big] = \sum_l (F_{0i} F_{lj} - F_{0j} F_{li})[x_0, x_l] = \sum_k \big(F_{0i}(AF)_{kj} - F_{0j}(AF)_{ki}\big) x_k,$$

$$\varphi\big([y_0, y_j]\big) = \sum_{l,\alpha} B_{lj} F_{\alpha l} x_\alpha = \Big(\sum_l F_{0l} B_{lj}\Big) x_0 + \sum_i (FB)_{ij} x_i,$$

$$\big[\varphi(y_0), \varphi(y_j)\big] = \sum_l (F_{00} F_{lj} - F_{l0} F_{0j})[x_0, x_l] = \sum_i \Big(F_{00}(AF)_{ij} + F_{0j} \sum_l A_{il} F_{l0}\Big) x_i,$$

from which item (1) follows. For item (2), one observes that $B$ is invertible over $\mathbb{Q}_p$ and applies item (1a). Item (3) follows directly from item (1). ■

**Corollary 2.4.** *Let $L$ be a $(d+1)$-dimensional $\mathbb{Z}_p$-Lie lattice with* $\dim[L, L] = d$, *and let $\varphi : M \to L$ be a virtual endomorphism of $L$. Let $\boldsymbol{x}$ be a good basis of $L$. Then the following holds.*

(1) *Let* $\mathbf{y}$ *be a basis of* $M$ *with the property* $M \cap \langle x_1, \ldots, x_d \rangle = \langle y_1, \ldots, y_d \rangle$. *Then* $FB = F_{00}AF$, *where* $A$, $B$, *and* $\overline{F}$ *are as in Lemma* 2.3.

(2) *Assume that* $\langle x_1, \ldots, x_d \rangle \subseteq M$. *Then* $\langle x_1, \ldots, x_d \rangle$ *is a* $\varphi$*-invariant ideal of* $L$.

**Remark 2.5.** Any 3-dimensional solvable $\mathbb{Z}_p$-Lie lattice admits a 2-dimensional abelian ideal.

### 2.2. Self-similarity results

When $\varphi : M \to L$ is a virtual endomorphism of a Lie lattice $L$, we denote by $D_n$, $n \in \mathbb{N}$, the domain of the power $\varphi^n$ and define $D_\infty = \bigcap_{n \in \mathbb{N}} D_n$. We recall that, by definition, $D_0 = L$ and $D_{n+1} = \{x \in M : \varphi(x) \in D_n\}$ (see, for instance, [19, Definition 1.1]).

**Proposition 2.6.** *Let* $k, d \geqslant 1$ *be integers, and let* $L$ *be a* $\mathbb{Z}_p$*-Lie lattice of dimension* $d+1$. *Assume that* $L$ *admits a* $d$*-dimensional abelian ideal. Then* $L$ *is self-similar of index* $p^{dk}$.

*Proof.* If $L$ is abelian, it is easy to see that $L$ is self-similar of index $p^m$ for all $m \geqslant 1$. Assume that $L$ is not abelian. There exists a basis $(x_0, x_1, \ldots, x_d)$ of $L$ such that $[x_i, x_j] = 0$ and $[x_0, x_i] = \sum_{l=1}^{d} A_{li} x_l$ for all $1 \leqslant i, j \leqslant d$, and some $A_{li} \in \mathbb{Z}_p$. We define $M = \langle x_0, p^k x_1, \ldots, p^k x_d \rangle$ and observe that $M$ is a subalgebra of $L$ of index $p^{dk}$. We define a homomorphism of algebras $\varphi : M \to L$ by $\varphi(x_0) = x_0$ and $\varphi(p^k x_i) = x_i$ for $1 \leqslant i \leqslant d$. We are going to show that $\varphi$ is simple. One sees that $D_\infty = \langle x_0 \rangle$. Let $I$ be a non-trivial ideal of $L$. We show that $I$ is not $\varphi$-invariant by proving the existence of $w \in I$ such that $w \notin D_\infty$. Indeed, there exists $0 \neq z = a_0 x_0 + \cdots + a_d x_d \in I$. If $a_i \neq 0$ for some $i > 0$, then one may take $w = z$. Otherwise, $z = a_0 x_0$ with $a_0 \neq 0$. Since $L$ is not abelian, there exists $i > 0$ such that $[x_0, x_i] \neq 0$. In this case one may take $w = [z, x_i]$. ∎

**Corollary 2.7.** *Let* $k \geqslant 1$ *be an integer, and let* $L$ *be a* 3*-dimensional solvable* $\mathbb{Z}_p$*-Lie lattice. Then* $L$ *is self-similar of index* $p^{2k}$.

*Proof.* Since $L$ admits a 2-dimensional abelian ideal, the corollary follows from Proposition 2.6. ∎

In order to have a more elegant proof of simplicity in Lemma 2.9 below, we observe that the following generalization of [16, Proposition 2.9.2] holds. Let $R$ be a principal ideal domain, and let $K$ be the field of fractions of $R$. We identify $R \subseteq K$. Let $d \in \mathbb{N}$, $\Phi : K^d \to K^d$ be a $K$-linear function, and let $p_\Phi(\lambda) \in K[\lambda]$ be the characteristic polynomial of $\Phi$. Let $M$ be the set of $x \in R^d$ such that $\Phi(x) \in R^d$. Then $M$ is a sub-$R$-module of $R^d$ and the restriction $\varphi : M \to R^d$ of $\Phi$ may be interpreted as a virtual endomorphism of the $R$-module $R^d$ (in the application below, $R^d$ is thought of as an abelian $R$-Lie lattice).

**Proposition 2.8.** *In the context described above,* $D_\infty = \{0\}$ *if and only if there are no monic irreducible factors of* $p_\Phi(\lambda)$ *with coefficients in* $R$.

*Proof.* The proof of [16, Proposition 2.9.2] works in this more general context. ∎

**Lemma 2.9.** *Let $k \geqslant 1$ be an integer. Then the following Lie lattices are self-similar of index $p^k$:*

(1) $L_6(a)$;

(2) $L_2(s, r, c)$ *with* $v_p(c) = 1$;

(3) $L_7(s, a, c)$ *with* $v_p(c) = 1$ *and* $v_p(a) \geqslant 1$, *or with* $v_p(4c + a^2) = 1$, $v_p(a) = 0$, *and* $v_p(c) = 0$;

(4) *for* $p \geqslant 3$, $L_7(s, 0, 1)$.

*Proof.* Let $(x_0, x_1, x_2)$ be the basis of the relevant Lie lattice as given by its presentation in Definition 2.1. We begin with $L = L_6(a)$, where we exhibit a simple virtual endomorphism $\varphi : M \to L$ of index $p^k$. Define $M = \langle x_0, x_1, p^k x_2\rangle$. For $a = 0$, the abelian case, define $\varphi(x_0) = x_1$, $\varphi(x_1) = x_2$, and $\varphi(p^k x_2) = x_0$. For $a \neq 0$, define $\varphi(x_0) = x_0$, $\varphi(x_1) = x_2$, and $\varphi(p^k x_2) = x_1$. Recall that $D_\infty$ is the intersection of the domains of the powers of $\varphi$. In the abelian case one shows that $D_\infty = \{0\}$, while in the non-abelian case one shows that $D_\infty = \langle x_0\rangle$. Since a $\varphi$-invariant subset of $L$ has to be a subset of $D_\infty$, in both cases one shows that a non-zero ideal of $L$ is not $\varphi$-invariant (cf. the proof of Proposition 2.6).

We now denote by $L$ any of the Lie lattices that remain to be analyzed. From Corollary 2.7, it is enough to treat the case where $k = 2l + 1$ is odd. We exhibit a simple virtual endomorphism $\varphi : M \to L$ of index $p^{2l+1}$. For $L_2(s, r, c)$, define $M = \langle x_0, p^l x_1, p^{l+1} x_2\rangle$ and $\varphi(x_0) = x_0$, $\varphi(p^l x_1) = x_1 + p^{-1} c x_2$, and $\varphi(p^{l+1} x_2) = x_1 + p x_2$. For $L_7(s, a, c)$ with $v_p(c) = 1$ and $v_p(a) \geqslant 1$, define $M = \langle x_0, p^l x_1, p^{l+1} x_2\rangle$ and $\varphi(x_0) = x_0$, $\varphi(p^l x_1) = p^{-1} c x_2$, and $\varphi(p^{l+1} x_2) = x_1 - a x_2$. For $L_7(s, a, c)$ with $v_p(4c + a^2) = 1$, $v_p(a) = 0$ and $v_p(c) = 0$ (necessarily $p \geqslant 3$), define $M = \langle x_0, p^l(x_1 - 2^{-1} a x_2), p^{l+1} x_2\rangle$ and $\varphi(x_0) = x_0$, $\varphi(p^l(x_1 - 2^{-1} a x_2)) = p^{-1}(c + 4^{-1} a^2) x_2$, and $\varphi(p^{l+1} x_2) = x_1 - 2^{-1} a x_2$. Finally, for $L_7(s, 0, 1)$, define $M = \langle x_0, p^l(x_1 - x_2), p^{l+1} x_2\rangle$ and $\varphi(x_0) = -x_0$, $\varphi(p^l(x_1 - x_2)) = x_1 + x_2$, and $\varphi(p^{l+1} x_2) = x_1 - (1 + p) x_2$. The proof of simplicity of $\varphi$ may go as follows. Let $\psi : M \cap \langle x_1, x_2\rangle \to \langle x_1, x_2\rangle$ be the restriction of $\varphi$. Let $D_\infty$ be as above, and let $E_\infty$ be the intersection of the domains of the powers of $\psi$. We have $D_\infty = \langle x_0\rangle \oplus E_\infty$ (indeed, $\varphi$ is the direct sum of $\psi$ and a homomorphism that sends $\langle x_0\rangle$ to $\langle x_0\rangle$). We claim that $E_\infty = \{0\}$, from which the simplicity of $\varphi$ follows. Observe that in each of the cases at hand $\psi$ is an isomorphism. Because of that, one can see that the virtual endomorphism associated with $\Phi := \psi \otimes \mathbb{Q}_p$ (as described above Proposition 2.8) may be identified with $\psi$. Hence, by the proposition itself, it suffices to show that the characteristic polynomial $p(\lambda) \in \mathbb{Q}_p[\lambda]$ of $\Phi : \mathbb{Q}_p x_1 \oplus \mathbb{Q}_p x_2 \to \mathbb{Q}_p x_1 \oplus \mathbb{Q}_p x_2$ has no monic irreducible factors with coefficients in $\mathbb{Z}_p$. We treat the case of $L_2(s, r, c)$; the other cases are similar and are left to the reader. We have $\Phi(x_1) = p^{-l} x_1 + p^{-l-1} c x_2$ and $\Phi(x_2) = p^{-l-1} x_1 + p^{-l} x_2$. Then $p(\lambda) = \lambda^2 - 2p^{-l}\lambda + p^{-2l} - c p^{-2l-2}$. Observe that $v_p(p^{-2l} - c p^{-2l-2}) = -2l - 1 < 0$ so that in case $p(\lambda)$ is irreducible there is nothing left to prove. Assume that $p(\lambda)$ is reducible. The proof of the lemma is concluded once we prove that this assumption leads to a contradiction. Indeed, $p(\lambda) = (\lambda - \mu)(\lambda - \nu)$ for some $\mu, \nu \in \mathbb{Q}_p$. We have $\mu + \nu = 2p^{-l}$ and $\mu\nu = p^{-2l} - c p^{-2l-2}$. Since $v_p(\mu) + v_p(\nu) = -2l - 1$, then, without loss of generality, we can assume that $v_p(\mu) < v_p(\nu)$, so that $v_p(\mu) \leqslant -l - 1$. It follows that $v_p(\mu + \nu) \leqslant -l - 1 < v_p(2p^{-l}) = v_p(\mu + \nu)$, a contradiction. ∎

### 2.3. Non-self-similarity results in dimension 3

The main results of this relatively long technical section are Propositions 2.13, 2.16, and 2.20.

