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Hyperbolic quotients of projection complexes

Matt Clay and Johanna Mangahas

Abstract. This paper is a continuation of our previous work with Margalit where we studied group actions on projection complexes. In that paper, we demonstrated sufficient conditions so that the normal closure of a family of subgroups of vertex stabilizers is a free product of certain conjugates of these subgroups. In this paper, we study both the quotient of the projection complex by this normal subgroup and the action of the quotient group on the quotient of the projection complex. We show that under certain conditions the quotient complex is δ-hyperbolic. Additionally, under certain circumstances, we show that if the original action on the projection complex was a non-elementary WPD action, then so is the action of the quotient group on the quotient of the projection complex. This implies that the quotient group is acylindrically hyperbolic.

1. Introduction

Projection complexes were originally defined by Bestvina–Bromberg–Fujiwara and were used to show that the mapping class group of an orientable surface has finite asymptotic dimension [1]. The motivating idea behind these complexes is the following. Start with a collection of subspaces $\{Z_i\}$ contained in some metric space X. We want these subspaces to satisfy some properties akin to negative curvature; in particular, we require that the nearest point projection from any one subset Z_i to another subset Z_j has uniformly bounded diameter. For example, one could take X to be the hyperbolic plane and the collection $\{Z_i\}$ to be the orbit of a geodesic in X under a discrete group of isometries of X. The projection complex built out of this data is the graph with vertex set $\{Z_i\}$ where two vertices Z_i and Z_j are joined by an edge if the diameter of the union of their projections to any other Z_k is small. A key feature of a projection complex is that, in general, it is a quasi-tree, in other words, it is quasi-isometric to a tree [1, Theorem 3.16]. Projection complexes have found several useful applications lately by many authors [2, 3, 8–12].

In a previous work with Margalit, we studied group actions on projection complexes [7]. We derived a structure theorem for normal subgroups generated by elliptic elements under some hypotheses; see Section 2.2 for the exact statement. We were able to apply our structure theorem to produce new examples of normal subgroups of mapping class groups of orientable surfaces that are isomorphic to right-angled Artin groups. In particular, we produced examples that were not free.

In this paper, we work in the general setting of a group acting on a projection complex with the same set of hypotheses as before and study both the quotients of the projection

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complex by such normal subgroups and the action of the quotient group on the corresponding quotient complex. These appear as Theorem 1.1 and Theorem 1.2, respectively. To state these results, we describe the set-up we studied before and continue to study in this paper.

Briefly, a projection complex is a graph $\mathcal{P}$ and a collection of functions

$$d_v\colon V \setminus \{v\} \times V \setminus \{v\} \to \mathbb{R}_{\geq 0},$$

where V is the set of vertices of $\mathcal{P}$ and $v \in V$. The full definition appears in Section 2.1. Following our previous work and as explained below, our definition is a mild modification of the original definition of Bestvina–Bromberg–Fujiwara.

Let $\mathcal{P}$ be a projection complex, and let G be a group that acts on $\mathcal{P}$. Further, for each vertex v of $\mathcal{P}$, let R_v be a subgroup of the stabilizer of v in G. Let $L > 0$. We say that the family of subgroups $\{R_v\}$ is an *equivariant L-spinning family* of subgroups of G if it satisfies the following two conditions

- *Equivariance*: if g lies in G and v is a vertex of $\mathcal{P}$, then

$$gR_vg^{-1} = R_{gv}.$$

- *Spinning*: for any distinct vertices v and w of $\mathcal{P}$ and any nontrivial $h \in R_v$ we have

$$d_v(w, hw) \geq L.$$

By the equivariance condition, for each vertex v the subgroup R_v is normal in $\mathrm{Stab}_G(v)$, and the subgroup H of G generated by the R_v is normal in G. If $\{v_i\}$ is a set of orbit representatives for the action of G on the vertices of $\mathcal{P}$, then H is the normal closure of the set $\{R_{v_i}\}$.

We can now state our theorem regarding the quotient complex.

Theorem 1.1. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. There exists a constant $L_{\mathrm{hyp}}(\mathcal{P})$ with the following property. If $L \geq L_{\mathrm{hyp}}(\mathcal{P})$ and if $\{R_v\}$ is an equivariant L-spinning family of subgroups of G, then $\mathcal{P}/\langle R_v\rangle$ is δ-hyperbolic.*

We also examine the action of the quotient group $G/\langle R_v\rangle$ on the quotient space $\mathcal{P}/\langle R_v\rangle$. Our result on the action briefly says that certain features of the action of G on $\mathcal{P}$ persist in the quotient action. Before we can state our result on this action, we need to define a number of notions.

Let X be a geodesic metric space and let G be a group that acts on X by isometries. An element f of G is *hyperbolic* if

$$\lim_{n\to\infty} \frac{d(x, f^n x)}{n}$$

is positive for some $x \in X$, equivalently, for any $x \in X$. Two hyperbolic elements, f_1 and f_2, of G are *independent* if $d(f_1^{n_1}x, f_2^{n_2}x) \to \infty$ as $n_1, n_2 \to \pm\infty$ for some $x \in X$, equivalently, for any $x \in X$.

An element f of G is a *WPD element* if f is hyperbolic and if for all points $x \in X$ and for all $D \geq 0$, there is an $M \geq 0$ such that the set

$$\{g \in G \mid d(x, gx) \leq D \text{ and } d(f^M x, g f^M x) \leq D\}$$

is finite. We remark that it suffices to demonstrate finiteness of the above set at a single point in X. The notion of a WPD element was introduced by Bestvina–Fujiwara as a tool for constructing quasi-morphisms [5]. There are several known examples of WPD elements: pseudo-Anosov mapping classes acting on the corresponding curve complex [5] and fully irreducible outer automorphisms of a free group acting on the corresponding free factor complex [4] for instance. If the action of G on X is properly discontinuous, then any hyperbolic element is a WPD element.

The action of G on X is a *non-elementary WPD action* if there exist two elements in G that are WPD elements and independent. We remark that if X is δ-hyperbolic, G is not virtually cyclic and there is one element f in G that is a WPD element, then for some element g in G, the elements f and gfg^{-1} are independent WPD elements. In fact, one can take any $g \in G$ such that $\langle f \rangle \cap g\langle f \rangle g^{-1}$ is finite.

We can now state our theorem on the action of the quotient group on the quotient complex.

Theorem 1.2. *Let $\mathcal{P}$ be a projection complex and let G be a group with a non-elementary WPD action on $\mathcal{P}$. There exists a constant $L_{\mathrm{WPD}}(\mathcal{P}, G)$ with the following property. If $L \geq L_{\mathrm{WPD}}(\mathcal{P}, G)$ and if $\{R_v\}$ is an equivariant L-spinning family of subgroups of G, then the action of $G/\langle R_v \rangle$ on $\mathcal{P}/\langle R_v \rangle$ is a non-elementary WPD action.*

Precisely, if f_1 and f_2 are independent WPD elements of G for its action on $\mathcal{P}$, then there is a constant $L_{\mathrm{WPD}}(\mathcal{P}, f_1, f_2)$ such that their images $\bar{f}_1$ and $\bar{f}_2$ in $G/\langle R_v \rangle$ are independent WPD elements for the action of $G/\langle R_v \rangle$ on $\mathcal{P}/\langle R_v \rangle$ when $L \geq L_{\mathrm{WPD}}(\mathcal{P}, f_1, f_2)$ and $\{R_v\}$ is an equivariant L-spinning family of subgroups of G.

Whereas the constant in Theorem 1.1 does not depend on G, the constant in Theorem 1.2 necessarily does. Indeed, if G is equal to $\langle R_v \rangle$, then the quotient group is trivial. Hence we must choose L after G—more precisely after choosing two independent WPD elements—to ensure that the quotient is as claimed.

There is a strengthening of the WPD condition called acylindricity that arises in several settings that we describe now.

Let X be a metric space and let G be a group acting on X by isometries. The action is *acylindrical* if for all $D \geq 0$ there exist $R \geq 0$ and $N \geq 0$ such that for all points x and y in X where $d(x, y) \geq R$, the set

$$\{g \in G \mid d(x, gx) \leq D \text{ and } d(y, gy) \leq D\}$$

contains at most N elements.

A group G is *acylindrically hyperbolic* if it admits an acylindrical action on a hyperbolic space for which there exist elements f_1 and f_2 in G that are hyperbolic and independent. Both the mapping class group of an orientable surface [6] and the outer automorphism group of a finitely generated free group are acylindrically hyperbolic [14].

There are several other examples and much is known about this class of groups. The paper by Osin contains a survey of examples and results for acylindrically hyperbolic groups [14].

Osin derived a number of conditions that are equivalent to acylindrical hyperbolicity, one of which is that the group is not virtually cyclic and admits an action on a δ-hyperbolic space where one element is a WPD element [14, Theorem 1.2]. Hence we obtain the following corollary of Theorems 1.1 and 1.2.

Corollary 1.3. *Let $\mathcal{P}$ be a projection complex and let G be a group with non-elementary WPD action on $\mathcal{P}$. There exists a constant $L_{\mathrm{WPD}}(\mathcal{P}, G)$ with the following property. If $L \geq L_{\mathrm{WPD}}(\mathcal{P}, G)$ and if $\{R_v\}$ is an equivariant L-spinning family of subgroups of G, then $G/\langle R_v \rangle$ is acylindrically hyperbolic.*

In Section 8, we describe new examples of acylindrically hyperbolic groups coming from this construction. Each of these examples is the quotient of the mapping class group of an orientable surface by a normal subgroup we produced in our earlier work.

