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Distal strongly ergodic actions

Eli Glasner and Benjamin Weiss

Abstract. Let η be an arbitrary countable ordinal. Using results of Bourgain, Gamburd, and Sarnak on compact systems with spectral gap, we show the existence of an action of the free group on three generators F_3 on a compact metric space X, admitting an invariant probability measure μ, such that the resulting dynamical system (X, μ, F_3) is strongly ergodic and distal of rank η. In particular, this shows that there is an F_3 system which is strongly ergodic but not compact. This result answers the open question whether such actions exist.

1. Introduction

In this note we construct a strongly ergodic, distal, non-compact system for the free group F_3 on three generators. This answers a question of Ibarlucía, Le Maitre, Tsankov, and Tucker-Drob. Moreover, we show that for an arbitrary countable ordinal η, there is a strongly ergodic, distal system of rank η.

2. Some preliminaries

2.1. Structure theory

Let Γ be a discrete countable infinite group. A Γ-*dynamical system* is a quadruple $\mathbf{X} = (X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a standard probability space and $\gamma \mapsto T_\gamma$ is a homomorphism from Γ into the Polish group $\mathrm{Aut}(X, \mu)$ of invertible measure preserving transformation of $(X, \mathcal{X}, \mu)$. When there is no room for confusion, we write γx instead of $T_\gamma x$. When $\mathbf{X}$ and $\mathbf{Y} = (Y, \mathcal{Y}, \nu, S)$ are two dynamical systems, we say that $\mathbf{Y}$ is a *factor* of $\mathbf{X}$ (or that $\mathbf{X}$ is an *extension* of $\mathbf{Y}$) if there is a measurable map $\pi : X \to Y$ such that $\pi_*(\mu) = \nu$ and such that $\pi(T_\gamma x) = S_\gamma \pi(x)$ for every $\gamma \in \Gamma$ and μ almost every $x \in X$. The map π is called a *factor map* (or an *extension*).

The system $\mathbf{X}$ is *ergodic* if every Γ-invariant set $A \in \mathcal{X}$ (i.e., $T_\gamma A = A \pmod{\mu}$ for every $\gamma \in \Gamma$) is trivial (i.e., $\mu(A)(1 - \mu(A)) = 0$).

Let $\mathbf{Y}$ be a dynamical system and $(U, \mathcal{U}, \rho)$ a standard probability space. Let $\alpha : \Gamma \times Y \to \mathrm{Aut}(U, \rho)$ be a measurable cocycle; that is α satisfies the *cocycle equation*

$$\alpha(\gamma\gamma', y) = \alpha(\gamma, \gamma' y)\alpha(\gamma', y).$$

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We define the *skew-product system* $\mathbf{Y} \times_\alpha (U, \rho)$ to be the system $(Y \times U, \mathcal{Y} \otimes \mathcal{U}, \mu \times \rho, \Gamma)$, where $\gamma(y, u) = (\gamma y, \alpha(\gamma, y)u)$. (Check that this indeed defines an action of Γ on $X = Y \times U$.) When $\Gamma = \mathbb{Z}$ with the measure-preserving invertible map T as the generator of the $\mathbb{Z}$-system, a cocycle α is completely determined by the map $\alpha(y) = \alpha(T, y)$ and we have

$$\alpha(n, y) = \begin{cases} \alpha(T^{n-1}y) \cdots \alpha(Ty)\alpha(y) & \text{for } n \geq 1, \\ \text{id} & \text{for } n = 0, \\ \alpha(T^n y)^{-1} \cdots \alpha(T^{-1}y)^{-1} & \text{for } n < 0. \end{cases}$$

In the special case where U is a compact group and ρ is its normalized Haar measure, any measurable function $\alpha : Y \to U$ defines a skew product by the formula

$$T(y, u) = \big(Ty, \alpha(y)u\big), \quad y \in Y,\ u \in U.$$

We have the following basic theorem.

Theorem 2.1 (Rohlin). *Let* $\mathbf{X} \to \mathbf{Y}$ *be a factor map of dynamical systems with* $\mathbf{X}$ *ergodic, then* $\mathbf{X}$ *is isomorphic to a skew-product over* $\mathbf{Y}$. *Explicitly, there exist a standard probability space* $(U, \mathcal{U}, \rho)$ *and a measurable cocycle* $\alpha : \Gamma \times Y \to \mathrm{Aut}(U, \rho)$ *with* $\mathbf{X} \cong \mathbf{Y} \times_\alpha (U, \rho) = (Y \times U, \mathcal{Y} \otimes \mathcal{U}, \nu \times \rho, \Gamma)$*, where* $\gamma(y, u) = (\gamma y, \alpha(\gamma, y)u)$.