**Remark 2.10.** Let $L$ be a 3-dimensional $\mathbb{Z}_p$-lattice endowed with a basis $(x_0, x_1, x_2)$. For $e, f \in \mathbb{Z}_p$ we define submodules of $L$ of index $p$ by $L^{()} = \langle px_0, x_1, x_2\rangle$, $L^{(e)} = \langle x_0 + ex_1, px_1, x_2\rangle$, and $L^{(e,f)} = \langle x_0 + ex_2, x_1 + fx_2, px_2\rangle$. Any submodule of $L$ of index $p$ is isomorphic to $L^{\xi}$ for some $\xi = (\,), (e), (e, f)$. By changing $e$ or $f$ modulo $p$, $L^{(e)}$ and $L^{(e,f)}$ do not change (cf. [19, Definition 2.22, Lemma 2.23]). Observe that when $L^{\xi}$ is displayed as above, it is endowed with a basis.

Assume that $M$ is a submodule of $L$ of index $p$ endowed with a basis $(y_0, y_1, y_2)$, and let $\varphi : M \to L$ be a homomorphism of modules. We denote by $\overline{F} = (F_{\alpha\beta}) \in \mathrm{gl}_3(\mathbb{Z}_p)$ the matrix of $\varphi$ relative to the respective bases, namely, $\varphi(y_\beta) = \sum_\alpha F_{\alpha\beta} x_\alpha$ (cf. Section 2.1).

First, we treat the case where $\dim[L, L] = 1$.

**Lemma 2.11.** *Let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie lattice with $\dim[L, L] = 1$. Then the following holds.*

(1) $\dim Z(L) = 1$.

(2) *Let $M \subseteq L$ be a subalgebra of index $p$. Then $Z(L) \subseteq M$ or $[M, M] = [L, L]$.*

*Proof.* There exist $s \in \mathbb{N}$, $r \in \mathbb{N} \cup \{\infty\}$ and a basis $(x_0, x_1, x_2)$ of $L$ such that $[x_1, x_2] = 0$, $[x_0, x_1] = p^s(p^r x_1 + x_2)$, and $[x_0, x_2] = 0$, where $p^\infty := 0$. For item (1), one easily checks that $Z(L) = \langle x_2\rangle$. For item (2), one observes that if $M$ is of type $L^{()}$ or $L^{(e)}$ (cf. Remark 2.10), then $Z(L) \subseteq M$. On the other hand, if $M$ is of type $L^{(e,f)}$, then it is a straightforward computation to show that $[M, M] = [L, L]$. ∎

**Lemma 2.12.** *Let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie lattice with $\dim[L, L] = 1$. Let $\varphi : M \to L$ be a virtual endomorphism of $L$. If $\varphi$ is simple, then $\varphi$ is injective.*

*Proof.* Assume that $\varphi$ is not injective. We exhibit a non-trivial $\varphi$-invariant ideal $I$ of $L$.

*Case 1:* $\ker\varphi \subseteq Z(L)$. Since $\dim Z(L) = 1$, then $\dim\ker\varphi = 1$, so that there exists $k \in \mathbb{N}$ such that $p^k Z(L) \subseteq \ker\varphi$. Thus, it suffices to take $I = p^k Z(L)$.

*Case 2:* $\ker\varphi \not\subseteq Z(L)$. There exists $z \in \ker\varphi$ such that $z \notin Z(L)$, so $[w, z] \neq 0$ for some $w \in L$. Since $M$ has finite index in $L$, there exists $k \in \mathbb{N}$ such that $p^k w \in M$. Hence, $p^k[w, z] \neq 0$ and $p^k[w, z] \in \ker\varphi$. By taking $x \in L$ such that $\mathrm{iso}_L[L, L] = \langle x\rangle$, one sees that $p^k[w, z] = ax$ for some $a \in \mathbb{Z}_p$ with $a \neq 0$. Thus, it suffices to take $I = \langle ax\rangle$. ∎

**Proposition 2.13.** *Let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie lattice with $\dim[L, L] = 1$. Then $L$ is not self-similar of index $p$.*

*Proof.* Let $\varphi : M \to L$ be a virtual endomorphism of $L$ of index $p$. We prove that $\varphi$ is not simple by either referring to a previous result or by exhibiting a non-trivial $\varphi$-invariant ideal $I$ of $L$. If $[M, M] = [L, L]$, then it suffices to take $I = [L, L]$. Otherwise, by item (2) of Lemma 2.11, we have $Z(L) \subseteq M$. Then $Z(L) = Z(M)$. Also, if $\varphi$ is not injective, then $\varphi$ is not simple (Lemma 2.12); hence, we can assume that $\varphi$ is injective. Then $\dim \varphi(M) = \dim L$, so that $\varphi(Z(M)) \subseteq Z(L)$. Thus, it suffices to take $I = Z(L)$. ■

Next, we treat the case where $\dim[L, L] = 2$.

**Lemma 2.14.** *In the context of Remark 2.10, assume that $F_{01} = F_{02} = 0$. Then the following holds.*

(1) *Assume that $M = L^{()}$. Then $\langle x_1, x_2\rangle$ is $\varphi$-invariant.*

(2) *Assume that $M = L^{(e)}$, $p|F_{11}$, and $p|F_{21}$. Then $\langle px_1, px_2\rangle$ is $\varphi$-invariant.*

(3) *Assume that $M = L^{(e,f)}$, $p|F_{12}$, and $p|F_{22}$. Then $\langle px_1, px_2\rangle$ is $\varphi$-invariant.*

(4) *Assume that $M = L^{(e,f)}$, $f \equiv_p 0$, $p|F_{21}$, and $p|F_{22}$. Then $\langle x_1, px_2\rangle$ is $\varphi$-invariant.*

*Proof.* We leave the simple proof to the reader. ■

**Lemma 2.15.** *Let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie lattice with $\dim[L, L] = 2$, and let $\boldsymbol{x} = (x_0, x_1, x_2)$ be a good basis of $L$. Let $M = \langle x_0 + ex_1, px_1, x_2\rangle$ for some $e \in \mathbb{Z}_p$, assume that $M$ is a subalgebra of $L$, and let $\varphi : M \to L$ be a homomorphism of algebras. Let*

$$A = p^s \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad s \in \mathbb{N}, \quad a, b, c, d \in \mathbb{Z}_p$$

*be the matrix of $L$ with respect to $\boldsymbol{x}$. Moreover, assume that one of the following conditions is true:*

(1) *$v_p(b) = 0$ or*

(2) *$a = d = 1$, $v_p(b) \leqslant v_p(c)$, and $b \neq 0$.*

*Then $\langle px_1, px_2\rangle$ is a $\varphi$-invariant ideal of $L$.*

*Proof.* Clearly, $I = \langle px_1, px_2\rangle$ is an ideal of $L$. Let $B$ be the matrix of $M$ with respect to the displayed basis, and let $\bar{F}$ be the matrix of $\varphi$. From item (1) of Corollary 2.4 it follows that $FB = F_{00}AF$, and this matrix equation is equivalent to the system of scalar equations

$$a(1 - F_{00})F_{11} + pcF_{12} - bF_{00}F_{21} = 0, \tag{2.1}$$

$$-cF_{00}F_{11} + (a - dF_{00})F_{21} + pcF_{22} = 0, \tag{2.2}$$

$$bF_{11} + p(d - aF_{00})F_{12} - pbF_{00}F_{22} = 0, \tag{2.3}$$

$$-pcF_{00}F_{12} + bF_{21} + pd(1 - F_{00})F_{22} = 0. \tag{2.4}$$

From item (2) of Lemma 2.14 it is enough to show that $p|F_{11}$ and $p|F_{21}$. Indeed, we claim that $p|F_{11}$ and $p|F_{21}$ and proceed to prove the claim. In case $v_p(b) = 0$, the claim follows from (2.3) and (2.4). Assume that $a = d = 1$ and $r := v_p(b) \leqslant v_p(c)$. If $v_p(1 - F_{00}) \geqslant r$, the claim follows again from (2.3) and (2.4). In case $v_p(1 - F_{00}) < r$ the claim follows from (2.1) and (2.2). ∎

**Proposition 2.16.** *Let $s, r \in \mathbb{N}$ with $r \geqslant 1$, and let $c \in \mathbb{Z}_p$ with $v_p(c) \neq 1$. Then $L_2(s, r, c)$ is not self-similar of index $p$.*

*Proof.* Observe that $\dim[L, L] = 2$. Let $\varphi : M \to L$ be a virtual endomorphism of $L$ of index $p$. We will show that there exists a non-trivial $\varphi$-invariant ideal $I$ of $L$, from which the proposition follows. Observe that $\langle x_1, x_2\rangle$, $\langle px_1, px_2\rangle$, and $\langle x_1, px_2\rangle$ are non-trivial ideals of $L$. The $\varphi$-invariance of the various $I$ defined below follows from Lemma 2.3 (2) and Lemma 2.14. Observe that the matrix equation $FB = F_{00}AF$ of item (1) of Corollary 2.4 holds. We divide the proof into four cases.

*Case 1: $M = L^{()}$.* It suffices to take $I = \langle x_1, x_2\rangle$.

*Case 2: $M = L^{(e)}$.* It suffices to take $I = \langle px_1, px_2\rangle$ (Lemma 2.15).

*Case 3: $M = L^{(e,f)}$ with $f \not\equiv_p 0$.* The matrix equation $FB = F_{00}AF$ implies that the following equations hold true:

$$p(1 + p^r f - F_{00})F_{11} + (-f - p^r f^2 + p^r c + p^2 f)F_{12} - p^{r+1}F_{00}F_{21} = 0,$$
$$-p^{r+1}cF_{00}F_{11} + p(1 + p^r f - F_{00})F_{21} + (-f - p^r f^2 + p^r c + p^2 f)F_{22} = 0,$$

from which we can see that $p|F_{12}$ and $p|F_{22}$. Thus, it suffices to take $I = \langle px_1, px_2\rangle$.

