Automaton groups and complete square complexes

Ievgen Bondarenko and Bohdan Kivva

Abstract. The first example of a non-residually finite group in the classes of finitely presented small-cancelation groups, automatic groups, and CAT(0) groups was constructed by Wise as the fundamental group of a complete square complex (CSC for short) with twelve squares. At the same time, Janzen and Wise proved that CSCs with at most three, five or seven squares have residually finite fundamental group. The smallest open cases were CSCs with four squares and directed complete $\mathcal{VH}$ complexes with six squares. We prove that the CSC with four squares studied by Janzen and Wise has a non-residually finite fundamental group. For the class of complete directed $\mathcal{VH}$ complexes, we prove that there are exactly two complexes with six squares having a non-residually finite fundamental group. In particular, this positively answers to a question of Wise. Our approach relies on the connection between square complexes and automata discovered by Glasner and Mozes, where complete $\mathcal{VH}$ complexes with one vertex correspond to bireversible automata. We prove that the square complex associated to a bireversible automaton with two states or over the binary alphabet generating an infinite automaton group has a non-residually finite fundamental group. We describe automaton groups associated to CSCs with four squares and get two simple automaton representations of the free group F_2.

1. Introduction

One of the outstanding open problems in geometric group theory is whether word-hyperbolic groups are residually finite, which is important in understanding the topology of hyperbolic spaces. Before the introduction of word-hyperbolic groups, Schupp and later Pride, Gersten, and others asked about the residual properties of groups in a neighborhood of word-hyperbolic groups: for finitely presented small-cancelation groups, automatic groups, CAT(0) groups. The first example of a non-residually finite group belonging to each of these classes was constructed by Wise in his dissertation [30]. Shortly after that Burger and Mozes [4, 5] constructed finitely presented torsion-free simple groups. All these examples are fundamental groups of complete square complexes (CSCs for short), that is non-positively curved square complexes covered by the direct product of two trees.

The construction of Wise is based on the special square complex Δ_W, which is called the main example in [30, 32]. This complex is obtained by gluing the six unit squares shown in Figure 1. Wise studied tiling properties of these squares and proved that they admit a special non-periodic tiling of the plane with periodically labeled axes (an anti-torus). This observation was the key ingredient in several interesting examples (see [13,14,
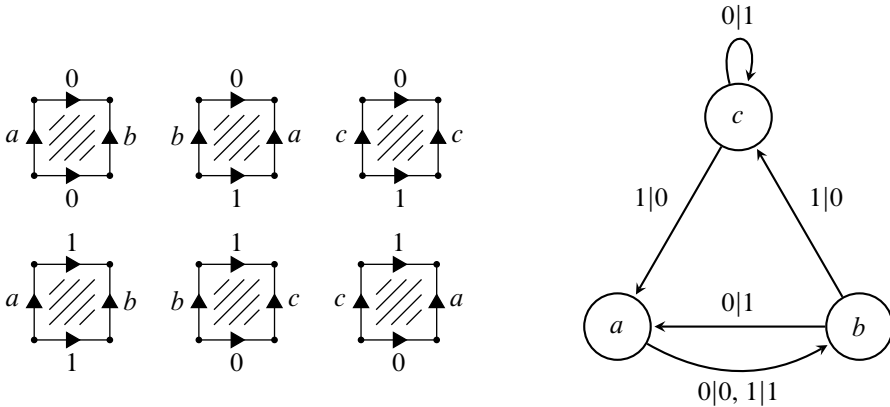
Figure 1. The Wise complex Δ_W and the Aleshin automaton.

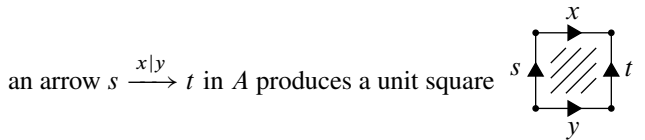an arrow $s \xrightarrow{x|y} t$ in A produces a unit square

Figure 2. A labeled unit square associated to an arrow in an automaton.

31]). In particular, Wise proved that the subgroup of $\pi_1(\Delta_W)$ generated by the loops a, b, c is not separable, and therefore the amalgamated free product $\pi_1(\Delta_W) *_{\langle a,b,c \rangle} \pi_1(\Delta_W)$ is a non-residually finite CAT(0) group. The question was raised [32, Problem 10.19] whether $\pi_1(\Delta_W)$ itself is non-residually finite. Below we give a positive answer to this question as a corollary of a more general statement.

Our approach and initial interest in square complexes lie through the theory of automaton groups, which is another fascinating topic of modern group theory. The theory of automaton groups deals with a special class of automata-transducers (see an example in Figure 1). The relation between such automata and square complexes was discovered by Glasner and Mozes in [9]. Let A be an automaton with the set of states S and an input-output alphabet X. The square complex Δ_A associated to A is obtained by gluing the squares given by the arrows in A as shown in Figure 2. The complex Δ_A belongs to the class of directed $\mathcal{VH}$ square complexes with one vertex. The fundamental group of Δ_A has presentation

$$\pi_1(\Delta_A) = \langle S, X \mid sx = yt \text{ for each arrow } s \xrightarrow{x|y} t \text{ in } A \rangle,$$

which immediately suggests that there should be a connection between geometric properties of Δ_A and the combinatorial structure of A. Indeed, Glasner and Mozes noticed that Δ_A is a CSC exactly when A belongs to the class of bireversible automata introduced in [19] in relation to commensurators of free groups. The bireversible automaton corresponding to the Wise complex Δ_W is shown in Figure 1.

Another algebraic object related to an automaton is the associated automaton group. An automaton group G_A is defined by the action of the automaton A on input words. Roughly speaking, a group is an automaton group if one can put an automaton structure on the group consistent with its group structure. The study of automaton groups was initially motivated by several examples, mainly the Grigorchuk group, that enjoy many fascinating properties: torsion, intermediate growth, amenable but not elementary amenable, non-uniformly exponential growth, finite width, just-infiniteness, etc. Further investigations showed that automaton groups naturally arise in diverse areas of mathematics (see [10, 12, 21]). In this paper we show that automaton groups are useful in the study of square complexes as well.

Theorem 4.5. *Let $A = (X, S, \lambda)$ be a bireversible automaton.*

(1) *If G_A is finite, then $\pi_1(\Delta_A)$ is virtually a direct product of two free groups and therefore residually finite.*

(2) *If G_A is infinite, then $\pi_1(\Delta_A)$ is not $\langle S \rangle$-separable and not $\langle X \rangle$-separable.*

Note that G_A is infinite if and only if $\pi_1(\Delta_A)$ is an irreducible lattice in the product of two trees (see Corollary 3.7).

Theorem 4.10. *Let $A = (X, S, \lambda)$ be a bireversible automaton with two states or over the binary alphabet. If G_A is infinite, then $\pi_1(\Delta_A)$ is non-residually finite.*

The key ingredient in the proof of Theorem 4.10 is a nontrivial endomorphism of $\pi_1(\Delta_A)$ that is trivial on all states or all letters. The existence of such an endomorphism follows from the fact that the automaton group G_A "contains" a subautomaton nontrivially isomorphic to the original automaton A. For example, for the Wise complex Δ_W the map

$$\phi : \begin{array}{l} a \mapsto a, \\ b \mapsto b, \\ c \mapsto c, \end{array} \quad \begin{array}{l} 0 \mapsto 01^{-1}01^{-1}0 \\ 1 \mapsto 10^{-1}10^{-1}1 \end{array}$$

extends to an endomorphism of $\pi_1(\Delta_W)$ with $\text{Fix}(\phi) = \langle a, b, c \rangle$. Since $\pi_1(\Delta_W)$ is not $\langle a, b, c \rangle$-separable, it is non-residually finite; the element $(0^{-1}1)^4$ belongs to the intersection of all finite index subgroups of $\pi_1(\Delta_W)$.

Remark 1.1. In [6], the non-residual finiteness of $\pi_1(\Delta_W)$ is proved independently via a more powerful approach.

Interestingly, the automaton associated to the Wise complex is well known in the theory of automaton groups. This automaton was constructed by Aleshin in [1] in the first attempt to generate a free non-abelian group by (initial) automata. The proof was considered incomplete, and the problem remained open for many years. The first automaton realization of free non-abelian groups was made by Glasner and Mozes in [9] based on the connection with square complexes and Burger-Mozes groups. The smallest automaton generating a free group constructed by Glasner and Mozes has 6 states over a 14-letter
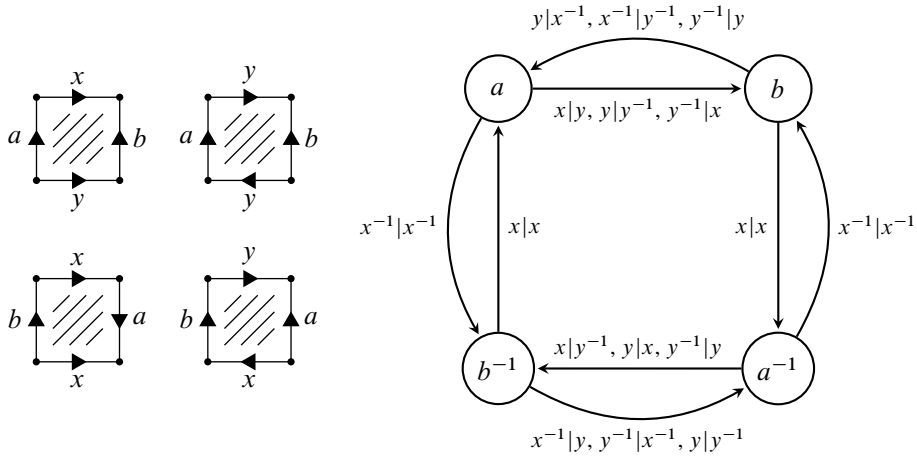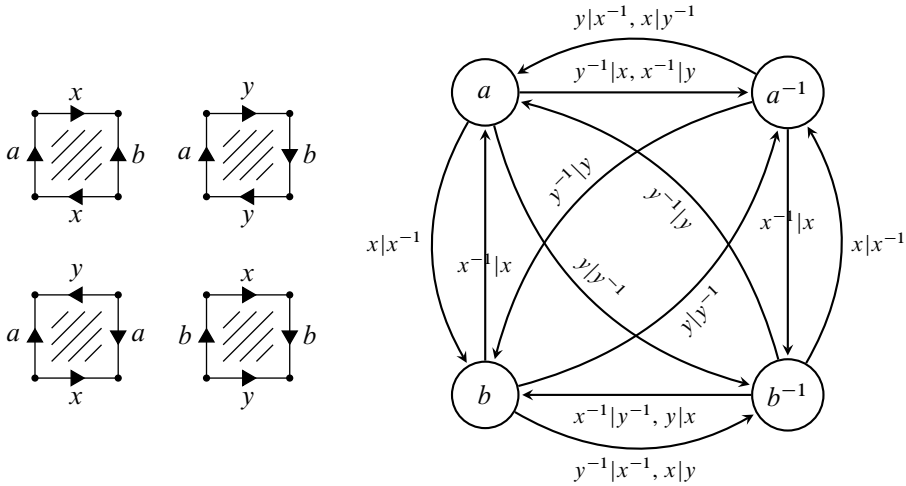
Figure 3. The smallest CSC Δ_D with non-residually finite fundamental group and the associated automaton with $G_A \cong C_3 * C_3$ and $G_{\partial A} \cong F_2$.

alphabet. Finally, Vorobets and Vorobets in [28] proved that the Aleshin automaton generates the free group of rank three. Surprisingly, among the hundreds of automata with three states over the binary alphabet only the Aleshin automaton generates a free non-abelian group (see [3]). Automaton realizations of free groups and free products of cyclic groups of order two are constructed in [25, 26, 29].

