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Length statistics of random multicurves on closed hyperbolic surfaces

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Abstract. A multicurve can be decomposed into components. The ratios of the length of each component to the total length give a hint of the shape of the multicurve. In this paper, we determine the distribution of these ratios of a random multicurve with a given topological type on a closed hyperbolic surface, using the methods of Margulis' thesis and Mirzakhani's equidistribution theorem for horospheres. This distribution admits a polynomial density whose coefficients can be expressed explicitly in terms of intersection numbers of psi-classes on the Deligne–Mumford compactification of the moduli space of complex curves, and in particular it does not depend on the hyperbolic metric. This result generalizes prior work of M. Mirzakhani in the case of random pants decompositions. Results very close to ours were obtained independently and simultaneously by F. Arana-Herrera.

1. Introduction

Let X be a connected closed oriented complete hyperbolic surface of genus $g \geq 2$. An *ordered multi-geodesic* on X is a finite ordered list $\gamma = (m_1\gamma_1, \dots, m_k\gamma_k)$, where $m_1, \dots, m_k \in \mathbb{Z}_{\geq 1}$, and the γ_i's are pairwise disjoint closed geodesics on X without self-intersections. Our definition of a random (ordered) multi-geodesic is as follows. Consider the set

$$s_{X,R,\gamma} := \left\{\alpha \in \mathrm{Mod}(X) \cdot \gamma : \ell_X(\alpha) \leq R\right\}$$

of ordered multi-geodesics of the same topological type as that of γ and of length at most R, where $\mathrm{Mod}(X)$ is the mapping class group of X, and

$$\ell_X(\alpha) := m_1\ell_X(\alpha_1) + \cdots + m_k\ell_X(\alpha_k)$$

is the total length of α. The set $s_{X,R,\gamma}$ is finite. Consider the uniform probability measure on this set, and the random variable that associates a multi-geodesic to its normalized length vector

$$\hat{\ell}_{X,R,\gamma}\colon s_{X,R,\gamma} \to \Delta^{k-1}, \quad (m_1\alpha_1, \dots, m_k\alpha_k) \mapsto \frac{1}{\ell_X(\alpha)}(m_1\ell_X(\alpha_1), \dots, m_k\ell_X(\alpha_k)),$$

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where

$$\Delta^{k-1} := \left\{(x_1, \dots, x_k) \in \mathbb{R}^k_{\geq 0} : x_1 + \dots + x_k = 1\right\}$$

is the standard simplex of dimension $k - 1$.

Our goal is to find the limiting distribution of $\hat{\ell}_{X,R,\gamma}$ as $R \to \infty$.

The study of the length partition of random multi-geodesics is initialed by M. Mirzakhani in [14], where she proves the following result.

Theorem 1.1 ([14, Theorem 1.2]). *If $\{\gamma_1, \dots, \gamma_{3g-3}\}$ is a pants decomposition of X, then as $R \to \infty$, the random variable $\hat{\ell}_{X,R,\gamma}$ converges in law to the Dirichlet distribution of order $3g - 3$ with parameters $1, \dots, 1$, namely the probability distribution that has a density function $(6g-7)! \cdot x_1 \cdots x_{3g-3}$ with respect to the Lebesgue measure on the standard simplex $\Delta^{3g-4} := \{(x_1, \dots, x_{3g-3}) \in \mathbb{R}^{3g-3}_{\geq 0} : x_1 + \dots + x_{3g-3} = 1\}$ of dimension $3g - 4$. In other words, for any open subset U of Δ^{3g-4},*

$$\lim_{R\to\infty} \mathbb{P}(\hat{\ell}_{X,R,\gamma} \in U) = (6g-7)! \int_U x_1 \cdots x_{3g-3}\, \lambda(dx),$$

where λ is the Lebesgue measure on Δ^{3g-4}.

Our main result is the following generation of the preceding theorem to any arbitrary topological type.

Theorem 1.2. *Let $\gamma = (m_1\gamma_1, \dots, m_k\gamma_k)$ be an ordered multi-geodesic on X. As $R \to \infty$, the random variable $\hat{\ell}_{X,R,\gamma}$ converges in law to a random variable which admits a polynomial density with respect to the Lebesgue measure on Δ^{k-1} given by, up to a normalizing constant,*

$$(x_1, \dots, x_k) \mapsto \overline{P}_\gamma(x_1/m_k, \dots, x_k/m_k),$$

where $\overline{P}_\gamma$ is the top-degree (homogeneous) part of the graph polynomial P_γ associated to γ defined by (2.2)

Remark 1.3. The function $\overline{P}_\gamma$ is a homogeneous polynomial of degree $6g - 7$ whose coefficients can be expressed in terms of the psi-classes on the Deligne–Mumford compactification of the moduli space of smooth complex curves $\overline{\mathcal{M}}_{g,n}$ (see Theorem 2.2). In particular, it depends only upon the topological type of γ, but not on the hyperbolic metric X.