*Case 4: $M = L^{(e,0)}$.* The matrix equation $FB = F_{00}AF$ is equivalent to the equations

$$(1 - F_{00})F_{11} + p^{r-1}cF_{12} - p^r F_{00}F_{21} = 0, \tag{2.5}$$
$$-p^r cF_{00}F_{11} + (1 - F_{00})F_{21} + p^{r-1}cF_{22} = 0, \tag{2.6}$$
$$p^{r+1}F_{11} + (1 - F_{00})F_{12} - p^r F_{00}F_{22} = 0, \tag{2.7}$$
$$-p^r cF_{00}F_{12} + p^{r+1}F_{21} + (1 - F_{00})F_{22} = 0. \tag{2.8}$$

If $v_p(1 - F_{00}) < r$, then (2.7) and (2.8) imply that $p|F_{12}$ and $p|F_{22}$, and we can take $I = \langle px_1, px_2\rangle$. Assume that $l := v_p(1 - F_{00}) \geqslant r$. Observe that, since $r \geqslant 1$, $F_{00} \in \mathbb{Z}_p^*$. We divide the proof into two cases, according to whether $v_p(c) \geqslant 2$ or $v_p(c) = 0$.

(1) Assume that $v_p(c) \geqslant 2$.
   (a) Assume that $l \geqslant r + 1$. From (2.7), we have $p|F_{22}$, so that $p|F_{21}$ (see (2.8)). Thus, it suffices to take $I = \langle x_1, px_2\rangle$.
   (b) Assume that $l = r$. From (2.8), we have $p|F_{22}$, so that $p|F_{12}$ (see (2.7)). Thus, it suffices to take $I = \langle px_1, px_2\rangle$.

(2) Assume that $v_p(c) = 0$. From (2.5), we have $p|F_{12}$; from (2.6), we have $p|F_{22}$. Thus, it suffices to take $I = \langle px_1, px_2\rangle$. ∎

**Lemma 2.17.** *Let $s \in \mathbb{N}$ and $a, c, e, f \in \mathbb{Z}_p$ with $c \neq 0$. Define $L = L_7(s, a, c)$, where $L$ is endowed with the basis $(x_0, x_1, x_2)$ given in Definition 2.1. Let $M = \langle x_0 + ex_2,$*

*$x_1 + fx_2, px_2\rangle$ and assume that $M$ is a subalgebra of $L$. Let $\varphi : M \to L$ be homomorphism of algebras, and let $F$ be the matrix of $\varphi$ with respect to the given bases. Then*

$$pF_{21} - F_{00}\big[pfF_{11} + (c - af - f^2)F_{12}\big] = 0, \tag{2.9}$$

$$F_{22} - F_{00}\big[pF_{11} - (a + f)F_{12}\big] = 0, \tag{2.10}$$

$$(F_{00} - 1)\big[-p(1 + F_{00})F_{11} + \big(f(1 + F_{00}) + aF_{00}\big)F_{12}\big] = 0, \tag{2.11}$$

$$(F_{00} - 1)\big[p\big(a + f(1 + F_{00})\big)F_{11} + (1 + F_{00})(c - af - f^2)F_{12}\big] = 0. \tag{2.12}$$

*Proof.* The result follows from Corollary 2.4 (1). ■

**Lemma 2.18.** *In the context of Lemma 2.17, the following holds.*

(1) *Assume that $p | F_{12}$. Then $\langle px_1, px_2\rangle$ is a $\varphi$-invariant ideal of $L$.*

(2) *Assume that $c - af - f^2 = 0$, $a \neq 0$, $2f + a \not\equiv_p 0$, and $F_{12} \in \mathbb{Z}_p^*$. Then $\langle x_1 + fx_2\rangle$ is a $\varphi$-invariant ideal of $L$.*

(3) *Assume that $p \geqslant 3$, $f = -2^{-1}a$, $v_p(a) = 0$, $v_p(4c + a^2) \geqslant 2$, and $F_{12} \neq 0$. Then $\langle x_1 - 2^{-1}ax_2, px_2\rangle$ is a $\varphi$-invariant ideal of $L$.*

*Proof.* (1) From (2.10) of Lemma 2.17 it follows that $p | F_{22}$. Now the item follows from item (3) of Lemma 2.14.

(2) Observe that $f \neq 0, -a$, since $c \neq 0$. One checks directly that $[x_0, x_1 + fx_2]$ is a $\mathbb{Z}_p$-multiple of $x_1 + fx_2$; hence, $I = \langle x_1 + fx_2\rangle$ is an ideal of $L$. We have

$$\varphi(x_1 + fx_2) = F_{11}(x_1 + fx_2) + (F_{21} - fF_{11})x_2.$$

(a) Case $F_{00} = 1$. From (2.9) of Lemma 2.17 we have $F_{21} = fF_{11}$; hence, $I$ is $\varphi$-invariant.

(b) Case $F_{00} \neq 1$. Since $F_{12} \in \mathbb{Z}_p^*$, (2.11) and (2.12) of Lemma 2.17 have a non-trivial solution in the variables $F_{11}$ and $F_{12}$. It follows that

$$\big[f(1 + F_{00}) + aF_{00}\big]\big[a + f(1 + F_{00})\big] = 0.$$

(i) Case $f(1 + F_{00}) + aF_{00} = 0$. We have

$$F_{00} = -\frac{f}{a + f}, \quad a + f(1 + F_{00}) = a\frac{a + 2f}{a + f} \neq 0.$$

Hence, $F_{11} = 0$ (see (2.12)), so that $F_{21} = 0$ (see (2.9)). Hence, $I$ is $\varphi$-invariant.

(ii) Case $a + f(1 + F_{00}) = 0$. We show that we have a contradiction. Indeed, $F_{00} = -\frac{a+f}{f}$, so that $a[pF_{11} - (2f + a)F_{12}] = 0$ (see (2.11)), and consequently $pF_{11} = (2f + a)F_{12} \in \mathbb{Z}_p^*$, which is a contradiction.

(3) By applying $[x_0, \cdot\,]$ to its generators, one sees that $I = \langle x_1 - 2^{-1}ax_2, px_2\rangle$ is an ideal of $L$. We have (cf. item (2) of Lemma 2.3)

$$\varphi(x_1 - 2^{-1}ax_2) = F_{11}(x_1 - 2^{-1}ax_2) + (F_{21} + 2^{-1}aF_{11})x_2,$$

$$\varphi(px_2) = F_{12}(x_1 - 2^{-1}ax_2) + (F_{22} + 2^{-1}aF_{12})x_2,$$

from which we see that to show that $I$ is $\varphi$-invariant is equivalent to show that $p|(F_{21}+2^{-1}aF_{11})$ and $p|(F_{22}+2^{-1}aF_{12})$. We claim that, indeed, $p|(F_{21}+2^{-1}aF_{11})$ and $p|(F_{22}+2^{-1}aF_{12})$. In fact, (2.9) and (2.10) of Lemma 2.17 are equivalent to

$$F_{21}+2^{-1}aF_{11} = -(F_{00}-1)2^{-1}aF_{11}+F_{00}p^{-1}(c+4^{-1}a^2)F_{12},$$
$$F_{22}+2^{-1}aF_{12} = pF_{00}F_{11}-(F_{00}-1)2^{-1}aF_{12}.$$

(a) Case $F_{00}=1$. The claim is obviously true.

(b) Case $F_{00}\neq 1$. Since $F_{12}\neq 0$, (2.11) and (2.12) of Lemma 2.17 have a non-trivial solution in the variables $F_{11}$ and $F_{12}$. Hence, the determinant of the coefficient matrix has to be zero, which implies that

$$a^2(F_{00}-1)^2-(F_{00}+1)^2(4c+a^2)=0.$$

It follows that $p|(F_{00}-1)$, so the claim is true. ∎

**Corollary 2.19.** *In the context of Lemma 2.17, assume that $a\neq 0$ and that $f$ is a simple root modulo $p$ of the polynomial $P(\kappa)=\kappa^2+a\kappa-c$. Then $\varphi$ is not simple.*

*Proof.* From Hensel's lemma it follows that there exists $\bar{f}\in\mathbb{Z}_p$ such that $\bar{f}\equiv_p f$ and $P(\bar{f})=0$. Clearly, $2\bar{f}+a\not\equiv_p 0$ and $M=\langle x_0+ex_2, x_1+\bar{f}x_2, px_2\rangle$. In other words, we can assume that $c-af-f^2=0$ and $2f+a\not\equiv_p 0$. In case $p|F_{12}$, $\langle px_1, px_2\rangle$ is a non-trivial $\varphi$-invariant ideal of $L$ by Lemma 2.18 (1). In case $p\nmid F_{12}$, $\langle x_1+fx_2\rangle$ is a non-trivial $\varphi$-invariant ideal of $L$ by Lemma 2.18 (2). Hence, $\varphi$ is not simple. ∎

**Proposition 2.20.** *Let $s\in\mathbb{N}$ and $a,c\in\mathbb{Z}_p$ with $c\neq 0$. Assume that one of the following conditions is satisfied:*

(1) $v_p(a)\geqslant 1$ *and* $v_p(c)\geqslant 2$*;*

(2) $v_p(a)=0$ *and* $v_p(c)\geqslant 1$*;*

(3) $p\geqslant 3$, $a\neq 0$, $v_p(a)\geqslant 1$, *and* $v_p(c)=0$*;*

(4) $a=0$, $v_p(c)=0$, *and $c$ is not a square modulo $p$;*

(5) $v_p(a)=0$, $v_p(c)=0$, *and* $v_p(4c+a^2)\neq 1$.

*Then $L_7(s,a,c)$ is not self-similar of index $p$.*

*Proof.* Observe that $\dim[L,L]=2$. Denote $L=L_7(s,a,c)$, and let $\varphi: M\to L$ be a virtual endomorphism of $L$ of index $p$. We will show that $\varphi$ is not simple by either applying a previously proven result or exhibiting a $\varphi$-invariant ideal $I$ of $L$. Recall Remark 2.10. If $M=\langle px_0, x_1, x_2\rangle$, then it suffices to take $I=\langle x_1,x_2\rangle$ (Corollary 2.4 (2)). If $M=\langle x_0+ex_1, px_1, x_2\rangle$, where $e\in\mathbb{Z}_p$, then it suffices to take $I=\langle px_1, px_2\rangle$ (Lemma 2.15). Assume that $M=\langle x_0+ex_2, x_1+fx_2, px_2\rangle$, where $e,f\in\mathbb{Z}_p$ (the last case to be treated). By Lemma 2.18 (1), we can assume that $F_{12}\in\mathbb{Z}_p^*$. We observe that this implies that $c-af-f^2\equiv_p 0$. We divide the proof into several cases, depending on which assumption of the statement holds.

(1) Assume that $v_p(a) \geqslant 1$ and $v_p(c) \geqslant 2$. Hence, $f \equiv_p 0$, and it follows that $p | F_{21}$ and $p | F_{22}$. Thus, it suffices to take $I = \langle x_1, px_2 \rangle$, which is an ideal of $L$ (since, in particular, $v_p(c) \geqslant 1$) and is $\varphi$-invariant by Lemma 2.14 (4).