The strategy to prove Theorem 1.1 and Theorem 1.2 is very similar to strategy of Dahmani–Hagen–Sisto in a recent paper [10]. In this paper, Dahmani–Hagen–Sisto consider the action of the subgroup of the mapping class group generated by kth powers of Dehn twists on the curve graph, i.e., the 1-skeleton of the curve complex, and they prove results similar to Theorem 1.1 and Theorem 1.2. They make use of the fact shown by Dahmani that the curve graph has the structure of a *composite projection graph* [8]. That is, there is a partition of the curve graph into finitely many pieces that behave like a projection complex, along with certain combatility conditions on how the pieces interact. Since we deal with a projection complex as opposed to a composite projection graph, some parts of their strategy can be simplified.

In order to prove Theorem 1.1, we show that geodesic triangles in $\mathcal{P}/\langle R_v \rangle$ lift to geodesic triangles in $\mathcal{P}$ (Proposition 4.3). As $\mathcal{P}$ is a quasi-tree, it is a δ-hyperbolic metric space and hence geodesic triangles are δ_0-thin for some δ_0. As the quotient map $p\colon \mathcal{P} \to \mathcal{P}/\langle R_v \rangle$ is 1-Lipschitz, this shows that the geodesic triangles in $\mathcal{P}/\langle R_v \rangle$ are δ_0-thin as well. The proof of Theorem 1.2 is similar except that it involves lifting geodesic quadrilaterals. A key fact needed here is that a geodesic in $\mathcal{P}$ for which the projection of any two of its vertices to any other vertex in $\mathcal{P}$ is uniformly bounded is isometrically embedded in the quotient (Lemma 7.1).

A closed path in $\mathcal{P}/\langle R_v \rangle$ can be lifted to a path in $\mathcal{P}$ with endpoints x and hx for some $h \in \langle R_v \rangle$. We describe a technique called *path bending* for replacing the lifted path with a new lift. There is a notion of *complexity* for an element in $\langle R_v \rangle$. We show that when $x \neq hx$, we can bend the given lift to get a lift from x to $h'x$ where h' has less complexity than that of h (Proposition 3.2). This is the technique known as *shortening* and it plays a key role in understanding both lifts (Proposition 4.3) and images (Lemma 6.1). This technique was introduced by Dahmani–Hagen–Sisto and is also essential to their work [10].

1.1. Outline of paper

Section 2 collects the necessary facts on projection complexes that are needed for the remainder. Starting in Section 3, we follow the strategy of Dahmani–Hagen–Sisto [10]. In Section 3, we prove the main technical tool of the paper: Proposition 3.2. This is the technique known as shortening and allows us to replace a lift of a path in the quotient of the projection complex with another lift that is simpler in a precise sense. We apply the shortening tool in Section 4 to show that geodesic quadrilaterals in the quotient of the projection complex lift to geodesic quadrilaterals. The proof of Theorem 1.1 appears in Section 5. In Section 6, we show that when vertices along a geodesic in the projection complex have bounded projections, the image of the geodesic in the quotient graph is still a geodesic. Using this, we can establish that certain WPD elements for the action of G on $\mathcal{P}$ have images in $G/\langle R_v \rangle$ that are still WPD elements for the action of $G/\langle R_v \rangle$ on $\mathcal{P}/\langle R_v \rangle$. In Section 7, we prove Theorem 1.2. Finally, in Section 8 we present some examples when G is the mapping class group of a surface.

2. Projection complexes, windmills, and pivot points

In this section, we provide the definitions of projection complexes, windmills, and pivot points. The majority of the discussion in this section appears in our previous work with Margalit [7]. The essential material that is needed for the sequel is recorded in Lemma 2.1.

2.1. Projection complexes

We begin with the definition of a projection complex. Let $\mathbb{Y}$ be a set and let $\theta \geq 0$ be a constant. Assume that for each $y \in \mathbb{Y}$ there is a function

$$d_y \colon \mathbb{Y} \setminus \{y\} \times \mathbb{Y} \setminus \{y\} \to \mathbb{R}_{\geq 0}$$

with the following properties.

Symmetry: $d_y(x,z) = d_y(z,x)$ for all $x,y,z \in \mathbb{Y}$.

Triangle inequality: $d_y(x,z) + d_y(z,w) \geq d_y(x,w)$ for all $x,y,z,w \in \mathbb{Y}$.

Inequality on triples: $\min\{d_y(x,z), d_z(x,y)\} \leq \theta$ for all $x,y,z \in \mathbb{Y}$.

Finiteness: $\#\{y \in \mathbb{Y} \mid d_y(x,z) > \theta\}$ is finite for all $x,z \in \mathbb{Y}$.

These conditions are known as the *projection complex axioms*. When we say that a set $\mathbb{Y}$ and a collection of functions $\{d_y\}_{y\in\mathbb{Y}}$ as above satisfy the projection complex axioms, the constant θ is implicit.

For a given $K \geq 0$, we will define a graph $\mathcal{P}_K(\mathbb{Y})$ with vertices corresponding to the elements in $\mathbb{Y}$. The edges are defined using the notion of modified distance functions.

Given the functions $\{d_y\}$, Bestvina–Bromberg–Fujiwara [1] constructed another collection of functions $\{d'_y\}_{y\in\mathbb{Y}}$, where each d'_y shares the same domain and target as d_y. Because the definition of the d'_y is technical and because we do not use the definition in this paper, we do not state it here. Bestvina–Bromberg–Fujiwara [1, Theorem 3.3B] showed that the modified functions are coarsely equivalent to the original functions: for $x \neq y \neq z \in \mathbb{Y}$, $d'_y(x,z) \leq d_y(x,z) \leq d'_y(x,z) + 2\theta$.

Fix $K \geq 0$. Then two vertices x, z of $\mathcal{P}_K(\mathbb{Y})$ are connected by an edge if $d'_y(x,z) \leq K$ for all $y \in \mathbb{Y} - \{x,z\}$. Let d denote the resulting path metric on $\mathcal{P}_K(\mathbb{Y})$.

Bestvina–Bromberg–Fujiwara showed that for K large enough relative to θ, there are constants C_{e}, C_{p}, and C_{g}, so that the following properties hold (see [1, Proposition 3.14 and Lemma 3.18]).

Bounded edge image. If $x \neq y \neq z$ are vertices of $\mathcal{P}_K(\mathbb{Y})$ and $d(x,z) = 1$, then $d_y(x,z) \leq C_{\mathrm{e}}$.

Bounded path image. If a path in $\mathcal{P}_K(\mathbb{Y})$ connects vertices x to z without passing through the 2-neighborhood of the vertex y, then $d_y(x,z) \leq C_{\mathrm{p}}$.

Bounded geodesic image. If a geodesic in $\mathcal{P}_K(\mathbb{Y})$ connects vertices x to z without passing through the vertex y, then $d_y(x,z) \leq C_{\mathrm{g}}$.

(The bounded edge image property follows from the definition of the edges of $\mathcal{P}_K(\mathbb{Y})$, with $C_{\mathrm{e}} = K + 2\theta$.) If K is large enough so that the graph $\mathcal{P}_K(\mathbb{Y})$ satisfies the bounded edge, path, and geodesic properties for some C_{e}, C_{p}, and C_{g}, then we say that $\mathcal{P}_K(\mathbb{Y})$ is a *projection complex.*

This is the same definition as we used in our previous work [7]. As mentioned there, we note that our terminology is not standard; in the papers by Bestvina–Bromberg–Fujiwara [1] and Bestvina–Bromberg–Fujiwara–Sisto [3], every $\mathcal{P}_K(\mathbb{Y})$ is called a projection complex.

Group actions on projection complexes. We say that a group G acts on a projection complex $\mathcal{P}_K(\mathbb{Y})$ if G acts on the set $\mathbb{Y}$ in such a way that the associated distance functions d_y are G-invariant; i.e., $d_{gy}(gx, gz) = d_y(x,z)$. We note that if the original distance functions d_y are G-invariant, then the modified distance functions are G-invariant as well—as is evident from the definition [1, Definition 3.1]—and so the action of G on $\mathbb{Y}$ extends an action of G on the graph $\mathcal{P}_K(\mathbb{Y})$ by simplicial automorphisms.

2.2. Windmills

To understand the action of $\langle R_v \rangle$ on $\mathcal{P}$, in our previous work we used the notion of a windmill. This tool is also necessary in this current work and we review the construction now.

Given an action of a group G on a projection complex $\mathcal{P}$ with an equivariant family of subgroups $\{R_v\}$ of G, we can inductively define a sequence of subgraphs W_i of $\mathcal{P}$, a sequence of subsets $\mathcal{O}_i$ of the set of vertices of $\mathcal{P}$, and a sequence of subgroups H_i of G as follows.

Let v_0 be some base point for $\mathcal{P}$. To begin the inductive definitions at $i = 0$, we define

- $H_0 = R_{v_0}$ and
- $W_0 = \mathcal{O}_0 = \{v_0\}$.

