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# Descriptive chromatic numbers of locally finite and everywhere two-ended graphs

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**Abstract.** We construct Borel graphs which settle several questions in descriptive graph combinatorics. These include "Can the Baire measurable chromatic number of a locally finite Borel graph exceed the usual chromatic number by more than one?" and "Can marked groups with isomorphic Cayley graphs have Borel chromatic numbers for their shift graphs which differ by more than one?" We also provide a new bound for Borel chromatic numbers of graphs whose connected components all have two ends.

## 1. Introduction

A *graph* on a set $X$ is a symmetric irreflexive relation $G \subseteq X \times X$. In this situation, the elements of $X$ are called the *vertices* of $G$. Vertices $x$ and $y$ are called *adjacent* if $(x, y) \in G$, and in this case, the pair $\{x, y\}$ is called an *edge* of $G$. The *degree* of a vertex is the number of other vertices adjacent to it. $G$ is called *locally finite* if every vertex has finite degree, is said to have *bounded degree* $d$ if every vertex has degree at most $d$, and is called $d$*-regular* if every vertex has degree exactly $d$, where $d$ is some natural number. A *connected component* of $G$ is an equivalence class of the equivalence relation generated by $G$.

A (proper) *coloring* of $G$ is a function, say, $c : X \to Y$ to some set $Y$ such that if $x$ and $y$ are adjacent, $c(x) \neq c(y)$. In this situation, the elements of $Y$ are called *colors*. The sets $c^{-1}(\{y\})$ for $y \in Y$ are called *color sets*. If $|Y| = k$, $c$ is called a $k$-coloring. The *chromatic number* of $G$, denoted by $\chi(G)$, is the least $k$ such that $G$ admits a $k$-coloring.

Descriptive graph combinatorics studies these notions in the descriptive setting: Let $X$ now be a Polish space. A graph $G$ on $X$ is called *Borel* if $G$ is Borel in the product space $X \times X$. A coloring $c : X \to Y$ is called *Borel* if $Y$ is also a Polish space and $c$ is a Borel function. The *Borel chromatic number* of $G$, denoted by $\chi_B(G)$, is the least $k$ such that $G$ admits a Borel $k$-coloring. Similarly, $c$ is called *Baire measurable* if it is a Baire measurable function, and the *Baire measurable chromatic number* of $G$, denoted by $\chi_{\mathrm{BM}}(G)$, is the least $k$ such that $G$ admits a Baire measurable $k$-coloring. For a survey covering this exciting emerging field, see [4].

For a Borel graph $G$, we of course have $\chi(G) \le \chi_{\mathrm{BM}}(G) \le \chi_B(G)$, but it is natural to ask just how large $\chi_{\mathrm{BM}}(G)$ and $\chi_B(G)$ can be compared to $\chi(G)$. There are many known

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examples [4] where $\chi(G) = 2$ while $\chi_{BM}(G)$ and $\chi_B(G)$ are infinite. However, for graphs of bounded degree $d$, Kechris, Solecki, and Todorcevic [6] proved $\chi_B(G) \leq d + 1$. We therefore restrict our attention to bounded degree graphs for the remainder of the paper.

In [7], Marks proved that the bound $\chi_B(G) \leq d + 1$ is sharp, even for acyclic $G$ (so in particular, $\chi(G) = 2$). Thus, $\chi_B(G)$ can be arbitrarily large compared to $\chi(G)$. On the other hand, for Baire measurable chromatic numbers, Conley and Miller proved the following [2, Theorem B].

**Theorem 1.1.** *Let $G$ be a locally finite Borel graph such that $\chi(G) < \aleph_0$. Then, $\chi_{BM}(G) \leq 2\chi(G) - 1$.*

The question "How close to this bound can we get?" still remains. Previously, not much has been known regarding this: In fact, Kechris and Marks pose the following problem [4, Problem 4.7].

**Problem 1.2.** *Is there a bounded degree Borel graph $G$ for which $\chi_{BM}(G) > \chi(G) + 1$?*

The graphs constructed by Marks in [7] are not hyperfinite (see Section 4 for a definition). Furthermore, an analogue of Theorem 1.1 holds for measure chromatic numbers if the extra assumption of hyperfiniteness is added (see Theorem 4.1). This led to the question of whether the $2\chi(G) - 1$ bound held for Borel chromatic numbers in the hyperfinite setting [4, Question 5.19]. In [1], though, Marks' techniques were adapted to the hyperfinite setting, giving a negative answer to this question.

In this paper, however, we note that a certain strengthening of the hyperfiniteness assumption *is* enough to get this bound. Using techniques similar to those in [2] and some results from [8], we prove in Section 2 the following analogue of Theorem 1.1.

**Theorem 1.3.** *Let $G$ be a locally finite Borel graph such that $\chi(G) < \aleph_0$ and such that every connected component of $G$ has two ends. Then, $\chi_B(G) \leq 2\chi(G) - 1$.*

See Section 2 for a definition of two-endedness. Also note that this condition is indeed a strengthening of hyperfiniteness [8].