Motivations. Theorem 1.2 is motivated by Theorem 1.1 of Mirzakhani. Another motivation originates from Theorem 1.25 in the section "Statistical geometry of square-tiled surfaces" of [4]. Namely, the statistics of perimeters of maximal cylinders of a "random" square-tiled surface associated to a given multicurve γ given by formula (1.38) from [4] coincide with statistics of hyperbolic lengths of different components of γ given by Theorem 1.2 above. Though formula (1.38) from [4] can be interpreted as a certain *average* of lengths statistics for individual hyperbolic surfaces X, it does not imply that such statistics do not change when X changes. The conjecture of non-varying of statistics of hyperbolic lengths for any fixed multicurve under arbitrary deformation of the hyperbolic metric, proved in Theorem 1.2, was one of our principal motivations.

Idea of the proof. The structure of the proof is similar to that of [14, Theorem 1.2]. The limiting distribution that we are after boils down to the asymptotics of multicurves counting under constraints, which can be transformed to a problem of approximating to the number of "lattice points" within a horoball in a covering space of the moduli space. By considering tiling of the covering space by translates of a fundamental domain for the action of the mapping class group, it would not be unreasonable to expect that this number might be proportional to the volume of the horoball divided by the volume of the moduli space, and finally this is not so far from the truth. We proceed using techniques that Margulis introduced in his thesis [8], and the equidistribution theorem for large horospheres initially established by Mirzakhani [10]. Similar methods were also applied in, e.g., [6].

Theorem 1.2 can be generalized to hyperbolic surfaces with cusps if Mirzakhani's work on the ergodicity of the earthquake flow can be generalized to such surfaces, which seems to be the case (see [14]).

Remark. While the author was finishing this paper, the paper [1] by F. Arana-Herrera appeared on the arXiv. Both papers are devoted to a similar circle of problems and use a similar circle of ideas, though they were written in parallel and completely independently. In particular, Arana-Herrera proves a much more general version of our Theorem 5.1 ([1, Theorem 1.3]), which is one of the key ingredients allowing to attack the counting problem and the length statistics. We learned from [1] that this kind of statistics was initially conjectured by S. Wolpert. Papers [1] and [2] established results closely related to Theorem 1.2.

Proposition 5.4 below is based on a theorem stated by M. Mirzakhani but presented without a detailed proof. The paper [1] contains a detailed proof of an even stronger estimate which implies, in particular, the statements of this theorem; see Remark 5.8 below.

2. Background

Throughout this paper, we use the symbol Σ_g to denote a connected, closed, oriented, topological surface of genus $g \geq 2$, the symbol $\Sigma_{g,n}$ to denote a connected closed oriented topological surface of genus g with n boundary circles labeled by $\{1, \ldots, n\}$ with $2g - 2 + n > 0$, and the letter d to denote $3g - 3 + n$.

2.1. Deformation spaces and mapping class group

Let us consider the set of orientation-preserving homeomorphisms $\varphi\colon \Sigma_g \to X$ where X is an oriented complete hyperbolic surface of genus g. Two such homeomorphisms $\varphi_1\colon \Sigma_g \to X_1$ and $\varphi_2\colon \Sigma_g \to X_2$ are said to be *equivalent* if $\varphi_2 \circ \varphi_1^{-1}$ is isotropic to an isometry. The *Teichmüller space*, denoted by $\mathcal{T}(\Sigma_g)$ or simply $\mathcal{T}_g$, is the set of such equivalence classes.

Let us denote by $\mathrm{Homeo}^+(\Sigma_g)$ the group of self-homeomorphisms of Σ_g that preserve the orientation, and write $\mathrm{Homeo}_0(\Sigma_g)$ for the subgroup of $\mathrm{Homeo}^+(\Sigma_g)$ consisting of homeomorphisms isotropic to the identity. The *mapping class group*, denoted by $\mathrm{Mod}(\Sigma_g)$ or simply Mod_g, is the quotient group $\mathrm{Homeo}^+(\Sigma_g)/\,\mathrm{Homeo}_0(\Sigma_g)$.

The group $\mathrm{Homeo}^+(\Sigma_g)$ acts (properly and discontinuously) from the right on $\mathcal{T}_g$ by precomposition, and $\mathrm{Homeo}_0(\Sigma_g)$ acts trivially. The *moduli space*, denoted by $\mathcal{M}(\Sigma_g)$ or $\mathcal{M}_g$, is the quotient $\mathcal{T}_g/\,\mathrm{Mod}_g$.

The Teichmüller space $\mathcal{T}_{g,n}(L_1,\dots,L_n)$ and moduli space $\mathcal{M}_{g,n}(L_1,\dots,L_n)$ of oriented complete hyperbolic surfaces of genus g with n (labeled) totally geodesic boundary components of lengths $L_1,\dots,L_n \geq 0$ respectively can be defined in a similar manner.

