Groups Geom. Dyn. 16 (2022), 341–361
DOI 10.4171/GGD/651

Hyperbolicity of $T(6)$ cyclically presented groups

Ihechukwu Chinyere and Gerald Williams

Abstract. We consider groups defined by cyclic presentations where the defining word has length 3 and the cyclic presentation satisfies the $T(6)$ small cancellation condition. We classify when these groups are hyperbolic. When combined with known results, this completely classifies the hyperbolic $T(6)$ cyclically presented groups.

1. Introduction

Groups defined by presentations that satisfy the $C(p)-T(q)$ (non-metric) small cancellation conditions where $1/p + 1/q < 1/2$ are hyperbolic [16, Corollary 3.3]. Therefore the cases $(p,q) = (3,6), (4,4), (6,3)$ present boundary cases and here both hyperbolic and non-hyperbolic groups can arise. For these cases, in [22, Corollary, p. 1860] the $C(p)-T(q)$ presentations that define hyperbolic groups are characterised as those for which there is no minimal flat over the presentation. In this article we consider groups defined by a class of presentations that admit a certain cyclic symmetry and satisfy $C(3)-T(6)$. We classify when the corresponding groups are hyperbolic in terms of the defining parameters of the presentations.

The *cyclically presented group* $G_n(w)$ is the group defined by the *cyclic presentation*

$$P_n(w) = \langle x_0, \dots, x_{n-1} \mid w, \theta(w), \dots, \theta^{n-1}(w) \rangle,$$

where $w(x_0, \dots, x_{n-1})$ is a cyclically reduced word in the free group F_n (of length $l(w)$) with generators $x_0, \dots, x_{n-1}$ and $\theta : F_n \to F_n$ is the *shift automorphism* of F_n given by $\theta(x_i) = x_{i+1}$ for each $0 \le i < n$ (subscripts mod n, $n \ge 2$).

If a presentation satisfies $T(6)$ then, as observed by Pride (see [28, Section 5] and [16, Lemma 3.1]), every piece has length 1 and so if $P_n(w)$ satisfies $T(6)$, then it satisfies $C(l(w)) - T(6)$. Thus if $l(w) > 3$, then the presentation $P_n(w)$ satisfies $C(4) - T(6)$, and hence $G_n(w)$ is hyperbolic by [16, Corollary 4.1] and, therefore, it is non-elementary hyperbolic by [10] or [12]. If the length $l(w) = 1$, then $G_n(w)$ is trivial, and if $l(w) = 2$, then $G_n(w)$ is the free product of copies of $\mathbb{Z}$ or $\mathbb{Z}_2$. Therefore we must consider the case $l(w) = 3$ (in which case the $C(3) - T(6)$ condition coincides with the $T(6)$ condition). If w is a positive (or negative) word, then we may assume that $w = x_0 x_k x_l$, and if w is non-positive (and non-negative), then we may assume $w = x_0 x_m x_k^{-1}$. Our main results consider these cases.

2020 Mathematics Subject Classification. 20F05, 20F06, 20F67.
Keywords. Cyclically presented group, hyperbolic group, small cancellation theory.

The groups $G_n(x_0x_mx_k^{-1})$ are known as the *groups of Fibonacci-type* and were introduced independently in [7, 23], for topological and algebraic reasons, respectively. This family of groups contains the *Fibonacci groups* $F(2,n) = G_n(x_0x_1x_2^{-1})$ of [11], the *Sieradski groups* $S(2,n) = G_n(x_0x_2x_1^{-1})$ of [29], and the *Gilbert–Howie groups* $H(n,m) = G_n(x_0x_mx_1^{-1})$ of [17]. They have been subsequently studied in [1, 8, 20, 21, 30]—see [31] for a survey. In particular, the $T(6)$ and $T(7)$ presentations $P_n(x_0x_mx_k^{-1})$ were classified in [20, Theorem 10] (see Corollary 3.2, below) and [20, Theorem 11] records that in the $T(7)$ case the groups $G_n(x_0x_mx_k^{-1})$ are non-elementary hyperbolic. The groups $G_n(x_0x_kx_l)$ were introduced in [9] and studied further in [4, 14, 27]. The $T(6)$ presentations $P_n(x_0x_kx_l)$ were classified in [14, Lemma 5.1] (see Lemma 2.1, below). Moreover, [27, Theorem 3.7] shows that for all but finitely many n the $T(6)$ groups $G_n(x_0x_kx_l)$ are hyperbolic.

Our main results are as follows.

Theorem A. *Let $n \geq 2$, $0 \leq k, l < n$, $(n,k,l) = 1$, and suppose that the cyclic presentation $P_n(x_0x_kx_l)$ is $T(6)$. Let $G = G_n(x_0x_kx_l)$. If $n = 7$ or 8 or*

(a) *$n = 21$ and ($l \equiv 5k$ or $k \equiv 5l \bmod n$) or*

(b) *$n = 24$ and ($l \equiv 5k$ or $k \equiv -4l$ or $l \equiv -4k \bmod n$),*

then G is not hyperbolic; otherwise G is non-elementary hyperbolic.

Theorem B. *Let $n \geq 2$, $0 \leq m, k < n$, $(n,m,k) = 1$, $m \neq k$, $k \neq 0$, and suppose that the cyclic presentation $P_n(x_0x_mx_k^{-1})$ is $T(6)$. Let $G = G_n(x_0x_mx_k^{-1})$. If $n = 8$ or ($n \geq 12$ is even and $2(2k - m) \equiv 0 \bmod n$), then G is not hyperbolic; otherwise G is non-elementary hyperbolic.*

The coprimality hypotheses of Theorems A and B are imposed to avoid the presentations and groups decomposing in canonical ways. Specifically, if $d = (n,k,l) > 1$, then the presentation $P_n(x_0x_kx_l)$ is the disjoint union of d copies of the presentation $P_{n/d}(x_0x_{k/d}x_{l/d})$ [9, Lemma 2.4] and so $G_n(x_0x_kx_l)$ is the free product of d copies of $G_{n/d}(x_0x_{k/d}x_{l/d})$ and $P_n(x_0x_kx_l)$ satisfies $T(6)$ if and only if $P_{n/d}(x_0x_{k/d}x_{l/d})$ satisfies $T(6)$. Similarly if $d = (n,m,k) > 1$, then $P_n(x_0x_mx_k^{-1})$ is the disjoint union of d copies of $P_{n/d}(x_0x_{m/d}x_{k/d}^{-1})$ [1, Lemma 1.2]; so the analogous conclusions can be drawn in this case. Since a free product $H * K$ is hyperbolic if and only if H and K are hyperbolic—see, for example, [2, Theorem H]—there is no loss in generality in assuming that such decompositions do not arise. The conditions $m \neq k$, $k \neq 0$ are imposed in Theorem B to ensure that the relators are cyclically reduced and, as otherwise, the group is trivial.

A relator of a presentation is *freely redundant* if it is freely equal to the conjugate of another relator or its inverse. The cyclic presentations $P_n(x_0x_mx_k^{-1})$ have no freely redundant relators, and if $(n,k,l) = 1$, the presentation $P_n(x_0x_kx_l)$ has a freely redundant relator if and only if $n = 3$ and $\{k,l\} = \{1,2\}$, in which case it defines the (non-elementary hyperbolic) group $\mathbb{Z} * \mathbb{Z}$. For this reason, throughout this article we will only consider presentations with no freely redundant relators.

A consequence of Theorem A and the results of [14] (see also [27, Example 10, Propositions 7.7 and 7.8]) is that the hyperbolicity status of cyclically presented groups G of the form $G_n(x_0x_kx_l)$ is known, except when G is isomorphic to $G_n(x_0x_1x_{n/2-1})$ for even $n \geq 10$, $n \neq 12, 18$.

We prove Theorem A in Section 2 and Theorem B in Section 3.

2. The positive case

Let $P = \langle \mathcal{X} \mid \mathcal{R} \rangle$ be a group presentation such that each relator $r \in \mathcal{R}$ is a cyclically reduced word in the generators. Let $\widetilde{\mathcal{R}}$ denote the symmetrized closure of $\mathcal{R}$; that is, the set of all cyclic permutations of elements in $\mathcal{R} \cup \mathcal{R}^{-1}$. The *star graph* of P is the undirected graph with vertex set $\mathcal{X} \cup \mathcal{X}^{-1}$, and with an edge joining vertices x, y for each word $xy^{-1}u$ in $\widetilde{\mathcal{R}}$. These words occur in pairs: $xy^{-1}u \in \widetilde{\mathcal{R}}$ implies that $yx^{-1}u^{-1} \in \widetilde{\mathcal{R}}$. Such pairs are called *inverse pairs* and the two corresponding edges are identified in Γ [25, p. 61]. Thus if Γ is the star graph of the cyclic presentation $P_n(x_0x_kx_l)$, then Γ has vertices x_i and x_i^{-1} and edges $x_i - x_{i+k}^{-1}$, $x_i - x_{i+l-k}^{-1}$, and $x_i - x_{i-l}^{-1}$ $(0 \leq i < n)$, which we will refer to as edges of type X, Y, and Z, respectively.

By [18] a presentation in which each relator has length at least 3 satisfies $T(q)$ $(q > 3)$ if and only if its star graph has no cycle of length less than q. As we are interested in presentations that satisfy $T(6)$, in Section 2.1 we analyse cycles of length at most 6 in the star graph Γ of $P_n(x_0x_kx_l)$. In particular, we note that Γ always contains a cycle of length at most 6. We show that if two additional cycle types of length 6 arise, then only a few small values of n are possible and $G_n(x_0x_kx_l)$ is isomorphic to one of only a few groups, one of which turns out to be hyperbolic. In Section 2.2 we prove that the remainder are not hyperbolic. In Section 2.3 we consider the case when at most one further cycle type of length 6 occurs and perform a detailed analysis of van Kampen diagrams (see [25, Chapter 5]) over the defining presentation to prove that $G_n(x_0x_kx_l)$ has a linear isoperimetric function, and hence is hyperbolic. We then combine these results to prove Theorem A in Section 2.4.