The topology on $\mathrm{Aut}(X, \mu)$ is induced by a complete metric

$$D(S, T) = \sum_{n \in \mathbb{N}} 2^{-n} \big(\mu(SA_n \triangle TA_n) + \mu(S^{-1}A_n \triangle T^{-1}A_n)\big),$$

with $\{A_n\}_{n \in \mathbb{N}}$ a dense sequence in the measure algebra $(\mathcal{X}, d_\mu)$, where $d_\mu(A, B) = \mu(A \triangle B)$. Equipped with this topology, $\mathrm{Aut}(X, \mu)$ is a Polish topological group and we say that the dynamical system $\mathbf{X}$ is *compact* if the image $\{T_\gamma : \gamma \in \Gamma\}$ is a precompact subgroup of $\mathrm{Aut}(X, \mu)$.

Example 2.2. Let $\Gamma = F_2$, the free group of rank 2. Let

$$X = \varprojlim \big\{\Gamma/N : N \lhd \Gamma,\ \text{with } [\Gamma : N] < \infty\big\}.$$

This is the *profinite completion* of Γ. It is a compact metrizable topological group and thus admits a unique normalized Haar measure μ. There is a canonical embedding $\phi : \Gamma \to X$ with a dense image and for $\gamma \in \Gamma$ we let $T_\gamma x = \phi(\gamma)x$, $x \in X$. With $\mathcal{X}$ the algebra of Borel subsets of X, $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ is an ergodic compact F_2 dynamical system.

It turns out that, in fact, every ergodic compact Γ system $\mathbf{X}$ has the form $X = K/H$, where K is a compact metrizable topological group, $H < K$ is a closed subgroup, μ is the induced Haar measure on X, and the action of Γ on X is via a homomorphism $\phi : \Gamma \to K$ with dense image so that $T_\gamma kH = \phi(\gamma)kH$, $k \in K$.

The notion of compactness can now be relativized as follows.

An extension $\pi : \mathbf{X} \to \mathbf{Y}$ is a *compact extension* if there is a compact metrizable group K, a closed subgroup $H < K$ and a cocycle $\alpha : \Gamma \times Y \to K$ such that

$$\mathbf{X} \cong \mathbf{Y} \times_\alpha (K/H, \rho) = (Y \times K/H, \mathcal{Y} \otimes \mathcal{K}, \nu \times \rho, \Gamma),$$

where ρ is the Haar measure on K/H and for each $\gamma \in \Gamma$, $\gamma(y, kH) = (\gamma y, \alpha(\gamma, y)kH)$.

This construction can be iterated and a dynamical system $\mathbf{X}$ is called *distal* if it is an iteration of countably many compact extensions, where in the possibly transfinite construction at a limit ordinal one takes an inverse limit. The so-called *distal tower* is unique if at each stage one takes the maximal compact extension (within $\mathbf{X}$). The height of this tower (a countable ordinal) is called the *rank* of the distal system $\mathbf{X}$, so the height of a compact action is 1.

An extension of dynamical systems $\pi : \mathbf{X} \to \mathbf{Y}$ is called a *weakly mixing extension* when the corresponding relative product $(X \underset{Y}{\times} X, \mu \underset{\nu}{\times} \mu, \Gamma)$ is ergodic. In particular, $\mathbf{X}$ is *weakly mixing* when the product system $\mathbf{X} \times \mathbf{X}$ is ergodic.

We now can state the following.

Theorem 2.3 (Furstenberg–Zimmer structure theorem). *Every ergodic system* $\mathbf{X}$ *has (uniquely) a largest distal factor* $\pi : \mathbf{X} \to \mathbf{Y}$ *and the extension* π *is a weakly mixing one.*

In [1] Beleznay and Foreman show that, for $\Gamma = \mathbb{Z}$, for every countable ordinal η there is an ergodic distal system of height η.

We refer, e.g., to [5] for more details on structure theory in ergodic theory.