(2) Assume that $v_p(a) = 0$ and $v_p(c) \geqslant 1$, or that $p \geqslant 3$, $a \neq 0$, $v_p(a) \geqslant 1$, and $v_p(c) = 0$. Then $f$ is a simple root of the polynomial $P(\kappa) = \kappa^2 + a\kappa - c$ modulo $p$. Applying Corollary 2.19, we see that $\varphi$ is not simple.

(3) Assume that $a = 0$, $v_p(c) = 0$, and $c$ is not a square modulo $p$. This case contradicts $c - af - f^2 \equiv_p 0$.

(4) Assume that $v_p(a) = 0$, $v_p(c) = 0$, and $v_p(4c + a^2) \neq 1$. Case 1: $v_p(4c + a^2) = 0$. Then $f$ is a simple root of the polynomial $P(\kappa) = \kappa^2 + a\kappa - c$ modulo $p$. Applying Corollary 2.19, we see that $\varphi$ is not simple. Case 2: $v_p(4c + a^2) \geqslant 2$. Then $p \geqslant 3$ and $f \equiv_p -2^{-1}a$. We can assume that $f = -2^{-1}a$. By Lemma 2.18 (3), we can take $I = \langle x_1 - 2^{-1}ax_2, px_2 \rangle$. ∎

### 2.4. Self-similarity of 3-dimensional solvable Lie lattices

**Remark 2.21.** Assume that $p \geqslant 3$. Any 3-dimensional solvable $\mathbb{Z}_p$-Lie lattice is isomorphic to exactly one of the Lie lattices in the list below (see Definition 2.1 for the notation and [7, Proposition 7.3] for the proof). We also give necessary and sufficient conditions for the respective Lie lattice to be residually nilpotent (cf. [7, p. 731]). For $p \geqslant 5$ the residually nilpotent Lie lattices in the list provide a classification of 3-dimensional solvable torsion-free $p$-adic analytic pro-$p$ groups (cf. [7, Theorem B]).

(0) $L_0$. It is abelian; hence, it is residually nilpotent.

(1) $L_1(s)$. It is residually nilpotent if and only if $s \geqslant 1$.

(2) $L_2(s, r, c)$ with $r \geqslant 1$. It is residually nilpotent if and only if $s \geqslant 1$.

(3) $L_3(s)$. It is nilpotent; hence, it is residually nilpotent.

(4) $L_4(s, t, \varepsilon)$. It is residually nilpotent if and only if $s \geqslant 1$ or $t \geqslant 1$.

(5) $L_5(s, r, c)$. It is residually nilpotent if and only if $s \geqslant 1$ holds, or $r \geqslant 1$ and $v_p(c) \geqslant 1$ hold.

Recall that the *self-similarity index* of a self-similar $\mathbb{Z}_p$-Lie lattice $L$ is the smallest power of $p$, say $p^k$, such that $L$ is self-similar of index $p^k$.

**Theorem 2.22.** *Assume that $p \geqslant 3$. Let $L$ be a 3-dimensional solvable $\mathbb{Z}_p$-Lie lattice, and let $\sigma$ be the self-similarity index of $L$. Then $\sigma = p$ or $\sigma = p^2$. Moreover, $\sigma = p$ if and only if $L$ is isomorphic to one of the Lie lattices that appear in the following sublist of the list given in Remark 2.21:*

(0) *$L_0$;*

(1) *$L_1(s)$;*

(2) *$L_2(s, r, c)$ with $v_p(c) = 1$ (and $r \geqslant 1$);*

(4) $L_4(s,t,\varepsilon)$ *with* $t = 1$*, or with* $t = 0$ *and* $\varepsilon = 0$*;*

(5) $L_5(s,r,c)$ *with* $r \geqslant 1$ *and* $v_p(c) = 1$*, or with* $r = 0$ *and* $v_p(4c+1) = 1$*.*

*Proof.* By Corollary 2.7, $\sigma = p$ or $\sigma = p^2$. Observe that $L_0 = L_6(0)$, $L_1(s) = L_6(p^s)$, $L_4(s,t,\varepsilon) = L_7(s,0,p^t\rho^\varepsilon)$, and $L_5(s,r,c) = L_7(s,p^r,c)$. The claim that the Lie lattices in the statement are self-similar of index $p$ follows from Lemma 2.9. The remaining Lie lattices of Remark 2.21 (the ones not in the statement) are not self-similar of index $p$ by Propositions 2.13, 2.16, and 2.20. ∎

### 2.5. Non-self-similarity results in higher dimension

The main results of this section are Proposition 2.25 and Corollary 2.26; the latter is a key ingredient in the proof of Theorem 2.34.

Let $d \geqslant 2$ be an integer. As in Section 2.1, Greek indices will take values in $\{0,\dots,d\}$, while Latin indices will take values in $\{1,\dots,d\}$. We denote the $p$-adic valuation by $v$ instead of $v_p$.

**Definition 2.23.** Let $a = (a_1,\dots,a_d) \in \mathbb{Z}_p^d$ and $b = (b_1,\dots,b_{d-1}) \in \mathbb{Z}_p^{d-1}$. We define an antisymmetric $(d+1)$-dimensional $\mathbb{Z}_p$-algebra $L(a,b)$ as follows. As a $\mathbb{Z}_p$-module, $L = \mathbb{Z}_p^{d+1}$. Denoting by $(x_0,\dots,x_d)$ the canonical basis of $L$, the bracket of $L(a,b)$ is induced by the commutation relations

$$\begin{aligned}[x_i,x_j] &= 0,\\ [x_0,x_1] &= \textstyle\sum_i a_i x_i,\\ [x_0,x_{i+1}] &= b_i x_i \quad \text{if } i < d.\end{aligned}$$

**Remark 2.24.** $L(a,b)$ is a metabelian (possibly abelian) Lie lattice.

We will prove the following proposition at the end of the section.

**Proposition 2.25.** *Let* $a \in \mathbb{Z}_p^d$ *and* $b \in \mathbb{Z}_p^{d-1}$ *be as in Definition 2.23. Assume that*

(1) $a_d \neq 0$,

(2) $v(b_i) < v(b_{i+1})$ *whenever* $i < d-1$,

(3) $v(b_i) < v(a_i)$ *whenever* $i < d$*, and*

(4) $v(b_{d-1}) + 1 < v(a_d)$.

*Then* $L(a,b)$ *is not self-similar of index* $p$*.*

**Corollary 2.26.** *Let* $a \in \mathbb{Z}_p^d$ *and* $b \in \mathbb{Z}_p^{d-1}$ *be as in Definition 2.23. Assume that* $a_d \neq 0$ *and* $b_1 = \dots = b_{d-1} = 1$*. Then* $L(a,b)$ *admits a finite-index subalgebra that is not self-similar of index* $p$*.*

*Proof.* Let $L = L(a,b)$ and take $k_0,\dots,k_d \in \mathbb{N}$ as follows. Choose

$$k_0 > \frac{d-1}{2}, \quad k_1 \geqslant \max\left((i-1)k_0 - \frac{(i-1)(i-2)}{2}\right)_{i=1,\dots,d}$$

and, for $i \geqslant 2$, define

$$k_i = k_1 - (i-1)k_0 + \frac{(i-1)(i-2)}{2}.$$

It is not difficult to show that $k_0 + k_1 - k_i > i - 1$ for all $i$. Define $M = \langle p^{k_0}x_0, \dots, p^{k_d}x_d \rangle$. Then $M$ is a finite-index subalgebra of $L$ which is isomorphic to $L(a', b')$, where

$$b'_i = p^{i-1} \quad \text{if } i < d,$$
$$a'_i = p^{k_0+k_1-k_i} a_i.$$

By Proposition 2.25, $M$ is not self-similar of index $p$. ∎

The remainder of the section is devoted to the proof of Proposition 2.25.

**Remark 2.27.** Let $a \in \mathbb{Z}_p^d$, $b \in \mathbb{Z}_p^{d-1}$, and $L = L(a, b)$; see Definition 2.23. We define $I_0 = \langle x_1, \dots, x_d \rangle$ and $I_i = \langle x_1, \dots, x_{i-1}, px_i, \dots, px_d \rangle$. Hence, $I_1 = pI_0$ and $I_1 \subset I_2 \subset \dots \subset I_d \subset I_0$. Moreover, $I_0$ and $I_1$ are ideals of $L$.

**Lemma 2.28.** *Let $i > 1$ and $I_i$ be defined as in Remark 2.27. Then $I_i$ is an ideal of $L$ if and only if $p | a_j$ for all $j \geqslant i$.*

*Proof.* It suffices to observe that $I_i$ is an ideal of $L$ if and only if $[x_0, y] \in I_i$ for all the generators $y$ of $I_i$ displayed in the definition of $I_i$. ∎

**Lemma 2.29.** *Let $a_1, \dots, a_d, b_1, \dots, b_{d-1} \in \mathbb{Z}_p$ and define $A = (A_{ij}) \in \mathrm{gl}_d(\mathbb{Z}_p)$ by*

$$A_{ij} = \begin{cases} a_i & \text{if } j = 1, \\ b_i & \text{if } j = i+1, \\ 0 & \text{if } j > 1 \text{ and } j \neq i+1. \end{cases}$$

*Let $i_0 \in \{1, \dots, d\}$ and $f_1, \dots, f_{i_0-1} \in \mathbb{Z}_p$ (no choice of coefficients "$f$" has to be made when $i_0 = 1$). Define $U = (U_{ij}) \in \mathrm{gl}_d(\mathbb{Z}_p)$ by*

$$U_{ij} = \begin{cases} p & \text{if } i = j = i_0, \\ 1 & \text{if } i = j \neq i_0, \\ -f_j & \text{if } i > j \text{ and } i = i_0, \\ 0 & \text{if } i > j \text{ and } i \neq i_0, \\ 0 & \text{if } i < j. \end{cases}$$

*Let $\widehat{U}$ be the cofactor matrix of $U$. Let $F_{00} \in \mathbb{Z}_p$ and $F = (F_{ij}) \in \mathrm{gl}_d(\mathbb{Z}_p)$. Assume that $F\widehat{U}^T A = F_{00} A F \widehat{U}^T$ and that the $a_i$'s and $b_j$'s satisfy the four assumptions in the statement of Proposition 2.25. Then the following holds.*

(1) *Assume that $f_k = 0$ for all $k < i_0$. Then $p | F_{ij}$ for $i \geqslant i_0$ and $j \leqslant i_0$.*

(2) *Assume that there exists $k_0 < i_0$ such that $f_{k_0} \not\equiv_p 0$ and $f_k = 0$ for all $k < k_0$. Then $p | F_{i,i_0}$ for $i \geqslant k_0$, and $p | F_{ij}$ for $i \geqslant k_0$ and $j < k_0$.*

*Proof.* We have

$$\hat{U}^T_{ij} = \begin{cases} 1 & \text{if } i = j = i_0, \\ p & \text{if } i = j \neq i_0, \\ f_j & \text{if } i > j \text{ and } i = i_0, \\ 0 & \text{if } i > j \text{ and } i \neq i_0, \\ 0 & \text{if } i < j. \end{cases}$$