2.1. Analysis of short cycles in the star graph of $P_n(x_0x_kx_l)$

The following classification of the $T(6)$ cyclic presentations $P_n(x_0x_kx_l)$ in terms of three types of congruences (B), (C), and (D) was obtained in [14]. As indicated in Table 1, the (B) conditions correspond to cycles (of length 2) of the form XY, YZ, and ZX; the (C) conditions correspond to cycles (of length 4) of the form $XZYZ$, $YXZX$, and $ZYXY$; and the (D) conditions correspond to cycles (of length 4) of the form $(XY)^2$, $(YZ)^2$, and $(ZX)^2$, as well as to cycles (of length 6) of the form $XYZYXZ$, $YZXZYX$, and $ZXYXZY$. Replacing parameter k by $l-k$ and l by $-k$ corresponds to replacing edge type X by Y, Y by Z, and Z by X and to replacing a condition $(*.j)$ of Table 1 by $(*.j + 1) \pmod 3$, and replacing the group $G_n(x_0x_kx_l)$ by the isomorphic copy $G_n(x_0x_{l-k}x_{-k})$. (To see that $G_n(x_0x_kx_l) \cong G_n(x_0x_{l-k}x_{-k})$ set $j = i + k$ in the relators $x_ix_{i+k}x_{i+l}$ of $P_n(x_0x_kx_l)$ and then cyclically permute to get the relators $x_jx_{j+l-k}x_{j-k}$

	j	**0**	**1**	**2**
$(B.j)$	congruence	$2k-l\equiv 0$	$2l-k\equiv 0$	$k+l\equiv 0$
	cycle type	XY	YZ	ZX
$(C.j)$	congruence	$l\equiv \pm\frac{n}{3}$	$k\equiv \pm\frac{n}{3}$	$k-l\equiv \pm\frac{n}{3}$
	cycle type	$XZYZ$	$YXZX$	$ZYXY$
$(D.j)$	congruence	$2k-l\equiv \frac{n}{2}$	$2l-k\equiv \frac{n}{2}$	$k+l\equiv \frac{n}{2}$
	cycle type	$(XY)^2$ or $XYZYXZ$	$(YZ)^2$ or $YZXZYX$	$(ZX)^2$ or $ZXYXZY$
$(E.j)$	congruence	$2k-l\equiv \pm\frac{n}{3}$	$2l-k\equiv \pm\frac{n}{3}$	$k+l\equiv \pm\frac{n}{3}$
	cycle type	$(XY)^3$	$(YZ)^3$	$(ZX)^3$
$(F1.j)$	congruence	$5k-l\equiv 0$	$5l-4k\equiv 0$	$k+4l\equiv 0$
	cycle type	$(XY)^2XZ$	$(YZ)^2YX$	$(ZX)^2ZY$
$(F2.j)$	congruence	$4l-5k\equiv 0$	$4k+l\equiv 0$	$5l-k\equiv 0$
	cycle type	$(YX)^2YZ$	$(XZ)^2XY$	$(ZY)^2ZX$

Table 1. Congruences $(\bmod\, n)$ corresponding to short cycles in the star graph of $P_n(x_0x_kx_l)$.

of $P_n(x_0x_{l-k}x_{-k})$.) Replacing parameter k by $l-k$ corresponds to interchanging the roles of edge types X and Y and so interchanging the roles of conditions $(F1.j)$ and $(F2.j)$, and replacing the group $G_n(x_0x_kx_l)$ by the isomorphic copy $G_n(x_0x_{l-k}x_l)$. (To see that $G_n(x_0x_kx_l)\cong G_n(x_0x_{l-k}x_l)$ replace the generators x_i by x_i^{-1}, negate the subscripts, and set $j=-i-l$ in the relators $x_ix_{i+k}x_{i+l}$ and then invert to get the relators $x_jx_{j+l-k}x_{j+l}$ of $P_n(x_0x_{l-k}x_l)$.)

Lemma 2.1 ([14, Lemma 5.1]). *Let $n\geq 2$ and suppose that $(n,k,l)=1$, $0\leq k,\ l<n$. Then $P_n(x_0x_kx_l)$ satisfies $T(6)$ if and only if none of the congruences $(B.j)$, $(C.j)$ or $(D.j)$ $(0\leq j\leq 2)$ of Table 1 holds.*

Observation 2.2 (see [27, Theorem 3.4]). *Suppose that $(n,k,l)=1$, $0\leq k,\ l<n$, and that none of the congruences $(B.j)$, $(C.j)$ or $(D.j)$ $(0\leq j\leq 2)$ of Table 1 holds. Then for each $0\leq i<n$ the sequence of vertices and edges $x_i - x_{i+k}^{-1} - x_{i+2k-l} - x_{i+2k-2l}^{-1} - x_{i+k-2l} - x_{i-l}^{-1} - x_i$ forms a cycle of length 6 of the form $(XYZ)^2$ in the star graph Γ.*

We now consider how other cycles of length 6 can arise in Γ.

Lemma 2.3. *Let $n\geq 2$. Suppose that $(n,k,l)=1$, $0\leq k,\ l<n$, and that none of the congruences $(B.j)$, $(C.j)$ or $(D.j)$ $(0\leq j\leq 2)$ of Table 1 holds. Then the star graph Γ contains a cycle of length 6 of cycle type other than $(XYZ)^2$ if and only if at least one of the congruences $(E.j)$, $(F1.j)$ or $(F2.j)$ $(0\leq j\leq 2)$ of Table 1 holds, in which case the corresponding entry of the table is a label of the cycle.*

Proof. Let C be a cycle of length 6 in Γ. Then there are no subpaths of C of the form XX, YY or ZZ. If C involves each of the edge types X, Y, Z twice, then C is a cycle of the form $(XYZ)^2$, $XYZYXZ$, $YZXZYX$ or $ZXYXZY$. But these last three cycles only occur if the congruence $(D.j)$ holds, contrary to hypothesis. If C does not involve an edge of type X (resp. Y, resp. Z), then C is a cycle of the form $(YZ)^3$ (resp. $(XZ)^3$, resp. $(XY)^3$), which correspond to the conditions $(E.j)$. If C involves exactly one edge of type X (resp. Y, resp. Z), then C is a cycle of the form $(YZ)^2YX$ or $(ZY)^2ZX$ (resp. $(ZX)^2ZY$ or $(XZ)^2XY$, resp. $(XY)^2XZ$ or $(YX)^2YZ$), which corresponds to the conditions $(F1.j)$ or $(F2.j)$.

Conversely, if any of the congruences $(E.j)$, $(F1.j)$ or $(F2.j)$ holds, then the corresponding entry of Table 1 is the label of a cycle of length 6 in Γ. ∎

Lemma 2.4. *Let $n \geq 2$. Suppose that $(n,k,l) = 1$, $0 \leq k,\ l < n$, and that none of the congruences $(B.j)$, $(C.j)$ or $(D.j)$ $(0 \leq j \leq 2)$ holds. If more than one of the congruences $(E.j)$, $(F1.j)$, and $(F2.j)$ $(0 \leq j \leq 2)$ hold, then one of the following holds:*

(a) *$n = 7$ and ($l \equiv 5k$ or $k \equiv 5l \bmod n$);*

(b) *$n = 8$ and ($l \equiv 5k$ or $k \equiv 5l \bmod n$);*

(c) *$n = 21$ and ($l \equiv 5k$ or $k \equiv 5l \bmod n$);*

(d) *$n = 24$ and ($l \equiv 5k$ or $k \equiv -4l$ or $l \equiv -4k \bmod n$);*

(e) *$n = 27$ and ($l \equiv 5k$ or $k \equiv 5l$ or $4k \equiv 5l$ or $4l \equiv 5k$ or $k \equiv -4l$ or $l \equiv -4k \bmod n$).*

In each case $G \cong G_n(x_0x_1x_5)$.

Proof. (Throughout this proof, the j value in a condition $(*.j)$ is to be taken mod 3.) If $(E.j)$ and $(F1.j)$ hold, then $(B.j+1)$ holds, a contradiction. If $(E.j)$ and $(F1.j+1)$ hold, then $(B.j+2)$ holds. If $(E.j)$ and $(F2.-j)$ hold, then $(B.j+2)$ holds. If $(E.j)$ and $(F2.1-j)$ hold, then $(B.j+1)$ or $(D.j+1)$ holds. If $(F1.j)$ and $(F2.-j)$ hold, then $(C.j)$ holds. If $(F1.j)$ and $(F2.1-j)$ hold, then $(B.j+1)$ holds. Suppose now that any two of the $(E.j)$ conditions hold; then all three of them hold. Since $(B.0)$ does not hold, condition $(E.0)$ implies $2k - l \equiv \pm n/3 \bmod n$ and since $(B.2)$ does not hold, condition $(E.2)$ implies $k + l \equiv \pm n/3 \bmod n$. Thus $2k - l \equiv \epsilon(k+l)$, where $\epsilon = \pm 1$. If $\epsilon = +1$, then $(B.1)$ holds, a contradiction; and if $\epsilon = -1$, then $(C.0)$ or $(C.1)$ holds, a contradiction.

Suppose that two of the $(F1.j)$ conditions hold. Then all of them hold so, in particular, $l \equiv 5k \bmod n$. Summing the congruences $(F1.0)$ and $(F1.1)$ gives that $k \equiv -4l \bmod n$ and so (by $(F1.0)$) $21l \equiv 0 \bmod n$. Moreover $1 = (n,k,l) = (n,-4l,l) = (n,l)$ so $n|21$. If $n = 3$, then $(F1.0)$ implies that $(B.0)$ holds, so $n = 7$ or 21. An analogous argument shows that if two of the $(F2.j)$ conditions hold, then $k \equiv 5l$ and $n = 7$ or 21, thus giving cases (a), (c) of the statement.

Suppose that $(F1.j)$ and $(F2.2-j)$ hold. We claim that $n = 8$ or 24; it then follows from one of the congruences that $l \equiv 5k$ or $k \equiv 5l \bmod n$ (by multiplying by 5, if necessary), giving cases (b) and (d). We prove this in the case $(F1.0)$ and $(F2.2)$, the other cases being similar. The congruence $(F1.0)$ implies $l \equiv 5k \bmod n$, so substituting into $(F2.2)$ gives $24k \equiv 0 \bmod n$, but $1 = (n,k,l) = (n,k)$ so $n|24$. If $n \leq 6$, then some condition $(B.j)$, $(C.j)$ or $(D.j)$ holds, and if $n = 12$, then $(B.2)$ or $(D.2)$ holds, a contradiction; thus $n = 8$ or 24, as claimed.