In [15] Janzen and Wise proved that CSCs with at most three, five or seven squares have residually finite fundamental group. Burger and Mozes in [5] proved that for every $n \geq 109$ and $m \geq 150$ there exists a complete $\mathcal{VH}$ complex with n and m elements in the vertical-horizontal decomposition having a non-residually finite (even virtually simple) fundamental group. The smallest open cases were CSCs with four squares and directed complete $\mathcal{VH}$ complexes with six squares. There are no bireversible automata with less than three states over the binary alphabet generating an infinite group, and only two bireversible automata with three states: the Aleshin automaton and the Bellaterra automaton (see [3]). By Theorems 4.5 and 4.10 these two automata produce the smallest possible complete directed $\mathcal{VH}$ square complexes with non-residually finite fundamental groups.

The developed technique also works for non-directed $\mathcal{VH}$ complexes. Using computations with GAP, based on Rattaggi's computations from [23], we have checked that there are only two complete $\mathcal{VH}$ complexes Δ_D and Δ_S with four squares and one vertex that could have a non-residually finite fundamental group (for all other complexes the associated automata generate finite groups). These two complexes are shown in Figures 3 and 4 together with the associated automata. Stix and Vdovina in [27] proved that $\pi_1(\Delta_S)$ is an irreducible lattice in $PGL_2(\mathbb{F}_3((t))) \times PGL_2(\mathbb{F}_3((t)))$ and, therefore, it is residually finite. The complex Δ_D was studied by Janzen and Wise in [15], who proved that it admits an anti-torus and, therefore, $\pi_1(\Delta_D)$ is an irreducible lattice in the direct product of two trees. The question was raised whether $\pi_1(\Delta_D)$ is residually finite and in the theorem

Figure 4. The smallest irreducible CSC Δ_S with residually finite fundamental group and the associated automaton with $G_A \cong G_{\partial A} \cong F_2$.

below we answer this question negatively. Interestingly, the corresponding automata provide new simple automaton representations of the free group F_2 and the first automaton representation of $C_3 * C_3$.

Theorem 5.1. *Let Δ_D be the square complex given by the four squares in Figure 3 and let A be the associated bireversible automaton. Then*

    (1) $\pi_1(\Delta_D)$ *is non-residually finite,*

    (2) $G_A \cong C_3 * C_3$ *and* $G_{\partial A} \cong F_2$.

Theorem 5.3. *Let Δ_S be the square complex given by the four squares in Figure 4 and let A be the associated bireversible automaton. Then*

    (1) $G_A \cong F_2$ *and* $G_{\partial A} \cong F_2$,

    (2) $\langle a, x \rangle$, $\langle a, y \rangle$, $\langle b, x \rangle$ *and* $\langle b, y \rangle$ *form anti-tori in* $\pi_1(\Delta_S)$.

Since the universal cover of Δ_D is the direct product of two regular trees of degree four, $\pi_1(\Delta_D)$ is a non-residually finite group isometric to $F_2 \times F_2$.

The paper is organized as follows. In Section 2 we recall basic facts about automaton groups (see [10, 12, 21] for more details). In Section 3 we describe the connection between automaton groups and square complexes. The residual properties of $\pi_1(\Delta_A)$ are studied in Section 4. In Section 5 we prove Theorems 5.1 and 5.3. In the last section, following the approach of Rattaggi [23], we construct finitely presented torsion-free simple groups which decompose into amalgamated free products $F_5 *_{F_{33}} F_5$, $F_7 *_{F_{37}} F_7$, and $F_7 *_{F_{49}} F_7$.

2. Automata and automaton groups

2.1. Automata

Let X be a nonempty set and X^* the free monoid over X. The elements of X^* are finite words $x_1 x_2 \ldots x_n$, where $x_i \in X$ and $n \in \mathbb{N}$, together with the empty word. The length of a word $v = x_1 x_2 \ldots x_n$ is $|v| = n$. The set X^n consists of all words of length n. By X^{-1} we denote the set of formal elements x^{-1} for $x \in X$. Let $X^{\pm *}$ be the set of all (non-reduced) words over $X \cup X^{-1}$. In contrast to general elements of $X^{\pm *}$, the nonempty elements of X^* are called positive words over X. The free group generated by X is denoted by F_X.

We consider complete deterministic automata-transducers (Mealy automata) with the same input and output alphabets. Hence, in this article the following definitions hold.

Definition 2.1. An *automaton* is a triple $A = (S, X, \lambda)$, where S and X are nonempty sets and $\lambda : S \times X \to X \times S$ is an arbitrary map. The set X is *the alphabet* of input and output letters, and the set S is *the set of states of A*.

Finite automata have finitely many states and finite alphabet. We identify an automaton $A = (S, X, \lambda)$ with a directed labeled graph on the vertex set S with the following edges:

$$s \xrightarrow{x|y} t \quad \text{whenever } \lambda(s, x) = (y, t).$$

Note that for every $s \in S$ and every $x \in X$ there exists a unique arrow passing from s and labeled by $x|y$ for some $y \in X$. The existence of such arrows indicates the completeness of A, while the uniqueness is that A is deterministic. In terms of automata theory, an arrow $s \xrightarrow{x|y} t$ means that if the automaton is initialized at state s and reads the letter x, then it outputs the letter y and changes its active state to t.

Two automata $A = (S_A, X_A, \lambda_A)$ and $B = (S_B, X_B, \lambda_B)$ are *isomorphic* if there exist bijections $\phi : S_A \to S_B$ and $\psi : X_A \to X_B$ such that $\lambda_A(s, x) = (y, t)$ if and only if $\lambda_B(\phi(s), \psi(x)) = (\psi(y), \phi(t))$. If A and B share the same alphabet $X = X_A = X_B$, then A and B are called *X-isomorphic* if there exists an isomorphism that is trivial on X.

There are two standard operations over automata: taking dual automaton and taking inverse automaton for invertible automata. The *dual of an automaton* $A = (S, X, \lambda)$ is the automaton $\partial(A) = (X^{-1}, S^{-1}, \delta)$, where

$$\delta(x^{-1}, s^{-1}) = (t^{-1}, y^{-1}) \quad \text{if } \lambda(s, x) = (y, t),$$

or in the graphical representation

$$x^{-1} \xrightarrow{s^{-1}|t^{-1}} y^{-1} \text{ in } \partial(A) \quad \text{if } s \xrightarrow{x|y} t \text{ in } A.$$

The reason to put an inverse sign will be clear in the next section. Since A is complete and deterministic, the dual $\partial(A)$ is complete and deterministic too, so it is always a well-defined automaton. Note that $\partial(\partial(A)) = A$. We say that A is *self-dual* if $\partial(A)$ is isomorphic to A.

The *inverse of an automaton* $A = (S, X, \lambda)$ is the tuple $i(A) = (S^{-1}, X, \delta)$, where

$$\delta(s^{-1}, y) = (x, t^{-1}) \quad \text{if } \lambda(s, x) = (y, t),$$

or in the graphical representation

$$s^{-1} \xrightarrow{y|x} t^{-1} \text{ in } i(A) \quad \text{if } s \xrightarrow{x|y} t \text{ in } A.$$

The $i(A)$ may be not deterministic/complete. The automaton A is called *invertible* if $i(A)$ is well defined, which is equivalent to the following property:

for every $s \in S$ and $y \in X$ there exists a unique arrow $s \xrightarrow{x|y} t$ in A

for some $x \in X$ and $t \in S$. Note that $i(i(A)) = A$.

Any automaton $A = (S, X, \lambda)$ can be naturally extended to an automaton $A^* = (S, X^*, \lambda^*)$ by consequently applying the rule

$$s \xrightarrow{x_1|y_1} p \quad \text{and} \quad p \xrightarrow{x_2|y_2} t \quad \text{produce } s \xrightarrow{x_1 x_2|y_1 y_2} t,$$

which in automata theory corresponds to the consecutive processing of input strings of letters. Also, we can extend A to an automaton ${}^*A = (S^*, X, \lambda^*)$ by consequently applying the rule

$$s_2 \xrightarrow{y|z} t_2 \quad \text{and} \quad s_1 \xrightarrow{x|y} t_1 \quad \text{produce } s_2 s_1 \xrightarrow{x|z} t_2 t_1,$$

which in automata theory corresponds to the (left) composition of automata. By applying consequently these rules we get a well-defined automaton ${}^*A^* = (S^*, X^*, \lambda^*)$.

If A is invertible, then we can consider the automaton ${}^{\pm*}A^* = {}^*(A \cup i(A))^*$ with the set of states $S^{\pm*}$ over the alphabet X^*. Note that the set of reduced words over S spans a subautomaton, which produces an automaton structure on the free group F_S.

2.2. Automaton groups

Every automaton $A = (S, X, \lambda)$ produces two semigroup actions—a left action $S^* \curvearrowright X^*$ and a right action $S^* \curvearrowleft X^*$ defined by the rule: for $g, h \in S^*$ and $u, v \in X^*$,

$$g(u) = v \quad \text{and} \quad (g)u = h \quad \text{if } g \xrightarrow{u|v} h \text{ in } {}^*A^*,$$

which is well defined, because ${}^*A^*$ is complete and deterministic. In other words, for each state $s \in S$, we put $s(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n$ for a path in A of the form

$$s \xrightarrow{x_1|y_1} s_2 \xrightarrow{x_2|y_2} s_3 \xrightarrow{x_3|y_3} \cdots \xrightarrow{x_n|y_n} s_{n+1},$$

and the transformation of X^* defined by a word over S is exactly the left composition of the respective transformations defined by the states. Similarly, the transformation of S^* defined by a word over X is the right composition of the transformations defined by the letters.

The transformations of X^* defined by the states of A are invertible if and only if A is invertible. The transformation defined by a state s^{-1} of $i(A)$ is inverse to the transformation defined by the state s of A. The automaton $^{\pm*}A^*$ defines an action of $S^\pm$* on X^*, and the transformation of X^* defined by a word $w \in S^{\pm*}$ is exactly the composition of transformations defined by the states of A and their inverses. Hence, every invertible automaton produces a natural action of the free group F_S on X^*.

Definition 2.2. Let $A = (S, X, \lambda)$ be a finite invertible automaton. The quotient of F_S by the kernel of its action on X^* is called the *automaton group G_A*.

From another point of view, the automaton group G_A is the group generated by the transformations of X^* defined by the states of A under composition.

The dual automaton $\partial(A) = (X^{-1}, S^{-1}, \delta)$ produces a left action $(X^{-1})^* \curvearrowright (S^{-1})^*$ and a right action $(X^{-1})^* \curvearrowleft (S^{-1})^*$. By taking formal inverses, these two actions are translated to the right action $S^* \curvearrowleft X^*$ and the left action $S^* \curvearrowright X^*$, respectively, associated to A:

$$ g \xrightarrow{u|v} h \text{ in } {}^*A^* \quad \text{if and only if } u^{-1} \xrightarrow{g^{-1}|h^{-1}} v^{-1} \text{ in } {}^*\partial(A)^*. $$

The dual automaton $\partial(A)$ is invertible if and only if the right action $S^* \curvearrowleft X^*$ is invertible. In this case, there is a natural automaton structure on F_X over the alphabet S and an action of F_X on the space S^*. The automaton group $G_{\partial(A)}$ is the quotient of F_X by the kernel of its action on S^*. The group $G_{\partial(A)}$ is generated by the transformations of S^* defined by the letters in X under composition.

Every automaton group G admits a natural sequence of finite index subgroups

$$ G = St_0(G) \geq St_1(G) \geq St_2(G) \geq \cdots, $$

where $St_n(G) = \{g \in G : g(v) = v \text{ for all } v \in X^n\}$ is the stabilizer of words of length n. Since their intersection is trivial, every automaton group is residually finite. Another property of all automaton groups is that they have solvable word problem.

3. Automata and square complexes

In this section we describe the connection between automata and square complexes discovered by Glasner and Mozes in [9]. We give a somewhat different presentation with emphasis on combinatorial and algebraic properties.

3.1. Square complexes

A *square complex* is a combinatorial 2-complex whose 2-cells are squares; i.e., they are attached by combinatorial paths of length four. We are interested in a special class of square complexes—complete $\mathcal{V}\mathcal{H}$ square complexes, introduced in [30].