2.2. Strong ergodicity, Kazhdan's property and expanders

Recall that a probability measure preserving action $(X, \mathcal{X}, \mu, G)$ is called *strongly ergodic* if there is no sequence of sets $A_n \in \mathcal{X}$ with $\mu(A_n) = 1/2, \forall n \in \mathbb{N}$ such that

$$\lim \mu(gA_n \triangle A_n) = 0, \quad \forall g \in G.$$

(Informally, we say that $\mathbf{X}$ admits no *almost invariant sets*.)

Theorems of Connes and Weiss [4]; Connes, Feldman, and Weiss [3]; and Schmidt [9, Theorems 2.4 and 2.5] assert that

(i) the group G is amenable iff every nontrivial ergodic G-system is not strongly ergodic [3, 9],

(ii) the group G has Kazhdan's property T iff every ergodic G-system is strongly ergodic [4].

The *Cheeger constant* of a finite k-regular graph $\mathcal{G}$ is defined as

$$h(\mathcal{G}) = \min \left\{ \frac{|\partial A|}{|A|} : A \subset V(\mathcal{G}),\ 0 < |A| \le \frac{|V(\mathcal{G})|}{2} \right\}.$$

The graph $\mathcal{G}$ is an *ε-expander* if $h(\mathcal{G}) > \varepsilon$.

A family $\{G_i\}$ of finite groups is called an *expander family* if for some k and $\varepsilon > 0$ there are generating sets Σ_i for G_i with $|\Sigma_i| \le k$, such that all the Cayley graphs $\mathcal{G}(G_i, \Sigma_i)$ are ε-expanders.

Example 2.4. Let S be a set of elements in $\mathrm{SL}_2(\mathbb{Z})$. If $\langle S\rangle$, the group generated by S, is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, Selberg's theorem [10] implies (see, e.g., [8, Theorem 4.3.2 and Example 4.3.3 (D)]) that $\mathcal{G}(\mathrm{SL}_2(\mathbb{Z}_p), S_p)$, the Cayley graphs of $\mathrm{SL}_2(\mathbb{Z}_p) = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ with respect to S_p, the natural projection of S modulo p, form a family of expanders as $p \to \infty$. For example we can take

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Remark 2.5. Suppose that G is a finite group generated by $S \subset G$, such that

$$h\big(\mathcal{G}(G, S)\big) > \varepsilon.$$

Then for $A \subset G$ with $|A| \approx \frac{|G|}{2}$ we have $\partial A \geq \varepsilon|A|$ and $|SA \,\triangle\, A| \geq \varepsilon|A|$. We can interpret this as the absence of almost invariant subsets.

In our construction we will have a finite set $S \subset \mathrm{SL}_2(\mathbb{Z})$ that generates a Zariski dense subgroup Λ and we will consider the diagonal action of Λ on products of the form

$$K_P = \prod \big\{ \mathrm{SL}_2(\mathbb{Z}_p) : p \in P \big\}$$

taken over an infinite set of primes P. In order to ensure that this action is strongly ergodic we need to know that finite products

$$K_F = \prod \big\{ \big(\mathrm{SL}_2(\mathbb{Z}_p), S_p \big) : p \in F \big\},$$

where $F \subset P$ is finite and S_p is the image of S in $\mathrm{SL}_2(\mathbb{Z}_p)$, form an expander family. In our situation S_p in the product will generate K_F and since such a finite product is isomorphic to $\mathrm{SL}_2(\mathbb{Z}_q)$, where $q = \prod\{p : p \in F\}$, we will need to know that the family $\{(\mathrm{SL}_2(\mathbb{Z}_q), S_q)\}$ is a family of expanders. This latter fact is established in the following theorem of Bourgain, Gamburd, and Sarnak [2, Theorem 1.2].

Theorem 2.6. *Let Λ be a Zariski dense subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let S be a finite symmetric set of generators for Λ. Then for q square-free the family of Cayley graphs $\mathcal{G}(\Lambda/\Lambda(q), S)$ is an expander family.*

In a sharp contrast to the result of Beleznay and Foreman [1] mentioned above, Chifan and Peterson (unpublished) and, independently, Ibarlucía and Tsankov [7] show that for Γ with Kazhdan's property T, every distal system is already compact. (In fact, they show that every distal extension of an ergodic system is a compact extension.)

This raises the question whether a group G which is neither amenable nor a Kazhdan group can admit a strongly ergodic distal action which is not compact. In the next section we will answer this question positively for the free group on three generators $G = F_3$.