A straightforward computation gives

$$(F\hat{U}^T A)_{ik} = \begin{cases} \sum_{j<i_0}(pa_j F_{ij} + f_j a_j F_{i,i_0}) & \\ \quad + a_{i_0} F_{i,i_0} + \sum_{j>i_0} pa_j F_{ij} & \text{if } k = 1, \\ pb_{k-1} F_{i,k-1} + f_{k-1} b_{k-1} F_{i,i_0} & \text{if } 1 < k < i_0 + 1, \\ b_{i_0} F_{i,i_0} & \text{if } k = i_0 + 1, \\ pb_{k-1} F_{i,k-1} & \text{if } k > i_0 + 1 \end{cases}$$

and

$$(F_{00} A F \hat{U}^T)_{ik} = \begin{cases} F_{00} pa_i F_{1k} + F_{00} f_k a_i F_{1,i_0} & \\ \quad + F_{00} pb_i F_{i+1,k} + F_{00} f_k b_i F_{i+1,i_0} & \text{if } i < d \text{ and } k < i_0, \\ F_{00} a_i F_{1,i_0} + F_{00} b_i F_{i+1,i_0} & \text{if } i < d \text{ and } k = i_0, \\ F_{00} pa_i F_{1k} + F_{00} pb_i F_{i+1,k} & \text{if } i < d \text{ and } k > i_0, \\ F_{00} pa_d F_{1k} + F_{00} f_k a_d F_{1,i_0} & \text{if } i = d \text{ and } k < i_0, \\ F_{00} a_d F_{1,i_0} & \text{if } i = d \text{ and } k = i_0, \\ F_{00} pa_d F_{1k} & \text{if } i = d \text{ and } k > i_0. \end{cases}$$

We denote by $F(i,k)$ the equality $(F\hat{U}^T A)_{ik} = (F_{00} A F \hat{U}^T)_{ik}$, which is true for all $i$ and $k$ by assumption. Observe that $b_i \neq 0$ whenever $i < d$. We divide the proof of the two items of the statement of the lemma into four cases. The fourth case will be treated in detail, while the details of the other cases are left to the reader.

(1) *Item* (2) *of the statement, proof of* $p|F_{i,i_0}$. The claim $p|F_{i,i_0}$ follows from $F(i, k_0+1)$. The proof has to be done by descending induction on $i$, since for $i = k_0$ one needs to use that $p|F_{k_0+1,i_0}$.

(2) *Item* (2) *of the statement, proof of* $p|F_{ij}$. The claim $p|F_{ij}$ follows from $F(i, j+1)$. We observe that for $i = k_0$ and $j = k_0 - 1$ one has also to use that $p|F_{k_0+1,i_0}$, which was proven in the previous item.

(3) *Item* (1) *of the statement, case* $i_0 < d$. The claim $p|F_{ij}$ follows from $F(i, j+1)$. The proof has to be done by descending induction on $j$, since for $i = i_0$ and $j = i_0 - 1$ one needs to use that $p|F_{i_0+1,i_0}$.

(4) *Item* (1) *of the statement, case* $i_0 = d$. We have to prove that $p | F_{dj}$ for all $j$.

(a) Assume that $j < d - 1$. The equation $F(d, j + 1)$ reads

$$pb_j F_{dj} = F_{00} pa_d F_{1,j+1}.$$

Hence, $v(F_{dj}) \geqslant v(a_d) - v(b_j)$. In particular, $p | F_{dj}$.

(b) Assume that $j = d - 1$. The equation $F(d, j + 1)$ reads

$$pb_{d-1} F_{d,d-1} = F_{00} a_d F_{1,d}.$$

Hence, $v(F_{d,d-1}) \geqslant v(a_d) - v(b_{d-1}) - 1$. In particular, $p | F_{d,d-1}$.

(c) Assume that $j = d$. The equation $F(d, 1)$ reads

$$\sum_{j<d} pa_j F_{dj} + a_d F_{dd} = F_{00} pa_d F_{11}.$$

For all $j < d - 1$, we have $v(pa_j F_{dj}) = 1 + v(a_j) + v(F_{dj}) \geqslant 1 + v(a_j) + v(a_d) - v(b_j) > v(a_d)$. For $j = d - 1$, we have

$$\begin{aligned} v(pa_{d-1} F_{d,d-1}) &= 1 + v(a_{d-1}) + v(F_{d,d-1}) \\ &\geqslant 1 + v(a_{d-1}) + v(a_d) - v(b_{d-1}) - 1 \\ &> v(a_d). \end{aligned}$$

Hence, $p | F_{dd}$. ■

**Remark 2.30.** Let $L$ be a $(d + 1)$-dimensional $\mathbb{Z}_p$-lattice endowed with a basis $(x_0, \dots, x_d)$, and let $M \subseteq L$ be a submodule of index $p$. Exactly one of the following cases holds (cf. [19, Lemma 2.23]):

(1) $(y_0, \dots, y_d)$ is a basis of $M$, where $y_0 = px_0$ and $y_i = x_i$;

(2) there exist $i_0 \in \{1, \dots, d\}$ and $f_0 \in \mathbb{Z}_p$ such that $(y_0, \dots, y_d)$ is a basis of $M$, where $y_0 = x_0 - f_0 x_{i_0}$ and

$$y_i = \begin{cases} x_i & \text{if } i \neq i_0, \\ px_{i_0} & \text{if } i = i_0; \end{cases}$$

(3) there exist $k_0, i_0 \in \{1, \dots, d\}$ and $f_0, f_{k_0}, \dots, f_{i_0-1} \in \mathbb{Z}_p$ such that $k_0 < i_0$, $f_{k_0}$ is invertible in $\mathbb{Z}_p$, and $(y_0, \dots, y_d)$ is a basis of $M$, where $y_0 = x_0 - f_0 x_{i_0}$ and

$$y_i = \begin{cases} x_i & \text{if } i < k_0, \\ x_i - f_i x_{i_0} & \text{if } k_0 \leqslant i < i_0, \\ px_{i_0} & \text{if } i = i_0, \\ x_i & \text{if } i > i_0. \end{cases}$$

**Lemma 2.31.** *Let $L$ be a $(d+1)$-dimensional lattice, let $M \subseteq L$ be a submodule of index $p$, and let $\varphi : M \to L$ be a homomorphism of modules. Let $(x_0, \dots, x_d)$ be a basis of $L$, let $(y_0, \dots, y_d)$ be a basis of $M$, and let $y_\beta = \sum_\alpha F_{\alpha\beta} x_\alpha$. Assume that $F_{0i} = 0$ for all $i$. Let $I_i := \langle z_1, \dots, z_d \rangle$, where*

$$z_j = \begin{cases} x_j & \text{if } j < i, \\ p x_j & \text{if } j \geqslant i \end{cases}$$

*(cf. Remark 2.27). Then the following holds.*

(1) *Assume that $(y_0, \dots, y_d)$ has the form displayed in case (2) of Remark 2.30. Then*
   (a) $I_{i_0} \subseteq M$,
   (b) $\varphi(I_{i_0}) \subseteq I_{i_0}$ *if and only if $p | F_{ij}$ for $i \geqslant i_0$ and $j \leqslant i_0$.*

(2) *Assume that $(y_0, \dots, y_d)$ has the form displayed in case (3) of Remark 2.30. Then*
   (a) $I_{k_0} \subseteq M$,
   (b) $\varphi(I_{k_0}) \subseteq I_{k_0}$ *if and only if $p | F_{i,i_0}$ for $i \geqslant k_0$, and $p | F_{ij}$ for $i \geqslant k_0$ and $j < k_0$.*

*Proof.* We prove item (2), leaving item (1), which is similar, to the reader. Since

$$z_j = \begin{cases} y_j & \text{if } j \leqslant k_0, \\ p y_j + f_j y_{i_0} & \text{if } k_0 \leqslant j < i_0, \\ y_j & \text{if } j = i_0, \\ p y_j & \text{if } j > i_0, \end{cases}$$

we have $I_{k_0} \subseteq M$. Observe that $\varphi(I_{k_0}) \subseteq I_{k_0}$ if and only if $\varphi(z_j) \in I_{k_0}$ for all $j$. Since

$$\varphi(z_j) = \begin{cases} \sum_i F_{ij} x_i & \text{if } j < k_0, \\ \sum_i (p F_{ij} + f_j F_{i,i_0}) x_i & \text{if } k_0 \leqslant j < i_0, \\ \sum_i F_{i,i_0} x_i & \text{if } j = i_0, \\ \sum_i p F_{ij} x_i & \text{if } j > i_0, \end{cases}$$

item (2) follows. ∎

*Proof of Proposition 2.25.* Let $L = L(a, b)$, let $M \subseteq L$ be a subalgebra of index $p$, and let $\varphi : M \to L$ be a homomorphism of algebras. We have to show that there exists a non-trivial $\varphi$-invariant ideal $I$ of $L$. Observe that any $b_i$ is non-zero and that $\dim[L, L] = d$. Moreover, $v(a_i) \geqslant 1$ for all $i$, and any $I_\alpha$ is a non-trivial ideal of $L$ (see Remark 2.27 and Lemma 2.28). Let $\boldsymbol{x} = (x_0, \dots, x_d)$ be the canonical basis of $L$, and let $\boldsymbol{y} = (y_0, \dots, y_d)$ be a basis of $M$ in one of the forms given in Remark 2.30. The bases $\boldsymbol{x}$ and $\boldsymbol{y}$ are good bases for $L$ and $M$, respectively (cf. Lemma 2.2). Let $A$ be the matrix of $L$ with respect to $\boldsymbol{x}$ (cf. Section 2.1), and observe that it is equal to the matrix $A$ of Lemma 2.29. Let $y_\beta = \sum_\alpha U_{\alpha\beta} x_\alpha$, and let $\varphi(y_\beta) = \sum_\alpha F_{\alpha\beta} x_\alpha$. By Lemma 2.3 (2), $F_{0i} = 0$ for all $i$. The proof is completed below by considering each one of the three cases of Remark 2.30. For the last two cases, in order to apply Lemma 2.29, we have to make some observations. In those cases, $U_{00} = 0$ and the $d \times d$ matrix $U = (U_{ij})$ has the format of the one of Lemma 2.29. The matrix of $M$ with respect to $\boldsymbol{y}$ is $B = U^{-1}AU$; moreover, $FB = F_{00}AF$ (Lemma 2.3 (3)). An easy computation gives $F\widehat{U}^T A = F_{00}AF\widehat{U}^T$, where $\widehat{U}$ is the cofactor matrix of $U$.

(1) For case (1) of Remark 2.30 we take $I = I_0$, which is invariant by Corollary 2.4 (2).

(2) For case (2) of Remark 2.30 we take $I = I_{i_0}$, which is invariant by Lemma 2.29 (1) and Lemma 2.31 (1).

(3) For case (3) of Remark 2.30 we take $I = I_{k_0}$, which is invariant by Lemma 2.29 (2) and Lemma 2.31 (2). ■

### 2.6. Strongly hereditarily self-similar Lie lattices

**Definition 2.32.** Let $L$ be a $\mathbb{Z}_p$-Lie lattice, and let $k \in \mathbb{N}$.

(1) $L$ is *hereditarily self-similar of index* $p^k$ if and only if any finite-index subalgebra of $L$ is self-similar of index $p^k$.