Suppose that either $((E.j)$ and $(F1.j+2))$ or $((E.j)$ and $(F2.2-j))$ hold. We claim that $n=27$. We prove this in the case where $(E.0)$ and $(F1.2)$ hold, the other cases being proved analogously. The congruence $(F1.2)$ implies $k \equiv -4l \bmod n$ so $(E.0)$ implies $27l \equiv 0 \bmod n$, but $1=(n,k,l)=(n,l)$ so $n|27$. If $n=3$ or 9, then $(B.0)$ holds, and hence $n=27$, as claimed.

The final assertion that $G_n(x_0x_kx_l) \cong G_n(x_0x_1x_5)$ in each case follows from [14, Lemma 2.1]. ∎

We now deal with the group arising in case (e) of Lemma 2.4.

Example 2.5 (The group $G_{27}(x_0x_1x_5)$). Using KBMAG [19], it is straightforward to show that the group $G_{27}(x_0x_1x_5)$ is hyperbolic, and since it contains a non-abelian free subgroup (by [14, Corollary 5.2]), it is non-elementary hyperbolic.

2.2. Non-hyperbolic groups $G_n(x_0x_kx_l)$

In this section, we consider the groups arising in cases (a)–(d) of Lemma 2.4. First we recall that the group $G_7(x_0x_1x_5)$ is not hyperbolic; see [27, Example 3.8] for a discussion.

Lemma 2.6 ([6, 13]). *The group $G_7(x_0x_1x_5)$ is not hyperbolic.*

We now show that the group $G_8(x_0x_1x_5)$ is not hyperbolic. We do this by an application of the Flat Plane Theorem [5] (an alternative approach would be to use [22, Corollary, p. 1860]).

Lemma 2.7. *The group $G_8(x_0x_1x_5)$ is not hyperbolic.*

Proof. Since the presentation $P_8(x_0x_1x_5)$ satisfies $C(3)-T(6)$ and each relator has length 3, each face in the geometric realisation $\widetilde{C}$ of the Cayley complex of P (obtained by assigning length 1 to each edge) is an equilateral triangle, and so $\widetilde{C}$ satisfies the CAT(0) inequality. Consider the geometric realisation Δ_0 of the reduced van Kampen diagram given in Figure 1 and for each $0 \le i < n$ let Δ_i be obtained from Δ_0 by applying the shift θ^i to each edge. Then placing $\Delta_0, \Delta_2, \Delta_4, \Delta_6$ one above the other gives the geometric realisation Δ of a reduced van Kampen diagram. Copies of Δ tile the Euclidean plane without cancellation of faces. Thus there is an isometric embedding of the Euclidean plane in $\widetilde{C}$, and so the result follows from the corollary to Theorem A in [5]. ∎

For later reference (in Section 3) we note that the relabelling of generators $y_0=x_0$, $y_1=x_7^{-1}$, $y_2=x_2$, $y_3=x_1^{-1}$, $y_4=x_4$, $y_5=x_3^{-1}$, $y_6=x_6$, and $y_7=x_5^{-1}$ shows that $G_8(x_0x_1x_5) \cong G_8(y_0y_4y_1^{-1})$, and so we have the following corollary.

Corollary 2.8. *The group $H(8,4)=G_8(x_0x_4x_1^{-1})$ is not hyperbolic.*

Remark 2.9. The van Kampen diagram arising in the proof of Lemma 2.7, and later the one in the proof of Lemma 3.6, provides a pair of commuting elements whose axes in the geometric realisation Δ meet at an angle $2\pi/3$. It follows that the groups considered in these results contain a free abelian subgroup of rank 2 (see, for example, [32, p. 446]).

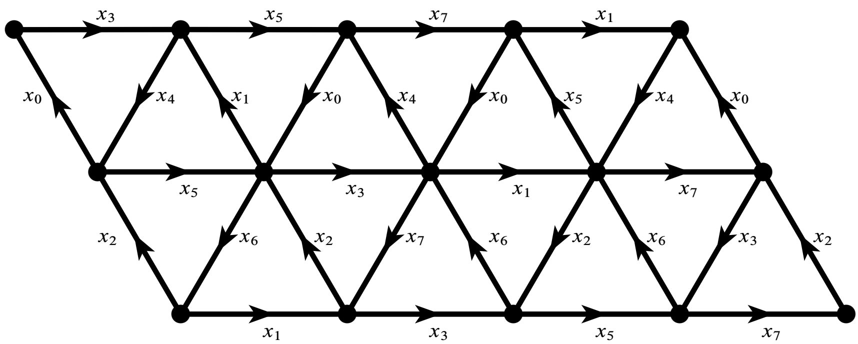

Figure 1. A van Kampen diagram over the presentation $P_8(x_0x_1x_5)$ with boundary label $(x_2x_0)(x_3x_5x_7x_1)(x_2x_0)^{-1}(x_1x_3x_5x_7)^{-1}$.

In Corollaries 2.10 and 2.11 we use Lemmas 2.6 and 2.7, respectively, to prove that the groups in cases (c) and (d) of Lemma 2.4 are not hyperbolic. To do this we first recall the shift extension of a cyclically presented group. The shift automorphism θ of a cyclically presented group $G_n(w)$ results in a $\mathbb{Z}_n$-action on $G_n(w)$ that determines the *shift extension* $E_n(w) = G_n(w) \rtimes_\theta \mathbb{Z}_n$, which admits a two-generator two-relator presentation of the form

$$E_n(W) = \big\langle x, t \mid t^n, W(x,t)\big\rangle,$$

where $W = W(x,t)$ is obtained by rewriting w in terms of the substitutions $x_i = t^i x t^{-i}$ (see, for example, [24, Theorem 4]). Thus there is a retraction $\nu^0 : E_n(W) \to \mathbb{Z}_n$ given by $\nu^0(t) = t$, $\nu^0(x) = t^0 = 1$ with kernel $G_n(w)$. Moreover, as shown in [3, Section 2], for certain values of f $(0 \le f < n)$ there may be further retractions ν^f. Specifically, by [3, Theorem 2.3] the kernel of a retraction $\nu^f : E_n(W) \to \mathbb{Z}_n$ given by $\nu^f(t) = t$, $\nu^f(x) = t^f$ is cyclically presented, generated by the elements $y_i = t^i x t^{-(i+f)}$ $(0 \le i < n)$. Since (non-elementary) hyperbolicity is preserved under taking finite index subgroups and finite extensions, the group $E_n(W)$ is (non-elementary) hyperbolic if and only if the kernel of any, and hence all, of its retractions ν^f is (non-elementary) hyperbolic.

In the case $w = x_0x_kx_l$ we have

$$E_n(W) = G_n(w) \rtimes_\theta \langle t \mid t^n\rangle = \langle x, t \mid t^n, xt^kxt^{l-k}xt^{-l}\rangle$$

which admits a retraction $\nu^f : E_n \to \langle t \mid t^n\rangle$ given by $\nu^f(t) = t$, $\nu^f(x) = t^f$ if and only if $3f \equiv 0 \bmod n$; the kernel of such a retraction is the cyclically presented group $G_n(x_0x_{f+k}x_{2f+l})$ (see [3, p. 158]).

Corollary 2.10. *The group $G_{21}(x_0x_1x_5)$ is not hyperbolic.*

Proof. The free product of three copies of $G_7(x_0x_1x_5)$ is the cyclically presented group $G_{21}(x_0x_3x_{15})$ with shift extension $E = \langle x, t \mid t^{21}, xt^3xt^{12}xt^{-15}\rangle$. The kernel of the retraction $\nu^7 : E \to \mathbb{Z}_n = \langle t \mid t^{21}\rangle$ given by $\nu^7(t) = t$, $\nu^7(x) = t^7$ is the group $G_{21}(x_0x_{10}x_8)$

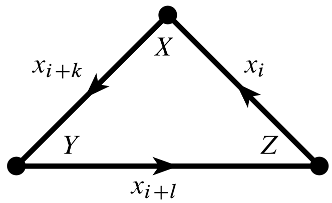

Figure 2. A typical face in a van Kampen diagram over the presentation $P_n(x_0x_kx_l)$.

which, by [14, Lemma 2.1 (iv), (v)], is isomorphic to $G_{21}(x_0x_1x_5)$. Since $G_7(x_0x_1x_5)$ is not hyperbolic, neither is $G_{21}(x_0x_3x_{15})$, nor E, and hence, nor is $G_{21}(x_0x_1x_5)$. ∎

Corollary 2.11. *The group $G_{24}(x_0x_1x_5)$ is not hyperbolic.*

Proof. The free product of three copies of $G_8(x_0x_1x_5)$ is the cyclically presented group $G_{24}(x_0x_3x_{15})$ with shift extension $E = \langle x, t \mid t^{24}, xt^3xt^{12}xt^{-15}\rangle$. The kernel of the retraction $\nu^8 : E \to \mathbb{Z}_{24} = \langle t \mid t^{24}\rangle$ given by $\nu^8(t) = t$, $\nu^8(x) = t^8$ is the group $G_{24}(x_0x_{11}x_7)$ which, by [14, Lemma 2.1 (v), (ii)], is isomorphic to $G_{24}(x_0x_1x_5)$. Since $G_8(x_0x_1x_5)$ is not hyperbolic, neither is $G_{24}(x_0x_3x_{15})$, nor E, and hence, nor is $G_{24}(x_0x_1x_5)$. ∎

2.3. Analysis of van Kampen diagrams over $P_n(x_0x_kx_l)$

In this section, we show that if the cyclic presentation $P = P_n(x_0x_kx_l)$ is $T(6)$ and at most one of the congruences $(E.j)$, $(F1.j)$ or $(F2.j)$ holds, then $G = G_n(x_0x_kx_l)$ is hyperbolic. Following the method of proof of [20, Theorem 13], we show that G has a linear isoperimetric function [15, Theorem 3.1]. That is, we show that there is a linear function $f : \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ and all freely reduced words $W \in F_n$ with length at most N that represent the identity of G we have $\mathrm{Area}(W) \leq f(N)$, where $\mathrm{Area}(W)$ denotes the minimum number of faces in a reduced van Kampen diagram over P with boundary label W. Without loss of generality, we may assume that the boundary of such a van Kampen diagram D is a simple closed curve. Note that each face in D is a triangle, as shown in Figure 2, where the corner labels X, Y, Z correspond to the edge types of the star graph of P. We say that a vertex of D is a *boundary vertex* if it lies on ∂D, and is an *interior vertex* otherwise. In order to obtain a linear isoperimetric function (in Lemma 2.16) we first carefully analyse degrees of vertices within D.