Similarly, little has been known regarding the sharpness of this bound. In fact, one of the goals of the project which led to this paper was to resolve the following.

**Problem 1.4.** *Is there a bounded degree Borel graph $G$ whose connected components all have two ends for which $\chi_B(G) > \chi(G) + 1$?*

In this paper, we answer Problems 1.2 and 1.4 as strongly as possible, proving the bounds in Theorems 1.1 and 1.3 are sharp.

**Theorem 1.5.** *Let $k \geq 3$. There is a Borel $3(k-1)^2$-regular graph, say, $G_k$, such that all the connected components of $G_k$ have two ends, $\chi(G_k) = k$, and $\chi_{BM}(G_k) = \chi_B(G_k) = 2k - 1$.*

The graphs $G_k$ will arise in the following way: A *marked group* (in this paper) is a pair $(\Gamma, S)$, where $\Gamma$ is a (typically infinite) finitely generated group and $S$ is a finite symmetric set of generators for it not containing the identity. When there is no confusion, we will sometimes refer to a marked group by its underlying group. Consider the group action $\Gamma \curvearrowright 2^\Gamma$ given by

$$(g \cdot x)(h) = x(g^{-1}h) \tag{1.1}$$

for $g, h \in \Gamma$ and $x \in 2^\Gamma$. This is called the *left shift action*. When $2^\Gamma$ is given the product topology, this action is clearly continuous. Let

$$F(2^\Gamma) = \{x \in 2^\Gamma \mid \forall g \in \Gamma \setminus \{\mathrm{id}\},\ g \cdot x \neq x\}. \tag{1.2}$$

This is a $G_\delta$ subspace of $2^\Gamma$, hence a Polish space. We can therefore form a Borel graph on $F(2^\Gamma)$ by putting an edge between $x$ and $y$ exactly when $s \cdot x = y$ for some $s \in S$. This is called the *shift graph* of $(\Gamma, S)$. We will always refer to the shift graph by its underlying set, $F(2^\Gamma)$. The graphs $G_k$ will all have the form $F(2^{\Gamma_k})$ for some marked group $\Gamma_k$.

Let $\mathrm{Cay}(\Gamma)$ be the *Cayley graph* of $(\Gamma, S)$. This is the graph on $\Gamma$ given by putting an edge between group elements $g$ and $h$ exactly when $sg = h$ for some $s \in S$. Clearly, as a (discrete) graph, $F(2^\Gamma)$ is isomorphic to a disjoint union of continuum many (if $\Gamma$ is infinite) copies of $2^\Gamma$. It is therefore natural to expect to get some information on the descriptive combinatorics of $F(2^\Gamma)$ from the graph $\mathrm{Cay}(\Gamma)$. However, in [9, Theorem 1], the author showed that $\mathrm{Cay}(\Gamma)$ is not enough to determine $\chi_B(F(2^\Gamma))$ or $\chi_{\mathrm{BM}}(F(2^\Gamma))$.

**Theorem 1.6** ([9]). *Let $k \geq 3$. There are marked groups $\Gamma$ and $\Delta$ with isomorphic Cayley graphs for which $\chi_B(F(2^\Delta)) = \chi_{\mathrm{BM}}(F(2^\Delta)) = k$ but $\chi_B(F(2^\Gamma)) = \chi_{\mathrm{BM}}(F(2^\Gamma)) = k+1$.*

This led to the natural question.

**Problem 1.7.** *Are there marked groups $\Gamma$ and $\Delta$ with isomorphic Cayley graphs for which $\chi_B(F(2^\Gamma)) - \chi_B(F(2^\Delta)) > 1$? What about for Baire measurable chromatic numbers?*

We answer this as well by producing for each $k$ a marked group $\Delta_k$ whose Cayley graph is isomorphic to that of $\Gamma_k$, but for which $\chi_B(F(2^{\Delta_k})) = \chi_{\mathrm{BM}}(F(2^{\Delta_k})) = k+1$. Thus we get the following result.

**Corollary 1.8.** *Let $k$ be a natural number. There are marked groups $\Gamma$ and $\Delta$ with isomorphic Cayley graphs but for which*

$$\chi_B\big(F(2^\Gamma)\big) - \chi_B\big(F(2^\Delta)\big) = \chi_{\mathrm{BM}}\big(F(2^\Gamma)\big) - \chi_{\mathrm{BM}}\big(F(2^\Delta)\big) = k.$$

In Section 3, we define the marked groups $\Delta_k$ and $\Gamma_k$ and compute their various chromatic numbers. In Section 4, we note that everything said in this paper about Baire measurable chromatic numbers can also be said about measure chromatic numbers in the hyperfinite setting.

## 2. Graphs whose connected components all have two ends

In this section, we prove Theorem 1.3. The proof uses little more than some results of Miller from [8], but nevertheless the result seems to be new and may be of interest to some.

Let $G$ be a graph on a set $X$. If $A \subseteq X$, we denote by $G \restriction A$ the graph $G \cap (A \times A)$ on $A$. We call $G$ *connected* if it has one connected component, and $A$ *connected* if $G \restriction A$ is connected.

