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Hyperbolic geometry of shapes of convex bodies

Clément Debin and François Fillastre

Abstract. We use the intrinsic area to define a distance on the space of homothety classes of convex bodies in the n-dimensional Euclidean space, which makes it isometric to a convex subset of the infinite dimensional hyperbolic space. The ambient Lorentzian structure is an extension of the intrinsic area form of convex bodies, and Alexandrov–Fenchel inequality is interpreted as the Lorentzian reversed Cauchy–Schwarz inequality. We deduce that the space of similarity classes of convex bodies has a proper geodesic distance with curvature bounded from below by -1 (in the sense of Alexandrov). In dimension 3, this space is homeomorphic to the space of distances with non-negative curvature on the 2-sphere, and this latter space contains the space of flat metrics on the 2-sphere considered by W. P. Thurston. Both Thurston's and the area distances rely on the area form. So the latter may be considered as a generalization of the "real part" of Thurston's construction.

1. Introduction

Let P be a non-empty space of flat metrics on the 2-sphere, with $n > 3$ prescribed angles $0 < \alpha_i < 2\pi$ at the cone singularities, up to orientation-preserving similarities, and with a labeling of the cone points. In a celebrated article [22], W. P. Thurston uses the area of the flat metrics to endow P with a *complex* hyperbolic structure. Among the multitude generalizations and adaptations of this construction, let us consider subspaces of P endowed with an isometric involution, studied in [2]. They are isometric to spaces of homothety classes of plane convex polygons with fixed direction of edges, endowed with real hyperbolic distances. This latter point of view was then extended to any dimension, using mixed volumes to hyperbolize some spaces of convex polytopes in $\mathbb{R}^n$. For $n = 3$, some of these spaces, which are isometric to (real!) hyperbolic polyhedra, isometrically embed into P [8, 9].

In the first part of the present article, we bring this real hyperbolization process to its full generality, by endowing the space of convex bodies in $\mathbb{R}^n$ with an "area distance," which appears to be hyperbolic in a sense clarified below. The idea behind the definition of the area distance is quite natural. Consider the convex combination $K_t = tK_1 + (1-t)K_2$ of two convex bodies, $t \in [0,1]$. In general, by Alexandrov–Fenchel inequality, there exists $t_0, t_1 \in \mathbb{R}$, $t_0 \leq 0 < 1 \leq t_1$ such that the formal area of K_t is zero. We then have two points (0 and 1) on the segment $[t_0, t_1]$, and, heuristically, t_0 and t_1 belong to the

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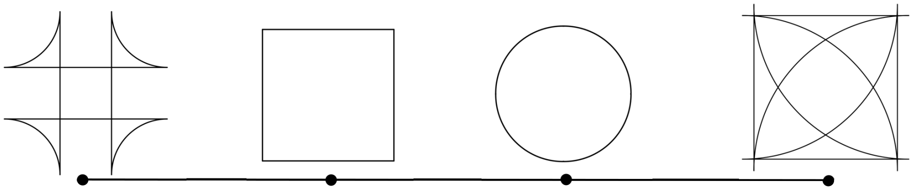

Figure 1. The convex segment between the disc and the square is extended until we arrive at two objects of zero (formal) area. The hyperbolic distance between the disc and the square is half of the logarithm of the cross-ratio of the four points.

isotropic cone of a quadratic form (the area). Mimicking the definition of the distance of the Klein model of the hyperbolic space, we define the area distance as half of the log of the cross-ratio of $t_0, 0, 1, t_1$; see Figure 1 and also Figure 2. The precise definition of the area distance will be given in Section 2.1.

Recall that two subsets A and B of $\mathbb{R}^n$ are *homothetic* if they differ by a translation and a positive scaling. If K is a convex body, we denote by $[K]$ its homothety class and by $\mathcal{H}\mathrm{om}^{n*}$ the space of homothety classes of all the convex bodies in $\mathbb{R}^n$, which are different from points and segments. The area distance introduced above is clearly invariant under homotheties. Let us denote by $d_{\mathcal{H}^n}$ the induced area distance on $\mathcal{H}\mathrm{om}^{n*}$. Note that it is not obvious that this is actually a distance.

Theorem 1.1. $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$ *is a metric space which*

(1) *is uniquely geodesic, and the unique shortest path between* $[K_1]$ *and* $[K_2]$ *is the class of the convex combination of* K_1 *and* K_2*,*

(2) *is of infinite Hausdorff dimension and infinite diameter,*

(3) *is proper,*

(4) *has curvature bounded from below and above by* -1 *in the sense of Alexandrov,*

(5) *has boundary homeomorphic to the real projective space of dimension* $(n-1)$*,*

(6) *any point in it is the endpoint of a shortest path that is not extendable beyond this point,*

(7) *is homeomorphic to the space of convex bodies of intrinsic area equal to one and Steiner point at the origin, endowed with the Hausdorff distance.*

As some definitions may depend on the authors, let us recall that a metric space is *geodesic* if any two points are joined by a shortest path, is *uniquely geodesic* if the shortest path is unique, and is *proper* if every bounded closed subset is compact. A proper metric space is locally compact and complete. A shortest path is *extendable* if it is strictly contained in another shortest path. The boundary of a metric space is the set of equivalence classes of geodesic rays at bounded distance, endowed with a natural topology; see [5] for details. In the present article, the definition of bounded curvature in the sense of Alexandrov is global.

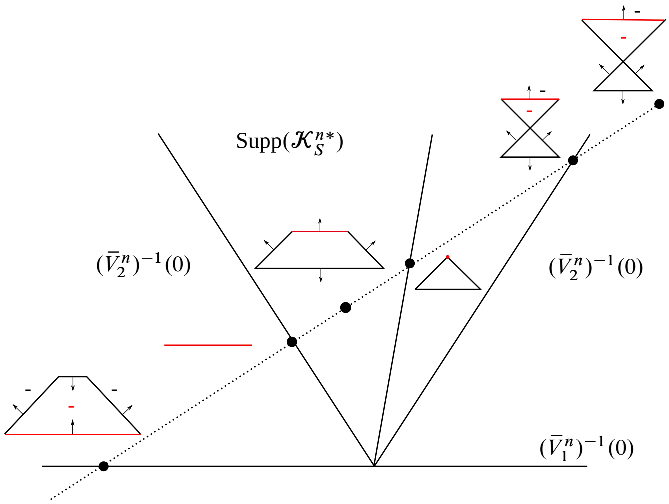

Figure 2. The minus sign outside of the polygons indicates edges with negative algebraic length, while the minus sign inside a polygon indicates a negative area $\bar{V}_2^n$.

The property (6) is proved in Section 2.5. The topological properties in Theorem 1.1 are consequences of a theorem of R. A. Vitale and the Blaschke's selection theorem; see Section 2.6. The other assertions in Theorem 1.1 are either straightforward or they come from the following extrinsic description of $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$.

Theorem 1.2. $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$ *is isometric to an infinite dimensional, unbounded, closed, convex subset, with empty interior, of the infinite dimensional hyperbolic space.*

Here, "the" infinite dimensional hyperbolic space is defined from a separable Hilbert space. The isometry in Theorem 1.2 is obtained by considering the support function of convex bodies. Under this identification, the area of convex bodies will give a bilinear form, that appears to have a Lorentzian signature. This is actually very natural as, for example, Alexandrov–Fenchel inequality for mixed area is then given by a reversed Cauchy–Schwarz inequality.

We say that the distance $d_{\mathcal{H}^n}$ is hyperbolic, because it is isometric to a totally geodesic subspace of a hyperbolic space, or because of the curvature property (4) in Theorem 1.1 (the latter being an immediate consequence of the former). Note that for metric spaces, it is meaningless to speak about "curvature equal to -1."

It was pointed out by Nicolas Monod to the second author that the present construction for $n = 2$ gives an explicit example of an exotic action of $\mathrm{PSL}(2, \mathbb{R})$ on the infinite dimensional hyperbolic space [18].

In the second part of the present article, we investigate $\mathcal{S}\mathrm{hape}^{n*}$, the quotient of $\mathcal{H}\mathrm{om}^{n*}$ by linear isometries of the Euclidean space $\mathbb{R}^n$: $\mathcal{S}\mathrm{hape}^{n*}$ is the space of convex bodies in $\mathbb{R}^n$ (not reduced to points or segments) up to Euclidean similarities (such an equivalence class is the "shape" of the convex body). It is endowed with the quotient distance $d_{\mathcal{S}^n}$. We obtain the following.

Theorem 1.3. $(\mathcal{S}\mathrm{hape}^{n*}, d_{\mathbb{S}^n})$ *is a proper geodesic metric space with curvature* ≥ -1 *and with boundary reduced to a single point. It is not uniquely geodesic. It contains many totally geodesic hyperbolic surfaces.*

There is another complex hyperbolic orbifold considered by Thurston, which is defined similarly to the space P introduced at the beginning of the present article, but where the singular points are not labeled. It is a subspace of $\mathcal{M}^1_{\geq 0}(\mathbb{S}^2)$, the space of metrics of non-negative curvature on the sphere, up to isometries, and with unit area. A natural generalization of Thurston construction would be to use the area of the metrics to endow $\mathcal{M}^1_{\geq 0}(\mathbb{S}^2)$ with a distance, and look at its properties. For example, one may look at curvature properties, or possible complex structure. From [3] and (7) in Theorem 1.1, it follows that $\mathcal{S}\mathrm{hape}^{3*}$ and $\mathcal{M}^1_{\geq 0}(\mathbb{S}^2)$ are homeomorphic if the latter space is endowed with the topology of uniform convergence of distances.[1] So Theorem 1.3, for $n = 3$, may be seen as a "real hyperbolization" of $\mathcal{M}^1_{\geq 0}(\mathbb{S}^2)$ with its natural topology. Here the word "hyperbolization" is used in a wide sense: as $(\mathcal{S}\mathrm{hape}^{n*}, d_{\mathbb{S}^n})$ is not uniquely geodesic, it is not of non-positive curvature, hence not with curvature ≤ -1. However that is an open question to know if it is locally of non-positive curvature.

We conclude the present article by a question about the space of shapes of *all* convex bodies (regardless of the dimension of the ambient space).

As we pointed out, the idea to consider convex bodies in an ambient hyperbolic space came from the observation that the Alexandrov–Fenchel inequality for the mixed area of convex bodies looks like the reversed Cauchy–Schwarz inequality in a Lorentzian vector space (see Remark 2.19). In dimension 2, Alexandrov–Fenchel inequality coincides with the Minkowski inequality. Also, mixed volumes were introduced by Minkowski. He also introduced Lorentzian vector spaces, which are now called Minkowski spaces. We are not aware if Minkowski knew a relation between the inequality and the spaces that both bear his name. But as far as we know, it seems that, in the meantime, this relation between the fundamental inequality of the theory of convex bodies and basic Lorentzian geometry was forgotten.

2. The area distance

2.1. Intrinsic area of convex bodies

A *convex body* is a non-empty compact convex subset of $\mathbb{R}^n$. In the present article, we set $n > 1$. For a plane convex body K (i.e., a convex body in $\mathbb{R}^2$), speaking about the "area" of K usually means to look at its volume (two-dimensional Lebesgue measure). Note that the area of plane convex bodies is positively homogeneous of degree 2: for $\lambda > 0$, $\mathrm{vol}_2(\lambda K) = \lambda^2 \mathrm{vol}_2(K)$. For a convex body in $\mathbb{R}^3$, the "area" usually refers to its surface

[1] For $n \geq 3$, the induced inner distance on the boundary of a convex body in $\mathbb{R}^n$ is (isometric to) a distance of non-negative curvature on $\mathbb{S}^{n-1}$ in the sense of Alexandrov. But not every such distance of non-negative curvature on $\mathbb{S}^{n-1}$ can arise in this way ([17], [1, Section 1.9]).

area, i.e., the two-dimensional total Hausdorff measure of its boundary ∂K. Here also, the surface area is positively homogeneous of degree two.