For $i \geq 1$, we denote by N_i the 1-neighborhood of W_{i-1}, we denote by L_i the vertices of $N_i \setminus W_{i-1}$, and we define

- $H_i = \langle R_v \mid v \in N_i \rangle$,
- $W_i = H_i \cdot N_i$, and
- $\mathcal{O}_i =$ a set of orbit representatives for the action of H_{i-1} on L_i.

The set $\{(H_i, W_i, \mathcal{O}_i)\}_{i=0}^{\infty}$ is called a set of *windmill data* for the equivariant family $\{R_v\}$. We observe that each W_i is connected.

The subgroup H of G generated by the R_v is the direct limit of the H_i. Let $\mathcal{O}$ be the union of the sets of representatives $\mathcal{O}_i$. In our previous work with Margalit [7, Theorem 1.6], we proved the existence of a constant $L(\mathcal{P})$ such that if $L \geq L(\mathcal{P})$ and $\{R_v\}$ is an equivariant L-spinning family of subgroups, then

$$H \cong \mathop{\ast}_{v \in \mathcal{O}} R_v.$$

For the remainder, we will always assume that $L \geq L(\mathcal{P})$ whenever we are discussing an equivariant L-spinning family so that this free product decomposition is valid. Each of the constants of the form L_* defined in the sequel is at least $L(\mathcal{P})$.

2.3. Pivot points

In our previous work, we introduced the notion of the set of pivot points for an element h of H in order to understand the group structure of H [7]. We review this notion now and state Lemma 2.1 which records the necessary technical facts required for the shortening argument in Section 3.

The *level* of a nontrivial element $h \in H$ is the minimal index i such that $h \in H_i$. We define the level of the identity element to be -1.

Each $h \in H$ with level i has a *syllable decomposition* $h_1 \cdots h_n$, where each syllable h_k is either a nontrivial element of H_{i-1} or a nontrivial element of R_{v_k} with $v_k \in \mathcal{O}_i$. Moreover, no two consecutive syllables are of the first type and consecutive syllables h_k and h_{k+1} of the second type have distinct corresponding fixed vertices v_k and v_{k+1}. We refer to n as the *syllable length* of h. As long as $L \geq L(\mathcal{P})$, which will be our standing assumption, this syllable decomposition is unique for an equivariant L-spinning family $\{R_v\}$.

Let $i \geq 1$ and fix some element h of H with level i and with syllable decomposition $h = h_1 \cdots h_n$. For $k \in \{1, \ldots, n\}$ with $h_k \notin H_{i-1}$ and with corresponding fixed vertex v_k we define a vertex w_k of $\mathcal{P}$ as follows:

$$w_k = h_1 \cdots h_{k-1} v_k.$$

Note that v_k and w_k are not defined for the syllables h_k that lie in H_{i-1}. Let $\mathrm{Piv}(h)$ be the ordered list of points w_k, and call these the *pivot points* for h. For $h \in H_0$ we define $\mathrm{Piv}(h)$ to be empty.

There are several key properties regarding windmills and pivot points that we recall now.

Lemma 2.1. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. There are constants L_0 and m with the following properties. Suppose $L \geq L_0$ and suppose $\{R_v\}$ is an equivariant L-spinning family of subgroups of G. Let $H = \langle R_v \rangle$ and choose windmill data $\{(H_i, W_i, \mathcal{O}_i)\}$.*

(1) *If h is an element of H and if $w \in \mathrm{Piv}(h)$, then $d_w(v_0, hv_0) > L/2$.*

(2) *If h is an element of H and if w, w' are pivot points for h with $w < w'$, then*

$$d_w(v_0, w') > L/2 - \theta \quad \textit{and} \quad d_{w'}(v_0, w) \leq \theta.$$

(3) *For all $i \geq 1$, if $x \in N_i$ and $v \notin W_{i-1}$ with $v \neq x$, then $d_v(v_0, x) \leq m$.*

(4) *For all $i \geq 1$, if h has level i, then no pivot point for h lies in W_{i-1}.*

Proof. Using the constants associated with $\mathcal{P}$, we set $m = 11C_e + 6C_g + 5C_p$ and $L_0 = 4(m + \theta) + 1$. We remark that m is the same constant as in the proof of Theorem 1.6 in our prior work [7] and that $L_0 \geq L(\mathcal{P})$ from that same theorem. The above listed facts follow from results and arguments appearing in the proof of Theorem 1.6 in that paper as we now explain.

***Proof of* (1).** Fix an element h in H with syllable decomposition $h = h_1 \cdots h_n$. If the level of h is less than 1, the statement is vacuous. Hence suppose that the level of h is at least 1. Consider a pivot point $w = h_1 \cdots h_{k-1} v_k$ for h. Equation (1) in the proof of Theorem 1.6 in our prior work states that

$$d_w(v_0, hv_0) \geq d_w(v_0, h_k v_0) - 2(m + \theta).$$

As $d_w(v_0, h_k v_0) \geq L$ and $L/2 > 2(m + \theta)$, the statement holds.

***Proof of* (2).** Again, fix an element h in H and assume that the level of h is at least 1 as the statement is vacuous otherwise. Let w and w' be pivot points for h with $w < w'$. Statement (B) of the inductive hypothesis in the proof of Theorem 1.6 implies that there is a geodesic from v_0 to w avoiding w'. Hence we have $d_{w'}(v_0, w) \leq C_g$. Therefore, using the first item, we have

$$d_{w'}(w, hv_0) \geq d_{w'}(v_0, hv_0) - d_{w'}(v_0, w) > L/2 - C_g > \theta.$$

Thus by the inequality on triples, we find $d_w(w', hv_0) \leq \theta$. From this, using the first item again, we conclude that

$$d_w(v_0, w') \geq d_w(v_0, hv_0) - d_w(w', hv_0) > L/2 - \theta.$$

As $d_w(v_0, w') > L/2 - \theta > \theta$, by the inequality on triples, we have $d_{w'}(v_0, w) \leq \theta$.

***Proof of* (3).** This is statement (C) of the inductive hypothesis in the proof of Theorem 1.6.

***Proof of* (4).** Fix $i \geq 1$ and let h be an element of H with level i. The first pivot point for h, w lies in L_i by definition. As L_i is disjoint from W_{i-1}, the statement holds for this pivot point. Let w' be another pivot point for h. By the second item, we have $d_w(v_0, w') >$

$L/2 - \theta > m$. If $w' \in W_{i-1} \subset N_i$, then as $w \notin W_{i-1}$, the third item would imply that $d_w(v_0, w') \le m$. This is a contradiction, hence $w' \notin W_{i-1}$. ∎

3. Shortening via pivot points

In this section, we introduce the key technical tool: *shortening*. The precise statement is given in Proposition 3.2. This proposition will allow us to bend paths in $\mathcal{P}$ without changing their images in the quotient $\mathcal{P}/\langle R_v \rangle$. The bent path has a lower complexity in a precise sense that we will explain. This will allow us to conclude that certain closed paths in $\mathcal{P}/\langle R_v \rangle$ lift to closed paths in $\mathcal{P}$.

Before we can state Proposition 3.2 we need to alter the notions of level and pivot points so that they are better suited for conjugacy classes. Assume that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G. Let $H = \langle R_v \rangle$ and choose windmill data $\{(H_i, W_i, \mathcal{O}_i)\}$.

Complexity of an element in H. The *complexity* of an element $h \in H$ is the ordered pair $(\mathsf{i}(h), \mathsf{n}(h))$, where $\mathsf{i}(h)$ is the minimal index of any H-conjugate of h and $\mathsf{n}(h)$ is the minimal syllable length of any H-conjugate of h that has level $\mathsf{i}(h)$. Lexicographical order on the pair $(\mathsf{i}(h), \mathsf{n}(h))$ gives a weak order on the elements in H. The only element with $\mathsf{i}(h) = -1$ is the trivial element. Also, we remark that if $\mathsf{i}(h) = 0$, then $\mathsf{n}(h) = 1$.

Essential pivot points. Given an element $h \in H$ with $\mathsf{i}(h) = i$ and $\mathsf{n}(h) = n$, we can express h as a reduced word

$$h = g(h_1 \cdots h_n)g^{-1},$$

where each h_k is either a nontrivial element of H_{i-1} or a nontrivial element of R_{v_k} with $v_k \in \mathcal{O}_i$ and $g \in H$. If $\mathsf{n}(h) > 1$, then minimality of $\mathsf{n}(h)$ implies that if $h_1 \in H_{i-1}$, then $h_n \notin H_{i-1}$, and that if $h_1 \in R_{v_1}$, then $h_n \notin R_{v_1}$. The subset of $\mathrm{Piv}(h)$ corresponding to the syllables h_k that lie in R_{v_k} for some $v_k \in \mathcal{O}_i$ are called *essential pivot points*. We denote this subset by $\mathrm{Piv}^*(h)$. This set is nonempty so long as $\mathsf{i}(h) \ge 1$.

The following lemma, whose proof is an easy exercise from the definitions, justifies calling these pivot points essential.

Lemma 3.1. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. Suppose that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G. Let $H = \langle R_v \rangle$ and choose windmill data $\{(H_i, W_i, \mathcal{O}_i)\}$. The following statements are true.*

(1) *If the syllable length of $h \in H$ equals $\mathsf{n}(h)$, then every pivot point is essential, i.e.,* $\mathrm{Piv}^*(h) = \mathrm{Piv}(h)$.