A *path* between vertices $x$ and $y$ is a finite sequence $x = x_0, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in G$ for all $i$ and $x_0, \ldots, x_{n-1}$ are all distinct. In this situation, $n$ is called the *length* of the path. Note that a graph is connected if and only if there is a path between any two of its vertices. The *path distance* between $x$ and $y$ is the smallest $n$ such that there is a path of length $n$ between $x$ and $y$, or $\infty$ if there is no path between $x$ and $y$. The *path distance* between two sets of vertices $A$ and $B$ is the smallest path distance between any pair of vertices $x \in A$ and $y \in B$. A graph is called acyclic if it admits no paths as above with $x_0 = x_n$.

An *independent* subset of a graph is a pairwise-non-adjacent set of vertices. Thus, a coloring is just a partition of the set of vertices into independent sets.

Now assume $G$ is connected and locally finite. We say a subset $F \subseteq X$ *divides* $G$ *into* $n$ *parts* if $G \restriction (X \setminus F)$ has $n$ infinite connected components. We say $G$ *has* $n$ *ends* if there is a finite set $F$ dividing $G$ into $n$ parts, but no such $F$ dividing $G$ into $m$ parts for any $m > n$. Note that if $G$ has $n$ ends, we can find a finite set $F$ dividing it into $n$ parts such that $F$ is furthermore connected. It should be noted that this definition is different in general from the one used in [8] but is equivalent in the locally finite case.

Now, let $G$ be a locally finite Borel graph on a space $X$ whose connected components all have two ends. Denote by $[G]^{<\infty}$ the standard Borel space of finite connected subsets of $X$. Let $\Phi \subseteq [G]^{<\infty}$ be the set of sets which divide their connected component into two parts. Miller proves [8, Lemma 5.3] (see also [5, Lemma 7.3]) that there is a maximal Borel set $\Psi' \subseteq \Phi$ whose members are pairwise disjoint. An easy modification of their proof shows that we can instead get a maximal Borel set $\Psi \subseteq \Phi$ such that the path distance between any two distinct members of $\Psi$ is at least 4. Fix such a $\Psi$.

Let $\mathcal{T}$ be the set of pairs $(S, T)$ with $S, T \in \Psi$ such that $S \neq T$ and there is a path from $S$ to $T$ which avoids all other points of $\bigcup \Psi$. Miller proves that $\mathcal{T}$ is an acyclic graph on $\Psi$, that $S$ and $T$ are connected in this graph if and only if they are subsets of the same connected component of $G$, and that every element of $\Psi$ is $\mathcal{T}$-adjacent to at most two other elements [8, Lemma 5.5]. (Strictly speaking, they prove these things for $\Psi'$, but the proofs clearly still apply to $\Psi$.)

**Lemma 2.1.** *Every $S \in \Psi$ is $\mathcal{T}$-adjacent to exactly two other elements.*

*Proof.* Suppose some $S \in \Psi$ has fewer than two $\mathcal{T}$-neighbors. Let $C$ be the connected component of $S$. Let $C_-$ and $C_+$ be the two infinite connected components of $G \restriction (C \setminus S)$. Without loss of generality, $C_+$ must contain no sets in $\Psi$. This follows from the fact that any $T \in \Psi$ with $T \subseteq C$ must be $\mathcal{T}$-connected to $S$.

Let $N$ be the set of points in $C_+$ whose path distance from $S$ is exactly 4. $N$ is finite since $G$ is locally finite. We claim $N$ divides $C$ into 2 parts: By König's lemma, we can find an injective sequence $\{x_n \mid n \in \omega\}$ of points in $C_+$ such that $(x_n, x_{n+1}) \in G$ for all $n$. Since $G$ is locally finite, there must be some $M$ for which for all $n \geq M$, the path distance between $x_n$ and $S$ is at least 5. Then, the sequence $\{x_n \mid n \geq M\}$ does not pass through $N$, so it is contained in an infinite connected component of $G \restriction (C \setminus N)$. Also, $C_-$ is contained in an infinite connected component of $G \restriction (C \setminus N)$, so it suffices to show there is no path from $C_-$ to $x_M$ avoiding $N$. This is clear, though, as any path from $C_-$ to $x_M$ must pass through $S$, say at the point $y$, since $x_M \in C_+$. Then, since the path distance from $S$ to $x_M$ is greater than 4, there must be some point in $C_+$ along our path from $y$ to $x_M$ whose path distance from $S$ is exactly 4.