2.2. Curves

In the introduction, the theorems are stated in terms of geodesics on a hyperbolic surface. Nevertheless, it is often more convenient to work with (the free homotopy classes of) the topological curves, and they are actually equivalent for our purposes. A curve in a topological space X is (the image of) a continuous application $\mathbb{S}^1 \to X$. In this paper, we are interested in curves up to free homotopy. A closed curve is said to be *simple* if it does not intersect itself. A *multicurve* is a finite multiset of disjoint simple curves, and a multicurve is *ordered* (or *labeled*) if its underlying set is labeled. We will often write an ordered multicurve γ as an ordered list $(m_1\gamma_1,\dots,m_k\gamma_k)$, and its unlabeled counterpart as a formal sum $\overline{\gamma} = m_1\gamma_1 + \cdots + m_k\gamma_k$ where $m_i \in \mathbb{Z}_{\geq 1}$, $1 \leq i \leq k$.

The group $\mathrm{Homeo}^+(\Sigma_g)$ acts on the set of closed curves on Σ_g by postcomposition, and the action of the subgroup $\mathrm{Homeo}_0(\Sigma_g)$ stabilizes sets of curves in the same free homotopy class. Thus the mapping class group acts on the set of free homotopy classes of closed curves on Σ_g. We say that two closed curves α and β have the same *topological type* if they lie in the same mapping class group orbit. The three following subgroups, associated to γ, of the mapping class group Mod_g will be useful later in the paper:

- $\mathrm{Stab}(\overline{\gamma})$ which fixes the multicurve $\overline{\gamma} = m_1\gamma_1 + \cdots + m_k\gamma_k$ (but the γ_i's can be permutated),
- $\mathrm{Stab}(\gamma)$ which fixes every γ_i for all $1 \leq i \leq k$, and
- $\mathrm{Stab}^+(\gamma)$ which fixes every γ_i and its orientation for all $1 \leq i \leq k$.

Let $X \in \mathcal{T}_g$. If a closed curve α on X is not homotopic to a point, then α is freely homotopic to a unique closed geodesic on X with the minimum length over all curves in the free homotopy class of α, and we write $\ell_X(\alpha)$ for the length of this geodesic.

The notions of topological type and length extend naturally to multicurves.

2.3. Fenchel–Nielsen coordinates

A *pair of pants* is a surface that is homeomorphic to a sphere with three holes. A *pants decomposition* of $\Sigma_{g,n}$ is a set of disjoint simple closed curves $\{\alpha_1,\dots,\alpha_{3g-3+n}\}$ on $\Sigma_{g,n}$ such that $\Sigma_{g,n} \smallsetminus \{\alpha_1,\dots,\alpha_{3g-3+n}\}$ is a disjoint union of pairs of pants.

Fix an ordered pants decomposition $(\alpha_1, \ldots, \alpha_{3g-3+n})$ of $\Sigma_{g,n}$. Given $X \in \mathcal{T}_{g,n}(L_1, \ldots, L_n)$ (or $X \in \mathcal{T}_g$), we can associate for each α_i two parameters: the length $\ell_{\alpha_i}(X) \in \mathbb{R}_{>0}$, and the *twist parameter* $\tau_{\alpha_i}(X) \in \mathbb{R}$ (corresponding to how much one turns before gluing two pairs of pants along α_i; see [3, Section 1.7] for a precise definition). These $6g - 6 + 2n$ parameters are called *Fenchel–Nielsen coordinates*. The application $\mathcal{T}_{g,n}(L_1, \ldots, L_n) \to (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3+n}$ given by $X \mapsto (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X))_{i=1}^k$ is a bijection (see [3, Chapter 6]).

2.4. Weil–Petersson volumes

The following theorem is often referred to as Wolpert's magical formula.

Theorem 2.1 ([15]). *Given a pants decomposition $\{\alpha_1, \ldots, \alpha_{3g-3+n}\}$, the term*

$$\sum_{i=1}^{3g-3+n} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}$$

defines a symplectic form which has the same expression in any other Fenchel–Nielsen coordinates. In particular, it is invariant under the action of the mapping class group.

The symplectic form thus defined is the so-called *Weil–Petersson symplectic form*, and we shall denote it by ω. See [15] for a more intrinsic definition.

Every symplectic form defines a volume form. The *Weil–Petersson volume* of the moduli space $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ is defined by

$$V_{g,n}(L_1, \ldots, L_n) := \int_{\mathcal{M}_{g,n}(L_1,\ldots,L_n)} \frac{\omega^{\wedge(3g-3+n)}}{(3g-3+n)!}.$$

The following fundamental result is due to Mirzakhani.