(2) *If h and g are elements of H, then* $\mathrm{Piv}^*(ghg^{-1}) = g\,\mathrm{Piv}^*(h)$.

(3) *If h is an element of H and $k \ge 1$, then*

$$\mathrm{Piv}^*(h^k) = \bigcup_{j=0}^{k-1} h^j\,\mathrm{Piv}^*(h)$$

as ordered sets.

Items (1) and (2) imply that if the syllable length of h equals $\mathsf{n}(h)$, then $\mathrm{Piv}^*(ghg^{-1}) = g\,\mathrm{Piv}(h)$. We remark that items (2) and (3) of Lemma 3.1 are false for the set of all pivot points. We now state and prove the shortening proposition.

Proposition 3.2. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. There is a constant L_{short} with the following properties. Suppose that $L \geq L_{\text{short}}$ and that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G. Let $H = \langle R_v \rangle$ and choose windmill data $\{(H_i, W_i, \mathcal{O}_i)\}$. Let x be a vertex in $\mathcal{P}$ and $h \in H$ such that $hx \neq x$. Then there exists a vertex v of $\mathcal{P}$ and element h_v of R_v such that*

(1) *either $v \in \{x, hx\}$ or $d_v(x, hx) > L/10$;*

(2) *$h_v h < h$.*

The first item roughly translates as stating that v lies on the geodesic from x to hx.

Proof. Let L_0 and m be the constants from Lemma 2.1. Set $L_{\text{short}} = \max\{L_0, 5m, 14\theta\}$. Take $L \geq L_{\text{short}}$ and suppose that G is acting on $\mathcal{P}$ and that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G.

Fix a vertex x of $\mathcal{P}$ and an element h of H such that $hx \neq x$. Let $i = \mathsf{i}(h)$, $n = \mathsf{n}(h)$ and express h as a reduced word:

$$h = gh_1 \cdots h_n g^{-1},$$

where each h_k is either a nontrivial element of H_{i-1} or a nontrivial element of R_{v_k} with $v_k \in \mathcal{O}_i$.

First, suppose that $i = 0$ and so $h = gh_1g^{-1}$ where $h_1 \in R_{v_0}$. In this case, we take $v = gv_0$ and $h_v = gh_1^{-1}g^{-1} \in R_v$. If $v \notin \{x, hx\}$, then $d_v(x, hx) = d_{v_0}(g^{-1}x, h_1g^{-1}x) \geq L > L/10$. As $h_v h$ is the identity, clearly $h_v h < h$.

Hence for the remainder, we assume that i is at least 1. In particular, the set $\mathrm{Piv}^*(h)$ is nonempty. Our strategy is to find an essential pivot point w for h and an integer p such that $v = h^p w$ satisfies the first item. Given such a pivot point $w = gh_\sigma v_k$, where $h_\sigma = h_1 \cdots h_{k-1}$, we take $h_v = h^p(gh_\sigma)h_k^{-1}(gh_\sigma)^{-1}h^{-p} \in R_v$. Then

$$\begin{aligned} h_v h &= \big(h^p(gh_\sigma)h_k^{-1}(gh_\sigma)^{-1}h^{-p}\big)h \\ &= h^p\big((gh_\sigma)h_k^{-1}(gh_\sigma)^{-1}h\big)h^{-p} \\ &= h^p(gh_1 \cdots h_{k-1}h_{k+1} \cdots h_n g^{-1})h^{-p}. \end{aligned}$$

Hence for this element we have $h_v h < h$, which is the second item.

If $\{x, hx\} \cap \mathrm{Piv}^*(h) \neq \emptyset$, we can take w to be an essential pivot point in this intersection and set $v = w$.

Thus we may assume that $\{x, hx\} \cap \mathrm{Piv}^*(h) = \emptyset$. There are two cases depending on whether $x \in gW_i$ or $x \notin gW_i$. Set $\bar{h} = h_1 \cdots h_n$ so that $h = g\bar{h}g^{-1}$. We observe that $\bar{h}$ has level i.

For the first case, we initially assume that $x \in gN_i \subset gW_i$. Let w be a pivot point for $\bar{h}$, thus gw is an essential pivot point for h. By Lemma 2.1 (1), we have that $d_w(v_0, \bar{h}v_0) > L/2$. By Lemma 2.1 (4), we have that $w \notin W_{i-1}$. Since $\bar{h}^{-1}w$ is a pivot point for $\bar{h}^{-1}$, Lemma 2.1 (4) also implies that $\bar{h}^{-1}w \notin W_{i-1}$ as well. Hence by Lemma 2.1 (3) as $g^{-1}x \in N_i$ and $w, \bar{h}^{-1}w \notin W_{i-1}$ we have that $d_w(g^{-1}x, v_0) \leq m$ and $d_w(\bar{h}g^{-1}x, \bar{h}v_0) = d_{\bar{h}^{-1}w}(g^{-1}x, v_0) \leq m$. Therefore

$$\begin{aligned} d_{gw}(x, hx) &= d_w(g^{-1}x, \bar{h}g^{-1}x) \geq d_w(v_0, \bar{h}v_0) - d_w(v_0, g^{-1}x) - d_w(\bar{h}v_0, \bar{h}g^{-1}x) \\ &> L/2 - 2m \geq L/10. \end{aligned}$$

Hence we may set $v = gw$.

Now suppose that $x \in gW_i - gN_i$. Then there is an $h_0 \in H_i$ such that $h_0x \in gN_i$. Let $h' = h_0hh_0^{-1}$ and $x' = h_0x$. We have $h'x' \neq x'$. Fix some pivot point w for $\bar{h}$ and so gw is an essential pivot point for h. By Lemma 3.1 (1), we have $h_0gw \in \mathrm{Piv}^*(h')$. As $x, hx \notin \mathrm{Piv}^*(h)$, we have that $h_0gw \neq x', h'x'$. Thus as $x' \in gN_i$, the above case applies and we have that

$$d_{gw}(x, hx) = d_{h_0gw}(x', h'x') > L/10.$$

Hence we may set $v = gw$.

Lastly, we deal with the second case that $x \notin gW_i$. In this case, we will be considering the projection of x to various points of the form $h^j w$, where w is an essential pivot point for h and j is an integer. As w lies in gW_i by definition and W_i is H_i-invariant, we have that $h^j w$ lies in gW_i. In particular, $x \neq h^j w$ for any essential pivot point w for h and any integer and therefore projections of x to such points are always defined.

Fix any essential pivot point w for h. By Lemma 3.1 (2) we have that $h^j w$ is an essential pivot point for h^k whenever $0 \leq j < k$ and additionally, such points are ordered $h^{j_1}w < h^{j_2}w$ if $j_1 < j_2$. By Lemma 2.1 (2), we have that for $1 \leq j_1 < j_2$

$$d_{h^{j_1}w}(w, h^{j_2}w) \geq d_{h^{j_1}w}(v_0, h^{j_2}w) - d_{h^{j_1}w}(v_0, w) \geq L/2 - 2\theta.$$

By a similar argument we have $d_{h^{j_1}w}(h^{j_0}w, h^{j_2}w) \geq L/2 - 2\theta$ for all integers $j_0 < j_1 < j_2$.

Claim. There is an integer J such that

$$d_{h^j w}(h^{j-1}w, x) > \theta \text{ for } j \leq J \quad \text{and} \quad d_{h^j w}(h^{j-1}w, x) \leq \theta \text{ for all } j > J.$$

We first show that the set $\{j \in \mathbb{Z} \mid d_{h^j w}(h^{j-1}w, x) \leq \theta\}$ has the form $(J, +\infty)$ for some $J \in \mathbb{Z} \cup \{-\infty, +\infty\}$. To this end, we suppose that $d_{h^j w}(h^{j-1}w, x) \leq \theta$. If $d_{h^{j+1}w}(h^j w, x) > \theta$, then by the inequality on triples we have $d_{h^j w}(h^{j+1}w, x) \leq \theta$. In this case, we find that

$$L/2 - 2\theta \leq d_{h^j w}(h^{j-1}w, h^{j+1}w) \leq d_{h^j w}(h^{j-1}w, x) + d_{h^j w}(h^{j+1}w, x) \leq 2\theta.$$

This is a contradiction as $L > 8\theta$ and therefore $d_{h^{j+1}w}(h^j w, x) \leq \theta$ too.

If $J = -\infty$, then $d_{h^j w}(h^{j-1}w, x) \leq \theta$ for all integers j. Thus for all $j \leq -1$ we find that

$$d_{h^j w}(w, x) \geq d_{h^j w}(h^{j-1}w, w) - d_{h^j w}(h^{j-1}w, x) \geq L/2 - 3\theta > \theta.$$

This contradicts the finiteness axiom.

If $J = +\infty$, then by the inequality on triples we have $d_{h^j w}(h^{j+1}w, x) \leq \theta$ for all integers j. Thus for all $j \geq 1$ we find that

$$d_{h^j w}(w, x) \geq d_{h^j w}(h^{j+1}w, w) - d_{h^j w}(h^{j+1}w, x) \geq L/2 - 3\theta > \theta.$$

Again, this contradicts the finiteness axiom. This completes the proof of the claim.