Let $D$ be the infinite connected component of $G \restriction (C \setminus N)$ not containing $S$. Let $N' \subseteq N$ be the set of elements of $N$ adjacent to a point in $D$. Then, $N'$ still divides $C$ into 2 parts. Furthermore, we can find a finite subset $A \subseteq D$ such that $N' \cup A$ is connected. Then, $N' \cup A \in \Phi$, and furthermore, since every point in $D$ has path distance at least 5 from $S$, the path distance between $S$ and $N' \cup A$ is 4. However, since we assumed $C^+$ contains no sets in $\Psi$, this contradicts the maximality of $\Psi$. ■

**Lemma 2.2.** *Every connected component of* $G \restriction (X \setminus \bigcup \Psi)$ *is finite.*

*Proof.* Let $x \in (X \setminus \bigcup \Psi)$. Let $C$ be the connected component of $x$ in the graph $G$, and let $D$ be the connected component of $x$ in the graph $G \restriction (C \setminus \bigcup \Psi)$. We want to show $D$ is finite.

By maximality, there is some element of $\Psi$ contained in $C$. Then, by Lemma 2.1 along with the fact that $\mathcal{T}$ is acyclic, we can label the elements of $\Psi$ contained in $C$ as $\{S_n \mid n \in \mathbb{Z}\}$, where the indices are chosen such that $(S_n, S_m) \in \mathcal{T}$ if and only if $|n - m| = 1$. By definition of $\Phi$, for each $n$ the graph $G \restriction (C \setminus S_n)$ has two infinite connected components; call them $C_{n,-}$ and $C_{n,+}$. By definition of $\mathcal{T}$, the sets $S_m$ for $m > n$ must all lie in the same connected component of $G \restriction (C \setminus S_n)$ and likewise for the sets $S_m$ for $m < n$. Therefore, by relabeling if necessary, we can assume $S_m \subseteq C_{n,+}$ for all $m > n$ and $S_m \subseteq C_{n,-}$ for all $m < n$.

Now, suppose $D$ is infinite. Then, for each $n$, either $D \subseteq C_{n,+}$ or $D \subseteq C_{n,-}$. Consider integers $n$, points $y \in S_n$, and paths from $x$ to $y$. Choose $n$, $y$, and such a path such that this path is of minimal length among all such choices. Then, this path cannot pass through any sets $S_m$ for $m \neq n$. Without loss of generality, assume $D \subseteq C_{n,+}$. We claim $D \subseteq C_{n+1,-}$. If not then $D \subseteq C_{n+1,+}$, but then $D$ and $S_n$ are in different connected components of $G \restriction (C \setminus S_{n+1})$, so there can be no path from $x$ to $S_n$ avoiding $S_{n+1}$, a contradiction. Therefore, $D \subseteq C_{n,+} \cap C_{n+1,-}$, so this intersection is infinite. This implies, however, that the finite set $S_n \cup S_{n+1}$ divides $G \restriction C$ into at least three parts, a contradiction. ■

We can now prove Theorem 1.3.

*Proof.* For each $S \in \Psi$, let $S^* = S \cup \{x \in X \mid \exists y \in S\ (x, y) \in G\}$. Since $G$ is locally finite and each $S$ is finite, each $S^*$ is finite. Let $B^* = \bigcup_{S \in \Psi} S^*$. $B^*$ is Borel since $\Psi$ is Borel. Since distinct $S$'s had path distances of at least 4 between them, distinct $S^*$'s have path distances of at least 2 between them. Thus, every connected component of $G \restriction B^*$ is a subset of some $S^*$. In particular, these connected components are all finite. Therefore, by the Lusin–Novikov uniformization theorem (see [3, Lemma 18.12]), there is a Borel function which picks out one of the finitely many $\chi(G)$-colorings of each connected component of $G \restriction B^*$. We can union these to get a Borel $\chi(G)$-coloring, say $c_1^* : B^* \to \{1, 2, \ldots, \chi(G)\}$ of $G \restriction B^*$. Let $B = B^* \setminus c_1^{*-1}(\{\chi(G)\})$ and $c_1 = c_1^* \restriction B$. Then, $B$ is Borel and $c_1$ is a Borel $(\chi(G) - 1)$-coloring of $G \restriction B$.

We claim that the connected components of $G \restriction (X \setminus B)$ are also all finite. Suppose to the contrary that $D \subseteq (X \setminus B)$ is some infinite connected component. Let $C$ be the connected component of $G$ containing $D$. We first claim that $D$ must contain infinitely many points not in $B^*$. If not, then $D$ contains infinitely points from $B^* \setminus B$ and only finitely many not in $B^*$. By construction, though, $B^* \setminus B$ is independent, so since $D$ is connected, for every $y \in (B^* \setminus B) \cap D$, there must be some $x \in D \setminus B^*$ with $(x, y) \in G$. Thus, there is some $x \in D \setminus B^*$ connected to infinitely many such $y$'s, contradicting local finiteness. Therefore, by Lemma 2.2, there are $x, y \in D \setminus B^*$ such that $x$ and $y$ are in different connected components of $G \restriction (C \setminus \bigcup \Psi)$. Let $x = x_0, x_1, \ldots, x_n = y$ be a path from $x$ to $y$ consisting of points in $D$. Then, there must be some $S \in \Psi$ and some $0 < i < n$ such that $x_i \in S$. Then, $x_{i-1}$, $x_i$, and $x_{i+1}$ are all in $S^*$. Since there are some edges between them, they cannot all be assigned the color $\chi(G)$ by $c_1^*$, but this means at least one of them is in $B$, a contradiction.