Lemma 2.12. *Let Δ be an interior face of D in which two of the vertices have label $(XYZ)^2$. Then the label of the third vertex contains a subword of the form aba, where b is the label of the corner of Δ at this vertex, and $a, b \in \{X, Y, Z\}$, $a \neq b$.*

Proof. Without loss of generality, we may assume that the edges of Δ are oriented in an anticlockwise manner. We name its vertices v_1, v_2, v_3, read in an anticlockwise manner, and suppose v_1, v_2 are labelled $(XYZ)^2$. If the corner of Δ at v_1 has label X (resp. Y, resp. Z), then the corner of Δ at v_2 has label Y (resp. Z, resp. X), in which case the label of v_3 has a subword YZY or XZX (resp. ZXZ or YXY, resp. XYX or ZYZ), as shown in Figure 3. ∎

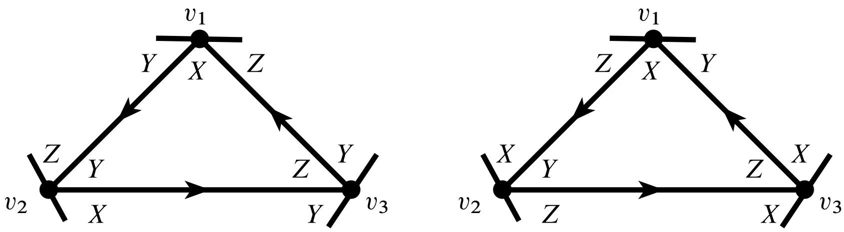

Figure 3. Possible configurations when two vertices have label $(XYZ)^2$.

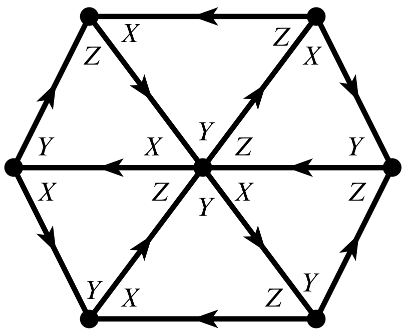

Figure 4. Neighbourhood of an interior vertex labelled $(XYZ)^2$.

Lemma 2.13. *If an interior vertex v of D of degree 6 has label $(XYZ)^2$, then two adjacent neighbours of v have XY as a cyclic subword of their labels, two adjacent neighbours have XZ as a cyclic subword of their labels, and two adjacent neighbours have YZ as a cyclic subword of their labels.*

Proof. If the label of v is $(XYZ)^2$ oriented clockwise, then the neighbourhood of v is as given in Figure 4, from which the conclusion can be observed. A similar figure deals with the case when the label of v is $(XYZ)^2$ oriented anticlockwise. ∎

Lemma 2.14. *Suppose that all interior vertices of D have degree at least 6 and all labels of interior vertices of degree 6 are either $(XYZ)^2$ or label $(E.j)$ for precisely one $j \in \{0, 1, 2\}$. If v is an interior vertex of degree 6 in D with label $(E.j)$ and where all the neighbours of v are interior vertices of degree 6 then every neighbour of v has two neighbours which are either boundary vertices or have degree at least 8.*

Proof. Consider first the case $(E.0)$, that is, a vertex label $(XY)^3$. As shown in Figure 5 all the neighbours of v must have label $(XYZ)^2$. Then each of the vertices $u_1, \ldots, u_6$ has a corner labelled Z. If a vertex u_i $(1 \le i \le 6)$ is interior, then if it is of degree 6, its label is not $(XYZ)^2$, by Lemma 2.12, and so it must be $(XY)^3$, a contradiction. Therefore u_i

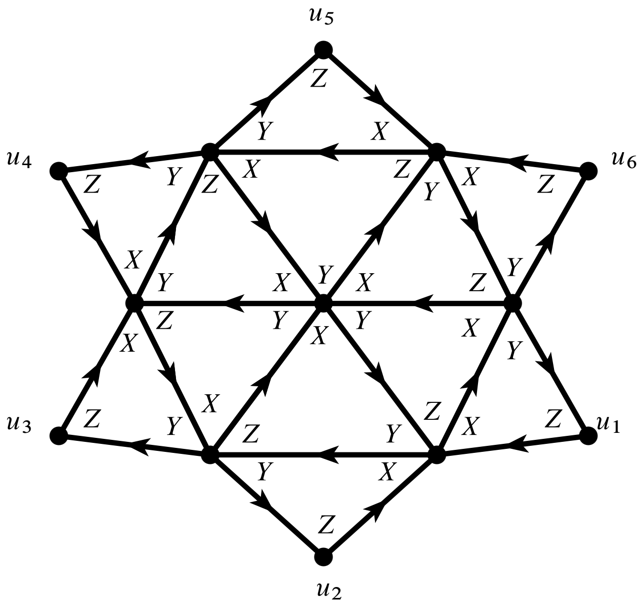

Figure 5. Neighbourhood of an interior vertex labelled $(E.0)$ and no boundary neighbours.

is either interior of degree at least 8, or a boundary vertex, as required. The cases $(E.1)$ and $(E.2)$ are dealt with by replacing X by Y, Y by Z, and Z by X, as explained in Section 2.1. ∎

Lemma 2.15. *Suppose that all interior vertices of D have degree at least 6 and all labels of interior vertices of degree 6 are either $(XYZ)^2$ or $(F1.j)$ (resp. $(F2.j)$) for precisely one $j \in \{0, 1, 2\}$. If v is an interior vertex of degree 6 in D with label $(F1.j)$ (resp. $(F2.j)$) and where all the neighbours of v are interior vertices, then v has a neighbour of degree at least 8.*

Proof. Consider the case $(F1.0)$, that is, v has label $(XY)^2XZ$ and suppose that all neighbours of v have degree 6. Then Figure 6 shows one of the two possible labellings of neighbours that can occur. Since two adjacent neighbours have YZ as a cyclic subword of their label, these must each be labelled $(XYZ)^2$, but this is impossible by Lemma 2.12; therefore v has a neighbour of degree at least 8. The same conclusion can be obtained if the second possible labelling of neighbours occurs. The cases $(F1.1)$ and $(F1.2)$ are dealt with by replacing X by Y, Y by Z, and Z by X. The cases $(F2.j)$ are obtained from the cases $(F1.j)$ by interchanging the roles of X and Y, as described in Section 2.1. ∎

We are now in a position to be able to establish the existence of a suitable isoperimetric function.

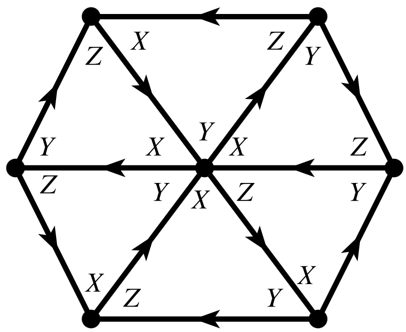

Figure 6. Neighbourhood of an interior vertex labelled $(F1.0)$.

Lemma 2.16. *Let $n \geq 2$ and suppose that none of the congruences $(B.j)$, $(C.j)$ or $(D.j)$ holds $(0 \leq j \leq 2)$ and that at most one congruence $(E.j)$, $(F1.j)$ or $(F2.j)$ holds $(j \in \{0,1,2\})$. Then $G_n(x_0x_kx_l)$ has a linear isoperimetric function.*

Proof. As at the beginning of this section, let $N \in \mathbb{N}$, let W be a freely reduced word in the free group F_n (with generators $x_0, \ldots, x_{n-1}$) of length at most N that represents the identity of G, and let D be a reduced van Kampen diagram whose boundary ∂D is a simple closed curve with label W. We let I denote the set of interior vertices of D, B the set of boundary vertices of D, and F the set of faces of D. Then $\mathrm{Area}(W) \leq |F|$. Writing π to denote 180, we define the curvature of a face f by $\kappa(f) = -\pi +$ (sum of angles in f), the curvature of an interior vertex v by $\kappa(v) = 2\pi -$ (sum of angles at v), and the curvature of a boundary vertex $\hat{v}$ by $\kappa(\hat{v}) = \pi -$ (sum of angles at $\hat{v}$). It follows from the Gauss–Bonnet theorem that

$$\sum_{v \in I} \kappa(v) + \sum_{\hat{v} \in B} \kappa(\hat{v}) + \sum_{f \in F} \kappa(f) = 2\pi \tag{1}$$

(see [26, Section 4] and the references therein).

Since none of the congruences $(B.j)$, $(C.j)$ or $(D.j)$ holds $(0 \leq j \leq 2)$, every interior vertex of D is of degree at least 6, and since at most one congruence $(E.j)$, $(F1.j)$ or $(F2.j)$ holds, the label of an interior vertex of degree 6 is either $(XYZ)^2$ or it is the label corresponding to that congruence, given in Table 1.

We assign angles to the corners of the faces in D as follows. If v is a boundary vertex or an interior vertex of degree at least 8, then assign angle 47 to every corner at v. Assume now that v is an interior vertex of degree 6 and consider a face f with vertices v and u, w: if u, w are interior of degree 6, then assign angle 59 to the corner of f at v; otherwise assign 66 to the corner of f at v.

Then, if a face f contains a boundary vertex, then $\kappa(f) \leq -\pi + (47+66+66) = -1$; if a face f has all its vertices interior and one vertex of degree at least 8, then $\kappa(f) \leq -\pi + (47+66+66) = -1$; if all the vertices of a face f are interior of degree 6, then $\kappa(f) \leq -\pi + (59+59+59) = -3$. Therefore $\kappa(f) \leq -1$ for all $f \in F$.