Theorem 2.2 ([11]). *The Weil–Petersson volume $V_{g,n}(L_1, \ldots, L_n)$ is a symmetric polynomial in $L_1^2, \ldots, L_n^2$ of degree $3g - 3 + 2n$. More precisely,*

$$V_{g,n}(L_1, \ldots, L_n) = \sum_{\substack{(d_0,d_1,\ldots,d_n)\in\mathbb{Z}_{\geq 0} \\ d_0+d_1+\cdots+d_n=3g-3+n}} \frac{(2\pi^2)^{d_0}}{2^{d_1+\cdots+d_n} d_0! d_1! \cdots d_n!} \cdot \left(\int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{d_0} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) L_1^{2d_1} \cdots L_n^{2d_n},$$

where $\overline{\mathcal{M}}_{g,n}$ is the Deligne–Mumford compactification, $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is the i-th psi-class, and $\kappa_1 = [\omega]/2\pi^2 \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is the first Mumford class.

2.5. Earthquakes

Multicurves can be regarded as "lattice points" in the space of *measured laminations* $\mathcal{ML}_g$. We will only need the following properties of this space. See, e.g., [7, Chapter 11] for more details.

(1) $\mathcal{ML}_g$ is a $(6g-6)$-dimensional real manifold equipped with a natural piece-wise integral linear structure, i.e., $\mathcal{ML}_g$ has a natural atlas whose transition functions are piece-wise in $\mathrm{GL}(6g-6,\mathbb{Z})$.

(2) The integral points in the coordinate charts of $\mathcal{ML}_g$, denoted by $\mathcal{ML}_g(\mathbb{Z})$, are in natural bijection with the (free homotopy classes of) integral multicurves on Σ_g.

(3) The action of the mapping class group on the set of multicurves extends to $\mathcal{ML}_g$.

(4) $(\mathbb{R}_{>0},\times)$ acts on $\mathcal{ML}_g$, and for any multicurve $\overline{\gamma}=m_1\gamma_1+\cdots+m_k\gamma_k$, and any $r\in\mathbb{R}_{>0}$, $r\cdot(m_1\gamma_1+\cdots+m_k\gamma_k)=r\,m_1\gamma_1+\cdots+r\,m_k\gamma_k$. We denote the quotient by $\mathbb{P}(\mathcal{ML}_g)$.

(5) Given $X\in\mathcal{T}_g$, the length function ℓ_X defined on the set of multicurves extends to $\mathcal{ML}_g$. Moreover, for any $\lambda\in\mathcal{ML}_g$, we have $\ell_{X\cdot h}(h^{-1}\cdot\lambda)=\ell_X(\lambda)$ for any $h\in\mathrm{Mod}_g$, and $\ell_X(r\cdot\lambda)=r\cdot\ell_X(\lambda)$ for any $r\in\mathbb{R}_{>0}$.

(6) The twist flow $\mathrm{tw}^t_{\overline{\gamma}}$ about a multicurve $\overline{\gamma}$ can be extended to any measured lamination $\lambda\in\mathcal{ML}_g$, and we have $(\mathrm{tw}^t_\lambda(X))\cdot h=\mathrm{tw}^t_{h^{-1}\lambda}(Xh)$ and $\mathrm{tw}^t_{r\lambda}(X)=\mathrm{tw}^{rt}_\lambda$ for all $t\in\mathbb{R}$, $h\in\mathrm{Mod}_g$, and $r\in\mathbb{R}_{>0}$.

(7) $\mathcal{ML}_g$ carries a natural mapping class group invariant measure μ_{Th} defined by asymptotic counting of integral points, called the *Thurston measure*. The Thurston measure is a Lebesgue measure in the coordinate charts of $\mathcal{ML}_g$, and for any open subset $U\subset\mathcal{ML}_g$, we have $\mu_{\mathrm{Th}}(t\cdot U)=t^{6g-6}\mu_{\mathrm{Th}}(U)$ for any $t\in\mathbb{R}_{>0}$.

Let $\mathcal{PT}_g:=\mathcal{T}_g\times\mathcal{ML}_g$ be the bundle of measured laminations over the Teichmüller space, and let $\mathcal{P}^1\mathcal{T}_g:=\{(X,\lambda)\in\mathcal{PT}_g:\ell_X(\lambda)=1\}$ be the unit sphere bundle of $\mathcal{PT}_g$ with respect to the length function.

The mapping class group acts on $\mathcal{PT}_g$ from the right via $(X,\lambda)\cdot h:=(X\cdot h,h^{-1}\cdot\lambda)$. This action is well-defined on $\mathcal{P}^1\mathcal{T}_g$ since it preserves the length function $\ell(X,\lambda):=\ell_X(\lambda)$. Write $\mathcal{PM}_g:=\mathcal{PT}_g/\mathrm{Mod}_g$ and $\mathcal{P}^1\mathcal{M}_g:=\mathcal{P}^1\mathcal{T}_g/\mathrm{Mod}_g$.