A square complex is called a *$\mathcal{V}\mathcal{H}$ complex* if its 1-cells can be partitioned into two classes V and H such that the attaching map of each 2-cell alternates between the edges

of V and H. If we are given an orientation on the edges of V and H and the attaching map of each 2-cell preserves this orientation, then the $\mathcal{VH}$ complex is called *directed*. The Gromov link condition implies that a $\mathcal{VH}$ complex is non-positively curved if there are no double edges in the links of vertices.

A square complex is called a CSC if the link of each vertex is a complete bipartite graph. A natural example of a complete $\mathcal{VH}$ complex is a direct product of two graphs. Moreover, a square complex is complete if and only if its universal cover is a direct product of two trees (see [30, Theorem 1.10]). Therefore, the fundamental group of a compact CSC acts freely and cocompactly on a CAT(0) space.

3.2. Square complexes associated to automata

Let $A = (S, X, \lambda)$ be a finite automaton. We associate to A a set of Wang tiles, unit squares with labeled edges, as follows:

$$W_A = \left\{ \begin{array}{c} x \\ s \; \boxed{\diagdown} \; t \\ y \end{array} \quad \text{for each arrow } s \xrightarrow{x|y} t \text{ in } A \right\}.$$

The set W_A contains $\#S \cdot \#X$ squares, whose horizontal sides are labeled by letters from X, while the vertical sides are labeled by states from S. In addition, all horizontal sides are oriented from left to right, and all vertical edges are oriented from bottom to top.

The *square complex Δ_A associated to A* has one vertex, a directed loop for every $s \in S$ and every $x \in X$, and a 2-cell for every square in W_A. The complex Δ_A is a directed $\mathcal{VH}$ square complex. The fundamental group of Δ_A has finite presentation

$$\pi_1(\Delta_A) = \langle S, X \mid sx = yt \text{ for each arrow } s \xrightarrow{x|y} t \text{ in } A \rangle.$$

It is direct to see that an arrow $g \xrightarrow{v|u} h$ in $^*A^*$ implies that W_A admits a tiling of a finite rectangle such that its left side is labeled by g, the top side by v, the bottom side by u, and the right side by h, which produces the relation $gv = uh$ in $\pi_1(\Delta_A)$.

The next statement follows from the fact that we consider complete automata (see similar statements in [23, Section 4.1] and [7]).

Proposition 3.1. *For any finite automaton A the set of Wang tiles W_A admits a periodic tiling of the plane and the group $\pi_1(\Delta_A)$ contains $\mathbb{Z}^2$ as a subgroup. Therefore, $\pi_1(\Delta_A)$ is never Gromov-hyperbolic.*

Proof. Let us construct the directed graph Γ on the vertex set $S \times X$, where we put the arrow $(s, x) \to (t, y)$ if $s \xrightarrow{x|y} t$ in A. Since A is finite, the graph Γ contains a directed cycle

$$(s_1, x_1) \to (s_2, x_2) \to \cdots \to (s_n, x_n) \to (s_1, x_1).$$

Therefore, we have the transitions in the automaton $^*A^*$:

$$s_n \ldots s_2 s_1 \xrightarrow{x_1|x_1} s_1 s_n \ldots s_2 \xrightarrow{x_2|x_2} s_2 s_1 \ldots s_3 \to \cdots \xrightarrow{x_n|x_n} s_n \ldots s_2 s_1.$$

Put $w = s_n \ldots s_2 s_1$ and $u = x_1 x_2 \ldots x_n$. Then $^*A^*$ contains a loop at w labeled by $u|u$, which corresponds to the relation $wu = uw$ in $\pi_1(\Delta_A)$. It follows that W_A admits a tiling of a rectangle with left/right labels w and top/bottom labels u, which extends to a periodic tiling of the plane.

Now we show that the subgroup $\langle w, u \rangle$ of $\pi_1(\Delta_A)$ is isomorphic to $\mathbb{Z}^2$. Since Δ_A is a directed $\mathcal{VH}$ complex, there exists a natural surjective homomorphism

$$\phi : \pi_1(\Delta_A) \to \mathbb{Z} \times \mathbb{Z}$$

which extends $s \mapsto (1, 0)$ for all $s \in S$ and $x \mapsto (0, 1)$ for all $x \in X$. Since we already know that $\langle w, u \rangle$ is abelian and $\phi(w) = (n, 0)$, $\phi(u) = (0, n)$, the statement follows. ∎

It seems to be an interesting problem to develop a method which, given a finite automaton A, describes all periodic tilings for the tileset W_A. These periodic tilings correspond to loops in the automaton $^*A^*$ labeled by $v|v$ for some $v \in X^*$. However, we do not see a nice "finite" description of all possible periodic tilings.

For general square complexes the following problem remains open.

Problem 3.2. *Is it true that if the fundamental group of a square complex is* CAT(0) *but not Gromov-hyperbolic, then it contains* $\mathbb{Z}^2$ *as a subgroup?*

This problem is a special case of a famous question by Gromov on whether each CAT(0) group which is not Gromov-hyperbolic contains $\mathbb{Z}^2$ as a subgroup. One of the approaches to get a negative answer to these problems would be to construct a set of Wang tiles which admits only non-periodic tilings of the plane with strong restrictions on side labels (see discussion in [16, Section 4], where the first 4-way deterministic aperiodic tileset is constructed).
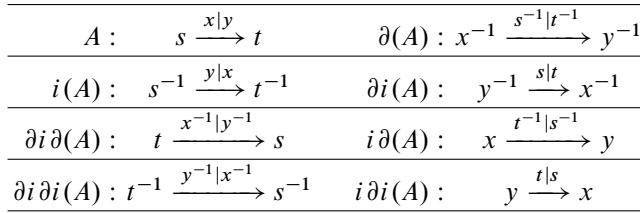
3.3. Bireversible automata and CSCs

Independently in automata theory, tiling theory, and in the study of square complexes people came up with similar classes of objects with strong deterministic properties. In automata theory this property is called bireversibility. It was introduced in [19] in relation to commensurators of free groups.

By applying the inverse and dual operations to any automaton A we get eight (not necessary deterministic and complete) automata:

$$A, \ \partial(A), \ i(A), \ i\partial(A), \ \partial i(A), \ \partial i \partial(A), \ i\partial i(A), \ i\partial i\partial(A) = \partial i\partial i(A). \tag{3.1}$$

The arrows in each of these automata corresponding to an arrow in A are shown in Figure 5.

$A:$	$s \xrightarrow{x\mid y} t$	$\partial(A):$	$x^{-1} \xrightarrow{s^{-1}\mid t^{-1}} y^{-1}$
$i(A):$	$s^{-1} \xrightarrow{y\mid x} t^{-1}$	$\partial i(A):$	$y^{-1} \xrightarrow{s\mid t} x^{-1}$
$\partial i\,\partial(A):$	$t \xrightarrow{x^{-1}\mid y^{-1}} s$	$i\,\partial(A):$	$x \xrightarrow{t^{-1}\mid s^{-1}} y$
$\partial i\,\partial i(A): t^{-1} \xrightarrow{y^{-1}\mid x^{-1}} s^{-1}$		$i\,\partial i(A):$	$y \xrightarrow{t\mid s} x$

Figure 5. Arrows in the eight automata obtained from A by passing to the dual and inverse automata.

Definition 3.3. A finite automaton A is called *bireversible* if all eight automata in (3.1) are well defined, i.e., complete and deterministic.

Since we start with a complete and deterministic automaton A, all eight automata in (3.1) are complete if and only if all of them are deterministic. Actually, since the complete and deterministic properties are preserved under passing to the dual automaton, an automaton A is bireversible if and only if $i(A)$, $i\,\partial(A)$, and $i\,\partial i(A)$ are deterministic (equiv., complete), or in other words, A, ∂A, and $\partial i(A)$ are invertible.

Notice that all the arrows in Figure 5 produce the same relation in the fundamental groups of the corresponding square complexes:

$$sx = yt, \quad s^{-1}y = xt^{-1}, \quad tx^{-1} = y^{-1}s, \quad t^{-1}y^{-1} = x^{-1}s^{-1}.$$

Therefore, the trivial map on the directed edges corresponding to S and X extends to an isomorphism between the square complexes associated to the eight automata.

The bireversibility of an automaton A can be checked using a finite bipartite graph Γ_A associated to A. The vertex set of Γ_A will be the disjoint union $(S \cup S^{-1}) \cup (X \cup X^{-1})$. Each arrow in A contributes four edges in Γ_A:

$$s \xrightarrow{x\mid y} t \text{ in } A \quad \Rightarrow \quad (s, x),\ (s^{-1}, y),\ (t, x^{-1}),\ (t^{-1}, y^{-1}) \text{ in } \Gamma_A.$$

The four edges represent corresponding arrows in A, $i(A)$, $i\,\partial(A)$, and $i\,\partial i(A)$. Since we want each of these automata to be complete, there should be edges (s, x), (s^{-1}, y), (t, x^{-1}), and (t^{-1}, y^{-1}) for all $s, t \in S$ and $x, y \in X$. Therefore, A is bireversible if and only if Γ_A is a complete bipartite graph. Note that Γ_A is the link of the unique vertex of Δ_A.

Bireversibility has a nice interpretation in terms of the tileset W_A. Note that the deterministic property of A implies that the colors of two edges adjacent to the top left corner uniquely determine a tile from W_A. The other three corners are responsible for deterministic properties of $i(A)$, $i\,\partial(A)$, and $i\,\partial i(A)$. Therefore, the bireversibility of automata corresponds to the 4-way deterministic property of Wang tilesets (this property means that the colors of any two adjacent edges uniquely determine a Wang tile).

Proposition 3.4. *Let A be a finite automaton. The following statements are equivalent:*

(1) *A is bireversible;*

(2) *Γ_A is a complete bipartite graph;*

(3) W_A *is* 4-*way deterministic;*

(4) Δ_A *is a CSC;*

(5) Δ_A *is non-positively curved;*

(6) *the universal cover of* Δ_A *is the direct product of two trees.*

Proof. The equivalence of items 1, 2, 3, and 4 is explained above. The equivalence of items 1, 5, and 6 is proved in [9], and the equivalence of items 4, 5, 6 follows from [30].   ∎

Every bireversible automaton $A = (S, X, \lambda)$ can be extended to an automaton $A^{\pm}$ with the state set $S^{\pm 1}$ and the alphabet $X^{\pm 1}$, in which the arrows are given by the first column of Figure 5. Basically, $A^{\pm}$ is the union of A, $i(A)$, $\partial i \partial (A)$, and $\partial i \partial i (A)$, while its dual $\partial(A^{\pm})$ is the union of $\partial(A)$, $\partial i (A)$, $i\partial(A)$, and $i\partial i(A)$. Note that the automaton $A^{\pm}$ is bireversible as well. Then we can naturally extend the state set of $A^{\pm}$ to words over $S^{\pm 1}$ and the alphabet to words over $X^{\pm 1}$ as in Section 2.1 and construct an automaton $^*(A^{\pm})^* = (S^{\pm *}, X^{\pm *}, \lambda^*)$. The left group action $F_S \curvearrowright X^*$ associated to A is extended to the left action $F_S \curvearrowright X^{\pm *}$ associated to $A^{\pm}$, while the right group action $S^* \curvearrowleft F_X$ is extended to the right action $S^{\pm *} \curvearrowleft F_X$. Note that the sets of reduced words $F_X \subset X^{\pm *}$ and $F_S \subset S^{\pm *}$ are invariant under these actions (they induce a subautomaton in $^*(A^{\pm})^*$).

3.4. The fundamental group of Δ_A for bireversible automata

The next statement contains some basic properties of the fundamental groups of complexes Δ_A known for all complete $\mathcal{V}\mathcal{H}$ square complexes with one vertex (see [30]).

Theorem 3.5. *Let* $A = (S, X, \lambda)$ *be a bireversible automaton.*

(1) *The group* $\pi_1(\Delta_A)$ *is torsion-free and* CAT(0).