We now consider curvature of the vertices. If v is an interior vertex of degree at least 8, then $\kappa(v) \leq 2\pi - 8(47) = -16$. If v is an interior vertex of degree 6 with a neighbour that is either on the boundary ∂D or has degree at least 8, then $\kappa(v) \leq 2\pi - 2(66) - 4(59) = -8$.

Now suppose that v is an interior vertex of degree 6 with all its neighbours interior of degree 6. Then $\kappa(v) = 2\pi - 6(59) = 6$ and by Lemma 2.15 the label of v is either $(XYZ)^2$ or $(E.j)$ for some $j \in \{0, 1, 2\}$. If the label of v is $(XYZ)^2$, then, since precisely one other label of degree 6 vertices is possible, Lemma 2.13 implies that v must have two adjacent neighbours, each labelled $(XYZ)^2$, but this is impossible by Lemma 2.12. If the label of v is $(E.j)$ (for some $j \in \{0, 1, 2\}$), then Lemma 2.14 implies that every neighbour v_i $(1 \leq i \leq 6)$ of v has two neighbours which are either boundary vertices or have degree at least 8. Therefore, for each $i \in \{1, \ldots, 6\}$ the curvature $\kappa(v_i) \leq 2\pi - 4(59) - 2(66) = -8$. In this situation, transfer curvature of -1 from each vertex v_i to vertex v; the resulting curvatures are $\kappa(v_i) \leq -8 + 1 = -7$ $(1 \leq i \leq 6)$ and $\kappa(v) = 6 - 6(1) = 0$. Since each vertex v_i has degree 6, the maximum number of times curvature can be transferred away from v_i is 6, so its curvature cannot exceed $\kappa(v) = -8 + 6(1) = -2$. Therefore for each interior vertex v we have $\kappa(v) \leq 0$.

Now (1) implies

$$\begin{aligned}
2\pi &= \sum_{v \in I} \kappa(v) + \sum_{\hat{v} \in B} \kappa(\hat{v}) + \sum_{f \in F} \kappa(f) \\
&\leq \sum_{v \in I} 0 + \sum_{\hat{v} \in B} (\pi - \text{sum of angles at } \hat{v}) + \sum_{f \in F} (-1) \\
&= |B|\pi - \sum_{\hat{v} \in B} (\text{sum of angles at } \hat{v}) - |F|
\end{aligned}$$

so

$$\sum_{\hat{v} \in B} (\text{sum of angles at } \hat{v}) \leq \big(|B| - 2\big)\pi - |F|.$$

On the other hand, the corner angle at any boundary vertex is 47, so the sum of angles over the boundary vertices is bounded below by $47|B|$. Therefore $47|B| \leq (|B| - 2)\pi - |F|$, so $|F| \leq 133|B| - 360$.

But $\text{Area}(W) \leq |F|$ and $|B| \leq N$ so $\text{Area}(W) \leq 133N - 360$, and so $f(N) = 133N - 360$ is a linear isoperimetric function. ■

We now have all the ingredients to prove Theorem A.

2.4. Proof of Theorem A

Suppose that $n \geq 2$, $0 \leq k, l < n$, $(n, k, l) = 1$ and that the cyclic presentation $P_n(x_0 x_k x_l)$ satisfies $T(6)$. If $P_n(x_0 x_k x_l)$ has a freely redundant relator, then $n = 3$ and $G \cong \mathbb{Z} * \mathbb{Z}$ (which is non-elementary hyperbolic) so we may assume that $P_n(x_0 x_k x_l)$ has no freely redundant relators. Then Lemma 2.1 implies that none of the congruences $(B.j)$, $(C.j)$ or $(D.j)$ $(0 \leq j \leq 2)$ (of Table 1) holds and so $n \geq 7$. If $n = 7$ or 8, then $G \cong G_n(x_0 x_1 x_5)$ (see, for example, [27, Table 2]) so is not hyperbolic by Lemmas 2.6 and 2.7. Assume then that $n > 8$. If more than one of the congruences $(E.j)$, $(F1.j)$, and $(F2.j)$ $(0 \leq j \leq 2)$ hold, then one of the cases (c), (d) or (e) of Lemma 2.4 holds. In cases (c) and (d), G is not hyperbolic by Corollaries 2.10 and 2.11 and in case (e), G is non-elementary hyperbolic, by Example 2.5. Thus we may assume that at most one of the congruences $(E.j)$, $(F1.j)$ or $(F2.j)$ $(0 \leq j \leq 2)$ holds, in which case Lemma 2.16 implies that $G_n(x_0x_kx_l)$ has a linear isoperimetric function, and hence is hyperbolic. By [14, Corollary 5.2] G contains a non-abelian free subgroup so it is non-elementary hyperbolic.

3. The non-positive case

As in [20], we express our arguments in terms of parameters $A = k$ and $B = k - m$.

Let Γ be the star graph of the cyclic presentation $P_n(x_0x_mx_k^{-1})$. Then Γ has vertices x_i and x_i^{-1} and edges $x_i - x_{i+m}^{-1}$, $x_i - x_{i+B}$, and $x_i^{-1} - x_{i+A}^{-1}$ $(0 \leq i < n)$, which we will refer to as edges of type X, Y, and Z, respectively. Replacing parameter k by $m - k$ corresponds to interchanging the roles of edges of types Y and Z, and so will correspond to interchanging the roles of conditions $(*.0)$ and $(*.1)$ in Table 2, and replacing the group $G_n(x_0x_mx_k^{-1})$ by the isomorphic copy $G_n(x_0x_mx_{m-k}^{-1})$. (To see that $G_n(x_0x_mx_k^{-1}) \cong G_n(x_0x_mx_{m-k}^{-1})$ replace the generators x_i by x_i^{-1}, invert the relators, negate the subscripts, and set $j = -i - m$ to get the relators $x_jx_{j+m}x_{j+m-k}^{-1}$ of $G_n(x_0x_mx_{m-k}^{-1})$.)

As in the positive case, we are interested in cycles of length at most 6 in Γ, so we analyse these in Section 3.1. We observe that if a particular cycle type of length 6 (which we refer to as $(\gamma+)$) occurs, then $G = G_n(x_0x_mx_k^{-1})$ is isomorphic to $G_n(x_0x_{n/2+2}x_1^{-1}) = H(n, n/2 + 2)$ which (in Section 3.2) we show is non-hyperbolic whenever its presentation satisfies $T(6)$. We then show that if two of the remaining cycle types of length 6 occur, then $G_n(x_0x_mx_k^{-1})$ is isomorphic to one of a few groups with low values of n, all but one of which turn out to be hyperbolic (the other, $G_8(x_0x_4x_1^{-1}) = H(8, 4)$, being non-hyperbolic). In Section 3.3 we consider the case when exactly one cycle type of length 6 occurs and perform a detailed analysis of van Kampen diagrams over the defining presentation to prove that $G_n(x_0x_mx_k^{-1})$ has a linear isoperimetric function, and hence is hyperbolic. We then combine these results to prove Theorem B in Section 3.4.

3.1. Analysis of short cycles in the star graph of $P_n(x_0x_mx_k^{-1})$

Short cycles in Γ were analysed in [20].

Lemma 3.1 ([20, Theorem 10]). *Let $n \geq 2$, $0 \leq m, k < n$, $m \neq k$, $k \neq 0$, and set $A = k$, $B = k - m$. Let Γ be the star graph of $P_n(x_0x_mx_k^{-1})$.*

(a) *Γ has a cycle of length less than 6 if and only if at least one of the congruences $(\rho.j)$, $(\sigma + .j)$, $(\sigma - .j)$, $(\tau + .j)$ or $(\tau - .j)$ of Table 2 holds, in which case a label of the cycle is the corresponding entry of the table.*

(b) *Γ has a cycle of length 6 if and only if at least one of the congruences $(\alpha.j)$, $(\beta + .j)$, $(\beta - .j)$, $(\gamma+)$ or $(\gamma-)$ of Table 2 holds, in which case a label of the cycle is the corresponding entry of the table.*

	j	**0**	**1**
$(\rho.j)$	m,k congruence	$k-m\equiv\frac{n}{2};\pm\frac{n}{3};\pm\frac{n}{4};\pm\frac{n}{5},\pm\frac{2n}{5}$	$k\equiv\frac{n}{2};\pm\frac{n}{3};\pm\frac{n}{4};\pm\frac{n}{5},\pm\frac{2n}{5}$
	A,B congruence	$B\equiv\frac{n}{2};\pm\frac{n}{3};\pm\frac{n}{4};\pm\frac{n}{5},\pm\frac{2n}{5}$	$A\equiv\frac{n}{2};\pm\frac{n}{3};\pm\frac{n}{4};\pm\frac{n}{5},\pm\frac{2n}{5}$
	cycle type	$Y^2;Y^3;Y^4;Y^5$	$Z^2;Z^3;Z^4;Z^5$
$(\sigma+)$	m,k congruence	$2k-m\equiv 0$	$2k-m\equiv 0$
	A,B congruence	$A+B\equiv 0$	$B+A\equiv 0$
	cycle type	$XYXZ$	$XZXY$
$(\sigma-)$	m,k congruence	$m\equiv 0$	$m\equiv 0$
	A,B congruence	$A-B\equiv 0$	$B-A\equiv 0$
	cycle type	$XYXZ$	$XZXY$
$(\tau+.j)$	m,k congruence	$3k-2m\equiv 0$	$3k-m\equiv 0$
	A,B congruence	$A+2B\equiv 0$	$B+2A\equiv 0$
	cycle type	$XZXYY$	$XYXZZ$
$(\tau-.j)$	m,k congruence	$2m-k\equiv 0$	$m+k\equiv 0$
	A,B congruence	$A-2B\equiv 0$	$B-2A\equiv 0$
	cycle type	$XZXYY$	$XYXZZ$
$(\alpha.j)$	m,k congruence	$k-m\equiv\pm\frac{n}{6}$	$k\equiv\pm\frac{n}{6}$
	A,B congruence	$B\equiv\pm\frac{n}{6}$	$A\equiv\pm\frac{n}{6}$
	cycle type	Y^6	Z^6
$(\beta+.j)$	m,k congruence	$4k-3m\equiv 0$	$4k-m\equiv 0$
	A,B congruence	$A+3B\equiv 0$	$B+3A\equiv 0$
	cycle type	$XZXYYY$	$XYXZZZ$
$(\beta-.j)$	m,k congruence	$3m-2k\equiv 0$	$2k+m\equiv 0$
	A,B congruence	$A-3B\equiv 0$	$B-3A\equiv 0$
	cycle type	$XZXYYY$	$XYXZZZ$
$(\gamma+)$	m,k congruence	$2k-m\equiv n/2$	$2k-m\equiv n/2$
	A,B congruence	$A+B\equiv n/2$	$B+A\equiv n/2$
	cycle type	$XZZXYY$	$XYYXZZ$
$(\gamma-)$	m,k congruence	$m\equiv n/2$	$m\equiv n/2$
	A,B congruence	$A-B\equiv n/2$	$B-A\equiv n/2$
	cycle type	$XZZXYY$	$XYYXZZ$

Table 2. Congruences $(\mathrm{mod}\, n)$ corresponding to short cycles in the star graph of $P_n(x_0x_mx_k^{-1})$. Here $A=k$, $B=k-m$.