3. An example of a strongly ergodic, distal but not compact F_3 dynamical system

Our goal is to construct an F_3 action which is strongly ergodic, distal but not compact.

We want to find compact metrizable groups K and L with the following properties:

- K contains three elements a, b, and c such that the subgroup $\Lambda = \langle a, b, c\rangle$ generated by them is free and dense in K, $\overline{\Lambda} = K$;
- the subgroup $\Lambda_0 = \langle a, b\rangle < \Lambda$ is also dense in K, $K = \overline{\Lambda}_0$;
- L contains two elements f, g such that the subgroup $\Sigma = \langle f, g\rangle$ generated by them is free and dense in L, $L = \overline{\Sigma}$.

Moreover, denoting by F_2, F_3 the free groups on two and three generators, respectively, all of the following actions (under left multiplication and with respect to the corresponding Haar measures) are strongly ergodic:

(1) the action of F_3 via Λ on K,

(2) the action of F_2 via Λ_0 on K,

(3) the action of F_2 via Σ on L,

(4) the diagonal action of F_2 via $\langle (a, f), (b, g)\rangle$ on $K \times L$.

(Of course conditions (1)–(3) follow from condition (4).)

Here is a construction of such groups.

Let $x = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ be two elementary matrices generating a free group, which we call H, in $\mathrm{SL}_2(\mathbb{Z})$ (it is actually of finite index). Let $a = x^2 = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ and $b = y^2 = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$, and let c be another element in H which together with a and b generate a free group Λ on 3 generators (such an element clearly exists as a and b generate a subgroup of infinite index in H). Now set

$$K = \prod \{\Lambda_p : p \text{ prime, and } p \equiv 1 \pmod 4\},$$

$$L = \prod \{\Lambda_p : p \text{ prime, and } p \equiv 3 \pmod 4\},$$

where Λ_p is the image of Λ in $\mathrm{SL}_2(\mathbb{Z}_p)$.

Note that, by the Chinese remainder theorem, as $a = x^2 = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ and $b = y^2 = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$, we have that, for any finite set of distinct primes $p_1, p_2, \dots, p_t$ ($p_i \neq 2$), denoting $q = p_1 p_2 \cdots p_t$, the image of Λ_0 in $\mathrm{SL}_2(\mathbb{Z}_q)$ is all of $\mathrm{SL}_2(\mathbb{Z}_q)$.

The strong ergodicity of F_2 via Λ_0 on K follows from Theorem 2.6. Indeed if $\mathbf{1}_A$ is the indicator function of an almost invariant set A with measure $\approx 1/2$ in K, then its projection onto a sufficiently large finite product will give rise to an almost invariant nontrivial function, contrary to the fact that the family is a family of expanders.

Of course the arguments for the actions on L and $K \times L$ are analogous.

Next let $K_1 = \overline{\langle c\rangle} < K$. By [11] (see also [6]) there is a cocycle $\phi_0 : K_1 \to L$ such that the corresponding $\mathbb{Z}$-action on $K_1 \times L$ given by

$$T_c(x, y) = \big(cx, \phi_0(x)y\big), \quad x \in K_1,\ y \in L,$$

is ergodic. Note that, as L is non-commutative, this distal $\mathbb{Z}$-action is necessarily not compact. To see this recall that T_c is ergodic and if it would be compact, as is well known, its centralizer would be commutative. In fact, ergodic compact $\mathbb{Z}$-actions are rotations of compact monothetic groups and the centralizer is the compact group itself.

Define $\phi : K \to L$ by the formula

$$\phi(x) = \phi_0(k_t^{-1}x) \quad \text{for } x \text{ in the coset } k_t K_1,$$

where $t \mapsto k_t$, $K/K_1 \to K$ is a Borel section for the map $K \to K/K_1$.

On $X \times Y := K \times L$ define an F_3 action $\{T_t\}_{t \in F_3} : K \times L \to K \times L$ by

$$\begin{aligned} T_a(x, y) &= (ax, fy), \\ T_b(x, y) &= (bx, gy), \\ T_c(x, y) &= \big(cx, \phi(x)y\big). \end{aligned}$$

With this data at hand we can now prove our main result.