(2) $L$ is *strongly hereditarily self-similar of index* $p^k$ if and only if $L$ is self-similar of index $p^k$ and any non-zero subalgebra of $L$ is self-similar of index $p^k$.

The main result of this section is as follows, and the proof of the theorem will be given at the end of the section.

**Definition 2.33.** Let $d \geqslant 2$ be an integer, and let $a \in \mathbb{Z}_p$. We define an antisymmetric $d$-dimensional $\mathbb{Z}_p$-algebra $L^d(a)$ as follows. As a $\mathbb{Z}_p$-module, $L^d(a) = \mathbb{Z}_p^d$. Denoting by $(x_0, \ldots, x_{d-1})$ the canonical basis of $\mathbb{Z}_p^d$, the bracket of $L^d(a)$ is induced by the commutation relations $[x_i, x_j] = 0$ and $[x_0, x_i] = ax_i$, where $i, j$ take values in $\{1, \ldots, d-1\}$.

**Theorem 2.34.** *Assume that $p \geqslant 3$. Let $d \geqslant 2$ be an integer, and let $L$ be a solvable $\mathbb{Z}_p$-Lie lattice of dimension $d$ that is strongly hereditarily self-similar of index $p$. Then $L \simeq L^d(p^s)$ for a unique $s \in \mathbb{N} \cup \{\infty\}$ (with $p^\infty := 0$).*

Before proving the theorem we provide some examples and make some remarks on hereditarily self-similar Lie lattices.

**Remark 2.35.** Let $L$ be a $\mathbb{Z}_p$-Lie lattice. Clearly, if $L$ is strongly hereditarily self-similar of index $p^k$, then $L$ is hereditarily self-similar of index $p^k$. From [19, Remark 2.2] it follows that if $L$ has dimension 1 or 2, then $L$ is strongly hereditarily self-similar of index $p^k$ for all $k \geqslant 1$. Consequently, if $L$ has dimension 3 and $L$ is hereditarily self-similar of index $p^k$, then $L$ is strongly hereditarily self-similar of index $p^k$. Proposition 2.41 below classifies, for $p \geqslant 3$, the 3-dimensional Lie lattices that are hereditarily self-similar of index $p$.

**Proposition 2.36.** *Let $m \geqslant 1$ be an integer, and let $L$ be a 3-dimensional solvable $\mathbb{Z}_p$-Lie lattice. Then $L$ is strongly hereditarily self-similar of index $p^{2m}$.*

*Proof.* The proposition follows from Corollary 2.7 and Remark 2.35. ■

Proposition 2.36 and [23, Proposition 3.1] have a consequence that we find worth to state explicitly. We recall that, by definition, two Lie lattices $L_1$ and $L_2$ are incommensurable if there are no finite-index subalgebras $M_1 \subseteq L_1$ and $M_2 \subseteq L_2$ such that $M_1 \simeq M_2$.

**Corollary 2.37.** *There exists a set $\mathcal{H}$ of the cardinality of the continuum such that any element of $\mathcal{H}$ is a $\mathbb{Z}_p$-Lie lattice that is strongly hereditarily self-similar of index $p^{2m}$ for each $m \geqslant 1$, and such that any two distinct elements of $\mathcal{H}$ are incommensurable.*

The next results are interesting on their own and they are a preparation for the proof of Theorem 2.34.

**Remark 2.38.** We list some properties of $L = L^d(a)$ that the reader may easily prove. The Lie lattice $L$ belongs to the class discussed in Section 2.1; in particular, $L$ is a Lie lattice and $\delta_2(L) = \{0\}$. We have $L^d(a) \simeq L^e(b)$ if and only if $d = e$ and $v_p(a) = v_p(b)$; moreover, $L$ is abelian if and only if $a = 0$. If $a \neq 0$, then $\mathrm{iso}_L[L, L] = \langle x_1, \dots, x_{d-1}\rangle$. Any submodule of $L$ is a subalgebra, and any 2-generated subalgebra of $L$ has dimension at most 2. Finally, note that $L_0 = L^3(0)$, $L_1(s) = L^3(p^s)$, and $L_6(a) = L^3(a)$ (see Definition 2.1).

**Lemma 2.39.** *Let $d \geqslant 2$ be an integer, let $a \in \mathbb{Z}_p$, and let $M$ be a subalgebra of $L^d(a)$ of dimension $e \geqslant 2$. Then $M \simeq L^e(p^s a)$ for some $s \in \mathbb{N} \cup \{\infty\}$.*

*Proof.* Denote $L = L^d(a)$ and recall that $L$ is endowed with the basis $(x_0, \dots, x_{d-1})$. Let $J_L = \mathrm{iso}_L[L, L]$. If $M \subseteq J_L$, then one takes $s = \infty$. Assume that $M \nsubseteq J_L$. Hence, $a \neq 0$ and $L/J_L \simeq \mathbb{Z}_p$, generated by the class of $x_0$. Let $\varphi : M \to L/J_L$ be the canonical map. Then $\varphi(M)$ is non-zero; hence, there exists $x \in M$ such that the class of $x$ in $L/J_L$ is a basis of $\varphi(M)$ over $\mathbb{Z}_p$. Also, $[L/J_L : \varphi(M)] = p^s$ for some $s \in \mathbb{N}$. Let $x = cx_0 + \sum_j c_j x_j$, where the index takes values in $\{1, \dots, d-1\}$. Observe that $v_p(c) = s$. One proves that $M = \langle x\rangle \oplus (M \cap J_L)$, from which the conclusion $M \simeq L_e(ca) \simeq L_e(p^s a)$ follows. ■

**Proposition 2.40.** *Let $d \geqslant 2$ and $k \geqslant 1$ be integers, and let $a \in \mathbb{Z}_p$. Then*

(1) *$L^d(a)$ is self-similar of index $p^k$,*

(2) *$L^d(a)$ is strongly hereditarily self-similar of index $p^k$.*

*Proof.* Item (2) is a consequence of item (1) and Lemma 2.39. We prove (1). For $d = 2$ see Remark 2.35. Assume that $d \geqslant 3$. Let $L = L^d(a)$, and let $M = \langle x_0, p^k x_1, x_2, \dots, x_{d-1}\rangle$. Then $M$ is a subalgebra of $L$ of index $p^k$. The module homomorphism $\varphi : M \to L$ determined by $\varphi(x_0) = x_0$, $\varphi(p^k x_1) = x_2$, $\varphi(x_i) = x_{i+1}$ for $2 \leqslant i < d-1$, and $\varphi(x_{d-1}) = x_1$ is a homomorphism of algebras. We prove that $\varphi$ is simple. Indeed, the intersection of the domains of the powers of $\varphi$ is $D_\infty = \langle x_0\rangle$. Let $I$ be a non-trivial ideal of $L$. Similarly to what has been done in the proof of Proposition 2.6, one shows that $L$ is not $\varphi$-invariant by proving the existence of $w \in I$ such that $w \notin D_\infty$. ■

**Proposition 2.41.** *Assume that $p \geqslant 3$, and let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie lattice. The following are equivalent.*

(1) *$L$ is hereditarily self-similar of index $p$.*

(2) *$L$ is isomorphic either to $L_0$ or to $L_1(s)$ for some $s \in \mathbb{N}$.*

*Proof.* Remark 2.38 and Proposition 2.40 (2) show that the implication "(2)⇒(1)" holds even in greater generality than stated here. For the other implication, we assume that (2) does not hold and show that there exists a finite-index subalgebra $M$ of $L$ that is not self-similar of index $p$. We divide the proof into two parts according to whether $L$ is solvable or unsolvable.

Assume that $L$ is solvable. The following observations are enough to cover all the cases (cf. Remark 2.21). If $\dim[L, L] = 1$, then $L$ itself is not self-similar of index $p$ (Proposition 2.13). If $r \geqslant 1$, then $M = \langle x_0, px_1, x_2 \rangle$ is a subalgebra of $L_2(s, r, c)$ and $M \simeq L_2(s, r-1, p^2c)$. If $r \geqslant 2$, then $M$ is not self-similar of index $p$ by Proposition 2.16. If $r = 1$, one shows that $L_2(s, 0, p^2c) \simeq L_7(s, 1, 4^{-1}(p^2c - 1))$, so that $M$ is not self-similar of index $p$ by item (5) of Proposition 2.20. Now, let $L = L_7(s, a, c)$ with $c \neq 0$. Observe that $pL$ is a subalgebra of $L$ and that $pL \simeq L_7(s+1, a, c)$; hence, we can assume that $s \geqslant 1$. Then $M = \langle x_0, px_1, x_2 \rangle$ is a subalgebra of $L$ and $M \simeq L_7(s-1, pa, p^2c)$, so that $M$ is not self-similar of index $p$ by item (1) of Proposition 2.20.

Now, assume that $L$ is unsolvable. There exists a basis $(x_0, x_1, x_2)$ of $L$ such that $[x_i, x_{i+i}] = a_{i+2}x_{i+2}$, where the index $i$ is interpreted in $\mathbb{Z}/3\mathbb{Z}$, and the $a_i$'s are non-zero $p$-adic integers with $v_p(a_0) \leqslant v_p(a_1) \leqslant v_p(a_2)$; see [19, Proposition 2.7]. It is not difficult to see that one can choose $k_0, k_1, k_2 \in \mathbb{N}$ such that, defining $y_i = p^{k_i}x_i$, one has $[y_i, y_{i+i}] = b_{i+2}y_{i+2}$, where the $b_i$'s are non-zero $p$-adic integers and $v_p(b_0) < v_p(b_1) < v_p(b_2)$. Hence, $M = \langle y_0, y_1, y_2 \rangle$ is a subalgebra of $L$ that is not self-similar of index $p$ by [19, Theorem 2.32]. ∎

*Proof of Theorem* 2.34. Uniqueness of $s$ is easy to prove (cf. Remark 2.38). The proof of existence is by induction on $d$. For $d = 2$ the theorem is easily proven, while for $d = 3$ it follows from Proposition 2.41 and Remark 2.38. For the induction step, let $d \geqslant 4$ and assume that the theorem holds with $d'$ in place of $d$, where $d' < d$. Let $\mathcal{L} = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Since $\mathcal{L}$ is a solvable Lie algebra over a field of characteristic 0, Lie's theorem implies that the $\mathbb{Q}_p$-Lie algebra $[\mathcal{L}, \mathcal{L}]$ is nilpotent. Hence, the $\mathbb{Z}_p$-Lie lattice $[L, L]$ is nilpotent as well.

We prove that $[L, L]$ is abelian. Denote temporarily $M = [L, L]$, and assume by contradiction that $M$ is not abelian. Hence, $M$ is a non-abelian nilpotent Lie lattice. Let $c$ be the nilpotency class of $M$; then $c \geqslant 2$. We claim that there exists $x, y \in M$ such that $[x, y] \neq 0$ and $[x, y] \in Z(M)$ (the center of $M$). Indeed, $\{0\} \neq \gamma_{c-1}(M) \subseteq Z(M)$. Hence, there exist $x \in M$ and $y \in \gamma_{c-2}(M)$ such that $[x, y] \neq 0$. Since $[x, y] \in \gamma_{c-1}(M)$, it follows that $[x, y] \in Z(M)$, and the claim is proven. Let $N$ be the subalgebra generated by $x$ and $y$. Then $N$ is a nilpotent non-abelian subalgebra of $L$ with $\dim[N, N] = 1$. The dimension of $N$ is either 2 or 3. Since no non-abelian Lie lattice of dimension 2 is nilpotent, we have $\dim N = 3$. Hence, $N$ is not self-similar of index $p$ by Proposition 2.13, a contradiction.