Corollary 3.2. *Let $n \geq 2$ and suppose that $(n, m, k) = 1$, $0 \leq m$, $k < n$, $m \neq k$, $k \neq 0$. Then*

(a) *$P_n(x_0 x_m x_k^{-1})$ satisfies $T(6)$ if and only if none of the congruences $(\rho.j)$, $(\sigma + .j)$, $(\sigma - .j)$, $(\tau + .j)$ or $(\tau - .j)$ of Table 2 holds;*

(b) *$P_n(x_0 x_m x_k^{-1})$ satisfies $T(7)$ if and only if none of the congruences $(\rho.j)$, $(\sigma + .j)$, $(\sigma - .j)$, $(\tau + .j)$, $(\tau - .j)$, $(\alpha.j)$, $(\beta + .j)$, $(\beta - .j)$, $(\gamma+)$ or $(\gamma-)$ of Table 2 holds.*

Note that the two $(\gamma+)$ conditions in Table 2 are identical conditions and the two $(\gamma-)$ conditions are identical; for this reason we do not add the "$.j$" to these conditions. We first identify the groups $G_n(x_0 x_m x_k^{-1})$ in the presence of a cycle of type $(\gamma+)$ of Table 2.

Lemma 3.3. *Let $n \geq 2$, $0 \leq m$, $k < n$ and let $A = k$, $B = k - m$. Suppose that $A + B \equiv n/2 \bmod n$ and that $(n, m, k) = 1$. Then $G_n(x_0 x_m x_k^{-1}) \cong G_n(x_0 x_{n/2+2} x_1^{-1}) = H(n, n+2)$.*

Proof. The hypotheses imply that $1 = (n, m, k) = (n, n/2 + 2k, k)$, which implies that $(n/2, k) = 1$ so either $(n, k) = 1$ or ($n/2$ is odd and $(n, k) = 2$). In the former case $G_n(x_0 x_m x_k^{-1}) \cong G_n(x_0 x_{n/2+2k} x_k^{-1}) \cong G_n(x_0 x_{n/2+2} x_1^{-1})$ (by [1, Lemma 1.3]); in the latter case $G_n(x_0 x_m x_k^{-1}) \cong G_n(x_0 x_{n/2+2k} x_k^{-1}) \cong G_n(x_0 x_{n/2+4} x_2^{-1})$ which is isomorphic to $G_n(x_0 x_{n/2+2} x_1^{-1})$ by [1, Lemma 1.3] and [30, Lemma 7]. ∎

In Lemma 3.6 we will show that the groups $H(n, n/2 + 2)$ are not hyperbolic for any even $n \geq 8$, $n \neq 10$. We now consider the groups that arise when more than one of the remaining length 6 cycle cases hold.

Lemma 3.4. *Let $n \geq 2$, $0 \leq m$, $k < n$, $m \neq k$, $k \neq 0$, where $(n, m, k) = 1$, and set $A = k$, $B = k - m$. Let Γ be the star graph of $P_n(x_0 x_m x_k^{-1})$. Suppose that none of the congruences $(\rho.j)$, $(\sigma + .j)$, $(\sigma - .j)$, $(\tau + .j)$ or $(\tau - .j)$ holds for $j \in \{0, 1\}$ and that $(\gamma+)$ does not hold. If more than one of the congruences $(\alpha.j)$, $(\beta + .j)$, $(\beta - .j)$, and $(\gamma-)$ hold, then $G_n(x_0 x_m x_k^{-1})$ is isomorphic to one of $H(8, 4)$, $H(8, 6)$, $H(10, 4)$, $H(18, 4)$ or $H(18, 16)$.*

Proof. If $(\alpha.0)$ and $(\alpha.1)$ hold, then $A \equiv \pm n/6$ and $B \equiv \pm n/6 \bmod n$, and so either $(\sigma+)$ or $(\sigma-)$ holds, a contradiction. If $(\alpha.0)$ and $(\gamma-)$ hold, then $(\rho.1)$ holds, a contradiction. If $(\alpha.1)$ and $(\gamma-)$ hold, then $(\rho.0)$ holds.

If $(\alpha.0)$ and $((\beta + .0)$ or $(\beta - .0))$ hold, then $2A \equiv 0 \bmod n$, and so $(\rho.1)$ holds, a contradiction; if $(\alpha.1)$ and $((\beta + .1)$ or $(\beta - .1))$ hold, then $2B \equiv 0 \bmod n$, and so $(\rho.0)$ holds. If $(\beta + .0)$ and $(\beta - .0)$ hold, then $2A \equiv 0 \bmod n$, and so $(\rho.1)$ holds, a contradiction; if $(\beta + .1)$ and $(\beta - .1)$ hold, then $2B \equiv 0 \bmod n$, and so $(\rho.0)$ holds.

If $(\beta - .0)$ and $(\gamma-)$ hold, then $A \equiv 3B \bmod n$ and $4B \equiv 0 \bmod n$, and so $(\rho.0)$ holds, a contradiction. Similarly, if $(\beta - .1)$ and $(\gamma-)$ hold, then $(\rho.1)$ holds.

If $(\beta + .0)$ and $(\beta + .1)$ hold, then $B \equiv -3A \bmod n$ and $8A \equiv 0 \bmod n$; moreover $1 = (n, A, B) = (n, A, -3A) = (n, A)$ so $n|8$, and if $n < 8$, then $(\rho.0)$ holds, a contradiction, so $n = 8$ and $G_n(x_0 x_m x_k^{-1})$ is isomorphic to $H(8, 4)$. Similarly, if $(\beta - .0)$ and $(\beta - .1)$

hold, then $G_n(x_0x_mx_k^{-1}) \cong H(8,6)$. If $(\beta+.0)$ and $(\gamma-)$ hold, then $A \equiv -3B$ mod n and $8B \equiv 0$ mod n; moreover $1=(n,A,B)=(n,-3B,B)=(n,B)$ so $n|8$, and if $n<8$, then $(\rho.0)$ holds, a contradiction, so $n=8$ and $G_n(x_0x_mx_k^{-1}) \cong H(8,4)$. Similarly, if $(\beta+.1)$ and $(\gamma-)$ hold, then $n=8$ and $G_n(x_0x_mx_k^{-1}) \cong H(8,4)$.

If $(\beta+.0)$ and $(\beta-.1)$ hold, then $B \equiv 3A$ mod n and $10A \equiv 0$ mod n; moreover $1=(n,A,B)=(n,A)$ so $n|10$, and if $n<10$, then $(\rho.0)$ holds, a contradiction, so $n=10$. Then $(k,n)=1$, and so by [1, Lemma 1.3] we may assume that $k=1$; so $A=1$ and $k-m=B=3$, and hence $m=8$. Thus $G_n(x_0x_mx_k^{-1}) \cong G_{10}(x_0x_8x_1^{-1})$. Similarly, if $(\beta+.1)$ and $(\beta-.0)$ hold, then $n=10$ and $G_n(x_0x_mx_k^{-1}) \cong G_{10}(x_0x_4x_1^{-1})$. By [8, Theorem 2] we have $G_{10}(x_0x_8x_1^{-1}) \cong G_{10}(x_0x_4x_1^{-1}) = H(10,4)$.

If $(\alpha.0)$ and $(\beta+.1)$ hold, then $B \equiv -3A$ and $18A \equiv 0$ mod n; moreover $1=(n,A,B)=(n,A)$ so $n|18$ and if $n \le 9$, then $(\rho.0)$ holds so $n=18$. Then $(k,n)=1$, and so we may assume that $k=1$ so $k-m=B=-3$, and hence $m=4$. Thus $G_n(x_0x_mx_k^{-1}) \cong G_{18}(x_0x_4x_1^{-1}) = H(18,4)$. If $(\alpha.1)$ and $(\beta+.0)$ hold, then $A \equiv -3B$ and $18B \equiv 0$ mod n; moreover $1=(n,A,B)=(n,B)$ so $n|18$ and again $n=18$. Then $(k-m,n)=1$, and so we may assume that $B=k-m=1$ so $k=A=-3$, and hence $m=-4$. Thus $G_n(x_0x_mx_k^{-1}) = G_{18}(x_0x_{-4}x_{-3}^{-1}) \cong G_{18}(x_0x_4x_1^{-1}) = H(18,4)$ by [1, Lemma 1.3] and [31, Lemma 7].

If $(\alpha.0)$ and $(\beta-.1)$ hold, then $B \equiv 3A$ and $18A \equiv 0$ mod n; moreover $1=(n,A,B)=(n,A)$ so $n|18$, and if $n \le 9$, then $(\rho.0)$ holds and so $n=18$. Then $(k,n)=1$, and so we may assume that $k=1$ so $k-m=B=3$, and hence $m=-2$. Thus $G_n(x_0x_mx_k^{-1}) \cong G_{18}(x_0x_{16}x_1^{-1}) = H(18,16)$. If $(\alpha.1)$ and $(\beta-.0)$ hold, then $A \equiv 3B$ and $18B \equiv 0$ mod n; moreover $1=(n,A,B)=(n,B)$ so $n|18$ and if $n \le 9$, then $(\rho.1)$ holds and so $n=18$. Then $(k-m,n)=1$ and so we may assume that $k-m=1$ so $k=A=3$, and hence $k=3$, $m=2$. Thus $G_n(x_0x_mx_k^{-1}) \cong G_{18}(x_0x_2x_3^{-1}) \cong H(18,16)$ by [1, Lemma 1.3] and [31, Lemma 7]. ∎

In Corollary 2.8 we showed that $H(8,4)$ is not hyperbolic; in Lemma 3.6 we will show that $H(8,6)$ is not hyperbolic. We now show that the remaining groups arising in Lemma 3.4 are hyperbolic.