Theorem 3.1. *The T action of F_3 on $K \times L$ is distal, not compact, and strongly ergodic.*

Proof. Clearly the T action on $K \times L$ is distal (of order 2). It cannot be compact because the restriction to the $\mathbb{Z}$-action $T_c : K_1 \times L \to K_1 \times L$ is not compact. Finally, by construction, the F_2-action on $K \times L$ is strongly ergodic and, a fortiori, so is the T-action. Our proof is complete. ∎

4. Distal strongly ergodic F_3-systems of arbitrary countable rank

With only some minor modifications the proof of Theorem 3.1 can be applied to prove the following stronger result.

Theorem 4.1. *Let η be an arbitrary countable ordinal η. Then there exists an action of the free group on three generators F_3 on a compact metric space X, admitting an invariant probability measure μ, such that the resulting dynamical system (X, μ, F_3) is strongly ergodic and distal of rank η.*

Proof. We keep the notations introduced in the previous section with the following modifications. Let $x = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and let H be the finite index subgroup $\langle x, y \rangle$. Let $a = x^2 = \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right)$ and $b = y^2 = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$, and let c be another element in H which together with a and b generate a free group Λ on 3 generators. Recall that for sufficiently large p, the image of $S_0 = \{a, b\}$ (and a fortiori that of the set $S = \{a, b, c\}$) in $\mathrm{SL}_2(\mathbb{Z}_p) \cong \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(p\mathbb{Z})$ generates $\mathrm{SL}_2(\mathbb{Z}_p)$. Let P be the set of primes for which this is true.

Let $P = \bigcup_{j=0}^{\infty} P_j$ be some partition of P, with each cell P_j being infinite. Choose an arbitrary bijection

$$\Phi : \{1, 2, \ldots\} \leftrightarrow \big\{\alpha < \eta : \alpha \text{ is a successor ordinal}\big\}$$

and set

$$\begin{aligned} K &= \prod \{\Lambda_p : p \in P_0\}, \\ K_{\Phi(j)} &= \prod \{\Lambda_p : p \in P_j\}, \end{aligned}$$

where Λ_p is the image of Λ in $\mathrm{SL}_2(\mathbb{Z}_p)$.

We now consider the action of S, induced by left multiplication, on K and on each K_α. As above all of these actions are strongly ergodic, and so is the action on the product space

$$X = K \times L = K \times \prod_{\alpha} K_\alpha. \tag{4.1}$$

We now recall the following result from [6] (adapted to our needs here).

Theorem 4.2. *Given an arbitrary countable ordinal η and a transfinite sequence*

$$\{G_\alpha : \alpha = 0 \text{ and } \alpha < \eta,\ \alpha \text{ a successor ordinal}\},$$

where for each α, G_α is an infinite compact second countable topological group, with the only requirement that G_0 be an infinite monothetic compact group (G_0, c), there exists an ergodic distal system $\mathbf{X} = (X, \mathcal{X}, \mu, A)$ of rank η such that, in its canonical distal tower, for each successor α the extension $\mathbf{X}_\alpha \to \mathbf{X}_{\alpha-1}$ is a $G_{\alpha-1}$-extension, and such that the induced action of A on G_0 is via multiplication by c.

We let $K_0 = \overline{\langle c \rangle} < K$, and we consider (K_0, c) as a (compact) $\mathbb{Z}$-dynamical system. Applying Theorem 4.2, with $G_\alpha = K_\alpha$ for all $0 \le \alpha < \eta$, we can now realize an action of the group $\langle c \rangle$ on the space $K_0 \times L$ by $T_c(x) = Ax$ (where for a limit ordinal $\xi \le \eta$ the corresponding system is determined as the inverse limit of the preceding systems directed by $\{\alpha : \alpha < \xi\}$).

Next we extend the T_c action to the space $X = K \times L$ in (4.1) by defining for $x = (k, l) \in K_0 \times L$,

$$T_c(k, l) = A(k_t^{-1} k, l), \quad \text{for } k \text{ in the coset } k_t K_0,$$

where $t \mapsto k_t$, $K/K_0 \to K_0$ is a Borel section for the map $K_0 \to K/K_0$.

Finally, on $K \times L$ define an F_3 action $\{T_t\}_{t \in F_3} : K \times L \to K \times L$ by

$$T_a(k, l) = (ak, al),$$
$$T_b(k, l) = (bk, gl),$$
$$T_c(k, l) \text{ as above.}$$

As in the previous section this completes our proof. ■

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Eli Glasner
Department of Mathematics, Tel Aviv University, Tel Aviv, Israel;
glasner@math.tau.ac.il

Benjamin Weiss
Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel;
weiss@math.huji.ac.il