Therefore, again by the Lusin–Novikov uniformization theorem, there is a Borel $\chi(G)$, coloring, say, $c_2 : (X \setminus B) \to \{\chi(G), \ldots, 2\chi(G) - 1\}$, of $G \restriction (X \setminus B)$. Since $c_1$ and $c_2$ use disjoint sets of colors, $c_1 \cup c_2$ is a Borel $(2\chi(G) - 1)$-coloring of $G$. ∎

## 3. The construction

Fix $k \geq 3$. In this section, we define the marked groups $\Gamma_k$ and $\Delta_k$ promised in Section 1.

We start with a finite marked group: Let $Z_k$ denote the cyclic group of order $k$, which we will identify with the integers modulo $k$. Consider the group $Z_k \times Z_k$ with generating set $S = \{(a, b) \mid 0 < a,\ b < k\}$. Let $H$ be the Cayley graph of this finite marked group. We will think of the vertices of $H$ as sitting on a $k$ by $k$ grid, with the horizontal axis corresponding to the first coordinate and the vertical to the second. Accordingly, by a *row* of $H$ we mean a set of the form $\{(a, b) \mid a \in Z_k\}$ for some fixed $b \in Z_k$, and by a *column* of $H$ we mean a set of the form $\{(a, b) \mid b \in Z_k\}$ for some fixed $a \in Z_k$.

Note that any independent subset of $H$ of size greater than one must be either completely contained in some row or completely contained in some column (and not both). Call such sets *horizontal* and *vertical*, respectively (see Figure 1).

**Lemma 3.1.** *Let $c : Z_k \times Z_k \to \{1, 2, \ldots, 2k - 2\}$ be a $(2k - 2)$-coloring of $H$. Exactly one of the following holds:*

- *Every row contains a horizontal color set.*
- *Every column contains a vertical color set.*

*Proof.* Since there are $k$ rows and $k$ columns, for both to hold simultaneously would require $2k$ colors. Therefore, at most one holds.

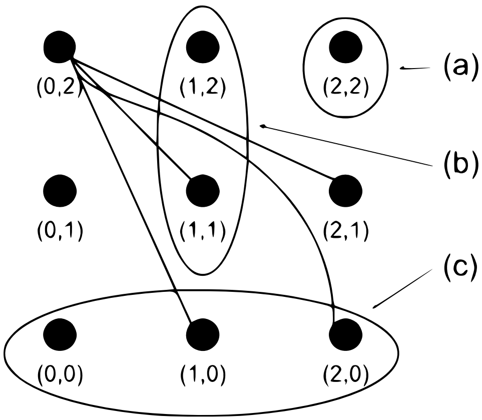


**Figure 1.** A drawing of the graph $H$ for $k = 3$. The edges shown are exactly those meeting $(0, 2)$. The three types of independent sets are shown in circles and labeled: cardinality one (a), vertical (b), and horizontal (c).

Suppose neither holds. Then, there is some column $C$ and some row $R$ such that $C$ does not contain a horizontal color set and $R$ does not contain a vertical color set. Then, every point in $R \cup C$ must have a different color, but $|R \cup C| = 2k - 1$. Therefore, at least one holds. ■

We call $c$ as in the lemma a *horizontal* coloring if the first condition holds and a *vertical* coloring if the second holds.

We can now define the marked group $\Delta_k$: It will be the group $(Z_k \times Z_k) \times \mathbb{Z}$, with generating set $S \times \{-1, 0, 1\}$. Let $G$ be the Cayley graph of $\Delta_k$. It is easy to see $\chi(G) = k$: A $k$-coloring is given by sending the element $((a, b), n)$ to $a$ for all $n \in \mathbb{Z}$ and $0 \leq a, b < k$. Also note that $G$ has two ends, as desired.

For each $n \in \mathbb{Z}$, the restriction of $G$ to the $(Z_k \times Z_k)$-orbit $(Z_k \times Z_k) \times \{n\}$ can be identified with $H$ in the obvious way. Thus, if $c : (Z_k \times Z_k) \times \mathbb{Z} \to \{1, 2, \ldots, 2k - 2\}$ is a $(2k - 2)$-coloring of $G$, the restriction of $c$ to the orbit $(Z_k \times Z_k) \times \{n\}$ is, for each $n$, either a horizontal coloring or a vertical coloring. In the $k$-coloring defined in the previous paragraph, all these restrictions were horizontal. The next lemma states that this was no accident.