Example 3.5. Using KBMAG [19], it is straightforward to show that the groups $H(10,4)$, $H(18,4)$, and $H(18,16)$ are hyperbolic, and since they contain a non-abelian free subgroup (by [20, Corollary 11]), they are non-elementary hyperbolic.

3.2. Non-hyperbolic groups $G_n(x_0x_mx_k^{-1})$

We now show that the $T(6)$ groups $H(n,n/2+2)$ arising in Lemma 3.3 are not hyperbolic. (Note that if $n=2,4,6$ or 10, then the presentation of $H(n,n/2+2)$ does not satisfy $T(6)$, by Corollary 3.2.) As in the proof of Lemma 2.7 we do this by an application of the Flat Plane Theorem.

Lemma 3.6. *Suppose that $n \ge 8$ is even, $n \ne 10$. Then $H(n,n/2+2) = G_n(x_0x_{n/2+2}x_1^{-1})$ is not hyperbolic.*

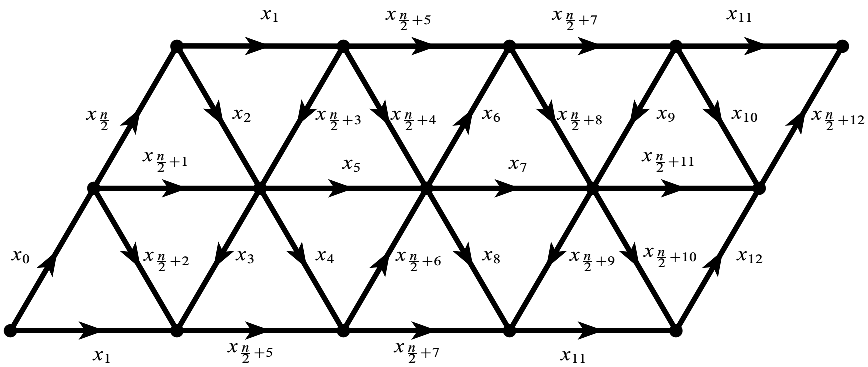

Figure 7. A van Kampen diagram over the presentation $P_n(x_0x_{n/2+2}x_1^{-1})$ with boundary label $(x_0x_{n/2})(x_1x_{n/2+5}x_{n/2+7}x_{11})(x_{12}x_{n/2+12})^{-1}(x_1x_{n/2+5}x_{n/2+7}x_{11})^{-1}$.

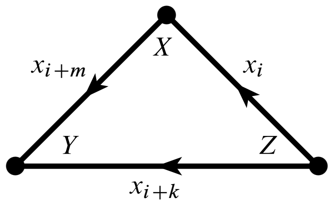

Figure 8. A typical face in a van Kampen diagram over the presentation $P_n(x_0x_mx_k^{-1})$.

Proof. Since the presentation $P_n(x_0x_{n/2+2}x_1^{-1})$ satisfies $C(3)-T(6)$ and each relator has length 3, each face in the geometric realisation $\widetilde{C}$ of the Cayley complex of P (obtained by assigning length 1 to each edge) is an equilateral triangle, and so $\widetilde{C}$ satisfies the CAT(0) inequality. Consider the geometric realisation Δ_0 of the reduced van Kampen diagram given in Figure 7 and for each $0 \le i < n$ let Δ_i be obtained from Δ_0 by applying the shift θ^i to each edge. Then placing $\Delta_0, \Delta_{12}, \Delta_{24}, \dots, \Delta_{6n-12}$ side by side gives the geometric realisation Δ of a reduced van Kampen diagram. Copies of Δ tile the Euclidean plane without cancellation of faces. Thus there is an isometric embedding of the Euclidean plane in $\widetilde{C}$, and so the result follows from the Corollary to Theorem A in [5]. ∎

3.3. Analysis of van Kampen diagrams over $P_n(x_0x_mx_k^{-1})$

In this section, we show that if the cyclic presentation $P = P_n(x_0x_mx_k^{-1})$ is $T(6)$ and precisely one of the congruences $(\alpha.j)$, $(\beta+.j)$, $(\beta-.j)$ or $(\gamma-)$ holds, then $G = G_n(x_0x_mx_k^{-1})$ is hyperbolic. As in Section 2.3 we do this by showing that there is a linear function $f : \mathbb{N} \to \mathbb{N}$ such that for all $N \in \mathbb{N}$ and all freely reduced words $W \in F_n$ with length at most N that represent the identity of G we have $\mathrm{Area}(W) \le f(N)$. Note that each face in D is a triangle, as shown in Figure 8, where the corner labels X, Y, Z correspond to the edge types of the star graph of P. In order to obtain a linear isoperimetric function (in Lemma 3.8) we first rule out certain configurations in D.

Lemma 3.7. *Suppose that all interior vertices of D have degree at least* 6 *and that all interior vertices of degree* 6 *of D correspond to precisely one of the congruences* $(\alpha.j)$, $(\beta+.j)$, $(\beta-.j)$ *or* $(\gamma-)$ *for* $j\in\{0,1\}$. *If v is an interior vertex of degree* 6 *where all the neighbours of v are interior vertices, then v has a neighbour of degree at least* 7.

Proof. If v is labelled Y^6 (resp. Z^6), then clearly none of its neighbours can be labelled Y^6 (resp. Z^6), so they must each have degree at least 7. If v is labelled $XZXYYY$ (resp. $XYXZZZ$, resp. $XZZXYY$), then the labels of the corners of the faces incident to v show that at least one of the neighbours of v does not have label $XZXYYY$ (resp. $XYXZZZ$, resp. $XZZXYY$), and hence has degree at least 7. ∎

We are now in a position to be able to establish the existence of a suitable isoperimetric function.

Lemma 3.8. *Let* $n\geq 2$, $0\leq m,k<n$, $m\neq k$, $k\neq 0$ *and set* $A=k$, $B=k-m$. *Let* Γ *be the star graph of* $P_n(x_0x_mx_k^{-1})$. *Suppose that none of* $(\rho.j)$, $(\sigma+.j)$, $(\sigma-.j)$, $(\tau+.j)$ *or* $(\tau-.j)$ *holds and that exactly one of the congruences* $(\alpha.j)$, $(\beta+.j)$, $(\beta-.j)$ *or* $(\gamma-)$ *of Table 2 holds* ($j\in\{0,1\}$). *Then* $G_n(x_0x_mx_k^{-1})$ *has a linear isoperimetric function.*

Proof. Let $N\in\mathbb{N}$, let W be a freely reduced word in the free group F_n of length at most N that represents the identity of G, and let D be a reduced van Kampen diagram whose boundary is a simple closed curve with label W. We let I denote the set of interior vertices of D, B the set of boundary vertices of D, and F the set of faces of D. Then Area$(W)\leq |F|$. Writing π to denote 180, we define the curvature of a face f by $\kappa(f)=-\pi+$(sum of angles in f), the curvature of an interior vertex v by $\kappa(v)=2\pi-$(sum of angles at v), and the curvature of a boundary vertex $\hat{v}$ by $\kappa(\hat{v})=\pi-$(sum of angles at $\hat{v}$). Again it follows from the Gauss–Bonnet theorem that (1) holds.

Since none of the congruences $(\rho.j)$, $(\sigma+.j)$, $(\sigma-.j)$, $(\tau+.j)$ or $(\tau-.j)$ holds, every interior vertex of D is of degree at least 6 and since exactly one of the congruences $(\alpha.j)$, $(\beta+.j)$, $(\beta-.j)$ or $(\gamma-)$ holds, then the label of an interior vertex of degree 6 is the corresponding label given in Table 2.

We assign angles to the corners of faces in D as follows. If v is a boundary vertex, then assign 47 to every corner at v; if v is an interior vertex of degree at least 7, then assign 52 to every corner at v. Assume now that v is an interior vertex of degree 6 and consider a face f with vertices v and u,w: if u,w are interior of degree 6, then assign 59 to the corner of f at v; otherwise assign 63.5 to the corner of f at v. If a face f contains a boundary vertex, then $\kappa(f)\leq -\pi+47+2(63.5)=-6$; if a face f contains only interior vertices of degree 6, then $\kappa(f)=-\pi+3(59)=-3$; if a face f contains only interior vertices of degree at least 7, then $\kappa(f)=-\pi+3(52)=-24$; if a face f contains an interior vertex of degree 6 and two interior vertices of degree at least 7, then $\kappa(f)=-\pi+63.5+2(52)=-12.5$; if a face f contains two vertices of degree 6 and one of degree at least 7 then $\kappa(f)=-\pi+2(63.5)+52=-1$. Therefore $\kappa(f)\leq -1$ for all $f\in F$.

We now turn to curvature of the vertices. If v is an interior vertex of degree at least 7, then $\kappa(v)\leq 2\pi-7(52)=-4$; if v is an interior vertex of degree 6 that has a neighbour that is either interior of degree at least 7 or is a boundary vertex, then $\kappa(v) \le 2\pi - 4(59) - 2(63.5) = -3$.

By Lemma 3.7 every interior vertex of degree 6 has a neighbour on the boundary or a neighbour that is interior of degree at least 7. Then $\kappa(v) \le -3$ for all interior vertices v and so (1) implies that

$$\begin{aligned} 2\pi &= \sum_{v \in I} \kappa(v) + \sum_{\hat{v} \in B} \kappa(\hat{v}) + \sum_{f \in F} \kappa(f) \\ &\le \sum_{v \in I} (-3) + \sum_{\hat{v} \in B} (\pi - \text{sum of angles at } \hat{v}) + \sum_{f \in F} (-1) \\ &= -3|I| + \sum_{\hat{v} \in B} (\pi - \text{sum of angles at } \hat{v}) - |F| \\ &\le |B|\pi - \sum_{\hat{v} \in B} (\text{sum of angles at } \hat{v}) - |F| \end{aligned}$$

so

$$\sum_{\hat{v} \in B} (\text{sum of angles at } \hat{v}) \le \big(|B| - 2\big)\pi - |F|.$$

On the other hand, the corner angle at any boundary vertex is 47, and so the sum of angles over the boundary vertices is bounded below by $47|B|$. Therefore $47|B| \le (|B|-2)\pi - |F|$ so $|F| \le 133|B| - 360$. But $\text{Area}(W) \le |F|$ and $|B| \le N$ so $\text{Area}(W) \le 133N - 360$, and hence $f(N) = 133N - 360$ is a linear isoperimetric function, as required. ∎

We now have all the ingredients to prove Theorem B.