For $n > 3$, there are two ways to generalize the notion of "area" to convex bodies in $\mathbb{R}^n$. Both are coming from the Steiner formula. Let B^n be the closed unit ball centered at the origin in $\mathbb{R}^n$, and let κ_n be its volume. Let us set $\kappa_0 = 1$ and $\kappa_1 = 2$. If K is a convex body in $\mathbb{R}^n$, then there exist non-negative real numbers $V_i(K)$, $i = 0, \ldots, n$, such that, for any $\varepsilon > 0$,

$$\mathrm{vol}_n(K + \varepsilon B^n) = \sum_{i=0}^{n} \varepsilon^{n-i} \kappa_{n-i} V_i(K). \tag{2.1}$$

Here vol_n is the Lebesgue measure of $\mathbb{R}^n$, and the sum is the Minkowski addition: $A + B = \{a + b \mid a \in A,\ b \in B\}$. It appears that $V_0(K) = 1$ and $V_n(K) = \mathrm{vol}_n(K)$.

The first way to generalize the notion of surface area of convex bodies in $\mathbb{R}^3$ is to consider $V_{n-1}(K)$ as the "area," given by the first-order variation of $\mathrm{vol}_n(K + \varepsilon B^n)$, seen as a function of ε. Note that this "area" is homogeneous of degree $(n-1)$, and that for $n = 2$, this is related to the perimeter of the convex body and not to its area.

In the present article, we consider another way to generalize the notion of surface area of convex bodies in $\mathbb{R}^3$, and we call $V_2(K)$ given by (2.1) the *intrinsic area* of K. Let us mention some relevant properties. The property (A6) explains the terminology "intrinsic."

- (A1) For any $\lambda > 0$, $V_2(\lambda K) = \lambda^2 V_2(K)$.
- (A2) $V_2(K) \geq 0$.
- (A3) $K_1 \subset K_2 \Rightarrow V_2(K_1) \leq V_2(K_2)$.
- (A4) $V_2(K) = 0$ if and only if K is a point or a segment.
- (A5) For any $A \in O(n)$ and $p \in \mathbb{R}^n$, $V_2(A(K) + \{p\}) = V_2(K)$.
- (A6) Let $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a linear isometric embedding. Then $V_2(\iota(K)) = V_2(K)$.

The (intrinsic) area can be "polarized," in the sense that there exists a function called the (intrinsic) *mixed area* $V_2(\cdot, \cdot)$, that can be defined as

$$V_2(K_1, K_2) = \frac{1}{2}\big(V_2(K_1 + K_2) - V_2(K_1) - V_2(K_2)\big) \tag{2.2}$$

and satisfies the following properties:

- (M1) $V_2(K_1, K_1) = V_2(K_1)$;
- (M2) $V_2(K_1, K_2) = V_2(K_2, K_1)$;
- (M3) $V_2(K_1 + K_2, K_3) = V_2(K_1, K_3) + V_2(K_2, K_3)$;
- (M4) for $\lambda > 0$, $V_2(\lambda K_1, K_2) = \lambda V_2(K_1, K_2)$;
- (M5) $K_1 \subset K_2 \Rightarrow V_2(K_1, K_3) \leq V_2(K_2, K_3)$;
- (M6) K is a point if and only if for any convex body Q, $V_2(K, Q) = 0$;
- (M7) $V_2(K_1, K_2) \geq 0$; and $V_2(K_1, K_2) = 0$ if and only if K_1 or K_2 is a point, or both are segments with the same direction;

(M8) we have

$$\delta(K_1, K_2) = V_2(K_1, K_2)^2 - V_2(K_1)V_2(K_2) \geq 0 \tag{2.3}$$

and if K_1 and K_2 are not points, then equality occurs if and only if K_1 and K_2 are homothetic.

All those properties are classical, as V_2 is a particular case of mixed volume: $V_2(K_1, K_2) = V(K_1, K_2, B^n, \dots, B^n)$ [20]. Property (M8) is Alexandrov–Fenchel inequality. In the present article, we will generalize the properties listed above, using some simple analysis of functions on the sphere. Before that, let us introduce the area distance on the space of homothety classes of convex bodies. We will give two equivalent definitions, both using Alexandrov–Fenchel inequality (M8).

In the sequel, we denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$, and by $\mathcal{K}^{n*}$ the subset of convex bodies of positive intrinsic area. In other terms, by (A2) and (A4), $\mathcal{K}^{n*}$ is $\mathcal{K}^n$ minus points and segments. By property (M8) of the mixed area, for any $K_1, K_2 \in \mathcal{K}^{n*}$, the quantity

$$\tilde{d}_1(K_1, K_2) = \operatorname{argch}\left(\frac{V_2(K_1, K_2)}{\sqrt{V_2(K_1)V_2(K_2)}}\right)$$

is well defined. This is also clear that $\tilde{d}_1(K_1, K_2)$ is invariant under positive scaling of K_1 and K_2. Moreover, by (A5) and (2.2), for all $p \in \mathbb{R}^n$,

$$V_2\big(K_1 + \{p\}, K_2\big) = V_2\big(K_1, K_2 + \{p\}\big) = V_2(K_1, K_2),$$

hence $\tilde{d}_1$ is invariant under translations of K_1 or K_2. By the case of equality in property (M8), $\tilde{d}_1(K_1, K_2) = 0$ if and only if K_1 differ from K_2 by a homothety.

Let us define the space $\mathcal{H}\mathrm{om}^n$ (resp. $\mathcal{H}\mathrm{om}^{n*}$) as the quotient of $\mathcal{K}^n$ (resp. $\mathcal{K}^{n*}$) by homotheties. For a convex body K, we denote by $[K]$ the set of homothetic copies of K. For any $[K_1], [K_2] \in \mathcal{H}\mathrm{om}^{n*}$ we set

$$d_1\big([K_1], [K_2]\big) = \tilde{d}_1(K_1, K_2).$$

Let us do it in a different way. Let $K_1, K_2 \in \mathcal{K}^{n*}$. Assume that $V_2(K_1) = V_2(K_2) = a > 0$ and that $[K_1] \neq [K_2]$. Consider the following equation:

$$V_2\big((1-t)K_1 + tK_2\big) = 0. \tag{2.4}$$

By properties of the mixed area, the left-hand side is a polynomial in t, and the coefficient of t^2 is $2a - 2V_2(K_1, K_2)$. Since $[K_1] \neq [K_2]$, by Alexandrov–Fenchel inequality (M8), we have $V_2(K_1, K_2) > a$: the coefficient of t^2 is negative, in particular this is a second-order polynomial. An easy calculation shows that its discriminant is equal to $4\delta(K_1, K_2) > 0$ (see (2.3)). Let $t_1 < 0 < 1 < t_2$ be the two real solutions of (2.4), and let us define

$$\tilde{d}_2(K_1, K_2) = \frac{1}{2} \ln[0, 1, t_1, t_2],$$

where $[0, 1, t_1, t_2] = \frac{t_1}{t_2}\frac{1-t_2}{1-t_1}$ is the cross-ratio.

By (2.4), it is clear that $\tilde{d}_2$ is invariant by translation of K_1 or K_2. Let $[K_1], [K_2] \in \mathcal{H}\mathrm{om}^{n*}$ and let K_1, K_2 be two representatives having the same intrinsic area. We can then define

$$d_2\big([K_1],[K_2]\big) = \tilde{d}_2(K_1, K_2)$$

if $[K_1] \neq [K_2]$, and zero otherwise.

Classical trigonometry computations from hyperbolic geometry show $d_1 = d_2$. We define the *area distance* on $\mathcal{H}\mathrm{om}^{n*}$ as

$$d_{\mathcal{H}^n} := d_1 = d_2.$$

(Note that we have not proved yet that it is a distance.)

Even if the space of convex bodies is not a vector space, from its properties the mixed area resembles a symmetric bilinear form, whose kernel is the space of points, and whose isotropic cone is the space of points and segments. Moreover, Alexandrov–Fenchel inequality (M8) resembles a reversed Cauchy–Schwarz inequality. To define d_1 and d_2 above, we mimicked the definitions of the hyperboloid model and the Klein model of the hyperbolic space. It is actually the way we will prove Theorem 1.1.

2.2. Spaces of support functions

The *support function* $\mathrm{Supp}(K)$ of a convex body K in $\mathbb{R}^n$ gives, at the point $x \in \mathbb{S}^{n-1}$, the distance from the origin of $\mathbb{R}^n$ to the support hyperplane of K with outward normal x. More precisely, $\mathrm{Supp}(K) : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined as

$$\mathrm{Supp}(K)(x) = \max_{p \in K} \langle x, p \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product of $\mathbb{R}^n$.

Let us denote by $\| \cdot \|_{L^2}$ the L^2 norm on the round sphere $\mathbb{S}^{n-1}$. Let $H^1(\mathbb{S}^{n-1})$ be the Sobolev space of $\mathbb{S}^{n-1}$, i.e., the space of functions $\mathbb{S}^{n-1} \to \mathbb{R}$ which are in $L^2(\mathbb{S}^{n-1})$, as well as their first-order derivatives in the weak sense. The space $H^1(\mathbb{S}^{n-1})$ is implicitly endowed with the norm

$$\|h\|_{H^1} = \big(\|h\|_{L^2}^2 + \|\nabla h\|_{L^2}^2\big)^{1/2} = \bigg(\int_{\mathbb{S}^{n-1}} h^2 + \|\nabla h\|^2\bigg)^{1/2},$$

where the gradient ∇ is the one of the round sphere.

If K is contained in the ball centered at the origin and with radius R, then $\mathrm{Supp}(K)$ is R-Lipschitz. Hence we get a map

$$\mathrm{Supp} : \mathcal{K}^n \to H^1(\mathbb{S}^{n-1}).$$

Let us recall some basic properties [10, 11, 20].

- A function $h : \mathbb{S}^{n-1} \to \mathbb{R}$ is the support function of a convex body in $\mathbb{R}^n$ if and only if its one homogeneous extension $\tilde{h} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $\tilde{h}(x) = \|x\| h(x/\|x\|)$, $\tilde{h}(0) = 0$, is a convex function.

- $\mathrm{Supp}(K_1 + K_2) = \mathrm{Supp}(K_1) + \mathrm{Supp}(K_2)$, $\mathrm{Supp}(\lambda K) = \lambda\, \mathrm{Supp}(K)$, $\lambda > 0$; in particular, $\mathrm{Supp}(\mathcal{K}^n)$ is a convex cone in $H^1(\mathbb{S}^{n-1})$.
- Supp is a bijection onto its image.
- If $K_1 \subset K_2$, then $\mathrm{Supp}(K_1) \leq \mathrm{Supp}(K_2)$.
- If $(\mathrm{Supp}(K_i))_i$ converges pointwise to $\mathrm{Supp}(K)$, then the convergence is uniform.
- If $(\mathrm{Supp}(K_i))_i$ converges to $\mathrm{Supp}(K)$, then almost everywhere $(\nabla\, \mathrm{Supp}(K_i))_i \to \nabla\, \mathrm{Supp}(K)$.

Remark 2.1. Let us warn the reader that if $\mathrm{Supp}(\lambda K) = \lambda\, \mathrm{Supp}(K)$, $\lambda > 0$, we do not have $\mathrm{Supp}(-K) = -\mathrm{Supp}(K)$ in general, where $-K = \{-x \mid x \in K\}$. Indeed, both $\mathrm{Supp}(-K)$ and $\mathrm{Supp}(K)$ are positive if the origin of $\mathbb{R}^n$ is in the interior of K. Actually, $\mathrm{Supp}(-K)(v) = \mathrm{Supp}(K)(-v)$ and $-\mathrm{Supp}(K)$ is like the support function of K, but with the support planes defined by their *inward* unit normals.