Let J be as defined in the claim. To complete the proof of the proposition, there are two cases based on $d_{h^J w}(h^{J-1}w, x)$. We will show that we can take v to be either $h^J w$ or $h^{J+1}w$.

First, suppose that $d_{h^J w}(h^{J-1}w, x) \leq L/4$. We have $d_{h^J w}(h^{J-1}w, x) > \theta$ and by the inequality on triples and invariance we have $d_{h^J w}(h^{J+1}w, hx) = d_{h^{J-1}w}(h^J w, x) \leq \theta$. Thus

$$\begin{aligned} d_{h^J w}(x, hx) &\geq d_{h^J w}(h^{J-1}w, h^{J+1}w) - d_{h^J w}(h^{J-1}w, x) - d_{h^J w}(h^{J+1}w, hx) \\ &\geq L/2 - \theta - L/4 - \theta \geq L/4 - 2\theta > L/10. \end{aligned}$$

Hence we can set $v = h^J w$.

Else, we have that

$$d_{h^J w}(h^{J-1}w, x) = d_{h^{J+1}w}(h^J w, hx) > L/4.$$

As $d_{h^{J+1}}(h^J w, x) \leq \theta$, we have

$$d_{h^{J+1}w}(x, hx) \geq d_{h^{J+1}w}(h^J w, hx) - d_{h^{J+1}w}(h^J w, x) \geq L/4 - \theta > L/10.$$

Hence we can set $v = h^{J+1}w$. ∎

4. Lifting quadrilaterals

In this section, we apply the shortening argument of Proposition 3.2 to show that geodesic quadrilaterals in the quotient of the projection complex $\mathcal{P}/\langle R_v \rangle$ lift to geodesic quadrilaterals in the projection complex $\mathcal{P}$. This is stated in Proposition 4.3. As mentioned in Section 1, the strategy to show that $\mathcal{P}/\langle R_v \rangle$ is δ-hyperbolic is to lift geodesic triangles in $\mathcal{P}/\langle R_v \rangle$ to geodesic triangles in $\mathcal{P}$. As a triangle is a degenerate quadrilateral where one side has length 0, Proposition 4.3 applies to geodesic triangles as well. The reason we work with quadrilaterals is to show that the action of $G/\langle R_v \rangle$ on $\mathcal{P}/\langle R_v \rangle$ is a non-elementary WPD action, so long as the action of G on $\mathcal{P}$ is and L, the spinning constant, is large enough.

There are two items we need to discuss before stating and proving Proposition 4.3.

Lifting geodesics. Throughout this section we will be lifting geodesics from $\mathcal{P}/\langle R_v \rangle$ to $\mathcal{P}$ and modifying the lifts. It will be important to have a way of certifying that these lifts and their modifications are geodesics. This is the content of the following lemma. Throughout the rest of the paper, we will always assume that paths are 1-Lipschitz.

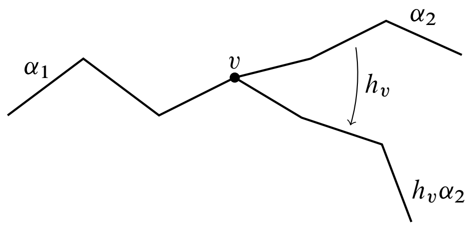

Figure 1. The paths α and $\alpha \vee_v h_v$.

Lemma 4.1. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. Suppose that H is a subgroup of G and let $p\colon \mathcal{P} \to \mathcal{P}/H$ be the quotient map. The following statements are true.*

(1) *If $\overline{\alpha}\colon [0,n] \to \mathcal{P}/H$ is a path and x is a point in $\mathcal{P}$ that satisfies $p(x) = \overline{\alpha}(0)$, then there exists a path $\alpha\colon [0,n] \to \mathcal{P}$ such that $p \circ \alpha = \overline{\alpha}$ and $\alpha(0) = x$.*

(2) *If $\alpha\colon [0,n] \to \mathcal{P}$ is a path and $n = d_{\mathcal{P}/H}(p(\alpha(0)), p(\alpha(n)))$, then α is a geodesic.*

Proof. The first statement is obvious.

The second statement follows as the map $p\colon \mathcal{P} \to \mathcal{P}/H$ is 1-Lipschitz. Indeed, if α is not a geodesic, then there is a geodesic $\alpha'\colon [0,n'] \to \mathcal{P}$, where $\alpha'(0) = \alpha(0)$, $\alpha'(n') = \alpha(n)$, and $n' < n$. As p is 1-Lipschitz, we find that

$$n = d_{\mathcal{P}/H}\big(p(\alpha'(0)), p(\alpha'(n'))\big) \leq n'.$$

This a contradiction and hence α is a geodesic. ■

Bending paths. Let v be a vertex in $\mathcal{P}$. Suppose that $\alpha\colon [0,n] \to \mathcal{P}$ is a path and that $v = \alpha(n_0)$ for some $n_0 \in \{0,\ldots,n\}$. For any $h_v \in R_v$ we define a new path $\alpha \vee_v h_v\colon [0,n] \to \mathcal{P}$ by

$$(\alpha \vee_v h_v)(t) = \begin{cases} \alpha(t), & \text{if } 0 \leq t \leq n_0, \text{ or} \\ h_v\alpha(t), & \text{if } n_0 \leq t \leq n. \end{cases}$$

As $\alpha(n_0) = h_v\alpha(n_0)$, this does define a path. We say that $\alpha \vee_v h_v$ is obtained by *bending α at v using h_v*. Writing α as the concatenation of two paths α_1 and α_2 where α_1 ends at v and α_2 begins at v, the bent path $\alpha \vee_v h_v$ is the concatenation of α_1 and $h_v\alpha_2$; see Figure 1.

Lemma 4.2. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. Suppose that $\{R_v\}$ is an equivariant family of subgroups of G. Let $H = \langle R_v \rangle$ and let $p\colon \mathcal{P} \to \mathcal{P}/H$ be the quotient map. Let $\alpha\colon [0,n] \to \mathcal{P}$ be a path and let v be a vertex in the image of α. Then for any $h_v \in R_v$ the following statements are true.*

(1) *The paths $p \circ \alpha$ and $p \circ (\alpha \vee_v h_v)$ are equal.*

(2) *For any $0 \leq t_1 < t_2 \leq n$, if $p \circ \alpha|[t_1,t_2]$ is a geodesic, then so is $(\alpha \vee_v h_v)|[t_1,t_2]$.*

Proof. The first statement follows immediately from the definitions.

The second statement follows from the first statement and Lemma 4.1 (2). ■

Let X be a graph considered as a metric space where every edge has length one. A *geodesic quadrilateral* Q in X consists of four geodesics and four points: α_k from x_k to $x_{k+1 \bmod 4}$ for $k = 0, 1, 2, 3$. We write $Q = \bigcup_{k=0}^{3} \alpha_k$.

Proposition 4.3. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. For any $B \geq 0$, there is a constant $L_{\mathrm{lift}}(B)$ with the following properties. Suppose that $L \geq L_{\mathrm{lift}}(B)$ and that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G. Let $H = \langle R_v \rangle$ and let $p\colon \mathcal{P} \to \mathcal{P}/H$ be the quotient map. For each geodesic quadrilateral $\bar{Q} = \bigcup_{k=0}^{3} \bar{\alpha}_k$ in $\mathcal{P}/H$ there exists a geodesic quadrilateral $Q = \bigcup_{k=0}^{3} \alpha_k$ in $\mathcal{P}$ so that $p(\alpha_k) = \bar{\alpha}_k$ for $k = 0, 1, 2, 3$.*

Additionally, Q satisfies the following property. If there are lifts $\tilde{\alpha}_0$ from $\tilde{x}_0$ to $\tilde{x}_1$ and $\tilde{\alpha}_2$ from $\tilde{x}_2$ to $\tilde{x}_3$ of $\bar{\alpha}_0$ and $\bar{\alpha}_2$, respectively, such that $d_v(\tilde{x}_0, \tilde{x}_1) \leq B$ and $d_v(\tilde{x}_2, \tilde{x}_3) \leq B$ when defined, then the geodesics α_0 and α_2 in Q are H-translates of $\tilde{\alpha}_0$ and $\tilde{\alpha}_2$, respectively.

Proof. Fix $B \geq 0$ and set $L_{\mathrm{lift}}(B) = \max\{L_{\mathrm{short}}, 40B, 40C_{\mathrm{g}}\}$. Take $L \geq L_{\mathrm{lift}}(B)$ and suppose that G is acting on $\mathcal{P}$ and that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G.

Let $\bar{x}_0$, $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ be the vertices of the geodesic quadrilateral $\bar{Q}$ in $\mathcal{P}/H$. By Lemma 4.1 (1), for any point $x_0 \in p^{-1}(\bar{x}_0)$, we can iteratively lift the geodesics $\bar{\alpha}_k$ to paths α_k from x_k to x_{k+1} where $p(x_k) = \bar{x}_k$. By Lemma 4.1 (2), the paths α_k are geodesics. If α_0 and α_2 as in the statement of the proposition exists, then we can ensure that α_0 and α_2 are H-translates of these geodesics. We denote the concatenation of the paths α_k by α and we say that α is a *special lift* of $\bar{Q}$.