3.4. Proof of Theorem B

Suppose that $n \ge 2$, $0 \le m, k < n$, $m \ne k$, $k \ne 0$, $(n, m, k) = 1$ and that the cyclic presentation $P_n(x_0 x_m x_k^{-1})$ satisfies $T(6)$. Then Lemma 3.1 implies that none of the congruences $(\rho.j)$, $(\sigma + .j)$, $(\sigma - .j)$, $(\tau + .j)$ or $(\tau - .j)$ holds. If $(\gamma+)$ holds, then $n = 8$ or $n \ge 12$ and G is not hyperbolic by Lemmas 3.3 and 3.6; so suppose that $(\gamma+)$ does not hold.

If Γ has no cycle of length less than 7, then $P_n(x_0 x_m x_k^{-1})$ satisfies $C(3) - T(7)$, and so $G_n(x_0 x_m x_k^{-1})$ is hyperbolic by [16, Corollary 4.1]. Thus we may assume that Γ has a cycle of length 6 so, by Lemma 3.1, at least one of the congruences $(\alpha.j)$, $(\beta + .j)$, $(\beta - .j)$ or $(\gamma-)$ holds ($j \in \{0, 1\}$). Suppose that more than one of them hold. Then G is one of the groups in the conclusion of Lemma 3.4. When $n = 8$, the group $G \cong G_8(x_0 x_4 x_1^{-1}) = H(8,4)$ or $G \cong G_8(x_0 x_6 x_1^{-1}) = H(8,6)$, which are non-hyperbolic by Corollary 2.8 and Lemma 3.6, respectively. In the remaining cases G is non-elementary hyperbolic by Example 3.5.

Suppose then that exactly one of the congruences $(\alpha.j)$, $(\beta + .j)$, $(\beta - .j)$ or $(\gamma-)$ holds. Then G has a linear isoperimetric function, and hence is hyperbolic, by Lemma 3.8. By [20, Corollary 11] G contains a non-abelian free subgroup so it is non-elementary hyperbolic.

Acknowledgements. The first author thanks Jim Howie for his hospitality during a research visit to the School of Mathematical and Computer Sciences, Heriot Watt University, in November 2019; that visit was central to the development of the Flat Plane arguments used in this paper. Both authors thank Jim Howie and William Bogley for insightful conversations relating to this work, and helpful comments on a draft of this article.

Funding. This work was supported by the Leverhulme Trust Research Project Grant RPG-2017-334. Part of the work was carried out during a research visit funded by Leverhulme Trust Emeritus Fellowship EM-2018-023\9.

References

[1] V. G. Bardakov and A. Y. Vesnin, On a generalization of Fibonacci groups. *Algebra Logika* **42** (2003), no. 2, 131–160, 255 Zbl 1031.57001 MR 2003626

[2] G. Baumslag, S. M. Gersten, M. Shapiro, and H. Short, Automatic groups and amalgams. *J. Pure Appl. Algebra* **76** (1991), no. 3, 229–316 Zbl 0749.20006 MR 1147304

[3] W. A. Bogley, On shift dynamics for cyclically presented groups. *J. Algebra* **418** (2014), 154–173 Zbl 1344.20042 MR 3250446

[4] W. A. Bogley and G. Williams, Coherence, subgroup separability, and metacyclic structures for a class of cyclically presented groups. *J. Algebra* **480** (2017), 266–297 Zbl 1371.20036 MR 3633308

[5] M. R. Bridson, On the existence of flat planes in spaces of nonpositive curvature. *Proc. Amer. Math. Soc.* **123** (1995), no. 1, 223–235 Zbl 0840.53033 MR 1273477

[6] D. I. Cartwright, A. M. Mantero, T. Steger, and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\tilde{A}_2$. I. *Geom. Dedicata* **47** (1993), no. 2, 143–166 Zbl 0784.51010 MR 1232965

[7] A. Cavicchioli, F. Hegenbarth, and D. Repovš, On manifold spines and cyclic presentations of groups. In *Knot Theory (Warsaw, 1995)*, pp. 49–56, Banach Center Publ. 42, Polish Acad. Sci. Inst. Math., Warsaw, 1998 Zbl 0904.57008 MR 1634446

[8] A. Cavicchioli, E. A. O'Brien, and F. Spaggiari, On some questions about a family of cyclically presented groups. *J. Algebra* **320** (2008), no. 11, 4063–4072 Zbl 1201.20027 MR 2464807

[9] A. Cavicchioli, D. Repovš, and F. Spaggiari, Families of group presentations related to topology. *J. Algebra* **286** (2005), no. 1, 41–56 Zbl 1066.57003 MR 2124807

[10] D. J. Collins, Free subgroups of small cancellation groups. *Proc. London Math. Soc. (3)* **26** (1973), 193–206 Zbl 0262.20035 MR 318323

[11] J. H. Conway, Advanced problem: 5327. *Amer. Math. Monthly* **72** (1965), no. 8, 915 MR 1533441

[12] M. Edjvet and J. Howie, Star graphs, projective planes and free subgroups in small cancellation groups. *Proc. London Math. Soc. (3)* **57** (1988), no. 2, 301–328 Zbl 0627.20020 MR 950593

[13] M. Edjvet and A. Vdovina, On the SQ-universality of groups with special presentations. *J. Group Theory* **13** (2010), no. 6, 923–931 Zbl 1211.20028 MR 2736165

[14] M. Edjvet and G. Williams, The cyclically presented groups with relators $x_i x_{i+k} x_{i+l}$. *Groups Geom. Dyn.* **4** (2010), no. 4, 759–775 Zbl 1243.20042 MR 2727662

[15] S. M. Gersten, Isoperimetric and isodiametric functions of finite presentations. In *Geometric Group Theory, Vol. 1 (Sussex, 1991)*, pp. 79–96, London Math. Soc. Lecture Note Ser. 181, Cambridge Univ. Press, Cambridge, 1993 Zbl 0829.20054 MR 1238517

[16] S. M. Gersten and H. B. Short, Small cancellation theory and automatic groups. *Invent. Math.* **102** (1990), no. 2, 305–334 Zbl 0714.20016 MR 1074477
[17] N. D. Gilbert and J. Howie, LOG groups and cyclically presented groups. *J. Algebra* **174** (1995), no. 1, 118–131 Zbl 0851.20025 MR 1332862
[18] P. Hill, S. J. Pride, and A. D. Vella, On the $T(q)$-conditions of small cancellation theory. *Israel J. Math.* **52** (1985), no. 4, 293–304 Zbl 0584.20026 MR 829360
[19] D. F. Holt, KBMAG (Knuth-Bendix in Monoids and Automatic Groups). 2000, http://www.warwick.ac.uk/staff/D.F.Holt/download/kbmag2/
[20] J. Howie and G. Williams, Tadpole Labelled Oriented Graph groups and cyclically presented groups. *J. Algebra* **371** (2012), 521–535 Zbl 1275.20027 MR 2975410
[21] J. Howie and G. Williams, Fibonacci type presentations and 3-manifolds. *Topology Appl.* **215** (2017), 24–34 Zbl 1365.20032 MR 3576437
[22] S. V. Ivanov and P. E. Schupp, On the hyperbolicity of small cancellation groups and one-relator groups. *Trans. Amer. Math. Soc.* **350** (1998), no. 5, 1851–1894 Zbl 0915.20016 MR 1401522
[23] D. L. Johnson and H. Mawdesley, Some groups of Fibonacci type. *J. Austral. Math. Soc.* **20** (1975), no. 2, 199–204 Zbl 0306.20039 MR 0387419
[24] D. L. Johnson, J. W. Wamsley, and D. Wright, The Fibonacci groups. *Proc. London Math. Soc. (3)* **29** (1974), 577–592 Zbl 0327.20014 MR 384941
[25] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*. Ergeb. Math. Grenzgeb. 89, Springer-Verlag, Berlin, 1977 Zbl 0368.20023 MR 0577064
[26] J. P. McCammond and D. T. Wise, Fans and ladders in small cancellation theory. *Proc. London Math. Soc. (3)* **84** (2002), no. 3, 599–644 Zbl 1022.20012 MR 1888425
[27] E. Mohamed and G. Williams, An investigation into the cyclically presented groups with length three positive relators. *Exp. Math.* (2019), DOI 10.1080/10586458.2019.1655817
[28] S. J. Pride, Star-complexes, and the dependence problems for hyperbolic complexes. *Glasgow Math. J.* **30** (1988), no. 2, 155–170 Zbl 0684.20022 MR 942986
[29] A. J. Sieradski, Combinatorial squashings, 3-manifolds, and the third homology of groups. *Invent. Math.* **84** (1986), no. 1, 121–139 Zbl 0604.57001 MR 830041
[30] G. Williams, The aspherical Cavicchioli-Hegenbarth-Repovš generalized Fibonacci groups. *J. Group Theory* **12** (2009), no. 1, 139–149 Zbl 1204.20044 MR 2488144
[31] G. Williams, Groups of Fibonacci type revisited. *Internat. J. Algebra Comput.* **22** (2012), no. 8, 1240002, 19 Zbl 1271.20043 MR 3010816
[32] D. T. Wise, Approximating flats by periodic flats in CAT(0) square complexes. *Canad. J. Math.* **57** (2005), no. 2, 416–448 Zbl 1085.20027 MR 2124924

Received 16 June 2020.

Ihechukwu Chinyere
Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, Essex CO4 3SQ, UK; ihechukwu.chinyere@essex.ac.uk

Gerald Williams
Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, Essex CO4 3SQ, UK; gerald.williams@essex.ac.uk