The *earthquake flow* tw^t on $\mathcal{PT}_n$ is defined by

$$\mathrm{tw}^t(X,\lambda):=(\mathrm{tw}^t_\lambda(X),\lambda).$$

The earthquake flow commutes with the action of the mapping class group, and therefore descends to $\mathcal{PM}_g$, and to $\mathcal{P}^1\mathcal{M}_g$ (since the earthquake preserves the length function).

The Thurston measure on $\mathcal{ML}_g$ induces a measure on $\{\lambda\in\mathcal{ML}_g:\ell_X(\lambda)=1\}$ in the following way: let $U\subset\{\lambda\in\mathcal{ML}_g:\ell_X(\lambda)=1\}$ be an open subset. The *Thurston measure* of U is defined to be

$$\mu_{\mathrm{Th}}\big\{s\cdot\lambda\in\mathcal{ML}_g:\lambda\in U,\ s\in[0,1]\big\}.$$

The measure ν_g on $\mathcal{P}^1\mathcal{T}_g$ defined by

$$\nu_g(U):=\int_{\mathcal{T}_g}\mu_{\mathrm{Th}}\big\{s\lambda\in\mathcal{ML}_g:(X,\lambda)\in U,\ s\in[0,1]\big\}\,dX$$

for any open subset $U \subset \mathcal{P}^1\mathcal{T}_g$ is invariant both under the earthquake flow (since μ_{WP} is) and under the action of the mapping class group (since μ_{Th} and μ_{WP} are), and hence descends to a measure on $\mathcal{P}^1\mathcal{M}_g$ that (by abuse of notation) we shall also denote by ν_g. The total mass of ν_g,

$$b_g = \int_{\mathcal{M}_g} B(X)\,dX,$$

where $B(X) := \mu_{\mathrm{Th}}\{\lambda \in \mathcal{ML}_g : \ell_X(\lambda) \le 1\}$, is finite [12, Theorem 3.3].

The following result is fundamental.

Theorem 2.3 ([12]). *The earthquake flow on $\mathcal{P}^1\mathcal{M}_g$ is ergodic with respect to ν_g.*

We recommend [17] for an expository survey on this topic.

2.6. Thurston distance

Let $X_1, X_2 \in \mathcal{T}_g$. Set

$$d(X,Y) := \sup_{\lambda \in \mathcal{ML}_g} \log \frac{\ell_{X_1}(\lambda)}{\ell_{X_2}(\lambda)}.$$

The *Thurston distance* between X_1 and X_2 is defined by

$$d_{\mathrm{Th}}(X_1, X_2) := \max\bigl\{d(X_1, X_2), d(X_1, X_2)\bigr\}.$$

The *Thurston distance ball* centered at $X \in \mathcal{T}_g$ of radius ε is defined to be

$$\mathbb{B}_X(\varepsilon) := \bigl\{Y \in \mathcal{T}_g : d_{\mathrm{Th}}(X,Y) \le \varepsilon/2\bigr\}.$$

The reason for this choice of radius is that, for small ε, e.g., $0 < \varepsilon < 1$,

$$e^{\varepsilon} < 1 + 2x, \quad e^{-\varepsilon} > 1 - 2x.$$

We have therefore, for any $\lambda \in \mathcal{ML}_g$ and any $Y \in \mathbb{B}_X(\varepsilon)$,

$$(1-\varepsilon)\cdot \ell_X(\lambda) \le \ell_Y(\lambda) \le (1+\varepsilon)\cdot \ell_X(\lambda). \tag{2.1}$$

Thurston distance balls are well-defined on $\mathcal{M}_g^{\gamma}$, and on $\mathcal{M}_g$, since the Thurston distance is Mod_g-invariant.

2.7. Stable graphs

Given a multicurve $m_1\gamma_1 + \cdots + m_k\gamma_k$, one can associate with it a *stable graph* in the following way. Cut the surface along $\gamma_1, \ldots, \gamma_k$. To each connected component S of $\Sigma_g \smallsetminus \{\gamma_1, \ldots, \gamma_k\}$, we associate a vertex, and we decorate this vertex with the genus of S. For each component γ_i of γ, we draw an edge that connects the two vertices (which could be the same) corresponding to the two connected components of $\Sigma_g \smallsetminus \{\gamma_1, \ldots, \gamma_k\}$ bounded by γ_i. See Figure 1 for an example. Note that the resulting graph does not depend on $m_1, \dots, m_k$. More formally, a stable graph consists of the data

$$\Gamma = (V,\ E,\ H,\ g\colon V \to \mathbb{Z}_{\geq 0},\ \iota\colon H \to H)$$

satisfying the following properties:

(1) The pair (V, E) defines a connected graph, with vertex set V and edge set E. The set H is the set of *half-edges*.