For each special lift α of $\bar{Q}$, with endpoints denoted by x_0 and x_4, there is an element $\mathsf{h}(\alpha) \in H$ with minimal complexity such that $x_4 = \mathsf{h}(\alpha)x_0$. Let α be a special lift of $\bar{Q}$ so that $\mathsf{h}(\alpha)$ has minimal complexity among all special lifts of $\bar{Q}$.

We claim that $x_0 = x_4$, which shows that α defines a geodesic quadrilateral Q as in the statement of the proposition. Indeed, if not we will show that we can bend α to a new path α' that is a special lift with $\mathsf{h}(\alpha') < \mathsf{h}(\alpha)$. This contradicts the minimality of $\mathsf{h}(\alpha)$.

To this end, suppose that $x_0 \neq x_4 = \mathsf{h}(\alpha)x_0$. Apply Proposition 3.2 to $x = x_0$ and $h = \mathsf{h}(\alpha)$ and let v be the corresponding vertex of $\mathcal{P}$ and $h_v \in R_v$ the corresponding element. We have that $h_v\mathsf{h}(\alpha) < \mathsf{h}(\alpha)$.

We claim that v lies in the image of α. Indeed, if $v \notin \{x_0, x_4\}$, then $d_v(x_0, x_4) > L/10$. If further $v \notin \{x_1, x_2, x_3\}$, then, by the triangle inequality, we have that

$$d_v(x_n, x_{n+1}) > L/40 \quad \text{for some } n.$$

As $L/40 \geq C_{\mathrm{g}}$, there is an n_0 such that $\alpha_n(n_0) = v$. Moreover, as $L/40 \geq B$, if lifts $\tilde{\alpha}_0$ and $\tilde{\alpha}_2$ as in the statement of the proposition exist, we must have that $n = 1$ or $n = 3$. This shows that v lies in the image of α. We consider the path $\alpha' = \alpha \vee_v h_v$.

By Lemma 4.2 (2), α' consists of four geodesic segments α'_k for $k = 0, 1, 2, 3$. Moreover, we observe that α' is a special lift of $\bar{Q}$ as if lifts $\tilde{\alpha}_0$ and $\tilde{\alpha}_2$ as in the statement of the proposition exist, then the segments α'_0 and α'_2 are H-translates of the segments α_0 and

α_2, respectively. Letting x_4' denote the terminal point of α' we find

$$x_4' = h_v x_4 = h_v \mathsf{h}(\alpha) x_0$$

so that $\mathsf{h}(\alpha') \leq h_v \mathsf{h}(\alpha) < \mathsf{h}(\alpha)$. This contradicts the minimality of $\mathsf{h}(\alpha)$. ■

5. Proof of Theorem 1.1

In this section, we prove the first of the two main results of this paper. Theorem 1.1 states that if a group G acts on a projection complex $\mathcal{P}$, then there exists a constant $L_{\text{hyp}}(\mathcal{P})$ so that if $L \geq L_{\text{hyp}}(\mathcal{P})$ and if $\{R_v\}$ is an equivariant L-spinning family of subgroups of G, then $\mathcal{P}/\langle R_v \rangle$ is δ-hyperbolic. The proof proceeds by showing that geodesic triangles in $\mathcal{P}/\langle R_v \rangle$ can be lifted to geodesic triangles in $\mathcal{P}$.

Proof of Theorem 1.1. Let $\mathcal{P}$ be a projection complex and set $L_{\text{hyp}}(\mathcal{P}) = L_{\text{lift}}(0)$. Bestvina–Bromberg–Fujiwara proved the $\mathcal{P}$ is a quasi-tree [1, Theorem 3.16]. Let δ be such that $\mathcal{P}$ is δ-hyperbolic. Take $L \geq L_{\text{hyp}}(\mathcal{P})$ and suppose that G is acting on $\mathcal{P}$ and that $\{R_v\}$ is an equivariant L-spinning family. Let $H = \langle R_v \rangle$.

Let $\overline{\alpha}_0$, $\overline{\alpha}_1$, and $\overline{\alpha}_2$ be the three sides of a geodesic triangle in $\mathcal{P}/H$. We set $\overline{\alpha}_3$ to be the trivial path at the endpoint of $\overline{\alpha}_2$. This gives a (degenerate) geodesic quadrilateral $\overline{Q} = \bigcup_{k=0}^{3} \overline{\alpha}_k$. By Proposition 4.3, there is a geodesic quadrilateral $Q = \bigcup_{k=0}^{3} \alpha_k$ so that $p(\alpha_k) = \overline{\alpha}_k$ for $k = 0, 1, 2, 3$. As α_3 is a trivial path, Q is in fact a geodesic triangle in $\mathcal{P}$.

As the map $p\colon \mathcal{P} \to \mathcal{P}/H$ is 1-Lipschitz and as Q is δ-thin, the geodesic triangle $\overline{Q}$ is δ-thin as well. Hence $\mathcal{P}/H$ is δ-hyperbolic. ■

6. Bounded projections

There are two key results in this section. First, we show that geodesics $\alpha\colon [0, n] \to \mathcal{P}$ with bounded projections are mapped by p to geodesics in $\mathcal{P}/\langle R_v \rangle$. This appears as Lemma 6.1. The proof of this lemma is very similar to the proof of Proposition 4.3 as it involves bending and shortening. Secondly, we apply Lemma 6.1 to show that given a WPD element in G where the orbit of some point has bounded projections, its image in $G/\langle R_v \rangle$ acts as a WPD element on $\mathcal{P}/\langle R_v \rangle$. This appears as Lemma 6.2. The proof of this lemma uses Proposition 4.3.

Lemma 6.1. *Let $\mathcal{P}$ be a projection complex and let G be a group acting on $\mathcal{P}$. For any $B \geq 0$, there is a constant $L_{\text{pro}}(B)$ with the following property. Suppose that $L \geq L_{\text{pro}}(B)$ and that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G. Let $H = \langle R_v \rangle$ and let $p\colon \mathcal{P} \to \mathcal{P}/H$ be the quotient map. If $\alpha\colon [0, n] \to \mathcal{P}$ is a geodesic, and $d_v(\alpha(0), \alpha(n)) \leq B$ for all vertices v of $\mathcal{P}$ other than $\alpha(0)$ and $\alpha(n)$, then $p \circ \alpha\colon [0, n] \to \mathcal{P}/H$ is a geodesic.*

Proof. Set $L_{\text{pro}}(B) = \max\{L_{\text{short}}, 10B + 10C_{\text{g}}\}$. Take $L \geq L_{\text{pro}}(B)$ and suppose that G is acting on $\mathcal{P}$ and that $\{R_v\}$ is an equivariant L-spinning family.

Let $\overline{\beta}\colon [0,n'] \to \mathcal{P}/H$ be a geodesic from $p(\alpha(0))$ to $p(\alpha(n))$. We will argue that $n = n'$, showing that $p \circ \alpha$ is a geodesic.

For each H-translate $h\alpha\colon [0,n] \to \mathcal{P}$ of α, we say a lift $\beta\colon [0,n'] \to \mathcal{P}$ of $\overline{\beta}$ is *compatible* with $h\alpha$ if $h\alpha(0) = \beta(0)$. In this situation, there is an element $\mathsf{h}(h\alpha,\beta)$ with minimal complexity such that $\beta(n') = \mathsf{h}(h\alpha,\beta)h\alpha(n)$. We replace α by an H-translate and let $\beta\colon [0,n'] \to \mathcal{P}$ be a compatible lift of $\overline{\beta}$ so that $\mathsf{h}(\alpha,\beta)$ minimizes complexity among all H-translates of α and compatible lifts.

We claim that $\alpha(n) = \beta(n')$, which shows that $n = n'$ as both α and β are geodesics. Indeed, if not we will show that we can find a translate α' of α and a compatible lift β' with $\mathsf{h}(\alpha',\beta') < \mathsf{h}(\alpha,\beta)$. The path β' is obtained by translating or bending β. This contradicts the minimality of $\mathsf{h}(\alpha,\beta)$.

To this end, suppose that $\alpha(n) \neq \beta(n')$. Apply Proposition 3.2 to $x = \alpha(n)$ and $h = \mathsf{h}(\alpha,\beta)$ and let v be the corresponding vertex and $h_v \in R_v$ the corresponding element. We have that $h_v\mathsf{h}(\alpha,\beta) < \mathsf{h}(\alpha,\beta)$.

There are two cases now depending on v.

If $v = \alpha(n)$, then for the H-translate $h_v\alpha$ and compatible lift $h_v\beta$, we have

$$h_v\beta(n') = h_v\mathsf{h}(\alpha,\beta)\alpha(n) = h_v\mathsf{h}(\alpha,\beta)h_v\alpha(n)$$

so that $\mathsf{h}(h_v\alpha, h_v\beta) \leq h_v\mathsf{h}(\alpha,\beta) < \mathsf{h}(\alpha,\beta)$. This contradicts the minimality of $\mathsf{h}(\alpha,\beta)$.

Else, we claim that v lies in the image of β. Indeed, if $v \neq \beta(n)$, then

$$d_v\big(\alpha(n), \beta(n')\big) > L/10.$$

If further $v \neq \beta(0)$, then as $d_v(\alpha(0),\alpha(n)) \leq B$ and $\alpha(0) = \beta(0)$, we have that

$$d_v\big(\beta(0), \beta(n')\big) \geq d_v\big(\alpha(n), \beta(n')\big) - d_v\big(\alpha(0), \alpha(n)\big) > L/10 - B > C_{\mathrm{g}}.$$

This shows that v lies in the image of β.