Let us set $\lambda_1 = n - 1$ and c_n be a given positive constant. For $h \in H^1(\mathbb{S}^{n-1})$, let us consider the quadratic form

$$\overline{V}_2^n(h) = c_n\big(\|h\|_{L^2}^2 - \lambda_1^{-1}\|\nabla h\|_{L^2}^2\big), \tag{2.5}$$

that comes from the following bilinear form: for $h, k \in H^1(\mathbb{S}^{n-1})$,

$$\overline{V}_2^n(h,k) = c_n\big((h,k)_{L^2} - \lambda_1^{-1}(\nabla h, \nabla k)_{L^2}\big).$$

To avoid confusion, let us emphasize that $\overline{V}_2^n(h,h) = \overline{V}_2^n(h)$. It is known (see, e.g., [11, Theorem 4.2a], [20, p. 298] or [10, Proposition 2.4.2]) that for any n there is a unique c_n such that, for any $K_1, K_2 \in \mathcal{K}^n$

$$V_2(K_1, K_2) = c_n \overline{V}_2^n\big(\mathrm{Supp}(K_1), \mathrm{Supp}(K_2)\big).$$

Let us first restrict $\overline{V}_2^n$ to a subspace where it is not degenerate. Hopefully, the kernel of $\overline{V}_2^n$ is exactly the image of points by Supp. Indeed, the support function of the point $p \in \mathbb{R}^n$ is the restriction to the sphere of the linear map $x \mapsto \langle p, x\rangle$. But the space of such maps is the eigenspace of the first non-zero eigenvalue of the Laplacian on the round sphere, and this eigenvalue is the λ_1 in (2.5), so we deduce easily the following fact.

Fact 2.2. *The kernel of $\overline{V}_2^n(\cdot,\cdot)$ on $H^1(\mathbb{S}^{n-1})$ is the eigenspace of λ_1.*

Proof. Let $h \in H^1(\mathbb{S}^{n-1})$. The function h belongs to the kernel of $\overline{V}_2^n(\cdot,\cdot)$ if and only if for any $k \in H^1(\mathbb{S}^{n-1})$ we have

$$\int_{\mathbb{S}^{n-1}} hk = \lambda_1^{-1}\int_{\mathbb{S}^{n-1}} \langle \nabla h, \nabla k\rangle.$$

By density of smooth functions on $\mathbb{S}^{n-1}$ for the H^1-norm and by Green formula, this is equivalent to the following property: for any smooth function k on $\mathbb{S}^{n-1}$ we have

$$\int_{\mathbb{S}^{n-1}} hk = \lambda_1^{-1}\int_{\mathbb{S}^{n-1}} h\Delta k,$$

and this means $h = \lambda_1^{-1}\Delta h$ in the weak (hence smooth) sense. ■

We will denote by $H^1(\mathbb{S}^{n-1})_1$ the subspace of $H^1(\mathbb{S}^{n-1})$ of functions L^2-orthogonal to the eigenspace of λ_1, i.e.,

$$\begin{aligned} H^1(\mathbb{S}^{n-1})_1 &= \left\{h \in H^1(\mathbb{S}^{n-1}) \mid (h, x^i)_{L^2} = 0,\ i = 1, \dots, n\right\} \\ &= \left\{h \in H^1(\mathbb{S}^{n-1}) \mid \int_{\mathbb{S}^{n-1}} h(x)x \, \mathrm{d}\mathbb{S}^{n-1}(x) = 0\right\}. \end{aligned}$$

In turn, $\overline{V}_2^n$ is non-degenerate on $H^1(\mathbb{S}^{n-1})_1$. This space has a clear geometric meaning for convex bodies. Recall that the Steiner point of a convex body K is the following point of $\mathbb{R}^n$:

$$\mathbf{stein}(K) = \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K)(x) x \, \mathrm{d}\mathbb{S}^{n-1},$$

so that

$$\mathbf{stein}(K) = 0 \Leftrightarrow \mathrm{Supp}(K) \in H^1(\mathbb{S}^{n-1})_1.$$

We have that for any $p \in \mathbb{R}^n$, $\mathbf{stein}(K + \{p\}) = \mathbf{stein}(K) + \{p\}$, hence a convex body with Steiner point at the origin is a representative of the class of this convex body up to translations.

Now we prove that $\overline{V}_2^n$ has a Lorentzian signature on $H^1(\mathbb{S}^{n-1})_1$: it is positive in one direction, and negative definite on the orthogonal (for a given scalar product, here the Sobolev one). Let L be the line of constant functions in $H^1(\mathbb{S}^{n-1})_1$. We denote by $H^1(\mathbb{S}^{n-1})_{01}$ the subspace of $H^1(\mathbb{S}^{n-1})_1$ of elements H^1 (or, equivalently, L^2) orthogonal to L.

Lemma 2.3. *For $h \in H^1(\mathbb{S}^{n-1})_{01}$,*

$$c_n \left(\frac{\lambda_2 - \lambda_1}{\lambda_1}\right) \|h\|_{L^2}^2 \leq -\overline{V}_2^n(h), \tag{2.6}$$

$$c_n \left(\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}\right) \|h\|_{H^1}^2 \leq -\overline{V}_2^n(h) \leq c_n \frac{1}{\lambda_1} \|h\|_{H^1}^2. \tag{2.7}$$

Proof. The space L is exactly the eigenspace of the zero eigenvalue of the spherical Laplacian. If we denote by $\lambda_2 (> \lambda_1)$ the second positive eigenvalue, then by Rayleigh's theorem, for $h \in H^1(\mathbb{S}^{n-1})_{01} \setminus \{0\}$ we have

$$\lambda_2 \leq \frac{\|\nabla h\|_{L^2}^2}{\|h\|_{L^2}^2}. \tag{2.8}$$

Now (2.6) is immediate from (2.8), and the right-hand side inequality in (2.7) follows from

$$-\overline{V}_2^n(h) \leq c_n \lambda_1^{-1} \|\nabla h\|_{L^2}^2 \leq c_n \lambda_1^{-1} \|h\|_{H^1}^2.$$

The left-hand side inequality in (2.7) follows by adding the two following inequalities: as $\lambda_2 > \lambda_1 = n - 1 \geq 1$, (2.6) gives

$$c_n \frac{1}{\lambda_2} \left(\frac{\lambda_2 - \lambda_1}{\lambda_1}\right) \|h\|_{L^2}^2 \leq -\overline{V}_2^n(h),$$

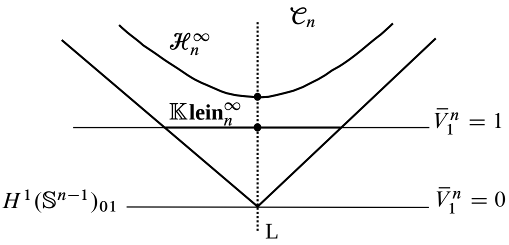

Figure 3. Notations for subspaces of $H^1(\mathbb{S}^{n-1})_1$.

and on the other hand, using again (2.8), the equality (2.5) gives

$$c_n\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\|\nabla h\|_{L^2}^2 \le -\bar{V}_2^n(h). \qquad \blacksquare$$

Clearly $\bar{V}_2^n$ is positive definite on L, and we have the following.

Proposition 2.4. $(H^1(\mathbb{S}^{n-1})_{01}, -\bar{V}_2^n(\cdot,\cdot))$ *is a separable Hilbert space.*

Proof. By (2.6) or (2.7), $-\bar{V}_2^n$ is a scalar product on $H^1(\mathbb{S}^{n-1})_{01}$. As $H^1(\mathbb{S}^{n-1})_{01}$ is orthogonal to a vector subspace, it is a closed subspace, hence complete and separable for the H^1 norm. The result follows from (2.7). $\blacksquare$

Note that as $\bar{V}_2^n$ is Lorentzian on $H^1(\mathbb{S}^{n-1})_1$, we obtain the reversed Cauchy–Schwarz inequality that generalizes Alexandrov–Fenchel inequality (M8):

$$\bar{V}_2^n(h,k)^2 \ge \bar{V}_2^n(h)\bar{V}_2^n(k), \tag{2.9}$$

for $h, k \in \mathscr{C}_n$ with (see Figure 3)

$$\mathscr{C}_n = \left\{h \in H^1(\mathbb{S}^{n-1})_1 \mid \bar{V}_2^n(h) > 0,\ \bar{V}_1^n(h) > 0\right\},$$

and where

$$\bar{V}_1^n(h) = \frac{1}{\kappa_{n-1}}\int_{\mathbb{S}^{n-1}} h,$$

and equality occurs in (2.9) if and only if $h = \lambda k$, $\lambda > 0$.

Let us mention that it is known that, for a convex body $K \subset \mathbb{R}^n$, if $V_1(K)$ is given by (2.1), then

$$V_1(K) = \bar{V}_1^n\big(\operatorname{Supp}(K)\big).$$

2.3. Infinite dimensional hyperbolic space

Let us introduce

$$\mathscr{H}_n^\infty = \left\{h \in \mathscr{C}_n \mid \bar{V}_2^n(h) = 1\right\}.$$

As the Hilbert structure on $H^1(\mathbb{S}^{n-1})_{01}$ is given by $\bar{V}_2^n$, the map $\bar{V}_2^n$ is smooth, and it is easy to see that $\mathscr{H}_n^\infty$ is the graph of a smooth map over $H^1(\mathbb{S}^{n-1})_{01}$, hence an infinite dimensional smooth manifold. We implicitly endow $\mathscr{H}_n^\infty$ with the restriction of $-\bar{V}_2^n(\cdot,\cdot)$ on its tangent spaces. The intersection of $\mathscr{H}_n^\infty$ with any vector subspace of finite dimension p of $H^1(\mathbb{S}^{n-1})_1$ containing a vector of $\mathcal{C}_n$ is clearly a hyperboloid model of the hyperbolic space of dimension $(p-1)$. In turn, $\mathcal{H}_n^\infty$ is a Riemannian manifold of constant sectional curvature -1. Moreover, it is not hard to see that the map $\mathbf{p}_{\mathcal{H}}$: $\mathcal{H}_n^\infty \to H^1(\mathbb{S}^{n-1})_{01}$, $\mathbf{p}_{\mathcal{H}}(h) = h - \frac{\bar{V}_1^n(h)}{\bar{V}_1^n(1)}$ is a bijection and locally bi-Lipschitz, so by Proposition 2.4, $\mathcal{H}_n^\infty$ is complete.

Let us denote by $d_{\mathcal{H}}$ the distance induced by the Riemannian structure, and we have, in the same way as in the finite dimensional case,

$$d_{\mathcal{H}}(h,k) = \operatorname{argch} \bar{V}_2^n(h,k).$$

We will also need the pull-back of the distance on the hyperboloid onto

$$\mathbb{K}\mathbf{lein}_n^\infty = \left\{h \in \mathcal{C}_n \mid \bar{V}_1^n(h) = 1\right\}$$

via a central projection, i.e., the hyperbolic distance on $\mathbb{K}\mathbf{lein}_n^\infty$ is defined by

$$d_{\mathbb{K}}(h,k) := d_{\mathcal{H}}\left(\bar{V}_2^n(h)^{-1/2}h, \bar{V}_2^n(k)^{-1/2}k\right). \tag{2.10}$$

Of course, it is possible to write $d_{\mathbb{K}}$ in an intrinsic way, as we did in Section 2.1 for the area distance, using (2.9) instead of (M8). For future references let us note the following non-surprising facts, whose proofs are left to the reader.