(2) The map v assigns each half-edge to its adjacent vertex.

(3) The map ι is an involution, such that the 2-cycles of ι are in bijection with E, and the fixed points of ι are in bijection with L.

(4) The *genus function* g assigns each vertex x to its genus (the genus of the surface corresponding to x), such that the *stability condition*

$$2g(x) - 2 + n(x) > 0$$

is satisfied, where $n(x)$ denotes the number of edges and legs adjacent to x.

2.8. Graph polynomials

Given a stable graph associated to the multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$, we associate to each edge e a variable x_e, and define the associated *graph polynomial* by the formula

$$P_\gamma(x_e : e \in E) = \prod_e x_e \cdot \prod_v V_{g(v),n(v)}(x_{e(h)} : h \in H,\ v(h) = v), \tag{2.2}$$

where e runs through the edge set E, v runs through the vertex set V, $V_{g(v),n(v)}$ is the Weil–Petersson volume polynomial of $\mathcal{M}_{g(v),n(v)}$ (see Theorem 2.2), $e(h)$ is the edge that contains the half-edge h, and $v(h)$ denotes the vertex incident to h. Note that P_γ is of degree $2d - k$. Finally, we write $\overline{P}_\gamma$ for the top-degree homogeneous part of P_γ, and $\overline{V}_{g,n}$ for that of $V_{g,n}$.

Example 2.4. If $\{\gamma_1, \dots, \gamma_{3g-3}\}$ is a pants decomposition, then $V_{g(v),n(v)} = 1$ for all $v \in V$, and $P_\gamma(x_1, \dots, x_{3g-3}) = \overline{P}_\gamma(x_1, \dots, x_{3g-3}) = x_1 \cdots x_{3g-3}$.

Example 2.5. Let $(\gamma_1, \gamma_2, \gamma_3)$ be an ordered multicurve on Σ_3 as in Figure 1. The Weil–Petersson volume polynomial $V_{1,3}(x_1, x_2, x_3)$ is equal to (see [5])

$$\Big(\frac{\mathrm{m}_{(3)}}{1152} + \frac{\mathrm{m}_{(2,1)}}{192} + \frac{\mathrm{m}_{(1,1,1)}}{96} + \frac{\pi^2\,\mathrm{m}_{(2)}}{24} + \frac{\pi^2\,\mathrm{m}_{(1,1)}}{8} + \frac{13\pi^4\,\mathrm{m}_{(1)}}{24}\Big)(x_1^2, x_2^2, x_3^2) + \frac{14\pi^6}{9},$$

where m stands for the monomial symmetric polynomial. For example,

$$\mathrm{m}_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_3^2 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2,$$
$$\mathrm{m}_{(1)}(x_1, x_2, x_3) = x_1 + x_2 + x_3.$$

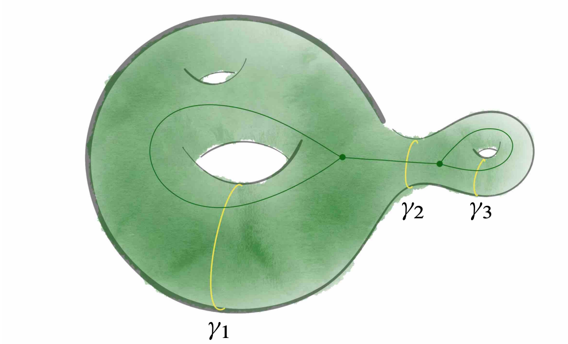

Figure 1. Example 2.5

So the top-degree part of $V_{1,3}$ is

$$\bar{V}_{1,3}(x_1, x_1, x_2) = \frac{2x_1^6 + x_2^6}{1152} + \frac{2x_1^6 + 2x_1^4 x_2^2 + 2x_1^2 x_2^4}{192} + \frac{x_1^4 x_2^2}{96},$$

and therefore,

$$\bar{P}_\gamma(x_1, x_2, x_3) = \frac{2x_1^6 x_2 x_3 + x_1 x_2^6 x_3}{1152} + \frac{x_1^7 x_2 x_3 + x_1^5 x_2^3 x_3 + x_1^3 x_2^5 x_3 + x_1^5 x_2^3 x_3}{96}.$$

3. Mirzakhani's covering spaces

Let $\gamma = (m_1\gamma_1, \dots, m_k\gamma_k)$ be an ordered multicurve on Σ_g. Recall that $\mathrm{Stab}(\gamma)$ denotes the subgroup of Mod_g that fixes each γ_i, $1 \le i \le k$. The quotient space

$$\mathcal{M}_g^\gamma := \mathcal{T}_g / \mathrm{Stab}(\gamma)$$

introduced by M. Mirzakhani in her thesis plays an important role in this paper.