(2) *The subgroups of* $\pi_1(\Delta_A)$ *generated by* S *and* X *are free of rank* #S *and* #X, *respectively.*

(3) *(Normal forms) The group* $\pi_1(\Delta_A)$ *admits an exact factorization by its free subgroups* $\langle S \rangle$ *and* $\langle X \rangle$. *In particular, every element* $\gamma \in \pi_1(\Delta_A)$ *can be uniquely written in the form* $\gamma = gv$ *and in the form* $\gamma = uh$ *for* $g, h \in \langle S \rangle$ *and* $v, u \in \langle X \rangle$.

There is a nice direct connection between the transitions in a bireversible automaton A and the two normal forms in $\pi_1(\Delta_A)$ given in Theorem 3.5. The completeness of the bipartite graph Γ_A means that the generators of $\pi_1(\Delta_A)$ satisfy

$$S^{\pm 1} X^{\pm 1} = X^{\pm 1} S^{\pm 1};$$

i.e., for any $s \in S^{\pm 1}$ and $x \in X^{\pm 1}$ there exists a unique pair $y \in X^{\pm 1}$ and $t \in S^{\pm 1}$ such that $sx = yt$ in $\pi_1(\Delta_A)$. This explains the normal form in $\pi_1(\Delta_A)$: given any word γ over generators, we can move every $s \in S^{\pm 1}$ to the left and every $x \in X^{\pm 1}$ to the right in order to find the representation $\gamma = gv$ for $g \in \langle S \rangle$ and $v \in \langle X \rangle$. We can also move every $s \in S^{\pm 1}$ to the right and every $x \in X^{\pm 1}$ to the left, and obtain another normal form $\gamma = uh$ for $u \in \langle X \rangle$ and $h \in \langle S \rangle$. Every relation $sx = yt$ for $s, t \in S^{\pm 1}$ and $x, y \in X^{\pm 1}$

corresponds to a transition in $A^{\pm}$. Therefore, for reduced words $g, h \in F_S$ and $v, u \in F_X$ we have

$$gv = uh \text{ in } \pi_1(\Delta_A) \quad \text{if and only if } g \xrightarrow{\; v|u \;} h \text{ in } {}^*(A^{\pm})^*. \tag{3.2}$$

In particular, we will frequently use the following relation between the action of F_S on $X^{\pm*}$ and the group $\pi_1(\Delta_A)$: for $g \in F_S$ and $v \in F_X$,

$$g(v) = v \quad \text{if and only if } v^{-1}gv \in F_S.$$

3.5. Automaton groups generated by bireversible automata

Every bireversible automaton $A = (S, X, \lambda)$ gives rise to eight invertible automata

$$A, \; \partial(A), \; i(A), \; i\partial(A), \; \partial i(A), \; \partial i \partial(A), \; i\partial i(A), \; i\partial i\partial(A) = \partial i \partial i(A),$$

an automaton $A^{\pm}$, and two group actions: a left action $F_S \curvearrowright X^{\pm*}$ and a right action $S^{\pm*} \curvearrowleft F_X$. The automaton group $G_{A^{\pm}}$ is the quotient of F_S by the kernel of its action on $X^{\pm*}$. The subsets of positive words X^* and negative words $(X^{-1})^*$ are invariant under the action of F_S. The corresponding restricted actions produce $G_A = G_{i(A)}$ and $G_{\partial i \partial(A)} = G_{i \partial i \partial(A)}$. Similarly, the dual automaton group $G_{\partial(A^{\pm})}$ is the quotient of F_X by the kernel of its action on $S^{\pm*}$, while $G_{\partial(A)} = G_{i \partial(A)}$ and $G_{\partial i(A)} = G_{i \partial i(A)}$ are the quotients of the corresponding actions on S^* and $(S^{-1})^*$. The next statement shows that in this way we get just two groups G_A and $G_{\partial(A)}$ and describes how to recover them from $\pi_1(\Delta_A)$.

Theorem 3.6. *Let $A = (S, X, \lambda)$ be a bireversible automaton. Then*

$$G_A = G_{i(A)} \cong G_{\partial i \partial(A)} = G_{i \partial i \partial(A)} \cong G_{A^{\pm}} \cong F_S / K,$$

where K is the maximal normal subgroup of $\pi_1(\Delta_A)$ that is contained in $F_S = \langle S \rangle$, and

$$G_{\partial(A)} = G_{i \partial(A)} \cong G_{\partial i(A)} = G_{i \partial i(A)} \cong G_{\partial(A^{\pm})} \cong F_X / K_{\partial},$$

where K_{∂} is the maximal normal subgroup of $\pi_1(\Delta_A)$ that is contained in $F_X = \langle X \rangle$.

Proof. We show that the action of $G_{A^{\pm}}$ on X^* is faithful. Let $g \in F_S$ act trivially on X^*, and let us show that g acts trivially on $X^{\pm*}$. For any $v \in X^*$ there exists a unique $g_1 \in F_S$ such that $gv = vg_1$ in $\pi_1(\Delta_A)$ and g_1 acts trivially on X^*. We can repeat this process and construct a sequence of elements $g_1, g_2, \ldots$ in F_S such that

$$gv = vg_1, \; g_1v = vg_2, \; g_2v = vg_3, \; \ldots \text{ in } \pi_1(\Delta_A),$$

which corresponds to the directed path

$$g \xrightarrow{\; v|v \;} g_1 \xrightarrow{\; v|v \;} g_2 \xrightarrow{\; v|v \;} g_3 \xrightarrow{\; v|v \;} \cdots \quad \text{in } {}^*(A^{\pm})^*.$$

Since all g_i have the same length, this sequence is eventually periodic. Moreover, the normal form in $\pi_1(\Delta_A)$ (or the deterministic properties of bireversible automata) implies that this sequence is periodic and there exists $n \in \mathbb{N}$ such that $g_n = g$. Hence, $gv^n = v^n g$

in $\pi_1(\Delta_A)$, which implies that $g v^{-n} = v^{-n} g$ and $g(v^{-1}) = v^{-1}$. Therefore, g acts trivially on $(X^{-1})^*$ as well and hence on $X^{\pm*}$. It follows that $G_A \cong G_{A^{\pm}}$. The other cases are analogous.

Let $K < \pi_1(\Delta_A)$ be the kernel of the action of $F_S \curvearrowright X^{\pm*}$ so that $G_{A^{\pm}} = F_S/K$. Then K is preserved under conjugation by elements of F_S. Since every element of K acts trivially on F_X, K is preserved under conjugation by elements of F_X. Therefore, K is a normal subgroup of $\pi_1(\Delta_A)$. It is maximal among normal subgroups that are contained in F_S, because all elements of such subgroups act trivially on words over $X^{\pm 1}$. The dual case is analogous. ∎

The following statement is a combination of [25, Proposition 2.2] and [5, Proposition 1.2]. A group $\pi_1(\Delta_A)$ is called *reducible* if it contains a finite index subgroup of the form $K \times H$ for $K < F_S$ and $H < F_X$.

Corollary 3.7. *Let $A = (S, X, \lambda)$ be a bireversible automaton. The following statements are equivalent:*

(1) *G_A is finite;*

(2) *$G_{\partial A}$ is finite;*

(3) *$\pi_1(\Delta_A)$ is reducible.*

Proof. If G_A is finite, then every orbit of the action $F_S \curvearrowright X^*$ contains at most $\#G_A$ elements. Therefore, every element $v \in X^{\pm*}$ belongs to a subautomaton of $^{\pm*}(\partial A)$ with at most $\#G_A$ states. Since there are only finitely many different automata with a fixed number of states, the group $G_{\partial A}$ is finite. Hence, the items 1 and 2 are equivalent.

If G_A and $G_{\partial A}$ are finite, then $K \times K_\partial$ has finite index in $\pi_1(\Delta_A)$. Conversely, if $\pi_1(\Delta_A)$ contains a finite index subgroup $K_1 \times K_2$ with $K_1 < F_S$ and $K_2 < F_X$, then the kernels $K > K_1$ and $K_\partial > K_2$ have finite index in F_S and F_X, respectively. Therefore, G_A and $G_{\partial A}$ are finite. ∎

Theorem 3.6 suggests two ways to generate free groups by automata.

Remark 3.8. If for some bireversible automaton A we had an infinite group G_A and a just-infinite[1] group $\pi_1(\Delta_A)$, then G_A and $G_{\partial A}$ would be free groups freely generated by S and X, respectively. However, $\pi_1(\Delta_A)$ cannot be just-infinite, because it projects onto $\mathbb{Z}^2$. Nevertheless, this approach works for non-directed complete $\mathcal{VH}$ complexes, some of which have a just-infinite fundamental group. For example, [5, Theorem 6.3] implies that for every $n \geq 15$ and $m \geq 19$ there exists a bireversible automaton A with $2n$ states over alphabet with $2m$ letters such that $G_A \cong F_n$ and $G_{\partial A} \cong F_m$.

Remark 3.9. It follows from the theorem that if $\{g_1, g_2, \ldots, g_n\}$ is an orbit of the action $S^{\pm*} \curvearrowright F_X$ and some g_i represents a nontrivial element of G_A, then all $g_1, \ldots, g_n$ represent nontrivial elements of G_A. In particular, if G_A is infinite and F_X acts transitively on

---

[1] An infinite group is called just-infinite if every nontrivial normal subgroup has finite index.
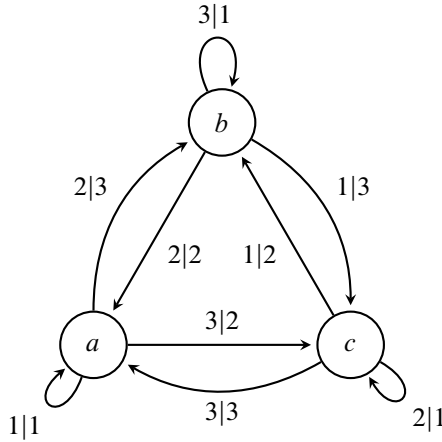
Figure 6. A self-inverse self-dual automaton generating the lamplighter group $\mathbb{Z}_3 \wr \mathbb{Z}$.

all reduced words of length n for each $n \in \mathbb{N}$, then G_A is a free group freely generated by S. However, F_X cannot have this transitivity property, because positive words over S are invariant under the action. Nevertheless, as in the previous remark, this approach works for some non-directed $\mathcal{VH}$ complexes and was used by Glasner and Mozes in [9] to construct the first examples of automata generating free groups.

Besides Corollary 3.7, it is not clear what the relation is between the groups G_A and $G_{\partial(A)}$. For all examples that we know, the groups G_A and $G_{\partial(A)}$ are either both finitely presented or both infinitely presented.

Question 3.10. *Which bireversible automata generate finitely presented groups? Is it true for a bireversible automaton A that G_A is virtually free if and only if $G_{\partial A}$ is virtually free? Is it true that groups generated by bireversible automata are linear?*

For these questions it seems useful to consider the quotient of $\pi_1(\Delta_A)$ by the kernels of the action of $\langle S \rangle$ on $\langle X \rangle$ and of the action of $\langle X \rangle$ on $\langle S \rangle$. We get a group

$$T_A = \pi_1(\Delta_A)/K \times K_\partial = G_A \cdot G_{\partial(A)},$$

which admits an exact factorization by both automaton groups. The group T_A brings information about G_A and $G_{\partial(A)}$, and their interconnection in $\pi_1(\Delta_A)$. It follows from the results of Y. Vorobets (private communication) that the group T_A for the Aleshin and Bellaterra automata is finitely presented. The first example of a bireversible automaton with non-finitely presented groups G_A and $G_{\partial(A)}$ is shown in Figure 6. It is self-dual and generates the lamplighter group $\mathbb{Z}_3 \wr \mathbb{Z}$ (see [2]).

Question 3.11. *For which bireversible automata A is the group T_A finitely presented?*

3.6. Anti-tori and tilings of the plane

Let A be a bireversible automaton and let $W_{A^\pm}$ be the set of Wang tiles associated to $A^\pm$. Since $W_{A^\pm}$ is complete and 4-way deterministic, all possible tilings of the plane by $W_{A^\pm}$ can be described as follows: for every pair of sequences $(s_i)_{i\in\mathbb{Z}}$, $s_i \in S^{\pm 1}$ and $(x_i)_{i\in\mathbb{Z}}$, $x_i \in X^{\pm 1}$ there exists a unique tiling $t : \mathbb{Z}^2 \to W_{A^\pm}$ of the plane such that the sequence $(s_i)_{i\in\mathbb{Z}}$ is read along the vertical axes, while the sequence $(x_i)_{i\in\mathbb{Z}}$ is read along the horizontal axes. For reduced sequences this tiling corresponds to a plane in the universal cover of Δ_A.