Fact 2.5. *On $\mathcal{H}_n^\infty$, $d_{\mathcal{H}}$ and d_{H^1} induce the same topology, where d_{H^1} is the distance induced by $\|\cdot\|_{H^1}$.*

Fact 2.6. *Let $h_i, k \in \mathbb{K}\mathbf{lein}_n^\infty$. Then*

$$\bar{V}_2^n(h_i) \to 0 \Leftrightarrow d_{\mathbb{K}}(h_i,k) \to +\infty.$$

Fact 2.7. *Let $(h_i)_i$ converge to h in $(\mathbb{K}\mathbf{lein}_n^\infty, d_{\mathbb{K}})$. Then $\bar{V}_2^n(h-h_i) \to 0$.*

Fact 2.8. *On $\mathbb{K}\mathbf{lein}_n^\infty$, $d_{\mathbb{K}}$ and d_{H^1} induce the same topology.*

2.4. Spaces of convex bodies

Recall that $\mathcal{K}^n$ (resp. $\mathcal{K}^{n*}$) is the set of convex bodies in $\mathbb{R}^n$ (resp. convex bodies with positive intrinsic area). We denote by $\mathcal{K}_S^n$ the space of convex bodies with Steiner point at the origin, and $\mathcal{K}_S^{n*} = \mathcal{K}_S^n \cap \mathcal{K}^{n*}$.

	up to positive scaling	**with $V_2 = 1$**	**with $V_1 = 1$**
up to translations	$\mathcal{H}\mathrm{om}^n, \mathcal{H}\mathrm{om}^{n*}$		
with Steiner point at the origin		$\mathcal{K}_{SV_2}^n, \mathcal{K}_{SV_2}^{n*}$	$\mathcal{K}_{SV_1}^n, \mathcal{K}_{SV_1}^{n*}$

Table 1. Convex bodies in $\mathbb{R}^n$.

In the sequel, a star as upper-script means that we consider only convex bodies with positive intrinsic area (that is, we exclude points and segments). In Table 1, it is obvious that all the sets without a star are in bijection, as well as all the sets with a star.

We have

$$\mathrm{Supp}(\mathcal{K}_S^{n*}) \subset \mathcal{C}_n, \quad \mathrm{Supp}(\mathcal{K}_{SV_2}^{n*}) \subset \mathcal{H}_n^{\infty}, \quad \mathrm{Supp}(\mathcal{K}_{SV_1}^{n*}) \subset \mathbb{K}\mathbf{lein}_n^{\infty}.$$

Clearly, $\mathcal{K}_{SV_2}^{n*}$ (resp. $\mathcal{K}_{SV_1}^{n*}$) is in bijection with $\mathcal{H}\mathrm{om}^{n*}$, and we denote by d_{SV_2} (resp. d_{SV_1}) the pull-back of $d_{\mathcal{H}^n}$ on $\mathcal{K}_{SV_2}^{n*}$ (resp. $\mathcal{K}_{SV_1}^{n*}$). By construction, the map Supp defines isometries

$$(\mathcal{K}_{SV_2}^{n*}, d_{SV_2}) \xrightarrow{\sim} (\mathrm{Supp}(\mathcal{K}_{SV_2}^{n*}), d_{\mathcal{H}}),$$

$$(\mathcal{K}_{SV_1}^{n*}, d_{SV_1}) \xrightarrow{\sim} (\mathrm{Supp}(\mathcal{K}_{SV_1}^{n*}), d_{\mathbb{K}}),$$

as all these sets are isometric to $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$. We immediately obtain some parts of Theorems 1.1 and 1.2: $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$ is a metric space, isometric to a convex subset of $\mathbb{H}_n^{\infty}$. In turn, it has curvatures ≤ -1 and ≥ -1, as this is clearly true for its isometric image in the hyperbolic space, and it is a uniquely geodesic metric space, as the hyperbolic space is uniquely geodesic. The unique shortest path is the convex combination, as the property occurs in $\mathbb{K}\mathbf{lein}_n^{\infty}$.

Let us check two easy facts that give other parts of Theorems 1.1 and 1.2. The first one implies that $\mathrm{Supp}(\mathcal{K}_{SH}^{n*})$ is unbounded.

Fact 2.9. $\mathrm{Supp}(\mathcal{K}_{SH}^{n*})$ *contains an entire geodesic of* $\mathbb{H}_n^{\infty}$.

Proof. In the plane, consider the following segments: $K_1 = [-1, 1] \times \{0\}$ and $K_2 = \{0\} \times [-1, 1]$. For $0 \leq t \leq 1$, the convex combination $(1-t)K_1 + tK_2$ is the rectangle $[-(1-t), 1-t] \times [-t, t]$, whose Steiner point is 0. This gives an entire geodesic of $\mathbb{H}_2^{\infty}$ contained in $\mathrm{Supp}(\mathcal{K}_{SH}^{2*})$. ■

The following fact implies that $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$ has infinite Hausdorff dimension.

Fact 2.10. *For any $s \in \mathbb{N}$, there is an open ball of the finite dimensional hyperbolic space $\mathbb{H}^s$ that isometrically embeds into* $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$.

Proof. The convex hyperbolic polyhedra constructed in [2] parametrize the similarity classes of convex polygons with fixed angles; by construction, they isometrically embed into $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$. The dimension of the hyperbolic polyhedra is $(s-3)$ if the polygons have s edges. ■

Fact 2.11. *The boundary of* $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$ *is homeomorphic to the real projective space of dimension* $(n-1)$.

Proof. The boundary is the space of segments, up to homotheties: indeed, for example by looking at the isometric model $(\mathrm{Supp}(\mathcal{K}_{SV_1}^{n*}), d_{\mathbb{K}})$, we see that the convex bodies K on the boundary are the one for which $V_2(K) = 0$ (see Fact 2.6) and $V_1(K) = 1$, and these are exactly unit length segments. Hence $\partial\,\mathcal{H}\mathrm{om}^{n*}$ is in bijection with $P^{n-1}(\mathbb{R})$, the real projective space of dimension $n-1$ (that is, the space of lines in $\mathbb{R}^n$).

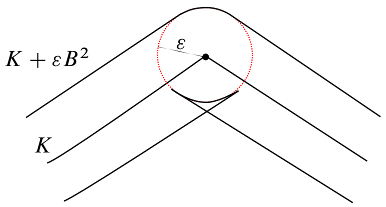

Figure 4. If a plane convex body K has a non-smooth point, then for any $\varepsilon > 0$, $\operatorname{Supp}(K) + \varepsilon \operatorname{Supp}(B^2)$ is the support function of a convex body, while $\operatorname{Supp}(K) - \varepsilon \operatorname{Supp}(B^2)$ is not.

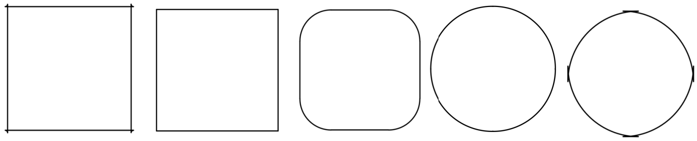

Figure 5. The disc and the square are both terminal points of the segment joining them.

We can endow $\partial\,\mathcal{H}\mathrm{om}^{n*}$ with the visibility metric from [B^n]: the distance between $a, b \in \partial\,\mathcal{H}\mathrm{om}^{n*}$, denoted by $\sphericalangle_B\,(a,b)$, is the angle (with value in $[0,\pi]$) between the two lines c_a and c_b from [B^n] and with endpoints a and b, respectively. But clearly, the element of $O(n)$ sending the line a to the line b is also a $d_{\mathcal{H}^n}$-isometry sending c_a to c_b. In turn, $\partial\,\mathcal{H}\mathrm{om}^{n*}$ endowed with the visibility metric is isometric to $P^{n-1}(\mathbb{R})$ endowed with its round metric. From [5, Proposition II.9.2], $\sphericalangle_B \colon \partial\,\mathcal{H}\mathrm{om}^{n*} \times \partial\,\mathcal{H}\mathrm{om}^{n*} \to \mathbb{R}$ is continuous for the classical topology on $\partial\,\mathcal{H}\mathrm{om}^{n*}$. Hence for this topology, $\partial\,\mathcal{H}\mathrm{om}^{n*}$ is homeomorphic to $P^{n-1}(\mathbb{R})$. ∎

In the two following sections we will prove the two remaining parts of Theorems 1.1 and 1.2: the assertion about terminal points of segments, and the topological properties.

2.5. Terminal points of segments

Let $K_1, K_2 \in \mathcal{K}^n_{SV_1}$. The *segment* between K_1 and K_2 is $\{(1-t)K_1 + tK_2,\ t \in [0,1]\}$. We say that $K_1 \in \mathcal{K}^n_{SV_1}$ is a *terminal point* of the segment if for any $t < 0$, $(1-t)\operatorname{Supp}(K_1) + t\operatorname{Supp}(K_2) \notin \operatorname{Supp}(\mathcal{K}^n_{SV_1})$. An *extreme point* K of $\mathcal{K}^n_{SV_1}$ is such that there does not exist $K_1, K_2 \in \mathcal{K}^n_{SV_1}$, $K_1 \neq K_2$, and $t \in (0,1)$ such that $\operatorname{Supp}(K) = (1-t)\operatorname{Supp}(K_1) + t\operatorname{Supp}(K_2)$. In the plane, extreme points of $\mathcal{K}^2_{SV_1}$ are segments and triangles [20, Theorem 3.2.14]. For $n \geq 3$, extreme points of $\mathcal{K}^n_{SV_1}$ are dense for the Hausdorff distance [20, Theorem 3.2.18].

Clearly, an extreme point is a terminal point for all the segments ending at this point. But there are much more terminal points. For example, one can find convex bodies with a non-smooth point on the boundary (i.e., a point of the convex body contained in more than one support plane) which are terminal points for the segment starting at the unit ball—this idea is illustrated in Figure 4.

In this section, we will use a different argument to prove that *any* convex body is the terminal point of some segment (Proposition 2.12); see Figure 5 for an example.

If a function $h \in \mathbb{K}\mathbf{lein}_n^\infty$ belongs to $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, then its one-homogeneous extension $\tilde{h}$ is convex, hence has non-negative Laplacian in the weak sense. This means that for every non-negative function $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \tilde{h}(x)\Delta_e\varphi(x)\mathrm{d}x \geq 0,$$

where $C_c^\infty(\mathbb{R}^n)$ is the set of smooth functions with compact support in $\mathbb{R}^n$.

For $1 \leq p < n$, we will denote by $B_{p,n}$ the p-dimensional ball with radius $r_1(p)$ in $\mathbb{R}^n$, which is the set of points $x \in \mathbb{R}^n$ with $x_1^2 + \cdots + x_p^2 \leq r_1(p)^2$ and $x_{p+1} = \cdots = x_n = 0$. The number $r_1(p)$ is such that a ball with such radius has $V_1 = 1$. We have $V_1(B_{p,n}) = 1$, hence $B_{p,n} \in \mathcal{K}^n_{SV_1}$ (note that $B_{p,n} \in \mathcal{K}^{n*}_{SV_1}$ if and only if $p \geq 2$). Let $b_{p,n} = \mathrm{Supp}(B_{p,n}) \in \mathrm{Supp}(\mathcal{K}^n_{SV_1})$ and let

$$\widetilde{b_{p,n}}(x) = r_1(p)\sqrt{x_1^2 + \cdots + x_p^2}$$

be the 1-homogeneous extension of $b_{p,n}$ (if $p = 1$, then $\widetilde{b_p}(x) = r_1(1)|x_1| = \frac{|x_1|}{2}$).