Write $\pi^\gamma : \mathcal{T}_g \to \mathcal{M}_g^\gamma$ and $\pi_\gamma : \mathcal{M}_g^\gamma \to \mathcal{M}_g$ for the two natural projections ("raising and lowering the index"). Let us consider the product space $P_\gamma := \mathcal{T}_g \times \mathrm{Mod}_g \cdot (\gamma_1, \dots, \gamma_k)$ of the Teichmüller space and the mapping class group orbit of $(\gamma_1, \dots, \gamma_k)$. The mapping class group Mod_g acts on P (from the right) via

$$(X; \alpha_1, \dots, \alpha_k) \cdot h = (X \cdot h; h^{-1}\alpha_1, \dots, h^{-1}\alpha_k).$$

Lemma 3.1. *The quotient $P_\gamma / \mathrm{Mod}_g$ is isomorphic to $\mathcal{M}_g^\gamma$ as symplectic orbifolds.*

Proof. Consider the map $P_\gamma \to \mathcal{M}_g^\gamma$ defined by $(X, h\gamma) \mapsto \pi^\gamma(Xh)$. This map is surjective, and descends to the quotient $P_\gamma / \mathrm{Mod}_g$. The resulting map $P_\gamma / \mathrm{Mod}_g \to \mathcal{M}_g^\gamma$ is a local isomorphism of symplectic orbifolds. All that remains now is to show that the map

$P_\gamma / \operatorname{Mod}_g \to \mathcal{M}_g^\gamma$ is injective. Let $(X_1, h_1\gamma), (X_2, h_2\gamma) \in P_\gamma$ such that $\pi^\gamma(X_1 h_1) = \pi^\gamma(X_2 h_2)$. By definition, there exists $s \in \operatorname{Stab}(\gamma)$ such that $X_1 h_1 s = X_2 h_2$. Therefore

$$(X_2, h_2\gamma) = (X_1 h_1 s h_2^{-1}, h_2\gamma) \sim (X_1 h_1 s, \gamma) \sim (X_1 h_1, \gamma),$$

which proves the injectivity. ∎

Remark 3.2. Let α be a simple closed curve on Σ_g. In general, $\ell_X(\alpha)$ is not well-defined for $X \in \mathcal{M}_g^\gamma$. However, it is if $\alpha = \gamma_i$ for some i.

The next lemma is a simple fact, but for our purposes it will be very important: it transforms the multicurves counting that we are after to a "lattice points" counting problem on $\mathcal{M}_g^\gamma$.

Lemma 3.3. *Let $\gamma = (m_1\gamma_1, \ldots, m_k\gamma_k)$ be an ordered multicurve, $X \in \mathcal{T}_g$, $R \in \mathbb{R}_{>0}$, and $A \subset \Delta^{k-1}$ be an open subset. The set*

$$\{h\gamma : h \in \operatorname{Mod}_g,\ \ell_X(h\gamma) \le R,\ \hat{\ell}_X(h\gamma) \in A\}$$

and the set

$$\{[(X, h\gamma)] \in \mathcal{M}_g^\gamma : h \in \operatorname{Mod}_g,\ \ell_X(h\gamma) \le R,\ \hat{\ell}_X(h\gamma) \in A\},$$

where by $[(X, h\gamma)]$ we mean the image of $(X, h\gamma)$ under $P \to P/\operatorname{Mod}_g$, are in bijection given by $h\gamma \mapsto [(X, h\gamma)]$,

Proof. The given map is obviously surjective. Suppose that $[(X, h_1\gamma)] = [(X, h_2\gamma)]$, then $h_1^{-1}h_2 \in \operatorname{Stab}(\gamma)$, and therefore $h_1\gamma = h_2\gamma$. The injectivity follows. ∎

Next, let us review another covering space of $\mathcal{M}_g$ that Mirzakhani introduced. By considering the Fenchel–Nielsen coordinates associated to a pants decomposition that contains $\gamma_1, \ldots, \gamma_k$, the Teichmüller space $\mathcal{T}_g$ can be written as

$$\begin{aligned}\{(\ell_e, \tau_e, X_v) : e \in E,\ v \in V,\ \ell_e \in \mathbb{R}_{>0},\ \tau_e \in \mathbb{R},& \\ X_v \in \mathcal{T}_{g(v),n(v)}(\ell_{e(h)} : h \in H,\ v(h) = v)\},& \end{aligned} \tag{3.1}$$

where V (resp. E; H) is the vertex (resp. edge; half-edge) set of the stable graph associated to γ. The group

$$G_\gamma := \prod_e \mathbb{Z} \times \prod_v \operatorname{Mod}_{g(v),n(v)}$$

acts naturally on $\mathcal{T}_g$ written in the form (3.1) (each copy of $\mathbb{Z}$ acts as the Dehn twist about a γ_i), and G_γ can be identified with $\operatorname{Stab}^+(\gamma)$. The quotient $C_\gamma := \mathcal{T}_g / G_\gamma$ is of the form

$$\begin{aligned}\{(\ell_e, \tau_e, X_v) : e \in E,\ v \in V,\ \ell_e \in \mathbb{R}_{>0},\ \tau_e \in \mathbb{R}/\ell_e\mathbb{Z},& \\ X_v \in \mathcal{M}_{g(v),n(v)}(\ell_{e(h)} : h \in H,\ v(h) = v)\}.&\end{aligned}$$