Let $a \in F_S$ and $b \in F_X$. Let us consider the tiling of the plane with the vertical axes labeled by $(a)_{i\in\mathbb{Z}}$ and the horizontal axes labeled by $(b)_{i\in\mathbb{Z}}$. If this tiling is periodic in both vertical and horizontal directions, then there exist nonzero $n, m \in \mathbb{Z}$ such that $a^n b^m = b^m a^n$ in $\pi_1(\Delta_A)$, which means that there is a torus in Δ_A. If this tiling is not periodic, then we come to the following definition.

Definition 3.12. For $a \in F_S$ and $b \in F_X$, the subgroup $\langle a, b \rangle$ is called an *anti-torus* in $\pi_1(\Delta_A)$ if $a^n b^m \neq b^m a^n$ for all $n, m \in \mathbb{Z} \setminus \{0\}$.

It seems to be an interesting and difficult problem, given a bireversible automaton A, to describe all anti-tori in $\pi_1(\Delta_A)$. This problem was studied in [24] for certain Burger-Mozes groups, where anti-tori correspond to non-commuting pairs of Hamilton quaternions.

Proposition 3.13. *Let $A = (S, X, \lambda)$ be a bireversible automaton. If $\pi_1(\Delta_A)$ admits an anti-torus, then G_A is infinite.*

Proof. Let $a \in F_S$ and $b \in F_X$ generate an anti-torus. We will prove that a represents an element of infinite order in G_A. Let us assume that $a^n(v) = v$ for all $v \in X^{\pm *}$. Then we have a cycle in the automaton

$$a^n \xrightarrow{b|b} a_2 \xrightarrow{b|b} a_3 \xrightarrow{b|b} \cdots \xrightarrow{b|b} a^n$$

(see the proof of Theorem 3.6). This means that $a^n b^m = b^m a^n$ in $\pi_1(\Delta_A)$ for some $m \in \mathbb{N}$, which contradicts the assumption of the statement. ∎

It is interesting whether the converse holds as follows.

Question 3.14. *Is it true that if G_A is infinite then $\pi_1(\Delta_A)$ contains an anti-torus?*

This question is related to the next one studied in the theory of automaton groups.

Question 3.15. *Can a bireversible automaton A generate an infinite torsion group G_A?*

By the proof of Proposition 3.13, a positive answer to Question 3.15 implies a negative answer to Question 3.14. However, we expect a negative answer to Question 3.15 and a positive answer to Question 3.14. It was shown in [18] that a bireversible automaton with two and three states cannot generate an infinite torsion group. Note, however, that there

are famous examples of infinite torsion groups generated by non-bireversible automata like the Grigorchuk group.

3.7. Automata from square complexes

Let Δ be a complete directed $\mathcal{VH}$ square complex with one vertex. Let S and X be the loops at the corresponding vertical-horizontal decomposition of Δ. Since Δ is complete and directed, for any $s \in S$ and $x \in X$ there exists a unique pair $y \in X$ and $t \in S$ such that $sx = yt$ in $\pi_1(\Delta)$. This property defines an automaton A over the alphabet X with the set of states S. This automaton is bireversible and the square complex Δ_A is exactly Δ.

The same construction works for non-directed complexes. Let Δ be a complete (not necessary directed) $\mathcal{VH}$ square complex with one vertex. Let S and X be the loops at the corresponding vertical-horizontal decomposition of Δ. The completeness of Δ implies that for any $s \in S^{\pm 1}$ and $x \in X^{\pm 1}$ there exists a unique pair $y \in X^{\pm 1}$ and $t \in S^{\pm 1}$ such that $sx = yt$ in $\pi_1(\Delta)$. As above, these relations produce a bireversible automaton A_Δ over the alphabet $X \cup X^{-1}$ and with the set of states $S \cup S^{-1}$. Two examples of non-directed square complexes and the associated automata are shown in Figures 3 and 4. Note that the square complex Δ_A associated to A_Δ is not Δ, because when we construct a square complex from automaton we treat every state as an independent color and do not take into account inverse elements. If we start with a directed complex $\Delta = \Delta_A$ associated to a bireversible automaton A, then A_Δ is exactly the automaton $A^\pm$ constructed in Section 3.3.

4. Residual finiteness of $\pi_1(\Delta_A)$

In this section we study the residual properties of the group $\pi_1(\Delta_A)$ for bireversible automata A. Our analysis follows the approach of Wise from [30] and relies on the following statement.

Let H be a subgroup of a group G. Then G is called H-*separable* if H is the intersection of finite index subgroups of G. A group G is residually finite if and only if it is separable with respect to the trivial subgroup.

Theorem 4.1 ([32, Theorem 7.2]). *Let ϕ be an endomorphism of a finitely generated residually finite group G, and let* $\mathrm{Fix}(\phi)$ *be the subgroup of elements fixed by ϕ. Then G is* $\mathrm{Fix}(\phi)$-*separable.*

Proof. We include the proof for completeness. Let $a \in G \setminus \mathrm{Fix}(\phi)$. Then $a^{-1}\phi(a) \neq e$ and $a^{-1}\phi(a) \notin N$ for some subgroup N of finite index n. We can assume that N is fully invariant by passing to the intersection of all subgroups of index $\leq n$. Then $\mathrm{Fix}(\phi)N$ has finite index and does not contain a. ∎

Corollary 4.2 ([32, Corollary 7.3]). *Let $D = G *_H G$ be the double of a group G along its subgroup H. If D is residually finite, then G is H-separable.*

In [30] Wise proved that $\pi_1(\Delta_W)$ is not $\langle a, b, c \rangle$-separable using the fact that it contains an anti-torus. We do not know which automata admit an anti-torus (see discussion

in Section 3.6); instead, we are using the following lemma, which relies on Zelmanov's solution of the restricted Burnside problem. Note that the conclusion of the lemma immediately follows from the existence of an anti-torus.

Let $A = (X, S, \lambda)$ be a bireversible automaton. For $m \in \mathbb{N}$, let P_m be the set of all pairs (x, y) of different letters $x, y \in X$ for which there exist $g \in S^*$ and $u \in X^*$ such that $g^m(ux) = uy$ (this means that there is a relation $g^m ux = uyh$ in the group $\pi_1(\Delta_A)$ for some $h \in S^*$). Note that P_m is empty only when g^m is trivial in G_A for every $g \in S^*$.

Lemma 4.3. *Let $A = (X, S, \lambda)$ be a bireversible automaton. If G_A is infinite, then $\bigcap_{m \in \mathbb{N}} P_m$ is nonempty.*

Proof. Let us assume that this intersection is empty. Then, for each pair of different letters (x, y), there exists $m_{xy} \in \mathbb{N}$ such that $(x, y) \notin P_{m_{xy}}$. Let m be the product of all these numbers m_{xy}. Note that if n divides m, then $P_m \subset P_n$, because we can rewrite the equality $g^m(ux) = uy$ as $(g^{m/n})^n(ux) = uy$. Since m_{xy} divides m for all pairs (x, y), the set P_m is empty. It follows that the element g^m is trivial in G_A for every $g \in S^*$. Hence the group G_A has finite exponent. Since G_A is finitely generated and residually finite, it should be finite by the solution of the restricted Burnside problem. We get a contradiction. ∎

Remark 4.4. For automata over the binary alphabet $X = \{0, 1\}$ it is straightforward to see that the infiniteness of G_A implies that $\bigcap_{m \in \mathbb{N}} P_m = \{(0, 1), (1, 0)\}$, and we do not need to rely on the restricted Burnside problem.

Theorem 4.5. *Let $A = (X, S, \lambda)$ be a bireversible automaton.*

(1) *If G_A is finite, then $\pi_1(\Delta_A)$ is virtually a direct product of two free groups and therefore residually finite.*

(2) *If G_A is infinite, then $\pi_1(\Delta_A)$ is not $\langle S \rangle$-separable and not $\langle X \rangle$-separable.*

Proof. The first item follows from Corollary 3.7.

Let G_A be infinite. If $\pi_1(\Delta_A)$ is $\langle S \rangle$-separable, then for each $g \in \pi_1(\Delta_A) \setminus \langle S \rangle$ there exists a subgroup $H < \pi_1(\Delta_A)$ of finite index such that $\langle S \rangle < H$ and $g \notin H$. Then there exists a normal subgroup $N \lhd \pi_1(\Delta_A)$ of finite index such that $g \notin \langle S \rangle N$.

Let $(x, y) \in \bigcap_{m \in \mathbb{N}} P_m$. We are going to prove that the element $x^{-1} y$ of the group $\pi_1(\Delta_A)$ belongs to $\langle S \rangle N$ for every normal subgroup N of finite index in $\pi_1(\Delta_A)$. Since $x^{-1} y \notin \langle S \rangle$ by Theorem 3.5, it will follow that $\pi_1(\Delta_A)$ is not $\langle S \rangle$-separable.

Let $N < \pi_1(\Delta_A)$ be a normal subgroup of index n. Then for every $g \in \langle S \rangle$ the element g^n belongs to N. Since $(x, y) \in P_n$, there exist words $g, h \in S^* \subset \langle S \rangle$ and $u \in X^*$ such that $g^n ux = uyh$ in $\pi_1(\Delta_A)$. Therefore

$$x^{-1} y = h^{-1}(hx^{-1}u^{-1}g^n uxh^{-1}) \in \langle S \rangle N,$$

which completes the proof. ∎

Corollary 4.6. *Let $A = (X, S, \lambda)$ be a bireversible automaton. The double $D_A = \pi_1(\Delta_A) *_S \pi_1(\Delta_A)$ of $\pi_1(\Delta_A)$ along $\langle S \rangle$ is residually finite if and only if G_A is finite.*

Proof. If G_A is infinite, then D_A is not residually finite by Theorem 4.5 and Corollary 4.2.

If G_A is finite, then $G_{A \cup \partial i \partial(A)} \cong G_A$ is finite. The group D_A is the fundamental group of the complex $\Delta_{A \cup \partial i \partial(A)}$, which is residually finite by Theorem 4.5. ∎

Remark 4.7. It follows from the proofs of Theorems 4.1 and 4.5 that for each $(x, y) \in \bigcap_{m \in \mathbb{N}} P_m$ the nontrivial element $(x^{-1}y)^{-1}\phi(x^{-1}y) = y^{-1}x\bar{x}^{-1}\bar{y}$ of $\pi_1(\Delta_A) *_S \pi_1(\Delta_A)$ belongs to the intersection of all finite index subgroups.

Remark 4.8. Theorem 4.5 holds for the fundamental groups of non-directed complete $\mathcal{VH}$ complexes Δ with one vertex and the associated bireversible automata, because the element $x^{-1}y$ from the proof remains nontrivial in $\pi_1(\Delta)$.

The above theorem does not tell us when the group $\pi_1(\Delta_A)$ is residually finite, and the next question seems to be difficult.

Question 4.9. *For which bireversible automata A is the group $\pi_1(\Delta_A)$ residually finite?*

We answer this question in some cases.

Theorem 4.10. *Let $A = (X, S, \lambda)$ be a bireversible automaton over an alphabet with two letters or with two states. If G_A is infinite, then $\pi_1(\Delta_A)$ is non-residually finite.*

Proof. We can assume that $X = \{x, y\}$. There exists $n \in \mathbb{N}$ such that $(y^{-1}x)^n$ and $(x^{-1}y)^n$ act trivially on $S \cup S^{-1}$; i.e., $s(y^{-1}x)^{\pm n}s^{-1} \in \langle X \rangle$ for every $s \in S \cup S^{-1}$. Let us show that the map

$$\phi(x) = x(y^{-1}x)^n, \quad \phi(y) = y(x^{-1}y)^n, \quad \text{and} \quad \phi(s) = s \quad \text{for all } s \in S$$

extends to an endomorphism of $\pi_1(\Delta_A)$.