Proposition 2.12. *Let $p \in \mathbb{N}$ such that $1 \leq p < n$. Then any $K \in \mathcal{K}^{n*}_{SV_1}$ is the terminal point of a segment in $\mathcal{K}^{n*}_{SV_1}$, which starts at some embedded p-dimensional ball in $\mathbb{R}^n$.*

Actually the proof will show that there are infinitely many such segments. If $p = 1$, this ball is in fact a segment and lies on the boundary of $\mathbb{K}\mathbf{lein}_n^\infty$.

To prove Proposition 2.12, we need the following theorem due to Alexandrov (see [4]).

Theorem 2.13. *A convex function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at almost every $\bar{x} \in \mathbb{R}^n$, which means that for almost every $\bar{x} \in \mathbb{R}^n$, there exists a quadratic polynomial $Q_{\bar{x}}$, and a function $R_{\bar{x}}$, such that*

$$f(x) = Q_{\bar{x}}(x) + R_{\bar{x}}(x) \quad \text{and} \quad \lim_{u\to 0} \frac{R_{\bar{x}}(\bar{x}+u)}{\|u\|^2} = 0.$$

Proof of Proposition 2.12. Let $k = \mathrm{Supp}(K) \in \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, and let $\tilde{k}$ be its 1-homogeneous extension. Let $\bar{x} \in \mathbb{R}^n$ be a point at which $\tilde{k}$ is twice differentiable, and let $Q_{\bar{x}}$ and $R_{\bar{x}}$ be as in Theorem 2.13. Since $n > p$, the vector space $\{x_1 = \cdots = x_p = 0\}$ has positive dimension, hence, up to a rotation of K, we may assume that the first components of $\bar{x}$ are $\bar{x}_1 = \cdots = \bar{x}_p = 0$.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a non-negative function, with support in the unit ball in $\mathbb{R}^n$, positive in a neighborhood of 0, and with $\int_{\mathbb{R}^n}\varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be the function $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x-\bar{x}}{\varepsilon})$: this function is non-negative, has support in $B(\varepsilon,\bar{x})$ (the ball centered at $\bar{x}$ and with radius ε), and $\int_{\mathbb{R}^n}\varphi_\varepsilon = 1$.

Let $t < 0$. We want to show that $(1-t)k + tb_{p,n} \notin \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$. We argue by contradiction: assume that $(1-t)k + tb_{p,n} \in \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$. Then $(1-t)\tilde{k} + t\widetilde{b_{p,n}}$ is a convex function on $\mathbb{R}^n$, hence its Laplacian is non-negative in the weak sense, so in particular we have

$$\int_{\mathbb{R}^n} \left((1-t)\tilde{k} + t\widetilde{b_{p,n}}\right)\Delta_e\varphi_\varepsilon \geq 0. \tag{2.11}$$

We will first show that we always have

$$\int_{\mathbb{R}^n} \tilde{k}\,\Delta_e\varphi_\varepsilon \xrightarrow[\varepsilon\to 0]{} +\infty. \tag{2.12}$$

Since t is negative, with (2.11) it is sufficient to show that

$$\int_{\mathbb{R}^n} \widetilde{b_{p,n}}\,\Delta_e\varphi_\varepsilon \xrightarrow[\varepsilon\to 0]{} +\infty. \tag{2.13}$$

Now we need to argue depending whether $p = 1$ or $p \geq 2$.

- If $p \geq 2$, we have

$$\Delta_e\widetilde{b_{p,n}}(x) = \frac{r_1(p)(p-1)}{\sqrt{x_1^2 + \cdots + x_p^2}},$$

  and since $\bar{x}_1 = \cdots = \bar{x}_p = 0$ we have $\sqrt{x_1^2 + \cdots + x_p^2} \leq \|x - \bar{x}\|$, hence $\Delta_e\widetilde{b_{p,n}}(x) \geq \frac{r_1(p)(p-1)}{\varepsilon}$ for every $x \in B(\varepsilon, \bar{x})$, so we have (by Green formula)

$$\int_{\mathbb{R}^n} \widetilde{b_{p,n}}\,\Delta_e\varphi_\varepsilon = \int_{B(\varepsilon,\bar{x})} \varphi_\varepsilon\Delta_e\widetilde{b_{p,n}} \geq \frac{r_1(p)(p-1)}{\varepsilon}\int_{B(\varepsilon,\bar{x})} \varphi_\varepsilon = \frac{r_1(p)(p-1)}{\varepsilon},$$

  and this gives (2.13).
- If $p = 1$, then we have

$$\begin{aligned}\int_{\mathbb{R}^n} \widetilde{b_{p,n}}(x)\Delta_e\varphi_\varepsilon(x)\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}^n} |x_1|\Delta_e\varphi_\varepsilon(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^{n-1}} \varphi_\varepsilon(0, x_2, \ldots, x_n)\mathrm{d}x_2\ldots\mathrm{d}x_n \\ &= \frac{1}{\varepsilon^n}\int_{\mathbb{R}^{n-1}} \varphi\left(0, \frac{x_2 - \bar{x}_2}{\varepsilon}, \ldots, \frac{x_n - \bar{x}_n}{\varepsilon}\right)\mathrm{d}x_2\ldots\mathrm{d}x_n \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^{n-1}} \varphi(0, y_2, \ldots, y_n)\mathrm{d}y_2\ldots\mathrm{d}y_n.\end{aligned}$$

  The second equality is a classical computation, the third is true because $\bar{x}_1 = 0$, and for the last one we use the change of variable $y_i = \frac{x_i - \bar{x}_i}{\varepsilon}$. Since φ is positive in a neighborhood of zero, we have $\int_{\mathbb{R}^{n-1}} \varphi(0, y_2, \ldots, y_n)\mathrm{d}y_2\ldots\mathrm{d}y_n > 0$, and this gives (2.13).

  Moreover, since $\tilde{k} = Q_{\bar{x}} + R_{\bar{x}}$, we have

$$\int_{\mathbb{R}^n} \tilde{k}\,\Delta_e\varphi_\varepsilon = \int_{\mathbb{R}^n} Q_{\bar{x}}\Delta_e\varphi_\varepsilon + \int_{\mathbb{R}^n} R_{\bar{x}}\Delta_e\varphi_\varepsilon.$$

The function $Q_{\bar{x}}$ is a quadratic polynomial, hence its Laplacian is equal to a constant $C \in \mathbb{R}$, which gives $\int_{\mathbb{R}^n} Q_{\bar{x}}\Delta_e\varphi_\varepsilon = \int_{\mathbb{R}^n} C\varphi_\varepsilon = C$. And since $\Delta_e\varphi_\varepsilon(x) = \frac{1}{\varepsilon^{n+2}}\Delta_e\varphi(\frac{x-\bar{x}}{\varepsilon})$, with the change of variable $y = \frac{x-\bar{x}}{\varepsilon}$, we have

$$\int_{\mathbb{R}^n} R_{\bar{x}}(x)\Delta_e\varphi_\varepsilon(x)\mathrm{d}x = \frac{1}{\varepsilon^{n+2}}\int_{B(\varepsilon,\bar{x})} R_{\bar{x}}(x)\Delta_e\varphi\left(\frac{x-\bar{x}}{\varepsilon}\right)\mathrm{d}x$$
$$= \frac{1}{\varepsilon^2}\int_{B(1,0)} R_{\bar{x}}(\bar{x}+\varepsilon y)\Delta_e\varphi(y)\mathrm{d}y.$$

Since $\frac{R_{\bar{x}}(\bar{x}+u)}{\|u\|^2} \to 0$ as $u \to 0$, there exists $M > 0$ such that $|R_{\bar{x}}(\bar{x}+u)| \leq M\|u\|^2$ for $\|u\|$ small enough, hence for ε small enough we have, for every $y \in B(1,0)$, $|R_{\bar{x}}(\bar{x}+\varepsilon y)| \leq M\varepsilon^2\|y\|^2$, hence we obtain

$$\left|\int_{\mathbb{R}^n} R_{\bar{x}}(x)\Delta_e\varphi_\varepsilon(x)\mathrm{d}x\right| \leq M\int_{B(1,0)} \|y\|^2\left|\Delta_e\varphi(y)\right|\mathrm{d}y.$$

The integral $\int_{\mathbb{R}^n} R_{\bar{x}}\Delta_e\varphi_\varepsilon$ does not go to $+\infty$ when ε goes to zero, and by (2.12) this is a contradiction. ∎

2.6. Comparison of topologies

We want to compare the topologies given by $d_{\mathbb{K}}$ and d_∞ on $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, where d_∞ is the distance given by the sup norm. As a tool, we will use the distances d_{L^2} and d_{H^1} induced by the L^2 and H^1 norms respectively on $H^1(\mathbb{S}^{n-1})_1$, as well as the following theorem; see [23] and [10, Proposition 2.3.1].

Theorem 2.14 (Vitale). *The distances d_∞ and d_{L^2} induce the same topology on* $\mathrm{Supp}(\mathcal{K}^n) \subset C^0(\mathbb{S}^{n-1})$.

The result is weaker than saying that the two norms are equivalent on the space of convex bodies, that is not true; see [23] for details.

Corollary 2.15. *The distances d_∞, d_{L^2}, and d_{H^1} induce the same topology on* $\mathrm{Supp}(\mathcal{K}^n)$.

Proof. We prove that d_{L^2} and d_{H^1} induce the same topology. If $h_i \to h$ for $\|\cdot\|_{H^1}$, then obviously $h_i \to h$ for $\|\cdot\|_{L^2}$. And if $h_i \to h$ for $\|\cdot\|_{L^2}$, then by Theorem 2.14 we have $h_i \to h$ for d_∞. Let us check that this implies the convergence for d_{H^1}. This is obvious that $h_i \to h$ in L^2. Moreover, let $R > 0$ be such that $h_i \leq R$ for every i. Then $(\nabla h_i)_i$ almost everywhere converges pointwise to ∇h, hence the convergence holds in L^2 via Lebesgue's dominated convergence theorem: these functions are uniformly bounded by R as the h_i are R-Lipschitz. Hence $h_i \to h$ for $\|\cdot\|_{H^1}$. ∎

A direct consequence of Fact 2.8 and Corollary 2.15 is the following corollary, which relates the distances d_∞ and $d_{\mathbb{K}}$.

Proposition 2.16. *On* $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$*, d_∞ and $d_{\mathbb{K}}$ (as well as d_{L^2} and d_{H^1}) induce the same topology.*

As d_∞ clearly induces the same topology on $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$ and $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_2})$, we obtain the last point of Theorem 1.1, as the Hausdorff distance for convex bodies is exactly d_∞ for the support functions.

Remark 2.17. Even if d_∞ and $d_{\mathbb{K}}$ induce the same topology, their behavior is quite different. First, similarly to the comparison between Euclidean and hyperbolic metric on the disc, we can see that $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_\infty)$ is bounded and $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}})$ is not. Also, if segments are also shortest paths for the Hausdorff distance, they are not unique in general; see note 11 in [20, Section 1.8].

Let us now check that $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}})$ is a proper metric space. It will be an immediate consequence of Blaschke's selection theorem together with Proposition 2.16.

Proposition 2.18. $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}})$ *is a proper metric space.*

Proof. Let A be a closed bounded subset of $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}})$. We want to show that A is compact for $d_{\mathbb{K}}$; by Proposition 2.16, it suffices to show that it is compact for d_∞. As $(\mathrm{Supp}(\mathcal{K}^{n}_{SV_1}), d_\infty)$ is compact (see [20, p. 165]), it suffices to show that A is closed in $(\mathrm{Supp}(\mathcal{K}^{n}_{SV_1}), d_\infty)$.