Let $m = \dim[L, L]$. Note that $m < d$, since otherwise $L$ would not be solvable. If $m = 0$ ($L$ abelian), then one takes $s = \infty$. Assume that $m > 0$ ($L$ not abelian). Let $J = \mathrm{iso}_L[L, L]$, which is an isolated abelian ideal of $L$. Hence, $\dim J = m$, and there exists a basis $(x_1, \dots, x_{d-m}, y_1, \dots, y_m)$ of $L$ such that $(y_1, \dots, y_m)$ is a basis of $J$. Let Greek indices take values in $\{1, \dots, d-m\}$, and Latin indices take values in $\{1, \dots, m\}$. We have $[y_i, y_j] = 0$, and any commutator in $L$ is a linear combination of the $y_i$'s. Let $M_\alpha = \langle x_\alpha, y_1, \dots, y_m \rangle$. Then $M_\alpha$ is a subalgebra of $L$ of dimension $m + 1 \geqslant 2$. For $z \in J$, $z \neq 0$, define $M'_z$ to be the subalgebra of $L$ generated by $x_1$ and $z$. Observe that $M'_z$ has dimension $n_z \geqslant 2$. Moreover, observe that all $M_\alpha$'s and $M'_z$'s are solvable and strongly hereditarily self-similar of index $p$. We divide the proof into two cases.

(1) Case $m < d - 1$. Then $m + 1 < d$ and $M_\alpha \simeq L^{m+1}(p^{s_\alpha})$ for some $s_\alpha \in \mathbb{N} \cup \{\infty\}$. Since $\langle x_\alpha, y_i \rangle$ is a subalgebra of $M_\alpha$, $[x_\alpha, y_i] = c_{\alpha i} y_i$ for some $c_{\alpha i} \in \mathbb{Z}_p$. By contradiction, assume that $c_{\alpha i} \neq c_{\alpha j}$ for some $i, j$. Since $\langle x_\alpha, y_i + y_j \rangle$ is a subalgebra of $M_\alpha$, $[x_\alpha, y_i + y_j] = c_{\alpha i}(y_i + y_j) + (c_{\alpha j} - c_{\alpha i})y_j \in \langle x_\alpha, y_i + y_j \rangle$, which is a contradiction. It follows that $[x_\alpha, y_i] = c_\alpha y_i$ for all indices $i$ and some $c_\alpha \in \mathbb{Z}_p$ with $v_p(c_\alpha) = s_\alpha$. Observe that $d - m \geqslant 2$ and that $c_{\alpha_0} \neq 0$ for some $\alpha_0$.

(a) Case $[x_\alpha, x_\beta] = 0$ for all $\alpha, \beta$. Let $N = \langle x_{\alpha_0}, x_{\alpha_1}, y_1 \rangle$ with $\alpha_1 \neq \alpha_0$. Then $N$ is a subalgebra of $L$ of dimension 3, and $\dim[N, N] = 1$. Hence, $N$ is not self-similar of index $p$, a contradiction.

(b) Case $[x_{\beta_0}, x_{\beta_1}] \neq 0$ for some $\beta_0, \beta_1$. Let $z = [x_{\beta_0}, x_{\beta_1}]$, and let $N = \langle x_{\beta_0}, x_{\beta_1}, z \rangle$. Then $N$ is a subalgebra of $L$ of dimension 3, and $\dim[N, N] = 1$ (observe that $[x_{\beta_j}, z] = c_{\beta_j} z$). Hence, $N$ is not self-similar of index $p$, a contradiction.

(2) Case $m = d - 1$. Recall the notation $n_z = \dim M'_z$.

(a) Case $n_z = d$ for some $z$. Let $M = M'_z$ and $J_M = \mathrm{iso}_M[M, M]$. Observe that $\dim J_M = d - 1$. Define by recursion $z_1 = z$ and $z_{i+1} = [x, z_i]$ for $i \geqslant 1$. One can show that $J_M = \langle z_i : i \geqslant 1 \rangle$. We claim that $\{z_1, \dots, z_{d-1}\}$ is a basis of $J_M$. Indeed, denoting by $\bar{w}$ the residue of $w \in J_M$ in $J_M/pJ_M$, we show that $\{\bar{z}_1, \dots, \bar{z}_{d-1}\}$ is linearly independent over $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$. If it was not independent, some $\bar{z}_{j_0}$ would be a linear combination of $\bar{z}_1, \dots, \bar{z}_{j_0-1}$, and one could prove (from the recursive definition of the $z_i$'s) that any $\bar{z}_i$, $i \geqslant j_0$, would be such a linear combination, so that the dimension of $J_M/pJ_M$ over $\mathbb{F}_p$ would be less than $d - 1$, a contradiction. The claim that $\{z_1, \dots, z_{d-1}\}$ is a basis of $J_M$ over $\mathbb{Z}_p$ follows, and from it we get a basis $\{x, z_1, \dots, z_{d-1}\}$ of $M$, where $[z_i, z_j] = 0$, $[x, z_i] = z_{i+1}$ for $i < d - 1$, and $[x, z_{d-1}] = \sum_{j=1}^{d-1} a_j z_j$ for some $a_j \in \mathbb{Z}_p$. We claim that $a_1 \neq 0$. By contradiction, assume that $a_1 = 0$. Then $N := \langle x, z_2, \dots, z_{d-1} \rangle$ is a subalgebra of $L$ of dimension $d - 1 \geqslant 3$. Moreover, $N$ is solvable and strongly hereditarily self-similar of index $p$. Thus, there exists $s \in \mathbb{N} \cup \{\infty\}$ such that $N \simeq L^{d-1}(p^s)$. Then $\langle x, z_2 \rangle$ is a subalgebra of $N$, a contradiction (since $d \geqslant 4$). Hence, $a_1 \neq 0$. By Corollary 2.26, there exists a non-zero subalgebra of $M$ that is not self-similar of index $p$, which gives a contradiction.

(b) Case $n_z < d$ for all $z$. Then $M'_{y_i} \simeq L^{n_{y_i}}(p^{s_i})$ for some $s_i \in \mathbb{N} \cup \{\infty\}$ (for all $i$). Hence, $\langle x_1, y_i \rangle$ is a subalgebra of $M'_{y_i}$, and so $[x_1, y_i] = b_i y_i$ for some $b_i \in \mathbb{Z}_p$. Assume by contradiction that $b_{j_0} \neq b_{j_1}$ for some $j_0, j_1$. Let $z_0 = y_{j_0} + y_{j_1}$. Then $M'_{z_0} \simeq L^{n_{z_0}}(p^t)$. On the other hand, $[x_1, z_0] = b_{j_0} z_0 + (b_{j_1} - b_{j_0}) y_{j_1}$ yields that the 2-generated algebra $L^{n_{z_0}}(p^t)$ has dimension greater than 2, which is a contradiction. Thus, $[x, y_i] = by_i$ for all indices $i$ and some $b \in \mathbb{Z}_p$ with $b \neq 0$. Hence, $L \simeq L^d(b) \simeq L^d(p^s)$, where $s = v_p(b)$. ■

## 3. Results on groups

In this section, we prove the main theorems of the paper, stated in Section 1. Essentially, the proofs follow from the results on Lie lattices of Section 2 and from Proposition 3.1 below, which is a slightly generalized version of [19, Proposition A]. Before stating the proposition we recall the notion of saturable pro-$p$ group and Lazard's correspondence.

A finitely generated pro-$p$ group is saturable if it admits a certain type of valuation map; for precise details we refer to [6, Section 3]. Saturable groups were introduced by Lazard [15] and play a central role in the theory of $p$-adic analytic groups: a topological group is $p$-adic analytic if and only if it contains an open finitely generated pro-$p$ subgroup which is saturable [15, Sections III (3.1) and III (3.2)]. With a saturable pro-$p$ group $G$ one may associate a saturable $\mathbb{Z}_p$-Lie lattice $L_G$ in the following way: $G$ and $L_G$ are identified as sets, and the Lie operations are defined by

$$g + h = \lim_{n\to\infty} (g^{p^n} h^{p^n})^{p^{-n}},$$
$$[g, h]_{\mathrm{Lie}} = \lim_{n\to\infty} [g^{p^n}, h^{p^n}]^{p^{-2n}} = \lim_{n\to\infty} (g^{-p^n} h^{-p^n} g^{p^n} h^{p^n})^{p^{-2n}}.$$

The assignment $G \mapsto L_G$ gives an isomorphism between the category of saturable pro-$p$ groups and the category of saturable $\mathbb{Z}_p$-Lie lattices; see [15, IV (3.2.6)], [11, Section 2], and [7] for more details.

**Proposition 3.1.** *Let $G$ be a torsion-free $p$-adic analytic pro-$p$ group. Assume that any closed subgroup of $G$ is saturable and that any 2-generated closed subgroup of $G$ has dimension at most $p$. Let $L_G$ be the $\mathbb{Z}_p$-Lie lattice associated with $G$, and assume that any 2-generated subalgebra of $L_G$ has dimension at most $p$. Then, for all $k \in \mathbb{N}$, the following holds.*

(1) *$G$ is a self-similar group of index $p^k$ if and only if $L_G$ is a self-similar Lie lattice of index $p^k$.*

(2) *$G$ is hereditarily self-similar of index $p^k$ (respectively, strongly hereditarily self-similar of index $p^k$) if and only if $L_G$ is hereditarily self-similar of index $p^k$ (respectively, strongly hereditarily self-similar of index $p^k$).*

*Proof.* The proposition follows from Lazard's correspondence, [7, Theorem E], the argument proving $[G : D] = [L_G : L_D]$ in the proof of [19, Theorem 3.1], and from [19, Proposition 1.3]. ■

**Remark 3.2.** Let $G$ be a torsion-free $p$-adic analytic pro-$p$ group. If $G$ is saturable and $\dim(G) \leqslant p$, then the hypotheses of Proposition 3.1 are satisfied; if $\dim(G) < p$, then the same hypotheses hold without assuming a priori that $G$ is saturable [7, Theorem A]. We will also use the fact that if $G$ is saturable and $L_G$ is the associated $\mathbb{Z}_p$-Lie lattice, then $G$ is solvable if and only if $L_G$ is solvable [6, Theorem B].