Since our alphabet is binary, a state $s \in S$ stabilizes x if and only if it stabilizes y. Let S^+ be the set of all states that stabilize x and y, and let S^- be the set of all states that stabilize x^{-1} and y^{-1}. Every defining relation for $\pi_1(\Delta_A)$ of the form $sx = xt$ or $sy = yt$ implies that $s \in S^+$ and $t \in S^-$, while a relation of the form $sx = yt$ or $sy = xt$ is possible only when $s \notin S^+$ and $t \notin S^-$. It follows that every $t \in S^-$ satisfies the relations $ty^{-1}x = y^{-1}xt_1$ and $tx^{-1}y = x^{-1}yt_2$ for some $t_1, t_2 \in S^-$, while every $t \notin S^-$ satisfies the relations $ty^{-1}x = x^{-1}yt_1$ and $tx^{-1}y = y^{-1}xt_2$ for some $t_1, t_2 \notin S^-$. Since $(y^{-1}x)^{\pm n}$ act trivially on S, we have relations

$$t(y^{-1}x)^{\pm n} = (y^{-1}x)^{\pm n}t \quad \text{for every } t \in S^-,$$
$$t(y^{-1}x)^{\pm n} = (x^{-1}y)^{\pm n}t \quad \text{for every } t \notin S^-.$$

Now for every defining relation of the form $sx = xt$ and $sy = yt'$ (here $s \in S^+$ and $t, t' \in S^-$), there are relations

$$sx(y^{-1}x)^n = xt(y^{-1}x)^n = x(y^{-1}x)^n t,$$
$$sy(x^{-1}y)^n = yt'(x^{-1}y)^n = y(x^{-1}y)^n t'.$$

For defining relations of the form $sx = yt$ or $sy = xt'$ (here $s \notin S^+$ and $t, t' \notin S^-$), there are relations

$$sx(y^{-1}x)^n = yt(y^{-1}x)^n = y(x^{-1}y)^n t,$$
$$sy(x^{-1}y)^n = xt'(x^{-1}y)^n = x(y^{-1}x)^n t'.$$

Thus, ϕ preserves all the defining relations of $\pi_1(\Delta_A)$ and induces an endomorphism of $\pi_1(\Delta_A)$.

Since $\phi(z)$ begins and ends on z for each $z \in \{x, y, x^{-1}, y^{-1}\}$, the image $\phi(w)$ of a reduced word $w \in \langle X \rangle$ is also reduced. The length of $\phi(w)$ is greater than the length of w; hence, no element in $\langle X \rangle$ can be fixed by ϕ. Therefore, $\mathrm{Fix}(\phi) = \langle S \rangle$ and $\pi_1(\Delta_A)$ is non-residually finite by Theorems 4.1 and 4.5. ∎

Remark 4.11. It follows from the proof that $(x^{-1}y)^n$ and $(y^{-1}x)^n$ are relations in the group $G_{\partial A}$. In particular, a bireversible automaton with two states cannot generate a free group of rank two.

Remark 4.12. The element

$$(x^{-1}y)^{-1}\phi(x^{-1}y) = (x^{-1}y)^{-1}\big(x(y^{-1}x)^n\big)^{-1}\big(y(x^{-1}y)^n\big) = (x^{-1}y)^{2n}$$

belongs to the intersection of all finite index subgroups of $\pi_1(\Delta_A)$. The normal closure of $(x^{-1}y)^n$ and $(y^{-1}x)^n$ has infinite index in $\pi_1(\Delta_A)$.

Corollary 4.13. *Let $A = (X, S, \lambda)$ be a bireversible automaton. Suppose that each connected component of A consists of two states and the automaton group G_A is infinite. Then $\pi_1(\Delta_A)$ is non-residually finite.*

Proof. By applying arguments from the proof of Theorem 4.10, we can construct a nontrivial endomorphism ϕ_i for each connected component $A_i = (X, S_i, \lambda)$ of A considering it as a separate automaton. By the construction, all ϕ_i map elements of X to themselves. Hence, we could extend them to a unique endomorphism ϕ of the whole group $\pi_1(\Delta_A)$. Similarly, the image of a reduced word $w \in \langle S \rangle$ is also reduced and is longer than the initial word. Therefore, $\mathrm{Fix}(\phi) = \langle X \rangle$ and the statement of the corollary follows from Theorems 4.1 and 4.5. ∎

Example 4.14. The smallest automata that satisfy the conditions of Theorem 4.10 are the Aleshin automaton A and its friendly version—the Bellaterra automaton B shown in Figure 7. The corresponding endomorphism of $\pi_1(\Delta_A)$ for the Aleshin automaton is shown in Section 1. The Bellaterra automaton possesses a simpler endomorphism. The states $\{a^{-1}, b^{-1}, c^{-1}\}$ of B^* span a subautomaton isomorphic to B. Hence, the map $\phi : \pi_1(\Delta_B) \to \pi_1(\Delta_B)$, which replaces each letter from $S \cup S^{-1}$ by its inverse, is an automorphism of $\pi_1(\Delta_B)$ with $\mathrm{Fix}(\phi) = \langle X \rangle$. The intersection of all finite index subgroup of $\pi_1(\Delta_B)$ contains, for example, the elements $a^2b^{-2}, b^2c^{-2}, c^2a^{-2}$.
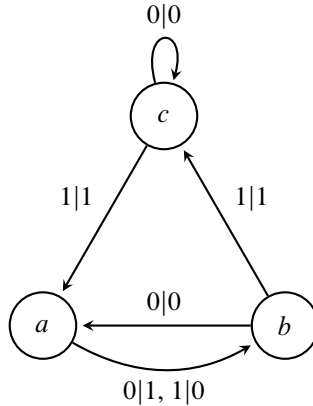
Figure 7. The Bellaterra automaton B generating $G_B \cong C_2 * C_2 * C_2$.

It is an open question whether the finiteness problem for automaton groups is algorithmically solvable. Probably not, because it is unsolvable for automaton semigroups (see [8]). However, the finiteness problem for automaton groups generated by bireversible automata over the binary alphabet (or with two states) is algorithmically solvable (see [17]). This fact together with Theorems 4.5 and 4.10 imply that there is an algorithm which for a given complete $\mathcal{V}\mathcal{H}$ complex Δ with one vertex and two edges in the vertical or horizontal part verifies whether its fundamental group $\pi_1(\Delta)$ is residually finite.

Corollary 4.15. *Let A be a connected bireversible automaton generating an infinite automaton group. If there is a nontrivial X-isomorphism between A and a subautomaton B of the automaton A^*, then $\pi_1(\Delta_A)$ is non-residually finite.*

Proof. The isomorphism between A and B extends to an endomorphism $\phi : \pi_1(\Delta_A) \to \pi_1(\Delta_A)$ with fixed subgroup $\text{Fix}(\phi)$ containing $\langle X \rangle$. By repeating the arguments from the proof of Theorem 4.5, one can show that there is a pair of states a, b in A such that $a^{-1}b \in \text{Fix}(\phi)N$ for every normal subgroup $N \lhd \pi_1(\Delta_A)$ of finite index. We just need to prove that $a^{-1}b \notin \text{Fix}(\phi)$. This will imply that $\pi_1(\Delta_A)$ is not $\text{Fix}(\phi)$-separable and, therefore, it is not residually finite by Theorem 4.1.

Let $\phi(a) = v$ and $\phi(b) = u$. Since B is connected, all words representing the states of B have the same length, in particular, $|v| = |u|$. Therefore, the equality $\phi(a^{-1}b) = v^{-1}u = a^{-1}b$ is possible only when v and u have a common beginning w so that $v = wa$ and $u = wb$. Since ϕ is nontrivial, the word w is nonempty. The first letter s of w is a state of A or A^{-1}, and we can assume that $s \in A$. For any state t of A there is a path from s to t, which induces two paths from wa and wb to two different words which start with the letter t. It follows that for each state of A there are at least two states of B. This contradiction completes the proof. ∎

An automaton is minimal if different states define different transformations of words.

Corollary 4.16. *Let A be a non-minimal connected bireversible automaton generating an infinite automaton group. Then $\pi_1(\Delta_A)$ is non-residually finite.*

The first automaton for which we cannot apply Corollary 4.15 is the self-dual automaton A with 3 states over a 3-letter alphabet shown in Figure 6. One can show that there is no subautomaton in A^* that is X-isomorphic to A.

5. CSCs with four squares

Every automaton group G_A possesses the following self-similarity property: for every $g \in G_A$ and $v \in X^*$ there exists a unique element $h \in G_A$ such that $g(vw) = g(v)h(w)$ for all $w \in X^*$. In the proof below we use notation $h = g|_v$.

Theorem 5.1. *Let Δ_D be the square complex given by the four squares in Figure 3 and let A be the associated bireversible automaton. Then*

(1) $\pi_1(\Delta_D)$ *is non-residually finite,*

(2) $G_A \cong C_3 * C_3$ *and* $G_{\partial A} \cong F_2$.

Proof. We have that $X = \{x, y, y^{-1}, x^{-1}\}$ and $S = \{a, b, b^{-1}, a^{-1}\}$. Since $\pi_1(\Delta_D)$ contains an anti-torus (see [15]), the automaton group G_A is infinite by Proposition 3.13, and the group $\pi_1(\Delta_D)$ is not $\langle X \rangle$-separable by Theorem 4.5 and Remark 4.8. It is direct to check that the map defined by

$$\phi(a) = a^4, \quad \phi(b) = b^4 \quad \text{and} \quad \phi(x) = x, \quad \phi(y) = y$$

extends to an endomorphism of $\pi_1(\Delta_D)$ with $\text{Fix}(\phi) = \langle X \rangle$. Therefore, $\pi_1(\Delta_D)$ is non-residually finite by Theorem 4.1. Further we indicate nontrivial elements in the intersection of its finite index subgroups.

Let us describe the group $G_{\partial A}$. One can directly check the following crucial property: any nontrivial orbit of the action of an element $g \in St_1(G_A)$ on X^2 consists of three elements, namely, these orbits are

$$\{xx, xy, xy^{-1}\}, \quad \{yx, yy, yx^{-1}\},$$
$$\{y^{-1}x, y^{-1}y^{-1}, y^{-1}x^{-1}\}, \quad \{x^{-1}y, x^{-1}y^{-1}, x^{-1}x^{-1}\}.$$

It follows that, for any $g \in G_A$ and $v \in X^*$ with the property $g(v) = v$ and $g|_v \in St_1(G_A)$, if $g(vz_1z_2) \neq vz_1z_2$ for some $z_1, z_2 \in X$, then the orbit of vz_1z_2 under g consists of three words vz_1z_3 for $z_3 \in X$, $z_3 \neq z_1^{-1}$. In particular, if $g \in St_n(G_A)$ and $g(vz) \neq vz$ for some word $v \in X^n$, $z \in X$ (note that such vz is necessary a (freely) reduced word), then the orbit of vz under g contains all three reduced words of length $n + 1$ with prefix v.

We will show by induction on n that the group G_A acts transitively on the set of (freely) reduced words over X of length n for each $n \in \mathbb{N}$. Let us assume that the statement holds for the words of length n. Since the group G_A is infinite, there exists $g \in G_A$ such that

$g \in St_n(G_A)$ and $g(vz_1) \neq vz_1$ for some $vz_1 \in X^{n+1}$. Note that vz_1 and $g(vz_1)$ are reduced words. Then the orbit of vz_1 under g consists of exactly three reduced words of length $n + 1$ with prefix v. Now let $wz_2 \in X^{n+1}$ be an arbitrary reduced word. By induction hypothesis there exists $h \in G_A$ such that $h(w) = v$. Then the word $h(wz_2)$ is reduced and has prefix v; therefore, it belongs to the orbit of vz_1 under g. This means that vz_1 and wz_2 belong to the same G_A-orbit. Our claim is proved.