So assume that $(h_i)_i$ is a sequence of elements of A converging to $h \in \mathrm{Supp}(\mathcal{K}^{n}_{SV_1})$ for d_∞; we want to show that $h \in A$. If $h \in \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, then this is true, because Proposition 2.16 implies that A is a closed subset of $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_\infty)$. Otherwise, $h \in \mathrm{Supp}(\mathcal{K}^{n}_{SV_1}) \setminus \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, hence $\bar{V}^n_2(h) = 0$ and it follows from Corollary 2.15 that $\bar{V}^n_2(h_i) \to 0$. Then by Fact 2.6, the distance in $(\mathbb{K}\mathbf{lein}^\infty_n, d_{\mathbb{K}})$ between h_i and any given point $k \in \mathbb{K}\mathbf{lein}^\infty_n$ goes to infinity, and that contradicts the fact that A is a bounded subset of $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}})$. ∎

Theorem 1.1 is now proved.

The two following facts conclude the proof of Theorem 1.2.

- Since $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$ is proper, it is complete, hence $(\mathrm{Supp}(\mathcal{K}^{n*}_{SH}), d_{\mathbb{H}})$ is also complete, so $\mathrm{Supp}(\mathcal{K}^{n*}_{SH}) \subset \mathbb{H}^\infty_n$ is a closed subspace.
- Now, let us prove that $\mathrm{Supp}(\mathcal{K}^{n*}_{SH})$ has empty interior. If this is not true, then there exists a ball B in $(\mathbb{H}^\infty_n, d_{\mathbb{H}})$ such that $B \subset \mathrm{Supp}(\mathcal{K}^{n*}_{SH})$; we can even assume that $\bar{B}$ (the closure of B) satisfies $\bar{B} \subset \mathrm{Supp}(\mathcal{K}^{n*}_{SH})$. Since $(\mathrm{Supp}(\mathcal{K}^{n*}_{SH}), d_{\mathbb{H}})$ is proper, closed balls are compact, hence $\bar{B}$ is compact. Hence there exists a non-empty relatively compact open set in $(\mathbb{K}\mathbf{lein}^\infty_n, d_{\mathbb{K}})$. But that would be true for the infinite-dimensional Banach space $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$, and that is impossible: a closed ball would be compact.

Remark 2.19. As far as we know, the idea to associate a hyperbolic metric to spaces of convex bodies *via* the area form and support function was more or less explicit in the 90s, for spaces of convex polygones. The main reference is [2]; see [7] for detailed references. This construction was extended to spaces of convex polytopes in [9].

The smallest vector space containing $\mathrm{Supp}(\mathcal{K}^n)$ as a convex cone is the vector space spanned by the cone:

$$\mathrm{Sonic}^n = \{h - k \mid h,\ k \in \mathrm{Supp}(\mathcal{K}^n)\},$$

the space of *n-dimensional hedgehogs*; see [20, Section 9.6], [21] and the references therein for more information. Let us say that the name was coined in [12], although they previously appeared in the literature under different names; see [19]. If $h \in \mathrm{Sonic}^n$, there is a way to associate a geometric object in $\mathbb{R}^n$ (see [16, 21]) that is illustrated in most of the figures of the present article. A description of Sonic^2 in $C^0(\mathbb{S}^1)$ is contained in [16]. But Sonic^n is not complete for any reasonable norm on it—it contains $C^2(\mathbb{S}^{n-1})$, so it is dense in both $H^1(\mathbb{S}^{n-1})$ and $C^0(\mathbb{S}^{n-1})$ endowed with their classical norms. Particular cases of the results of the present article were achieved in this setting (mostly in the regular case) in [13–15].

3. The space of shapes $\mathcal{S}\mathrm{hape}^{n*}$

3.1. Immediate properties

Let $\mathcal{S}\mathrm{hape}^{n*}$ be the quotient of $\mathcal{H}\mathrm{om}^{n*}$ by linear isometries of the Euclidean space $\mathbb{R}^n$: the action of $O(n)$ on $\mathcal{H}\mathrm{om}^{n*}$ is defined by $\Phi[K] := [\Phi K]$. For $K \in \mathcal{K}^{n*}$, we will denote by $[\![K]\!]$ the set of convex bodies differing from K by positive scaling and Euclidean isometries.

Since V_2 is $O(n)$-invariant, $d_{\mathcal{H}^n}(\Phi[K_1], \Phi[K_2])$ and $d_{\mathcal{H}^n}([K_1], [K_2])$ are equal, so $O(n)$ acts by isometries on $\mathcal{H}\mathrm{om}^{n*}$. Moreover, the action of $O(n)$ is clearly continuous on support functions for d_∞, hence by Proposition 2.16, the action is continuous on $(\mathcal{S}\mathrm{hape}^{n*}, d_{\mathcal{H}^n})$. Let us introduce

$$d_{\mathcal{S}^n}\big([\![K_1]\!], [\![K_2]\!]\big) = \inf_{\Phi, \Phi' \in O(n)} d_{\mathcal{H}^n}\big(\Phi[K_1], \Phi'[K_2]\big). \tag{3.1}$$

Noting that by continuity and compactness, the infimum is actually a minimum, it is not hard to deduce that $d_{\mathcal{S}^n}$ is a distance.

Proposition 3.1. *$(\mathcal{S}\mathrm{hape}^{n*}, d_{\mathcal{S}^n})$ is a proper geodesic metric space with curvature ≥ -1.*

Proof. It is a general fact that the quotient will be geodesic and with curvature ≥ -1; see for example [6, Proposition 10.2.4]. The fact that the quotient is proper is also very general. Indeed, suppose that $([\![K_i]\!])_{i\in\mathbb{N}}$ is a bounded sequence in $(\mathcal{S}\mathrm{hape}^{n*}, d_{\mathcal{S}^n})$. There are $\Phi_i \in O(n)$ such that $(\Phi_i[K_i])_{i\in\mathbb{N}}$ is a bounded sequence in $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$. Since $(\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n})$ is proper, up to extract a subsequence, there exists $[K] \in \mathcal{H}\mathrm{om}^{n*}$ such that $d_{\mathcal{H}^n}(\Phi_i[K_i], [K]) \to 0$. As $d_{\mathcal{S}^n}([\![K_i]\!], [\![K]\!]) \leq d_{\mathcal{H}^n}(\Phi_i[K_i], [K])$, we have

$$d_{\mathcal{S}^n}\big([\![K_i]\!], [\![K]\!]\big) \to 0.$$

∎

3.2. Non-uniqueness of shortest paths in $\mathcal{S}\mathrm{hape}^{n*}$

The aim of this section is to prove that shortest paths are not unique in $\mathcal{S}\mathrm{hape}^{n*}$. Obviously, since $\mathcal{S}\mathrm{hape}^{2*}$ isometrically embeds into $\mathcal{S}\mathrm{hape}^{n*}$ for $n \geq 2$, it is sufficient to prove this property for $n = 2$. Hence, in this section, we consider convex bodies in $\mathbb{R}^2$. We will produce a handmade example.

Let K be the intersection of the half-space $[0,\infty) \times \mathbb{R}$ with the ellipse with center 0, width $2\sqrt{2}$, and height $\frac{2}{\sqrt{2}}$. The support function of K is a function on $\mathbb{S}^1$, and with the parametrization $x = (\cos s, \sin s) \in \mathbb{S}^1$, for $s \in [0, 2\pi]$, we will actually define the support function k of K on $[0, 2\pi]$. Namely,

$$k(s) = \sqrt{2\cos^2 s + \frac{1}{2}\sin^2 s} \quad \text{for } s \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big],$$
$$k(s) = \frac{1}{\sqrt{2}}|\sin s| \quad \text{for } s \in \Big[\frac{\pi}{2}, \frac{3\pi}{2}\Big].$$

Let $(\beta, 0)$ be the Steiner point of K, and let $\alpha = V_1(K) = \frac{1}{2}\int_0^{2\pi} k \simeq 2.4$. Then the convex body $K_1 = \alpha^{-1}K + (-\alpha^{-1}\beta, 0)$ has Steiner point 0, and $V_1(K_1) = 1$: hence $K_1 \in \mathcal{K}^{2*}_{SV_1}$. Its support function $k_1 \in \mathrm{Supp}(\mathcal{K}^{2*}_{SV_1})$ is given by

$$k_1(s) = \alpha^{-1}\left(\sqrt{2\cos^2 s + \frac{1}{2}\sin^2 s} - \beta\cos s\right) \quad \text{for } s \in \Big[\frac{-\pi}{2}, \frac{\pi}{2}\Big],$$
$$k_1(s) = \alpha^{-1}\left(\frac{1}{\sqrt{2}}|\sin s| - \beta\cos s\right) \quad \text{for } s \in \Big[\frac{\pi}{2}, \frac{3\pi}{2}\Big].$$

Let K_2 be the rectangle $[-\frac{2}{5}, \frac{2}{5}] \times [-\frac{1}{10}, \frac{1}{10}]$. Obviously, 0 is the Steiner point of K_2. Its support function is defined for any $s \in [0, 2\pi]$ by

$$k_2(s) = \frac{2}{5}|\cos s| + \frac{1}{10}|\sin s|,$$

and since $K_2 = [-\frac{2}{5}, \frac{2}{5}] \times \{0\} + \{0\} \times [-\frac{1}{10}, \frac{1}{10}]$, we have $V_1(K_2) = \mathrm{length}([-\frac{2}{5}, \frac{2}{5}]) + \mathrm{length}([-\frac{1}{10}, \frac{1}{10}]) = 1$. Hence $K_2 \in \mathcal{K}^{2*}_{SV_1}$ and $k_2 \in \mathrm{Supp}(\mathcal{K}^{2*}_{SV_1})$.

Let $[\![K_1]\!]$ and $[\![K_2]\!]$ be the corresponding equivalent classes in $\mathcal{S}\mathrm{hape}^{2*}$. Since K_2 is invariant by the symmetry with respect to the horizontal line, the distance between $[\![K_1]\!]$ and $[\![K_2]\!]$ is given by

$$d_{\mathcal{S}^2}\big([\![K_1]\!], [\![K_2]\!]\big) = \min_{\theta\in\mathbb{R}} d_{\mathcal{H}^2}\big([K_1], R_\theta[K_2]\big),$$

where we denote by R_θ the rotation of angle θ in $\mathbb{R}^2$. We will prove the following.

Proposition 3.2. *The minimum is obtained for $\theta = 0$ and $\theta = \frac{\pi}{2}$; that is one has*

$$d_{\mathcal{S}^2}\big([\![K_1]\!], [\![K_2]\!]\big) = d_{\mathcal{H}^2}\big([K_1], [K_2]\big) = d_{\mathcal{H}^2}\big([K_1], R_{\frac{\pi}{2}}[K_2]\big).$$

Let us state the following fact. Note that in general, this is not true that every shortest path in a quotient space is obtained as the projection of a shortest path.

Lemma 3.3. *Let* $[K_1],[K_2]\in\mathcal{H}\mathrm{om}^{n*}$, *and let* $\Phi\in O(n)$ *be such that* $d_{\mathcal{S}^n}(\llbracket K_1\rrbracket,\llbracket K_2\rrbracket)=d_{\mathcal{H}^n}([K_1],\Phi[K_2])$. *Suppose that* $[\gamma]$ *is the shortest path between* $[K_1]$ *and* $\Phi[K_2]$. *Then the projection* $\llbracket\gamma\rrbracket$ *is a shortest path between* $\llbracket K_1\rrbracket$ *and* $\llbracket K_2\rrbracket$. *Moreover, the projection is an isometry from* $[\gamma]$ *to* $\llbracket\gamma\rrbracket$.