We define $\beta' = \beta \vee_v h_v$. By Lemma 4.2, β' is a compatible lift. Next, we find that

$$\beta'(n') = h_v\beta(n') = h_v\mathsf{h}(\alpha,\beta)\alpha(n)$$

so that $\mathsf{h}(\alpha,\beta') \leq \mathsf{h}(\alpha,\beta) < \mathsf{h}(\alpha,\beta)$. This contradicts the minimality of $\mathsf{h}(\alpha,\beta)$. ∎

Lemma 6.2. *Let $\mathcal{P}$ be a projection complex, G a group acting on $\mathcal{P}$, and $B \geq 0$. Suppose that $L \geq \max\{L_{\mathrm{lift}}(B), L_{\mathrm{pro}}(B)\}$ and that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G. Let $H = \langle R_v \rangle$. If $f \in G$ is a hyperbolic isometry of $\mathcal{P}$ so that $d_v(x_0, f^n x_0) \leq B$ for all $n \in \mathbb{Z}$ when defined, then its image $\bar{f} \in G/H$ is a hyperbolic isometry of $\mathcal{P}/H$. Additionally, if f is a WPD element, then so is $\bar{f}$.*

Proof. Fix $B \geq 0$ and suppose that G is acting on $\mathcal{P}$ and that $\{R_v\}$ is an equivariant L-spinning family where $L \geq \max\{L_{\mathrm{lift}}(B), L_{\mathrm{pro}}(B)\}$. Suppose that $f \in G$ is a hyperbolic isometry of $\mathcal{P}$ and x_0 is a vertex of $\mathcal{P}$ so that $d_v(x_0, f^n x_0) \leq B$ for all $n \in \mathbb{Z}$ when defined.

Let $\bar{x}_0 = p(x_0)$. As $L \geq L_{\mathrm{pro}}(B)$, by Lemma 6.1, we have that

$$d_{\mathcal{P}/H}(\bar{x}_0, \bar{f}^n \bar{x}_0) = d_{\mathcal{P}}(x_0, f^n x_0).$$

Hence as f is hyperbolic, $\bar{f}$ is also hyperbolic.

Now assume further that f is a WPD element. Fix $D \geq 0$ and let $M \geq 0$ be such that the set

$$\{g \in G \mid d_{\mathcal{P}}(x_0, g x_0) \leq D \text{ and } d_{\mathcal{P}}(f^M x_0, g f^M x_0) \leq D\}$$

is finite. Let K denote the cardinality of this set.

Suppose that $\{\bar{g}_1, \dots, \bar{g}_{K'}\}$ is a set of elements of G/H so that

$$d_{\mathcal{P}/H}(\bar{x}_0, \bar{g}_j \bar{x}_0) \leq D \quad \text{and} \quad d_{\mathcal{P}/H}(\bar{f}^M \bar{x}_0, \bar{g}_j \bar{f}^M \bar{x}_0) \leq D.$$

Fix elements $g_j \in G$ whose images are the $\bar{g}_j$s.

We consider the geodesic quadrilateral $\bar{Q}_j = \bigcup_{k=0}^{3} \bar{\alpha}_k$, where $\bar{\alpha}_0$ is a geodesic from $\bar{x}_0$ to $\bar{f}^M \bar{x}_0$, $\bar{\alpha}_1$ is a geodesic from $\bar{f}^M \bar{x}_0$ to $\bar{g}_j \bar{f}^M \bar{x}_0$, $\bar{\alpha}_2$ is a geodesic from $\bar{g}_j \bar{f}^M \bar{x}_0$ to $\bar{g}_j \bar{x}_0$, and $\bar{\alpha}_3$ is a geodesic from $\bar{g}_j \bar{x}_0$ to $\bar{x}_0$.

As $L \geq L_{\mathrm{lift}}(B)$ for each $1 \leq j \leq K'$, there is a geodesic quadrilateral $Q_j = \bigcup_{k=0}^{3} \alpha_k$ so that $p(\alpha_k) = \bar{\alpha}_k$. Moreover, there are elements $h_0, h_2 \in H$ such that α_0 is a geodesic from $h_0 \bar{x}_0$ to $h_0 f^M x_0$ and α_2 is a geodesic from $h_2 g_j f^M x_0$ to $h_2 g_j x_0$. In particular, for each $1 \leq j \leq K'$ we find that

$$d_{\mathcal{P}}(x_0, h_0^{-1} h_2 g_j) \leq D \quad \text{and} \quad d_{\mathcal{P}}(f^M x_0, h_0^{-1} h_2 g_j f^M x_0) \leq D.$$

This shows that $K' \leq K$.

As it suffices to check finiteness at a single point, this shows that $\bar{f}$ is a WPD element. ■

7. Proof of Theorem 1.2

In this section, we give the proof of the second of the main results in this paper. Theorem 1.2 states that if a group G admits a non-elementary WPD action on a projection complex $\mathcal{P}$, then there exists a constant $L_{\mathrm{WPD}}(\mathcal{P}, G)$ so that if $L \geq L_{\mathrm{WPD}}(\mathcal{P}, G)$ and if $\{R_v\}$ is an equivariant L-spinning family of subgroups of G, then the action of $G/\langle R_v \rangle$ on $\mathcal{P}/\langle R_v \rangle$ is a non-elementary WPD action.

Isometries have bounded projections. In order to apply the results of Section 6, we need to know that hyperbolic isometries of a projection complex have bounded projections. This is an application of the finiteness axiom of a projection complex as we now show.

Lemma 7.1. *Let $\mathcal{P}$ be a projection complex and let f be a hyperbolic isometry of $\mathcal{P}$. Then for any vertex x_0 of $\mathcal{P}$, there is a constant B_f such that $d_v(x_0, f^n x_0) \leq B_f$ for all $n \in \mathbb{Z}$ when defined.*

Proof. Let $M_1 = \max\{d_v(x_0, fx_0) \mid v \notin \{x_0, fx_0\}\}$ and $M_2 = d_{x_0}(f^{-1}x_0, fx_0)$. We remark that M_1 is finite by the finiteness axiom. Set $M = \max\{M_1, M_2\}$. Fix a geodesic α from x_0 to fx_0. Let N be such that $d(x, f^n y) > 4$ if x and y lie on α, and $n \geq N$. Define $B_f = NM + 2C_{\mathrm{p}}$.

By equivariance, it suffices to prove the lemma for non-negative integers. Fix an $n \in \mathbb{N}$ and suppose that $v \notin \{x_0, f^n x_0\}$. If v does not lie in the 2-neighborhood of the path $\alpha \cup f\alpha \cup \cdots \cup f^{n-1}\alpha$, then $d_v(x_0, f^n x_0) \leq C_{\mathrm{p}} \leq B_f$.

Else, there are indices $0 \leq i_0 \leq i_1 \leq n-1$ such that $i_1 - i_0 < N$ and v lies in the 2-neighborhood of $f^j\alpha$ only if $i_0 \leq j \leq i_1$. Thus as v does not lie in the 2-neighborhood of $\alpha \cup \cdots \cup f^{i_0-1}\alpha$ nor in the 2-neighborhood of $f^{i_1+1}\alpha \cup \cdots \cup f^n\alpha$, we have $d_v(x_0, f^{i_0}x_0) \leq C_{\mathrm{p}}$ and $d_v(f^{i_1+1}x_0, f^n x_0) \leq C_{\mathrm{p}}$.

Suppose that $v \neq f^j x_0$ for any $i_0 < j \leq i_1$ (by the definition of i_0 and i_1 these are the only possible indices). Then we find that

$$d_v(f^{i_0}x_0, f^{i_1+1}x_0) \leq \sum_{j=i_0}^{i_1} d_v(f^j x_0, f^{j+1}x_0) = \sum_{j=i_0}^{i_1} d_{f^{-j}v} d(x_0, fx_0) \leq NM_1 \leq NM.$$

Else, we have that $v = f^{j_0}x_0$ for some $i_0 < j_0 \leq i_1$. In this case, we find

$$\begin{aligned} d_v(f^{i_0}x_0, f^{i_1+1}x_0) &\leq \sum_{j=i_0}^{j_0-2} d_v(f^j x_0, f^{j+1}x_0) + d_{f^{j_0}x_0}(f^{j_0-1}x_0, f^{j_0+1}x_0) \\ &\quad + \sum_{j=j_0+1}^{i_1} d_v(f^j x_0, f^{j+1}x_0) \\ &\leq (j_0 - 1 - i_0)M_1 + M_2 + (i_1 - j_0)M_1 \\ &\leq (N-2)M_1 + M_2 \leq NM. \end{aligned}$$

Therefore

$$\begin{aligned} d_v(x_0, f^n x_0) &\leq d_v(x_0, f^{i_0}x_0) + d_v(f^{i_0}x_0, f^{i_1+1}x_0) + d_v(f^{i_1+1}x_0, f^n x_0) \\ &\leq NM + 2C_{\mathrm{p}} = B_f. \end{aligned}$$

■

Proof of Theorem 1.2. Let $\mathcal{P}$ be a projection complex and let G be a group with a non-elementary WPD action on $\mathcal{P}$.