**Lemma 3.2.** *Let $c : (Z_k \times Z_k) \times \mathbb{Z} \to \{1, 2, \ldots, 2k - 2\}$ be a $(2k - 2)$-coloring of $G$. Exactly one of the following holds:*

- *The restriction of $c$ to every $(Z_k \times Z_k)$-orbit is horizontal.*
- *The restriction of $c$ to every $(Z_k \times Z_k)$-orbit is vertical.*

*Proof.* By symmetry, it suffices to show that if the restriction of $c$ to $(Z_k \times Z_k) \times \{n\}$ is horizontal, then so is the restriction to $(Z_k \times Z_k) \times \{n + 1\}$. Suppose instead that it is vertical. For $1 \leq i \leq k$, let $R_i$ be a horizontal color set contained in the $i$th row

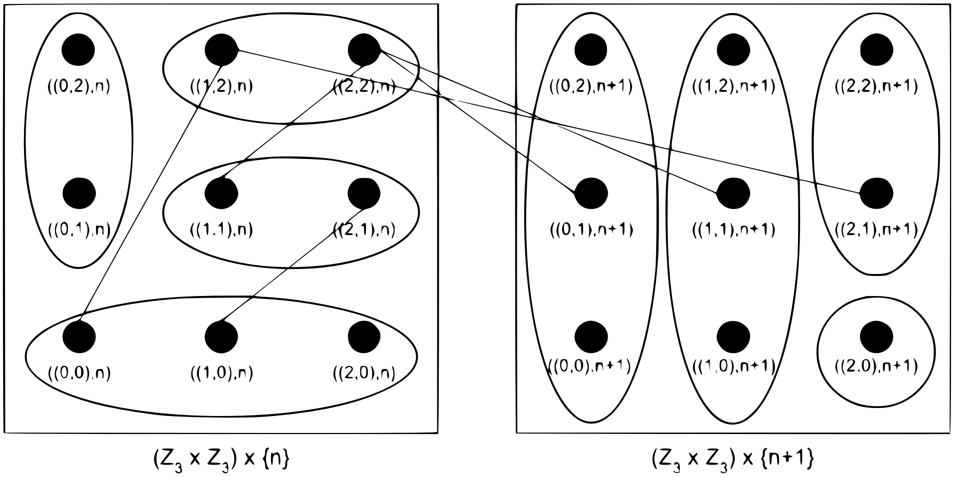


**Figure 2.** A visual explanation of the proof of Lemma 3.2 in the case $k = 3$. The two squares enclose neighboring $(Z_k \times Z_k)$-orbits. The circles represent color sets within each orbit. Most edges are omitted, but some are included to show any horizontal color set from the first orbit must admit an edge to every vertical color set from the second orbit. Others are included to show that horizontal color sets in different rows of a single orbit always have edges between them. The same is true for vertical color sets in different columns.

of $(Z_k \times Z_k) \times \{n\}$, and let $C_i$ be a vertical color set contained in the $i$th column of $(Z_k \times Z_k) \times \{n + 1\}$. Observe that, for every $1 \leq i,\ j \leq k$, there is at least one edge between $R_i$ and $C_j$ (see Figure 2). This is because $R_i$ must contain some vertex whose first coordinate is not $j$ and $C_j$ must contain some vertex whose second coordinate is not $i$. Furthermore, if $i \neq j$, there is at least one edge between $R_i$ and $R_j$, as well as between $C_i$ and $C_j$ (again see Figure 2). Therefore, each $R_i$ and $C_j$ must have a distinct color, but this requires $2k$ colors. ∎

This leads us to a natural definition of the marked group $\Gamma_k$: Let $\varphi \in \text{Aut}(Z_k \times Z_k)$ be the coordinate swapping map: $\varphi(a, b) = (b, a)$. $\Gamma_k$ will be the semi-direct product $(Z_k \times Z_k) \rtimes_{1 \mapsto \varphi} \mathbb{Z}$, again with generating set $S \times \{-1, 0, 1\}$. Observe that the following gives an isomorphism between the Cayley graphs of $\Gamma_k$ and $\Delta_k$:

$$\big((a, b), n\big) \mapsto \begin{cases} \big((a, b), n\big) & \text{for } n \text{ even}, \\ \big((b, a), n\big) & \text{for } n \text{ odd}, \end{cases} \tag{3.1}$$

where $a, b \in Z_k$ and $n \in \mathbb{Z}$. Thus, we still have $\chi(F(2^{\Gamma_k})) = \chi(\text{Cay}(\Gamma_k)) = k$, and this Cayley graph still has two ends as desired. We now compute the Borel and Baire measurable chromatic numbers of $F(2^{\Gamma_k})$, proving Theorem 1.5.

**Proposition 3.3.** $\chi_B(F(2^{\Gamma_k})) = \chi_{\text{BM}}(F(2^{\Gamma_k})) = 2k - 1$.

*Proof.* Theorem 1.3 gives us the upper bound $\chi_B(F(2^{\Gamma_k})) \leq 2k - 1$, so it remains to show there is no Baire measurable $(2k-2)$-coloring of $F(2^{\Gamma_k})$.