Proof. Let us suppose that $[\gamma]:[0,1]\to X$ is affinely parametrized. Then, for any $0\le s\le t\le 1$,

$$\begin{aligned} d_{\mathcal{S}^n}\big(\llbracket\gamma(s)\rrbracket,\llbracket\gamma(t)\rrbracket\big) &\le d_{\mathcal{H}^n}\big([\gamma(s)],[\gamma(t)]\big)\\ &=(t-s)d_{\mathcal{H}^n}\big([K_1],\Phi[K_2]\big)\\ &=(t-s)d_{\mathcal{S}^n}\big(\llbracket K_1\rrbracket,\llbracket K_2\rrbracket\big). \end{aligned}$$

Using three times this inequality, we obtain

$$\begin{aligned} d_{\mathcal{S}^n}\big(\llbracket K_1\rrbracket,\llbracket K_2\rrbracket\big) &\le d_{\mathcal{S}^n}\big(\llbracket\gamma(0)\rrbracket,\llbracket\gamma(s)\rrbracket\big)+d_{\mathcal{S}^n}\big(\llbracket\gamma(s)\rrbracket,\llbracket\gamma(t)\rrbracket\big)+d_{\mathcal{S}^n}\big(\llbracket\gamma(t)\rrbracket,\llbracket\gamma(1)\big)\rrbracket\\ &\le\big(s+(t-s)+(1-t)\big)d_{\mathcal{S}^n}\big(\llbracket x\rrbracket,\llbracket y\rrbracket\big)\\ &=d_{\mathcal{S}^n}\big(\llbracket x\rrbracket,\llbracket y\rrbracket\big). \end{aligned}$$

All these inequalities are equalities, so in particular

$$d_{\mathcal{S}^n}\big(\llbracket\gamma(s)\rrbracket,\llbracket\gamma(t)\rrbracket\big)=(t-s)d_{\mathcal{S}^n}\big(\llbracket K_1\rrbracket,\llbracket K_2\rrbracket\big).\qquad\blacksquare$$

Proposition 3.2 is sufficient to prove the non-uniqueness of shortest paths in $\mathcal{S}\mathrm{hape}^{2*}$. Indeed, Lemma 3.3 shows that the projections of the shortest paths in $\mathcal{H}\mathrm{om}^{2*}$ between $[K_1]$ and $[K_2]$, and between $[K_1]$ and $R_{\frac{\pi}{2}}[K_2]$, are again shortest paths in $\mathcal{S}\mathrm{hape}^{2*}$. But these two shortest paths are different: the first shortest path contains the point $\llbracket\frac12K_1+\frac12K_2\rrbracket$, and this point is not on the second shortest path $t\mapsto\llbracket(1-t)K_1+tR_{\frac{\pi}{2}}(K_2)\rrbracket$: $\frac12K_1+\frac12K_2$ is not the image by a rotation of $(1-t)K_1+tR_{\frac{\pi}{2}}(K_2)$, which is equivalent to say that $\frac12\alpha^{-1}K+\frac12K_2$ is not the image by a rotation and a translation of $(1-t)\alpha^{-1}K+tR_{\frac{\pi}{2}}(K_2)$; see Figure 6.

Since $R_\pi[K_2]=[K_2]$, to compute the minimum it is sufficient to consider $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$. Moreover, let T be the symmetry with respect to the x axis: we have $T[K_1]=[K_1]$, hence we have

$$\begin{aligned} d_{\mathcal{H}^2}\big([K_1],R_\theta[K_2]\big) &= d_{\mathcal{H}^2}\big(T[K_1],R_\theta[K_2]\big)\\ &=d_{\mathcal{H}^2}\big([K_1],T\circ R_\theta[K_2]\big)\\ &=d_{\mathcal{H}^2}\big([K_1],R_{-\theta}[K_2]\big). \end{aligned}$$

This shows that in fact we need only to consider $\theta\in[0,\frac{\pi}{2}]$.

Let k_2^θ be the support function of $R_\theta[K_2]$, that is $k_2^\theta(s)=k_2(s-\theta)$. We have

$$\cosh\big(d_{\mathcal{H}^2}([K_1],R_\theta[K_2])\big)=\frac{V_2(k_1,k_2^\theta)}{\sqrt{V_2(k_1)V_2(k_2^\theta)}}=\frac{f(\theta)}{2\sqrt{V_2(k_1)V_2(k_2)}},$$

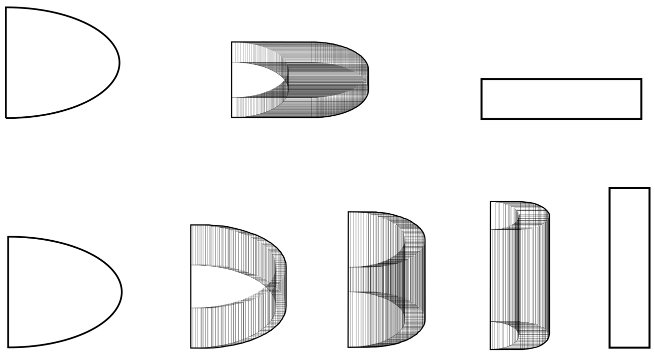

Figure 6. The convex body $\frac{1}{2}\alpha^{-1}K + \frac{1}{2}K_2$ (middle of the upper line) is not the image by a rotation and a translation of $(1-t)\alpha^{-1}K + tR_{\frac{\pi}{2}}(K_2)$ (represented on the bottom line for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$).

where we denote by $f(\theta)$ the function defined by

$$f(\theta) = \int_0^{2\pi} \big(k_1(s)k_2(s-\theta) - k_1'(s)k_2'(s-\theta)\big)\mathrm{d}s.$$

Proposition 3.2 is a direct consequence of the following lemma.

Lemma 3.4. *On* $[0, \frac{\pi}{2}]$, f *attains its minimum at the points* $\theta = 0$ *and* $\theta = \frac{\pi}{2}$.

Proof. Fix $\theta \in (0, \frac{\pi}{2})$ and consider the function $s \mapsto k_1(s)k_2'(s-\theta)$. This function is piecewise $\mathscr{C}^1$ but is not continuous: the function $k_2'(s-\theta)$ has jumps, with height $\frac{1}{5}$ at the points $s = \theta$ and $s = \pi + \theta$, and with height $\frac{4}{5}$ at the points $s = \frac{\pi}{2} + \theta$ and $s = \frac{3\pi}{2} + \theta$. Hence we have

$$\begin{aligned}\int_0^{2\pi} \big(k_1(s)k_2'(s-\theta)\big)'\,\mathrm{d}s &= -\frac{1}{5}k_1(\theta) - \frac{1}{5}k_1(\pi+\theta) - \frac{4}{5}k_1\left(\frac{\pi}{2}+\theta\right) - \frac{4}{5}k_1\left(\frac{3\pi}{2}+\theta\right)\\ &= -\frac{1}{5\alpha}\sqrt{2\cos^2\theta + \frac{1}{2}\sin^2\theta} - \frac{4}{5\alpha}\sqrt{2\sin^2\theta + \frac{1}{2}\cos^2\theta}\\ &\quad - \frac{1}{5\sqrt{2}\alpha}\sin\theta - \frac{4}{5\sqrt{2}\alpha}\cos\theta.\end{aligned}$$

The equality $(k_1k_2')' = k_1'k_2' + k_1k_2''$ gives $-k_1'k_2' = k_1k_2'' - (k_1k_2')'$, so

$$-\int_0^{2\pi} k_1'(s)k_2'(s-\theta)\mathrm{d}s = \int_0^{2\pi} \Big(k_1(s)k_2''(s-\theta) - \big(k_1(s)k_2'(s-\theta)\big)'\Big)\mathrm{d}s,$$

and since $k_2(s-\theta) + k_2''(s-\theta) = 0$ for almost every $s \in [0, 2\pi]$, we finally obtain

$$\begin{aligned}f(\theta) &= \int_0^{2\pi} \big(k_1(s)k_2(s-\theta) - k_1'(s)k_2'(s-\theta)\big)\mathrm{d}s\\ &= \int_0^{2\pi} \Big(k_1(s)\big(k_2(s-\theta) + k_2''(s-\theta)\big) - \big(k_1(s)k_2'(s-\theta)\big)'\Big)\mathrm{d}s\end{aligned}$$

$$= \frac{1}{5\alpha}\sqrt{2\cos^2\theta + \frac{1}{2}\sin^2\theta} + \frac{4}{5\alpha}\sqrt{2\sin^2\theta + \frac{1}{2}\cos^2\theta}$$
$$+ \frac{1}{5\sqrt{2}\alpha}\sin\theta + \frac{4}{5\sqrt{2}\alpha}\cos\theta.$$

We easily check that $f(0) = f(\frac{\pi}{2}) = \frac{\sqrt{2}}{\alpha}$ (the parameters of the ellipse and the segment have been chosen so that this property holds). And a direct computation shows that $f'(0) = \frac{1}{5\sqrt{2}\alpha} > 0$ and $f'(\frac{\pi}{2}) = -\frac{4}{5\sqrt{2}\alpha} < 0$. Moreover, let $g : [0,1] \to [0,\infty)$ be defined by

$$g(u) = \frac{1}{5\alpha}\sqrt{\frac{3}{2}u + \frac{1}{2}} + \frac{4}{5\alpha}\sqrt{2 - \frac{3}{2}u} + \frac{1}{5\sqrt{2}\alpha}\sqrt{1-u} + \frac{4}{5\sqrt{2}\alpha}\sqrt{u}.$$

With the identity $\cos^2 + \sin^2 = 1$, we easily check that $g(\cos^2\theta) = f(\theta)$ for any $\theta \in [0, \frac{\pi}{2}]$. Hence $f'(\theta) = -2g'(\cos^2\theta)\sin\theta\cos\theta$. But g is strictly concave, hence g' has at most one zero on $[0,1]$, hence f' has also at most one zero on $(0, \frac{\pi}{2})$. And this ends the proof: if the minimum of f on $[0, \frac{\pi}{2}]$ was attained at a point $\theta \notin \{0, \frac{\pi}{2}\}$, since $f'(0) > 0$ and $f'(\frac{\pi}{2}) < 0$, f' would have at least 3 zeros on $(0, \frac{\pi}{2})$, and that is impossible. ∎

3.3. Embedding of hyperbolic planes

Trivially, for any $\Phi \in O(n)$ we have $\Phi[B^n] = [B^n]$. Apart from the fact that the action of $O(n)$ on $\mathcal{H}\mathrm{om}^{n*}$ is not proper, this says that for any $[K] \in \mathcal{H}\mathrm{om}^{n*}$,

$$d_{\mathcal{S}^n}\big([\![K]\!], [\![B^n]\!]\big) = d_{\mathcal{H}^n}\big([K], [B^n]\big). \tag{3.2}$$

From this we first deduce the following fact.

Fact 3.5 (Uniqueness of shortest paths starting from B^n). *Let $[\![K]\!] \in \mathcal{S}\mathrm{hape}^{n*}$. Then there is a unique shortest path from $[\![B^n]\!]$ to $[\![K]\!]$, which is the projection of the shortest path in $\mathcal{H}\mathrm{om}^{n*}$ between $[B^n]$ and $[K]$.*

Proof. Let $\overline{\delta} : [0, d_{\mathcal{S}^n}([\![B^n]\!], [\![K]\!])] \to \mathcal{S}\mathrm{hape}^{n*}$ be an arc-length parametrized shortest path between $[\![B^n]\!]$ and $[\![K]\!]$, and let $[\delta(t)] \in \mathcal{H}\mathrm{om}^{n*}$ be such that $\overline{\delta}(t) = [\![\delta(t)]\!]$. Let $t \mapsto [\gamma(t)]$ be the (unique) arc-length parametrized shortest path in $\mathcal{H}\mathrm{om}^{n*}$ between $[B^n]$ and $[K]$: we want to show that $[\![\delta(t)]\!] = [\![\gamma(t)]\!]$.