We are ready to show that $G_{\partial A}$ is freely generated by X. Let w be any reduced word over X. The transitivity of the action of G_A on reduced words over X implies that if w represents the trivial element of $G_{\partial A}$, then all reduced words of length $|w|$ represent the trivial element (see Remark 3.9). However, this would mean that $G_{\partial A}$ is finite, but it is not.

Similarly, we show that $G_A = \langle a \rangle * \langle b \rangle = C_3 * C_3$. One checks directly that $a^3 = b^3 = e$ in G_A. We have to prove that every word of the form $[a^{\pm 1}]b^{\pm 1}a^{\pm 1} \ldots a^{\pm 1}[b^{\pm 1}]$ represents a nontrivial element of G_A. Note that the generators x and y map $a^{\pm 1}$ to $b^{\pm 1}$ and $b^{\pm 1}$ to $a^{\pm 1}$, and therefore $G_{\partial A}$ preserves the set of alternating words. As above, it is enough to show that the dual group $G_{\partial A}$ acts transitively on all such alternating words of fixed length n for each $n \in \mathbb{N}$. The proof goes in the same way as above and relies on the following property of $G_{\partial A}$: any nontrivial orbit of the action of an element $g \in St_1(G_{\partial A})$ on S^2 consists of two elements, namely, these orbits are

$$\{ab, ab^{-1}\}, \quad \{ba, ba^{-1}\}, \quad \{b^{-1}a, b^{-1}a^{-1}\}, \quad \{a^{-1}b, a^{-1}b^{-1}\}. \tag{5.1}$$

Then, for any element $g \in G_{\partial A}$ and a word $v \in S^*$ with the property $g(v) = v$ and $g|_v \in St_1(G_{\partial A})$, if $g(vz_1z_2) \neq vz_1z_2$ for some $z_1, z_2 \in S$, then $g(vz_1z_2) = vz_1z_2^{-1}$ and $g(vz_1z_2^{-1}) = vz_1z_2$. Let us assume by induction that $G_{\partial A}$ acts transitively on alternating words of length n. Since $G_{\partial A}$ is infinite, there exist $g \in St_n(G_{\partial A})$ and $vz_1 \in S^{n+1}$ such that $g(vz_1) \neq vz_1$; therefore, $g(vz_1) = vz_1^{-1}$. Note that vz_1 is necessary an alternating word, since otherwise vz_1 could be represented by a shorter word and would be fixed by g. Now let $wz_2 \in S^{n+1}$ be an arbitrary alternating word of length $n + 1$. By induction hypothesis, there exists $h \in G_{\partial A}$ such that $h(w) = v$. Then $h(wz_2)$ is equal to either vz_1 or vz_1^{-1}. In any case, we get that vz_1 and wz_2 belong to the same $G_{\partial A}$-orbit. Our claim is proved.

Now we indicate an element in the intersection N of all finite index subgroups of $\pi_1(\Delta_D)$ and compute the intersection $\bigcap_{m \in \mathbb{N}} P_m$ defined in Lemma 4.3 for the dual automaton ∂A. For every $m \in \mathbb{N}$, there exists an element $g \in G_{\partial A}$ such that $g^m \neq e$ and $g^m \in St_1(G_{\partial A})$. Then $g^m|_v \in St_1(G_{\partial A})$ and $g^m|_v \notin St_2(G_{\partial A})$ for some word $v \in S^*$. The only possible nontrivial orbits of the action of $g^m|_v$ on S^2 are listed in (5.1), which imply that P_m may contain only pairs $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$, and one of these pairs belongs to P_m. Therefore, either $a^{-2}\phi(a^2) = a^6$ or $b^{-2}\phi(b^2) = b^6$ belongs to N. However, there is an automorphism $\gamma : \pi_1(\Delta_D) \to \pi_1(\Delta_D)$ defined by

$$\gamma(a) = b, \quad \gamma(b) = a, \quad \gamma(x) = x^{-1}, \quad \gamma(y) = y^{-1}.$$

It follows that if one of a^6 and b^6 is in N, then they both are. Another implication is that $\bigcap_{m \in \mathbb{N}} P_m = \{\{a, a^{-1}\}, \{b, b^{-1}\}\}$. ∎

Remark 5.2. Since a^3 and b^3 are relations in G_A and G_A is infinite, the normal closure of a^3 and b^3 in $\pi_1(\Delta_D)$ has infinite index. Also, the intersection of all finite index subgroups of $\pi_1(\Delta_D)$ has infinite index (see [22, Theorem 34]).

Theorem 5.3. *Let Δ_S be the square complex given by the four squares in Figure 4 and let A be the associated bireversible automaton. Then*

(1) *A is self-dual and $G_A \cong F_2$,*

(2) *$\langle a, x \rangle$, $\langle a, y \rangle$, $\langle b, x \rangle$ and $\langle b, y \rangle$ form anti-tori in $\pi_1(\Delta_S)$.*

Proof. We have that $X = \{x, y, y^{-1}, x^{-1}\}$ and $S = \{a, b, b^{-1}, a^{-1}\}$. Notice that the map

$$a \mapsto y, \quad b \mapsto x, \quad x \mapsto b^{-1}, \quad y \mapsto a^{-1}$$

is an isomorphism between A and ∂A. Therefore, A is self-dual and $G_A \cong G_{\partial A}$.

The first item follows from Remark 3.8 and the result proved in [27] that $\pi_1(\Delta_S)$ is just-infinite. Below we give a direct proof.

Let us show that $G_{\partial A}$ is freely generated by X. As in Theorem 5.1, it is sufficient to prove that G_A acts transitively on reduced words over X of length n for each $n \in \mathbb{N}$. Since G_A enjoys the crucial property from the proof of Theorem 5.1 about the structure of G_A-orbits on X^2, the proof goes the same way once we show that G_A is infinite. We notice another cute property of G_A: the elements of $S_3 = \{a^3, b^3, b^{-3}, a^{-3}\}$ preserve the set $X_3 = \{x^3, y^3, y^{-3}, x^{-3}\}$, and the map

$$a \mapsto a^{-3}, \quad b \mapsto b^{-3}, \quad x \mapsto x^3, \quad y \mapsto y^3$$

is an isomorphism between A and the subautomaton of $^*A^*$ formed by the states S_3 over the alphabet X_3. Since $a^4(x) = x$ and $a^4(x^2) \neq x^2$, we get $a^{4 \cdot 3^n}(x^{3^n}) = x^{3^n}$ and $a^{4 \cdot 3^n}(x^{2 \cdot 3^n}) \neq x^{2 \cdot 3^n}$ for all $n \in \mathbb{N}$. It follows that the size of the orbit of x^m under a is not bounded independently on m by an absolute constant. Thus a has infinite order and G_A is infinite. The claim is proved.

Another implication is that a and x viewed as elements of $\pi_1(\Delta_S)$ generate an anti-torus. Indeed, let us assume that $a^{n_0} x^{m_0} = x^{m_0} a^{n_0}$ in $\pi_1(\Delta_S)$ for some $n_0, m_0 \in \mathbb{N}$. Then $a^{n_0}(x^m) = x^m$ for all $m \in \mathbb{N}$, which implies that the orbit of x^m under a contains at most n_0 elements. We get a contradiction. The same arguments work for $\langle a, y \rangle$, $\langle b, x \rangle$, and $\langle b, y \rangle$. ∎

6. Finitely presented torsion-free simple groups

In [4, 5] Burger and Mozes constructed the first examples of finitely presented torsion-free simple groups. Their construction relies on two sufficient conditions for the fundamental groups of complete $\mathcal{V}\mathcal{H}$ complexes with one vertex: one guarantees non-residual finiteness [5, Proposition 2.1], while the other one implies just-infiniteness [5, Theorem 4.1]. The non-residually finite groups coming from the first condition always have a nontrivial normal subgroup of infinite index. However, one can embed a group satisfying the

first condition into a group satisfying the second condition and construct virtually simple groups. The smallest simple group constructed in this way in [4, 5] has amalgam decomposition of the form $F_{7919} *_{F_{380065}} F_{7919}$ and $F_{47} *_{F_{364321}} F_{47}$.

In [23] Rattaggi followed this approach, but started with the non-residually finite group $\pi_1(\Delta_W) *_{\langle a,b,c \rangle} \pi_1(\Delta_W)$ constructed by Wise. In this way Rattaggi constructed a finitely presented torsion-free simple group that possesses amalgam decompositions $F_7 *_{F_{73}} F_7$ and $F_{11} *_{F_{81}} F_{11}$. We get somewhat "smaller" simple groups by starting with "smaller" non-residually finite groups $\pi_1(\Delta_A)$ and $\pi_1(\Delta_B)$ for the Aleshin and Bellaterra automata and the group $\pi_1(\Delta_D)$.

Example 6.1. The group $\Gamma = \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 | R_{4,4} \rangle$, where

$$R_{4,4} = \begin{cases} a_1b_1= b_1a_2, & a_1b_2= b_2a_2, & a_1b_3 = b_3a_1, & a_1b_4 = b_4^{-1}a_1, \\ a_2b_1= b_2a_1, & a_2b_2= b_1a_3, & a_2b_3 = b_3a_2, & a_2b_4 = b_4^{-1}a_2, \\ a_3b_1= b_2a_3, & a_3b_2= b_1a_1, & a_3b_3 = b_4a_3, & a_3b_4 = b_3a_4, \\ a_4b_1 = b_3a_4^{-1}, & a_4b_2 = b_4^{-1}a_4^{-1}, & a_4b_3 = b_4a_3^{-1}, & a_4b_2^{-1} = b_1a_4^{-1} \end{cases}$$

is non-residually finite and just-infinite. The intersection Γ_0 of all subgroups of finite index is a subgroup of index 4. The group Γ_0 is a finitely presented torsion-free simple group, and it decomposes into the amalgamated product $F_7 *_{F_{49}} F_7$.

Proof. The first six relations indicate that Γ contains $\pi_1(\Delta_A)$ for the Aleshin automaton. Hence, Γ is non-residually finite and Γ_0 contains $(b_1^{-1}b_2)^4$. One can directly check that Γ satisfies the other conditions of [5, Corollary 5.4] (here $P_h \cong P_v \cong A_8$), which implies that Γ_0 is simple. One can directly check with GAP that adding the relation $(b_1^{-1}b_2)^4$ to the presentation of Γ leads to a finite group of order 4. It follows that Γ_0 is the normal closure of $(b_1^{-1}b_2)^4$ and has index 4 in Γ. The amalgam decomposition of Γ_0 follows from [23, Proposition 1.4]. ∎

Example 6.2. The group $\Gamma = \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 | R_{4,4} \rangle$, where

$$R_{4,4} = \begin{cases} a_1b_1= b_2a_2, & a_1b_2= b_1a_2, & a_1b_3 = b_3a_1, & a_1b_4 = b_4^{-1}a_1, \\ a_2b_1= b_1a_1, & a_2b_2= b_2a_3, & a_2b_3 = b_3a_2, & a_2b_4 = b_4^{-1}a_2, \\ a_3b_1= b_1a_3, & a_3b_2= b_2a_1, & a_3b_3 = b_4a_4, & a_3b_4 = b_3a_3^{-1}, \\ a_4b_1 = b_1a_4, & a_4b_2 = b_3a_4, & a_4b_3 = b_4a_3^{-1}, & a_4b_4 = b_2a_4 \end{cases}$$

is non-residually finite and just-infinite. The intersection Γ_0 of all subgroups of finite index is a subgroup of index 4. The group Γ_0 is a finitely presented torsion-free simple group, and it decomposes into the amalgamated product $F_7 *_{F_{49}} F_7$.