Suppose first that $c : (Z_k \times Z_k) \rtimes_{1 \mapsto \varphi} \mathbb{Z} \to \{1, 2, \dots, 2k-2\}$ is a $(2k-2)$-coloring of $\mathrm{Cay}(\Gamma_k)$. Note that the isomorphism (3.1) sends $(Z_k \times Z_k)$-orbits to $(Z_k \times Z_k)$-orbits but preserves the notions of "horizontal" and "vertical" for those with even $\mathbb{Z}$-coordinate and flips those notions for those with odd $\mathbb{Z}$-coordinate. Thus, Lemma 3.2 has the following consequence for $\Gamma_k$: If for some $n$ the restriction of $c$ to $(Z_k \times Z_k) \times \{n\}$ is horizontal, the restriction to $(Z_k \times Z_k) \times \{n+1\}$ must be vertical, and vice versa.

Now suppose $c : F(2^{\Gamma_k}) \to \{1, 2, \dots, 2k-2\}$ is a Baire measurable $(2k-2)$-coloring. Define the map $d : F(2^{\Gamma_k}) \to \{1, 2\}$ by sending a point $x$ to 1 if the restriction of $c$ to the $(Z_k \times Z_k)$-orbit of $x$ is horizontal and 2 if it is vertical. It is clear that $d$ is Baire measurable since $c$ was. By the previous paragraph, $d(x) \neq d(((0,0),1) \cdot x)$ for all $x$.

Thus, $d$ is a Baire measurable 2-coloring of the graph induced by the action of $((0,0),1)$ (that is, the graph for which two points are adjacent exactly if one is sent to the other by this element). In [6], it was established that $\chi_{\mathrm{BM}}(F(2^{\mathbb{Z}})) > 2$, where $\mathbb{Z}$ is given the usual generators, and we can follow the argument used there to reach a contradiction from the existence of $d$: First observe that $d^{-1}(\{1\})$ and $d^{-1}(\{2\})$ are both invariant under the generically ergodic action of $((0,0),2)$, so, since they partition the space, one must be meager and the other comeager. The action of $((0,0),1)$ gives a homeomorphism sending one of these sets to the other, though, a contradiction. ∎

Finally, we compute the Borel and Baire measurable chromatic numbers of $F(2^{\Delta_k})$, which gives Corollary 1.8 as promised.

**Proposition 3.4.** $\chi_B(F(2^{\Delta_k})) = \chi_{\mathrm{BM}}(F(2^{\Delta_k})) = k+1$.

*Proof.* We first show there is no Baire measurable $k$-coloring $c : F(2^{\Delta_k}) \to \{1, 2, \dots, k\}$. Suppose we had such a coloring. Observe that all $k$-colorings of the Cayley graph of $\Delta_k$ look essentially like the one defined before Lemma 3.2: Up to a relabeling of the colors, they assign either the color $a$ to $((a,b),n)$ for all $b$ and $n$ or the color $b$ to $((a,b),n)$ for all $a$ and $n$. In particular, the elements $g$ and $((0,0),1) \cdot g$ always have the same color.

Therefore, if we let $C_i = c^{-1}(\{i\})$ for each $i$, each $C_i$ is sent to itself by the action of the element $((0,0),1)$. Since the order of this element is infinite, a standard argument (see [3, Theorem 8.46]) shows each $C_i$ is either meager or comeager. Since the $C_i$'s partition $F(2^{\Delta_k})$, at least one (in fact, all but one), say, $C_{i_0}$, must be meager. The sets $((a,b),0) \cdot C_{i_0}$ for $a, b \in Z_k$ cover $F(2^{\Delta_k})$, though, so this is a contradiction.

It remains to construct a Borel $(k+1)$-coloring $c : F(2^{\Delta_k}) \to \{1, 2, \dots, k+1\}$.

A subset of $x$ is called *$r$-discrete*, for $r$ a natural number, if the path distance between any two points in $A$ is greater than $r$. It is an easy corollary of [6, Proposition 4.2] that if $G$ is a Borel graph of bounded degree, then $X$ contains a Borel maximal $r$-discrete subset for every $r$.

Applying this, let $A \subseteq F(2^{\Delta_k})$ be a Borel maximal $3k$-discrete set. Then, every $(Z_k \times Z_k)$-orbit contains at most one element of $A$. For every $x \in A$, color the $(Z_k \times Z_k)$-orbit of $x$ by setting $c(((a,b),0) \cdot x) = a$ for $1 \leq a, b \leq k$.

We now color the $(Z_k \times Z_k)$-orbits between those meeting $A$. Let $x \in A$ with $(Z_k \times Z_k)$-orbit $E$, and let $N$ be the smallest positive number such that $((0,0), N) \cdot E$ contains a point of $A$. Call that point $y$. Also note $N > 3k$ by definition of $A$. There are elements $1 \le a_0, b_0 \le k$ such that $y = ((a_0, b_0), N) \cdot x$. Also let $E_n$ denote the orbit $((0,0), n) \cdot E$ for $n \in \mathbb{Z}$, so, for example, $y \in A \cap E_N$. We need to extend $c$ by coloring all the $E_n$'s for $0 < n < N$. We will proceed one $n$ at a time.