For any $t \in [0, d_{\mathcal{S}^n}([\![B^n]\!], [\![K]\!])]$, let $\Phi_t \in O(n)$ be such that

$$d_{\mathcal{S}^n}\big([\![K]\!], [\![\delta(t)]\!]\big) = d_{\mathcal{H}^n}\big([K], \Phi_t[\delta(t)]\big).$$

Since $t \mapsto [\![\delta(t)]\!]$ is a geodesic in $\mathcal{S}\mathrm{hape}^{n*}$, we have

$$\begin{aligned} d_{\mathcal{H}^n}\big([B^n], \Phi_t\big[\delta(t)\big]\big) + d_{\mathcal{H}^n}\big(\Phi_t\big[\delta(t)\big], [K]\big) &= d_{\mathcal{S}^n}\big([\![B^n]\!], [\![\delta(t)]\!]\big) + d_{\mathcal{S}^n}\big([\![\delta(t)]\!], [\![K]\!]\big) \\ &= d_{\mathcal{S}^n}\big([\![B^n]\!], [\![K]\!]\big) d_{\mathcal{H}^n}\big([B^n], [K]\big). \end{aligned}$$

Hence $\Phi_t[\delta(t)]$ is on the shortest path between $[B^n]$ and $[K]$ in $\mathcal{H}\mathrm{om}^{n*}$. Moreover, we have $d_{\mathcal{H}^n}([B^n], \Phi_t[\delta(t)]) = d_{\mathcal{S}^n}([\![B^n]\!], [\![\delta(t)]\!]) = t$ (the geodesic $t \mapsto [\![\delta(t)]\!]$ is arc-length parametrized), so $\Phi_t[\delta(t)] = [\gamma(t)]$ (remember that the geodesic $t \mapsto [\gamma(t)]$ is also arc-length parametrized). Finally, this gives $[\![\delta(t)]\!] = [\![\gamma(t)]\!]$. ∎

In turn, we can construct totally geodesic hyperbolic surfaces in $\mathcal{S}\mathrm{hape}^{n*}$. Interestingly, many properties in this section are very general, but this one uses Alexandrov–Fenchel inequality.

Proposition 3.6. *Let $[\![P]\!], [\![Q]\!] \in \mathcal{S}\mathrm{hape}^{n*}$ be such that $[\![P]\!]$, $[\![Q]\!]$, and $[\![B^n]\!]$ are three different points. Let $A \in O(n)$ be such that $d_{\mathcal{S}^n}([\![P]\!], [\![Q]\!]) = d_{\mathcal{H}^n}([P], A[Q])$. Then the projection $\mathcal{H}\mathrm{om}^{n*} \to \mathcal{S}\mathrm{hape}^{n*}$, when restricted to the (plain) geodesic triangle with vertices $[B^n]$, $[P]$, and $A[Q]$, is an isometry onto its image.*

Proof. Without loss of generality, we may assume that A is the identity (that is, $d_{\mathcal{S}^n}([\![P]\!], [\![Q]\!]) = d_{\mathcal{H}^n}([P], [Q])$). Let $[K_1]$ and $[K_2]$ be in the geodesic triangle with vertices $[B^n]$, $[P]$, and $[Q]$: since geodesics in $\mathcal{H}\mathrm{om}^{n*}$ are convex combinations, we can write

$$[K_1] = [\alpha_1 B^n + \beta_1 P + \gamma_1 Q] \quad \text{and} \quad [K_2] = [\alpha_2 B^n + \beta_2 P + \gamma_2 Q],$$

where the $\alpha_i, \beta_i, \gamma_i$ are non-negative real numbers, with $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 1$. We want to prove that $d_{\mathcal{S}^n}([\![K_1]\!], [\![K_2]\!]) = d_{\mathcal{H}^n}([K_1], [K_2])$, which means that for any $\Phi \in O(n)$ we have

$$d_{\mathcal{H}^n}\big([K_1], [K_2]\big) \le d_{\mathcal{H}^n}\big([K_1], \Phi[K_2]\big).$$

Since V_2 is $O(n)$-invariant, we only need to show that

$$V_2(K_1, K_2) \le V_2\big(K_1, \Phi(K_2)\big) \tag{3.3}$$

(K_1 and K_2 denote two convex bodies in the equivalent classes $[K_1]$ and $[K_2]$). We have

$$\begin{aligned} V_2(K_1, K_2) = {} & \alpha_1\alpha_2 V_2(B^n) + \alpha_1\beta_2 V_2(B^n, P) + \alpha_1\gamma_2 V_2(B^n, Q) \\ & + \beta_1\alpha_2 V_2(P, B^n) + \beta_1\beta_2 V_2(P) + \beta_1\gamma_2 V_2(P, Q) \\ & + \gamma_1\alpha_2 V_2(Q, B^n) + \gamma_1\beta_2 V_2(Q, P) + \gamma_1\gamma_2 V_2(Q). \end{aligned}$$

Moreover, $\Phi(K_2) = \alpha_2 B^n + \beta_2 \Phi(P) + \gamma_2 \Phi(Q)$, hence

$$\begin{aligned} V_2\big(K_1, \Phi(K_2)\big) = {} & \alpha_1\alpha_2 V_2(B^n) + \alpha_1\beta_2 V_2\big(B^n, \Phi(P)\big) + \alpha_1\gamma_2 V_2\big(B^n, \Phi(Q)\big) \\ & + \beta_1\alpha_2 V_2(P, B^n) + \beta_1\beta_2 V_2\big(P, \Phi(P)\big) + \beta_1\gamma_2 V_2\big(P, \Phi(Q)\big) \\ & + \gamma_1\alpha_2 V_2(Q, B^n) + \gamma_1\beta_2 V_2\big(Q, \Phi(P)\big) + \gamma_1\gamma_2 V_2\big(Q, \Phi(Q)\big). \end{aligned}$$

And we obviously have $V_2(B^n, P) = V_2(B^n, \Phi(P))$ and $V_2(B^n, Q) = V_2(B^n, \Phi(Q))$. Moreover, Alexandrov–Fenchel inequality (2.3) gives

$$\begin{aligned} V_2(P) &= \sqrt{V_2(P)V_2(\Phi(P))} \le V_2\big(P, \Phi(P)\big), \\ V_2(Q) &= \sqrt{V_2(Q)V_2(\Phi(Q))} \le V_2\big(Q, \Phi(Q)\big). \end{aligned}$$

And $d_{\mathcal{S}^n}([\![P]\!], [\![Q]\!]) = d_{\mathcal{H}^n}([P], [Q])$ gives $V_2(P, Q) \le V_2(P, \Phi(Q))$ and $V_2(Q, P) \le V_2(Q, \Phi(P))$. Since all the real numbers $\alpha_i, \beta_i, \gamma_i$ are non-negative, this gives inequality (3.3). ∎

3.4. Proof of Theorem 1.3

Proposition 3.1 and Sections 3.2 and 3.3 give part of Theorem 1.3. It remains to prove the assertion about the boundary of $\mathcal{S}\mathrm{hape}^{n*}$. It obviously contains only one point: indeed, the boundary of $\mathcal{H}\mathrm{om}^{n*}$ is the set of segments up to homotheties, so the boundary of $\mathcal{S}\mathrm{hape}^{n*}$ is the set of segments, up to translations, positive scaling, and rotations of $\mathbb{R}^n$, and there is only one equivalence class.

4. The space of all the (oriented) shapes

This section is an opening to the study of spaces of convex bodies, considered without making distinction between dimensions. For $p \geq 0$, let us denote by $\iota_{n,p}$ the canonical isometric embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+p}$ which is given by $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}^p \subset \mathbb{R}^{n+p}$. Due to the intrinsic nature of V_2, we have that the map

$$\iota_{n,p} : (\mathcal{H}\mathrm{om}^{n*}, d_{\mathcal{H}^n}) \to (\mathcal{H}\mathrm{om}^{(n+p)*}, d_{\mathcal{H}^{n+p}})$$

defined by $\iota_{n,p}([K]) = [\iota_{n,p}(K)]$ is an isometry. Let $\mathcal{H}\mathrm{om}^{\infty*}$ be the union over n of $\mathcal{H}\mathrm{om}^{n*}$, quotiented by the following equivalence relation: $[K_1]$ is equivalent to $[K_2]$ if and only if there exist $i, j \leq p$ such that $K_1 \subset \mathbb{R}^i$, $K_2 \subset \mathbb{R}^j$, and $[\iota_{i,p-i}(K_1)] = [\iota_{j,p-j}(K_2)]$. We will denote by $[K]_\infty$ an element of $\mathcal{H}\mathrm{om}^{\infty*}$. For two representatives of $[K_1]_\infty, [K_2]_\infty \in \mathcal{H}\mathrm{om}^{\infty*}$ in $\mathbb{R}^n$, let us define

$$d_{\mathcal{H}^\infty}\big([K_1]_\infty, [K_2]_\infty\big) = d_{\mathcal{H}^n}\big([K_1], [K_2]\big).$$

It is easy to see that $d_{\mathcal{H}^\infty}$ is well defined and that it is actually a distance on $\mathcal{H}\mathrm{om}^{\infty*}$. The isometric embeddings $\iota_{n,p}$ induce isometric maps from $(\mathcal{S}\mathrm{hape}^{n*}, d_{\mathcal{S}^n})$ to $(\mathcal{S}\mathrm{hape}^{(n+p)*}, d_{\mathcal{S}^{n+p}})$, so in the same way we can define the set $\mathcal{S}\mathrm{hape}^{\infty*}$ and the metric space $(\mathcal{S}\mathrm{hape}^{\infty*}, d_{\mathcal{S}^\infty})$.

It follows from Theorems 1.1 and 1.3 that $(\mathcal{H}\mathrm{om}^{\infty*}, d_{\mathcal{H}^\infty})$ and $(\mathcal{S}\mathrm{hape}^{\infty*}, d_{\mathcal{S}^\infty})$ are geodesic metric spaces. But two facts occur:

1. (1) it may happen that a sequence of convex bodies with non-empty interior in $\mathbb{R}^p$ converges to a convex body in $\mathcal{H}\mathrm{om}^\infty$ when p goes to infinity. Actually, for $(\varepsilon_p)_p$ a sequence of real numbers such that $\sqrt{p}\varepsilon_p \to 0$, one can check that the sequence $([\iota_{n,p}(K) + \varepsilon_p B^{n+p}]_\infty)_p$ converges in $\mathcal{H}\mathrm{om}^{\infty*}$ to $[K]_\infty$. In particular, there may exist other shortest paths than the convex combinations;
2. (2) one can check that the sequence of balls $([B^n]_\infty)_n$ (resp. $([\![B^n]\!]_\infty)_n$) is a diverging Cauchy sequence.

So we address the following.

Question 1. *Describe the completion of* $(\mathcal{H}\mathrm{om}^{\infty*}, d_{\mathcal{H}^\infty})$ *and* $(\mathcal{S}\mathrm{hape}^{\infty*}, d_{\mathcal{S}^\infty})$.

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Clément Debin
Mathématiques Générales, Université Grenoble Alpes, 38400 Saint-Martin-d’Hères, Grenoble; and Lycée Berthollet, Académie de Grenoble, 74000 Annecy, France; clement.debin@gmail.com

François Fillastre
Laboratoire AGM 2, CYU Cergy Paris Université, 95302 Cergy–Pontoise; and IMAG, Université de Montpellier, 34090 Montpellier, France; francois.fillastre@umontpellier.fr