**Remark 3.3.** This remark is the analogue of Remark 2.35 in the context of groups. Let $G$ be a finitely generated pro-$p$ group. For $k \in \mathbb{N}$, if $G$ is strongly hereditarily self-similar of index $p^k$, then $G$ is hereditarily self-similar of index $p^k$. Assume, moreover, that $G$ is torsion-free and $p$-adic analytic. From [19, Proposition 1.5] it follows that if $\dim(G) = 1, 2$, then $G$ is strongly hereditarily self-similar of index $p^k$ for all $k \geqslant 1$. Consequently, if $G$ has dimension 3 and $G$ is hereditarily self-similar of index $p^k$, then $G$ is strongly hereditarily self-similar of index $p^k$.

**Proposition 3.4.** *Let $m \geqslant 1$, and let $G$ be a 3-dimensional solvable torsion-free $p$-adic analytic pro-$p$ group. Assume that either "$p \geqslant 5$" or "$p = 3$ and $G$ is saturable". Then $G$ is strongly hereditarily self-similar of index $p^{2m}$.*

*Proof.* The proposition follows from Propositions 3.1 and 2.36. ∎

*Proof of Theorem* C. Let $L$ be the $\mathbb{Z}_p$-Lie lattice associated with $G$. Then $L$ is a residually nilpotent 3-dimensional solvable Lie lattice [7, Theorem B]. From Corollary 2.7, $L$ is self-similar of index $p^2$. Hence, by Proposition 3.1, $G$ is self-similar of index $p^2$. The statement on self-similarity of index $p$ follows from Proposition 3.1, Theorem 2.22, and Remark 2.21. ∎

*Proof of Theorem* D. The theorem follows from Remark 3.3, Proposition 3.1, and Proposition 2.41. ∎

**Remark 3.5.** A similar result to Theorem C holds for $p = 3$. Let $G$ be a 3-dimensional solvable *saturable* 3-adic analytic pro-3 group. Then $G$ is self-similar of index 9. Let $L$ be the $\mathbb{Z}_3$-Lie lattice associated with $G$. Then $G$ is self-similar of index 3 if and only if $L$ is isomorphic to a Lie lattice appearing in the list of Theorem 2.22.

**Remark 3.6.** Let $G$ be one of the groups in the list below, where $d$ is an integer. Observe that this list extends the one appearing in the statement of Theorem A (here there is no assumption $p > d$).

(1) For $d \geqslant 1$, the abelian pro-$p$
vspace-1pt group $\mathbb{Z}_p^d$.

(2) For $d \geqslant 2$, the metabelian pro-$p$ group $G^d(s) = \mathbb{Z}_p \ltimes \mathbb{Z}_p^{d-1}$, where the canonical generator of $\mathbb{Z}_p$ acts on $\mathbb{Z}_p^{d-1}$ by multiplication by the scalar $1 + p^s$ for some integer $s$ such that $s \geqslant 1$ if $p \geqslant 3$, and $s \geqslant 2$ if $p = 2$.

Then $G$ is a uniformly powerful $p$-adic analytic pro-$p$ group of dimension $d$. Let $L_G$ be the $\mathbb{Z}_p$-Lie lattice associated with $G$. Observe that if $G$ is abelian, then $L_G \simeq L^d(0)$, while if $G = G^d(s)$, then $L_G \simeq L^d(p^s)$. One can show that any subgroup of $G$ generated by two elements is powerful. It follows that any closed subgroup of $G$ is uniformly powerful, hence, saturable. Clearly, any 2-generated closed subgroup of $G$ has dimension at most 2.

**Proposition 3.7.** *Let $k \geqslant 1$ be an integer, and let $G$ be a group isomorphic to one of the groups in the list of Remark 3.6. Then $G$ is strongly hereditarily self-similar of index $p^k$.*

*Proof.* If $d := \dim(G) = 1$, then $G \simeq \mathbb{Z}_p$ and the result is clear. Assume that $d \geqslant 2$. The result follows from Remark 3.6, Remark 2.38, Proposition 3.1, and Proposition 2.40. ■

Under the assumption that $p > \dim(G)$ we can prove the converse of Proposition 3.7, which is the main result of the paper.

*Proof of Theorem A.* The "if" part follows from Proposition 3.7. For the "only if" part, if $d = 1$, then $G \simeq \mathbb{Z}_p$. Assume that $d \geqslant 2$. Observe that in this case $p \geqslant 3$. By Remark 3.2 we can apply Proposition 3.1. Let $L_G$ be the $\mathbb{Z}_p$-Lie lattice associated with $G$, which is residually nilpotent. From Theorem 2.34, $L_G \simeq L^d(p^s)$ for some $s \in \mathbb{N} \cup \{\infty\}$, while from residual nilpotency we deduce that $s \geqslant 1$. Now, the theorem follows from Remark 3.6. ■

Assume that $p$ is odd, and let $K$ be a field that contains a primitive $p$th root of unity (necessarily, $K$ has a characteristic different from $p$). In [26], Roger Ware proved that if $G_K(p)$ is finitely generated and it does not contain a non-abelian free pro-$p$ subgroup, then $G_K(p)$ is either a free abelian pro-$p$ group of finite rank, or it is isomorphic to $G^d(s)$ for some integers $d \geqslant 2$ and $s \geqslant 1$. In particular, the same conclusion holds if $G_K(p)$ is solvable or $p$-adic analytic. Indeed, Ware proved this result under the additional assumption that $K$ contains a primitive $p^2$th root of unity and conjectured that the result should be true without this assumption. The conjecture was proved by Quadrelli [20, Corollary 4.9]. As a direct consequence of Proposition 3.7 and the result of Ware, we have the following.

**Proposition 3.8.** *Assume that $p \geqslant 3$, and let $K$ be a field that contains a primitive $p$th root of unity. Suppose that $G_K(p)$ is a non-trivial finitely generated pro-$p$ group that does not contain a non-abelian free pro-$p$ subgroup. Then $G_K(p)$ is strongly hereditarily self-similar of index $p$.*

Conversely, for $p$ odd, it is shown in [26] that any group in the list of Remark 3.6 is isomorphic to $G_K(p)$ for some field $K$ that contains a primitive $p$th root of unity. We recall the construction of $K$ for the non-abelian groups $G^d(s)$, in which case $d \geqslant 2$ and $s \geqslant 1$. Let $r$ be a prime with $r \equiv_p 1$, and let $F = \mathbb{F}_r(\omega_s)$, where $\mathbb{F}_r$ is a finite field with $r$ elements and $\omega_s$ is a primitive $p^s$th root of unity. Then one may take $K = F((x_1)) \cdots ((x_{d-1}))$, the field of iterated formal Laurent series.

*Proof of Theorem B.* For $p > 2$ the result follows from Theorem A and the above discussion. When $p = 2$, we observe that $G_{\mathbb{F}_q}(2) \simeq \mathbb{Z}_2$ for any finite field $\mathbb{F}_q$ with $q$ elements; this follows from the well-known fact that the absolute Galois group of $\mathbb{F}_q$ is isomorphic to $\widehat{\mathbb{Z}} = \prod_r \mathbb{Z}_r$, where the product ranges over all primes $r$. ■

As mentioned in Section 1, during the last decade the groups listed in Theorem A have been object of study. We recall the related results and complement them with the results of this paper. A pro-$p$ group $G$ is said to have a constant generating number on open subgroups if $d(H) = d(G)$ for all open subgroups $H$ of $G$, where $d(G)$ is the minimum number of elements of a topological generating set for $G$. Pro-$p$ groups with constant generating number on open subgroups were classified by Klopsch and Snopce in [12]. A Bloch–Kato pro-$p$ group is a pro-$p$ group $G$ with the property that the $\mathbb{F}_p$-cohomology ring of every closed subgroup of $G$ is quadratic. In [20], Quadrelli described explicitly all finitely generated Bloch–Kato pro-$p$ groups that do not contain a free non-abelian pro-$p$ group. A pro-$p$ group $G$ is said to be hereditarily uniform if every open subgroup of $G$ is uniform. Hereditarily uniform pro-$p$ groups were classified by Klopsch and Snopce in [13]. Finally, a pro-$p$ group $G$ is said to be Frattini-injective if distinct finitely generated subgroups of $G$ have distinct Frattini subgroups. Frattini-injective pro-$p$ groups were introduced and studied by Snopce and Tanushevski in [24]. The results of Klopsch–Snopce, Quadrelli, and Snopce–Tanushevski ([12, Corollary 2.4], [13, Corollary 1.13], [20, Theorem B], and [24, Theorem 1.2]) together with Theorem B yield the following.

**Theorem 3.9.** *Let $G$ be a non-trivial solvable torsion-free $p$-adic analytic pro-$p$ group, and suppose that $p > \dim(G)$. Then the following are equivalent.*

(1) *$G$ is strongly hereditarily self-similar of index $p$.*

(2) *$G$ is isomorphic to the maximal pro-$p$ Galois group of some field that contains a primitive $p$th root of unity.*

(3) *$G$ has constant generating number on open subgroups.*

(4) *$G$ is a Bloch–Kato pro-$p$ group.*

(5) *$G$ is a hereditarily uniform pro-$p$ group.*

(6) *$G$ is a Frattini-injective pro-$p$ group.*

## 4. Open problems

This paper deals with as-yet-unexplored directions about self-similar groups, so there are many interesting open problems that one may consider. The following two problems are natural.

**Problem 1.** Classify the strongly hereditarily self-similar pro-$p$ groups of index $p$.

**Problem 2.** Classify the hereditarily self-similar pro-$p$ groups of index $p$.

All the examples of strongly hereditarily self-similar pro-$p$ groups of index $p$ that we know are $p$-adic analytic.

**Problem 3.** Is there a finitely generated strongly hereditarily self-similar pro-$p$ group of index $p$ which is not $p$-adic analytic?

Let $K$ be a $p$-adic number field, that is, a finite extension of $\mathbb{Q}_p$. It is well known (see [18, Theorem 7.5.11]) that if $K$ does not contain a primitive $p$th root of unity, then $G_K(p)$ is a free pro-$p$ group of finite rank. On the other hand, if $K$ contains a primitive $p$th root of unity, then $G_K(p)$ is a Demushkin group, that is, a Poincaré duality pro-$p$ group of dimension 2. Pro-$p$ completions of surface groups are also Demushkin groups. It would be interesting to find a proof of Proposition 3.8 using Galois theory. Such a proof would shed some light on how to approach the following two problems.

**Problem 4.** Does a free pro-$p$ group of finite rank admit a faithful self-similar action on a $p$-ary tree?

**Problem 5.** Does a Demushkin pro-$p$ group admit a faithful self-similar action on a $p$-ary tree?

Note that an affirmative answer to Problem 4 would imply that a free pro-$p$ group of finite rank is strongly hereditarily self-similar of index $p$. On the other hand, since every open subgroup of a Demushkin group is also Demushkin, an affirmative answer to Problem 5 would imply that Demushkin groups are hereditarily self-similar of index $p$. Moreover, since every infinite index subgroup of a Demushkin group is free pro-$p$ an affirmative answer to both problems would imply that Demushkin groups are strongly hereditarily self-similar of index $p$. Note that if $G$ is a Demushkin group with $d(G) = 2$, then it is a torsion-free $p$-adic analytic pro-$p$ group of dimension 2, and therefore it is strongly hereditarily self-similar of index $p$. Thus Problem 5 is open only for Demushkin groups $G$ with $d(G) > 2$.

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