Let f_1 and f_2 be independent WPD elements in G. Fix some point x_0 in $\mathcal{P}$ and let B_{f_1} and B_{f_2} be the constants from Lemma 7.1. Let $B_0 = \max\{d_v(f_1x_0, f_2x_0) \mid v \notin \{f_1x_0, f_2x_0\}\}$. Set $B = B_0 + B_{f_1} + B_{f_2}$. Let v be a vertex of $\mathcal{P}$ and suppose that $d_v(f_1^{n_1}x_0, f_2^{n_2}x_0)$ is defined for some integers n_1 and n_2. If $v \neq x_0$, then

$$d_v(f_1^{n_1}x_0, f_2^{n_2}x_0) \leq d_v(f_1^{n_1}x_0, x_0) + d_v(x_0, f_2^{n_2}x_0) \leq B_{f_1} + B_{f_2} \leq B.$$

Else, if $v = x_0$, then

$$\begin{aligned} d_v(f_1^{n_1}x_0, f_2^{n_2}x_0) &\leq d_v(f_1^{n_1}x_0, f_1x_0) + d_v(f_1x_0, f_2x_0) + d_v(f_2x_0, f_2^{n_2}x_0) \\ &\leq B_{f_1} + B_0 + B_{f_2} = B. \end{aligned}$$

Let $L_{\mathrm{WPD}}(\mathcal{P}, G) = \max\{L_{\mathrm{lift}}(B), L_{\mathrm{pro}}(B)\}$. Suppose that $\{R_v\}$ is an equivariant L-spinning family of subgroups of G where $L \geq L_{\mathrm{WPD}}(\mathcal{P}, G)$. Let $H = \langle R_v \rangle$.

By Lemma 6.2, the images $\bar{f}_1$ and $\bar{f}_2$ are WPD elements of G/H acting on $\mathcal{P}/H$. Additionally, by Lemma 6.1, we have that $d_{\mathcal{P}/H}(\bar{f}_1^{n_1}\bar{x}_0, \bar{f}_2^{n_2}\bar{x}_0) = d_{\mathcal{P}}(f_1^{n_1}x_0, f_2^{n_2}x_0)$ for integers n_2 and n_2. As f_1 and f_2 are independent, this shows that $\bar{f}_1$ and $\bar{f}_2$ are independent as well. ∎

8. Examples

In this final section, we present two examples when G is $\mathrm{Mod}(S)$, the mapping class group of an orientable surface S. In the first example, the subgroup H is the normal closure of a pseudo-Anosov mapping class; in the second example the subgroup H is the normal closure of a partial pseudo-Anosov defined on an orbit-overlapping subsurface. The first example in fact applies more generally, whenever G is a group acting on a δ-hyperbolic metric space and g is a WPD element for this action. The relevant background material and definitions relating to the mapping class group that appear in this section can be found in our previous paper with Margalit [7].

Before we give the examples, we first recall the criteria of Bestvina–Bromberg–Fujiwara for showing that an element g of G acts on a projection complex $\mathcal{P}$ as a WPD element.

WPD criterion. Suppose that $\mathcal{P}$ is a projection complex. Bestvina–Bromberg–Fujiwara proved the existence of a constant C_{WPD} which can be used to ensure that an element acting on $\mathcal{P}$ is a WPD element [1, Proposition 3.27]. The set-up is as follows. Assume that G is a group that acts on $\mathcal{P}$ and that g is an element of G that satisfies the following two conditions.

(1) There is a vertex v in $\mathcal{P}$ and $n > 0$ such that $d_v(g^{-n}v, g^n v) > C_{\mathrm{WPD}}$.

(2) There is an $m > 0$ such that the subgroup of G that fixes $v, gv, \ldots, g^m v$ is finite.

Then g is a WPD element of G.

8.1. First example

Let S be an orientable surface where $\chi(S) < 0$ and let f be a pseudo-Anosov mapping class of S. There is a projection complex $\mathcal{P}$ built using f and its action on the curve complex $\mathcal{C}(S)$. We briefly recall this construction here; full details can be found in our previous paper [7, Section 3.2].

Fix a point x in $\mathcal{C}(S)$ and consider the *quasi-axis bundle* $\beta = \mathrm{EC}(f) \cdot x$, where $\mathrm{EC}(f)$ is the *elementary closure of* f. In this context, $\mathrm{EC}(f)$ is the stabilizer of the set of transverse measured foliations associated to f considered in the space of projectivized measured foliations on S.

The vertex set of $\mathcal{P}$ consists of the $\mathrm{Mod}(S)$-translates of β. Next we define the distance functions. Given three vertices α_1, α_2, and β of $\mathcal{P}$, we define $d_\beta(\alpha_1, \alpha_2)$ to be diameter of the union of the projections of α_1 and α_2 to β.

Let g be a mapping class of S that does not lie in $\mathrm{EC}(f)$. We claim that gf^n is a WPD element for the action on $\mathcal{P}$ for n sufficiently large. As g does not lie in $\mathrm{EC}(f)$, we have that $\beta \notin \{g\beta, g^{-1}\beta\}$. Next, we have that

$$d_\beta\big((gf^n)^{-1}\beta, gf^n\beta\big) = d_\beta(g^{-1}\beta, f^n g\beta) \geq d_\beta(g\beta, f^n g\beta) - d_\beta(g\beta, g^{-1}\beta)$$

and thus $d_\beta((gf^n)^{-1}\beta, gf^n\beta)$ is bounded below by $An - B$ for some constants $A, B > 0$. In particular, $d_\beta((gf^n)^{-1}\beta, gf^n\beta) > C_{\mathrm{WPD}}$ for sufficiently large n. Further, as g does not stabilize the measured foliations associated to f, the stabilizer of β and $g\beta$ is finite. By the Bestvina–Bromberg–Fujiwara WPD criterion we have that $f_1 = gf^n$ is a WPD element for some fixed sufficiently large enough n.

Given an element h that does not lie in $\mathrm{EC}(f_1)$, the element $f_2 = hf_1h^{-1}$ is a WPD element and f_1 and f_2 are independent. Thus the action of $\mathrm{Mod}(S)$ on $\mathcal{P}$ is a non-elementary WPD action.

As explained in the proof of Theorem 1.7 in our previous work, for each $L \geq 0$, there is an $p > 0$ such that the collection of subgroups $R_{h\beta} = \langle hf^p h^{-1}\rangle$ is an equivariant L-spinning family of subgroups. Therefore, by Theorem 1.2, the elements of $\bar{f}_1$ and $\bar{f}_2$ in $\mathrm{Mod}(S)/\langle\langle f^p\rangle\rangle$ are independent WPD elements for its action on $\mathcal{P}/\langle\langle f^p\rangle\rangle$ for a certain large enough p.

As mentioned at the beginning of this section, this above discussion works in the larger context of a group G acting on a δ-hyperbolic space using a WPD element f of G.

8.2. Second example

Again, let S be an orientable surface where $\chi(S) < 0$. Let X be a connected subsurface of S so that for all $h \in \mathrm{Mod}(S)$, either $X = hX$ or X and hX have nontrivial intersection (what is called an *orbit-overlapping* subsurface in our previous work [7]). There is a projection complex $\mathcal{P}$ built using X and the curve complex $\mathcal{C}(S)$. We briefly recall this construction here; full details can be found in our previous paper [7, Section 3.3].

The vertices of $\mathcal{P}$ are the $\mathrm{Mod}(S)$-translates of X. Given three vertices Y_1, Y_2, and X of $\mathcal{P}$, the distance $d_X(Y_1, Y_2)$ is the diameter in $\mathcal{C}(X)$ of the Masur–Minsky subsurface projections of Y_1 and Y_2 to X [13].

There is a well-defined map $\mathrm{Mod}(X) \to \mathrm{Mod}(S)$; fix an element f in $\mathrm{Mod}(S)$ that is the image of a pseudo-Anosov element on X. Let g be a mapping class of S such that ∂X and $g\partial X$ fill S. We claim that gf^n is a WPD element for the action on $\mathcal{P}$ for sufficiently large n. The proof is similar to the first example and left to the reader.

Hence, as above, there are elements f_1 and f_2 in $\mathrm{Mod}(S)$ that are independent WPD elements for the action on $\mathcal{P}$. By taking certain p sufficiently large, we can ensure that the equivariant family of subgroups $R_{hX} = \langle\langle hf^p h^{-1}\rangle\rangle_{\mathrm{Stab}(hX)}$ is L-spinning for arbitrary L. Hence we can ensure that the images $\bar{f}_1$ and $\bar{f}_2$ are independent WPD elements for the action of $\mathrm{Mod}(S)/\langle\langle f^p\rangle\rangle$ on $\mathcal{P}/\langle\langle f^p\rangle\rangle$.

Similar arguments apply to the other subgroups of $\mathrm{Mod}(S)$ constructed in our previous work.

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Matt Clay
Department of Mathematical Sciences, 309 SCEN, University of Arkansas, Fayetteville, AR 72701, USA; mattclay@uark.edu

Johanna Mangahas
Department of Mathematics, University at Buffalo, 244 Mathematics Building, Buffalo, NY 14260, USA; mangahas@buffalo.edu