Proof. The proof is as above. The group Γ contains $\pi_1(\Delta_B)$ for the Bellaterra automaton and satisfies [5, Corollary 5.4]. It follows that Γ_0 is simple and contains $a_1^2a_2^{-2}$ (see Example 4.14). Adding the relation $a_1^2a_2^{-2}$ to the presentation of Γ leads to a finite group of order 4. Hence, Γ_0 is the normal closure of $a_1^2a_2^{-2}$ and has index 4 in Γ. ∎

$$\left\{ \begin{array}{llll}
s_{18}s_{15}s_3, & s_{11}s_3^{-1}s_{20}, & s_9s_1s_7, & s_{10}s_9^{-1}s_1^{-1}, \\
s_{21}s_{19}s_8, & s_{10}s_8s_7^{-1}, & s_{20}s_5s_{13}, & s_{11}s_{13}s_5, \\
s_{14}s_9s_6^{-1}, & s_{17}s_9s_{16}, & s_{17}s_{12}s_4, & s_{16}s_4s_{12}, \\
s_8^{-1}s_3s_8^{-1}s_2^{-1}, & s_7s_3^{-1}s_6^{-1}s_2, & s_{14}s_8^{-1}s_6^{-1}s_1, & s_{13}s_8s_4s_7, \\
s_{17}s_2^{-1}s_{16}s_1, & s_{10}s_2s_7^{-1}s_3^{-1}, & s_8s_2^{-1}s_6^{-1}s_3, & s_{14}s_2^{-1}s_8s_3, \\
s_{17}s_3^{-1}s_{16}s_2, & s_{21}s_4^{-1}s_{19}s_9^{-1}, & s_{12}s_8s_5s_7, & s_8^{-1}s_9^{-1}s_8^{-1}s_1^{-1}, \\
s_{17}s_{13}s_{16}s_3, & s_{13}s_{10}^{-1}s_4s_8, & s_{18}s_4^{-1}s_6s_{12}^{-1}, & s_{12}s_{10}^{-1}s_5s_8, \\
s_{18}s_2^{-1}s_{15}s_2, & s_{18}s_1^{-1}s_{15}s_9^{-1}, & s_{15}s_{12}^{-1}s_{14}^{-1}s_4^{-1}, & s_{14}s_3^{-1}s_{10}^{-1}s_2, \\
s_{17}s_8^{-1}s_{16}s_5, & s_{21}s_5^{-1}s_6s_{13}^{-1}, & s_{18}s_5^{-1}s_{15}s_8, & s_{21}s_3^{-1}s_{19}s_3, \\
s_{21}s_2^{-1}s_{19}s_{12}^{-1}, & s_{21}s_1^{-1}s_{19}s_2, & s_{11}s_9s_{20}s_4, & s_{11}s_8^{-1}s_{20}s_1, \\
s_{18}s_3^{-1}s_{15}s_{13}^{-1}, & s_{11}s_2^{-1}s_{20}s_3, & s_{11}s_{12}s_{20}s_2, & s_{19}s_{13}^{-1}s_{14}^{-1}s_5^{-1}
\end{array} \right\}$$

Table 1. Relators of the torsion-free simple group Γ_0 from Example 6.3.

The following example was suggested by Rattaggi (private communication).

Example 6.3. The group $\Gamma = \langle a_1, a_2, a_3, b_1, b_2, b_3, b_4 | R_{3,4} \rangle$, where

$$R_{3,4} = \left\{ \begin{array}{llll}
\underline{a_1b_1 = b_2a_2}, & \underline{a_1b_2 = b_2^{-1}a_2}, & a_1b_3 = b_4^{-1}a_2^{-1}, & a_1b_4 = b_3^{-1}a_1, \\
\underline{a_2b_1 = b_1a_1^{-1}}, & \underline{a_2b_2 = b_1^{-1}a_1}, & a_2b_3 = b_4^{-1}a_1, & a_2b_4^{-1} = b_3a_3^{-1}, \\
a_3b_1 = b_4^{-1}a_3^{-1}, & a_3b_2 = b_2a_3, & a_3b_3 = b_1a_3^{-1}, & a_3b_4^{-1} = b_3a_2^{-1}
\end{array} \right\}$$

is non-residually finite and just-infinite. The intersection Γ_0 of subgroups of finite index is a subgroup of index 4. The group Γ_0 is a finitely presented torsion-free simple group, and it decomposes into amalgamated products $F_5 *_{F_{33}} F_5$ and $F_7 *_{F_{37}} F_7$. The group Γ_0 has a finite presentation with 21 generators and 44 relations of total length 164 given in Table 1.

Proof. The proof is as above. The group Γ contains $\pi_1(\Delta_D)$ and satisfies [5, Corollary 5.4] (here $P_h \cong A_8$ and $P_v \cong A_6$). It follows that Γ_0 is simple and contains a_1^6 (see the proof of Theorem 6.3). Adding the relation a_1^6 to the presentation of Γ leads to a finite group of order 4. Hence, Γ_0 is the normal closure of a_1^6 and has index 4 in Γ. ∎

Remark 6.4. According to [11], the class of finitely generated just-infinite groups splits into just-infinite branch groups, hereditary just-infinite groups, and just-infinite virtually simple groups. Many automaton groups, including the Grigorchuk group, provide examples of just-infinite branch groups. As we saw above, the groups $\pi_1(\Delta_A)$ associated to some bireversible automata provide examples of just-infinite virtually simple groups. It would be interesting if there is a natural way to construct hereditary just-infinite groups using automata.

References

[1] S. V. Aleshin, A free group of finite automata. *Mosc. Univ. Math. Bull.* **38** (1983), no. 4, 10–13 Zbl 0522.68054   MR 713968

[2] I. Bondarenko, D. D'Angeli, and E. Rodaro, The lamplighter group $\mathbb{Z}_3 \wr \mathbb{Z}$ generated by a bireversible automaton. *Comm. Algebra* **44** (2016), no. 12, 5257–5268   Zbl 1368.20033 MR 3520274

[3] I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, V. Nekrashevych, D. Savchuk, and Z. Šunić, On classification of groups generated by 3-state automata over a 2-letter alphabet. *Algebra Discrete Math.* **2008** (2008), no. 1, 1–163   Zbl 1164.20004   MR 2432182

[4] M. Burger and S. Mozes, Finitely presented simple groups and products of trees. *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), no. 7, 747–752   Zbl 0966.20013   MR 1446574

[5] M. Burger and S. Mozes, Lattices in product of trees. *Publ. Math. Inst. Hautes Études Sci.* **92** (2000), 151–194   Zbl 1007.22013   MR 1839489

[6] P.-E. Caprace and P. Wesolek, Indicability, residual finiteness, and simple subquotients of groups acting on trees. *Geom. Topol.* **22** (2018), no. 7, 4163–4204   Zbl 1439.20025 MR 3890774

[7] D. D'Angeli, T. Godin, I. Klimann, M. Picantin, and E. Rodaro, Boundary action of automaton groups without singular points and Wang tilings. 2016, arXiv:1604.07736

[8] P. Gillibert, The finiteness problem for automaton semigroups is undecidable. *Internat. J. Algebra Comput.* **24** (2014), no. 1, 1–9   Zbl 1292.20040   MR 3189662

[9] Y. Glasner and S. Mozes, Automata and square complexes. *Geom. Dedicata* **111** (2005), 43–64 Zbl 1088.20037   MR 2155175

[10] R. Grigorchuk and Z. Šunić, Self-similarity and branching in group theory. In *Groups St. Andrews 2005. Vol. 1*, pp. 36–95, London Math. Soc. Lecture Note Ser. 339, Cambridge Univ. Press, Cambridge, 2007   Zbl 1185.20044   MR 2327317

[11] R. I. Grigorchuk, Just infinite branch groups. In *New Horizons in Pro-p Groups*, pp. 121–179, Progr. Math. 184, Birkhäuser, Boston, MA, 2000   Zbl 0982.20024   MR 1765119

[12] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchansky, Automata, dynamical systems, and groups. *Proc. Steklov Inst. Math.* **231** (2000), 128–203   Zbl 1155.37311   MR 1841755

[13] T. Hsu and D. T. Wise, A non-residually finite square of finite groups. In *Groups St. Andrews 1997 in Bath, I*, pp. 368–378, London Math. Soc. Lecture Note Ser. 260, Cambridge Univ. Press, Cambridge, 1999   Zbl 0940.20028   MR 1676633

[14] T. Hsu and D. T. Wise, Groups with infinitely many types of fixed subgroups. *Israel J. Math.* **144** (2004), 93–107   Zbl 1098.20033   MR 2121535

[15] D. Janzen and D. T. Wise, A smallest irreducible lattice in the product of trees. *Algebr. Geom. Topol.* **9** (2009), no. 4, 2191–2201   Zbl 1220.20039   MR 2558308

[16] J. Kari and P. Papasoglu, Deterministic aperiodic tile sets. *Geom. Funct. Anal.* **9** (1999), no. 2, 353–369   Zbl 0938.05024   MR 1692474

[17] I. Klimann, The finiteness of a group generated by a 2-letter invertible-reversible Mealy automaton is decidable. In *30th International Symposium on Theoretical Aspects of Computer*

Science, pp. 502–513, LIPIcs. Leibniz Int. Proc. Inform. 20, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2013   Zbl 1354.68159   MR 3090008

[18] I. Klimann, M. Picantin, and D. Savchuk, A connected 3-state reversible Mealy automaton cannot generate an infinite Burnside group. In *Developments in Language Theory*, pp. 313–325, Lecture Notes in Comput. Sci. 9168, Springer, Cham, 2015   Zbl 1386.68106   MR 3440681

[19] O. Macedońska, V. Nekrashevych, and V. I. Sushchansky, Commensurators of groups and reversible automata. *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki* **2000** (2000), no. 12, 36–39   Zbl 0977.20022   MR 1841119

[20] Y. Muntyan and D. Savchuk, AutomGrp – GAP package for computations in self-similar groups and semigroups (Version 1.1.2). 2008

[21] V. Nekrashevych, *Self-similar Groups*. Math. Surveys Monogr. 117, American Mathematical Society, Providence, RI, 2005   Zbl 1087.20032   MR 2162164

[22] D. Rattaggi, Examples of square complexes. 2003, ETH Zürich

[23] D. Rattaggi, *Computations in groups acting on a product of trees: normal subgroup structures and quaternion lattices*. Ph.D. thesis, ETH Zürich, 2004

[24] D. Rattaggi, Anti-tori in square complex groups. *Geom. Dedicata* **114** (2005), 189–207   Zbl 1147.20039   MR 2174099

[25] D. Savchuk and Y. Vorobets, Automata generating free products of groups of order 2. *J. Algebra* **336** (2011), 53–66   Zbl 1239.20032   MR 2802530

[26] B. Steinberg, M. Vorobets, and Y. Vorobets, Automata over a binary alphabet generating free groups of even rank. *Internat. J. Algebra Comput.* **21** (2011), no. 1-2, 329–354   Zbl 1239.20033   MR 2787464

[27] J. Stix and A. Vdovina, Simply transitive quaternionic lattices of rank 2 over $\mathbb{F}_q(t)$ and a non-classical fake quadric. *Math. Proc. Cambridge Philos. Soc.* **163** (2017), no. 3, 453–498   Zbl 1405.11089   MR 3708519

[28] M. Vorobets and Y. Vorobets, On a free group of transformations defined by an automaton. *Geom. Dedicata* **124** (2007), 237–249   Zbl 1183.20024   MR 2318547

[29] M. Vorobets and Y. Vorobets, On a series of finite automata defining free transformation groups. *Groups Geom. Dyn.* **4** (2010), no. 2, 377–405   Zbl 1227.20027   MR 2595096

[30] D. T. Wise, *Non-positively curved squared complexes: Aperiodic tilings and non-residually finite groups*. Ph.D. thesis, Princeton University, 1996

[31] D. T. Wise, A flat plane that is not the limit of periodic flat planes. *Algebr. Geom. Topol.* **3** (2003), 147–154   Zbl 1042.20034   MR 1997317

[32] D. T. Wise, Complete square complexes. *Comment. Math. Helv.* **82** (2007), no. 4, 683–724   Zbl 1142.20025   MR 2341837

Ievgen Bondarenko
Mechanics and Mathematics Faculty, National Taras Shevchenko University of Kyiv, vul. Volodymyrska 64, 01033 Kyiv, Ukraine; ievgbond@gmail.com

Bohdan Kivva
Department of Mathematics, The University of Chicago, 5734 S University Ave, Chicago, IL 60637, USA; bkivva@math.uchicago.edu