Given a $(Z_k \times Z_k)$-orbit $E'$ colored already by $c$ and a positive integer $n$, let $c_n(E')$ denote the coloring on $((0,0), n) \cdot E'$ given by $c_n(E')(z) = c(((0,0), -n) \cdot z)$. We could try to extend our coloring $c$, by coloring $E_1$ with $c_1(E)$, then $E_2$ with $c_1(E_1)$, and so on. If we happened to have $c \restriction E_N = c_N(E)$, this would work out, but otherwise we will have a conflict. We can use our additional color to fix this.

Now, color $E_1$ by using $c_1(E)$, but then swapping the color $k$ with the color $k+1$. Since $c \restriction E$ does not use the color $k+1$, this does not create a conflict. Then, $c \restriction E_1$ does not use the color $k$, so we can color $E_2$ by using $c_1(E_1)$, but then swapping the color $a_0$ with the color $k$. Then, $c \restriction E_2$ does not use the color $a_0$, so we can color $E_3$ by using $c_1(E_2)$, but then swapping the color $k+1$ with the color $a_0$. Now, $c \restriction E_3$ does not use the color $k+1$, and furthermore, it looks like $c_3(E)$, but with the colors $k$ and $a_0$ swapped. Note that, by performing this swap, we have arranged that $c_{N-3}(E_3)$ agrees with $c \restriction E_N$ on the $a_0$th row.

We can repeat this process $k$ times, so that, for each $i \le k$, $E_{3i}$ will not use the color $k+1$ and $c_{N-3i}(E_{3i})$ will agree with $c \restriction E_N$ on $i$ rows. In particular, we will have $c_{N-3k}(E_{3k}) = c \restriction E_N$. Thus, we can color the remaining orbits $E_{3k+i}$ for $0 < i < N - 3k$ using $c_i(E_{3k})$. Thus, we have a $(k+1)$-coloring $c$ as desired. Since $A$ was Borel, it is clear that $c$ is Borel, so we are done. ∎

## 4. Measure chromatic numbers

In this section, we extend our results to the measurable setting.

Let $G$ be a Borel graph on a space $X$, now equipped with a Borel probability measure $\mu$. Just as we defined Borel and Baire measurable colorings, we can define *$\mu$-measurable* colorings and the *$\mu$-measurable chromatic number*, denoted by $\chi_\mu(G)$. The *measure chromatic number* of $G$, denoted by $\chi_M(G)$, is the supremum of $\chi_\mu(G)$ over all Borel probability measures $\mu$ on $X$.

An equivalence relation $E$ on $X$ is called *Borel* if it is Borel as a subset of $X \times X$. $E$ is called *finite* if its equivalence classes are all finite. $E$ is called *hyperfinite* if it can be written as $E = \bigcup_{n \in \omega} E_n$ for some increasing sequence $E_n$ of finite Borel equivalence relations. $G$ is called *hyperfinite* if its connected component equivalence relation is hyperfinite.

In [2, Theorem A], Conley and Miller prove an analogue of Theorem 1.1 for measure chromatic numbers with the added assumption of hyperfiniteness.

**Theorem 4.1.** *Let $G$ be a hyperfinite locally finite Borel graph such that $\chi(G) < \aleph_0$. Then, $\chi_M(G) \le 2\chi(G) - 1$.*

As in the Baire measurable situation, the sharpness of this bound was previously unknown. All of the arguments we made in Section 3 in the Baire measurable setting still work in the measurable setting. Most crucially, we have an ergodicity argument establishing $\chi_M(F(2^{\mathbb{Z}})) > 2$ just as we did for the Baire measurable chromatic number, and as in the proof of Proposition 3.3, this can be adapted to give a lower bound for $\chi_M(F(2^{\Gamma_k}))$. Also, the arguments regarding $\chi_{\mathrm{BM}}(F(2^{\Delta_k}))$ in the proof of Proposition 3.4 still go through in the measure theoretic setting upon replacing "meager" and "comeager" with "measure 0" and "measure 1," respectively. Therefore, we have:

**Proposition 4.2.** *For all $k \geq 3$, $\chi_M(F(2^{\Gamma_k})) = 2k - 1$ and $\chi_M(F(2^{\Delta_k})) = k + 1$.*

As was noted in the introduction, these graphs are hyperfinite since their connected components all have two ends. Thus, the bound in Theorem 4.1 is indeed sharp.

**Theorem 4.3.** *Let $k \geq 3$. There is a Borel hyperfinite $3(k-1)^2$-regular graph, say, $G_k$, for which $\chi(G_k) = k$ but $\chi_M(G_k) = 2k - 1$.*

Similarly, alongside Theorem 1.6, Weilacher [9] proves that there are marked groups with isomorphic Cayley graphs can have measure chromatic numbers which differ by one but notes that it is open whether or not these numbers can differ by more than one. By Proposition 4.2, we have resolved this as well.

**Corollary 4.4.** *Let $k$ be a natural number. There are marked groups $\Gamma$ and $\Delta$ with isomorphic Cayley graphs but for which $|\chi_M(F(2^{\Gamma})) - \chi_M(F(2^{\Delta}))| > k$.*

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