Since $G_\gamma \simeq \mathrm{Stab}^+(\gamma)$ is a subgroup of $\mathrm{Stab}(\gamma)$, $\mathcal{T}_g \to \mathcal{M}_g^\gamma$ factors through a (ramified) covering map $C_\gamma \to \mathcal{M}_g^\gamma$. The degree of this covering map is

$$\kappa_\gamma = 2^{M(\gamma)} \cdot [\mathrm{Stab}(\gamma) : \langle \mathrm{Stab}^+(\gamma), \mathrm{Stab}_0(\gamma)\rangle],$$

where $M(\gamma)$ is the number of i such that γ_i bounds a surface homeomorphic to $\Sigma_{1,1}$, and $\langle \mathrm{Stab}^+(\gamma), \mathrm{Stab}_0(\gamma)\rangle$ stands for the subgroup of $\mathrm{Stab}(\gamma)$ generated by $\mathrm{Stab}^+(\gamma)$ and the kernel $\mathrm{Stab}_0(\gamma)$ of the action of $\mathrm{Stab}(\gamma)$ on $\mathcal{T}_g$. Note that $\mathrm{Stab}_0(\gamma)$ is trivial when $g \geq 3$, and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $g = 2$ (generated by the hyperelliptic involution which fixes the free homotopy class of every simple closed curve on Σ_2). For more details, see the footnote on pp. 369–370 of [16].

Integrating functions over C_γ (and $\mathcal{M}_g^\gamma$) is far less delicate than integrating function over $\mathcal{M}_g$. Starting from this observation, Mirzakhani was able to calculate the integrals of an important class of functions defined on $\mathcal{M}_g$, which she called "geometric functions".

Theorem 3.4 (Mirzakhani's integration formula). *Let $\gamma = (\gamma_1, \dots, \gamma_k)$ be an ordered multicurve, and $f\colon \mathbb{R}^k_{>0} \to \mathbb{R}$ be a measurable function. Let $X \in \mathcal{M}_g$, and choose an $\widetilde{X} \in \pi^{-1}(X) \in \mathcal{T}_g$. We define $f_\gamma\colon \mathcal{M}_g \to \mathbb{R}$ by the formula*

$$f_\gamma(X) := \sum_{(\alpha_1,\dots,\alpha_k)\in \mathrm{Mod}_g \cdot(\gamma_1,\dots,\gamma_k)} f(\ell_{\widetilde{X}}(\alpha_1), \dots, \ell_{\widetilde{X}}(\alpha_k)).$$

Note that $f_\gamma(X)$ does not depend on the choice of $\widetilde{X}$. We have

$$\begin{aligned}\int_{\mathcal{M}_g} f_\gamma(X)\, dX &= \int_{\mathcal{M}_g^\gamma} f(\ell_X(\gamma_1), \dots, \ell_X(\gamma_k))\, dX \\ &= \kappa_\gamma \int_{\mathbb{R}^k_{>0}} f(x_1, \dots, x_k) \cdot P_\gamma(x_1, \dots, x_k)\, dx_1 \cdots dx_k.\end{aligned}$$

4. Horospheres

Let $\gamma = (m_1\gamma_1, \dots, m_k\gamma_k)$ be an ordered multicurve, $\overline{\gamma} = m_1\gamma_1 + \cdots + m_k\gamma_k$ be its unlabeled counterpart, and A be an open subset of the standard simplex $\Delta^{k-1} := \{(x_1, \dots, x_k) \in \mathbb{R}^k_{\geq 0} : x_1 + \cdots + x_k = 1\}$ of dimension $k-1$.

The *horosphere* of radius R associated to γ and A on $\mathcal{T}_g$ is defined by

$$\widetilde{\mathcal{S}}^A_{R,\gamma} := \big\{X \in \mathcal{T}_g : \ell_X(\gamma) = R,\ \hat{\ell}_X(\gamma) \in A\big\}.$$

Similar notions can be defined on $\mathcal{M}_g^\gamma$ and on $\mathcal{M}_g$ by

$$\mathcal{S}^A_{R,\gamma} := \pi^\gamma(\widetilde{\mathcal{S}}^A_{R,\gamma}) \subset \mathcal{M}_g^\gamma, \quad \overline{\mathcal{S}}^A_{R,\gamma} := \pi(\widetilde{\mathcal{S}}^A_{R,\gamma}) \subset \mathcal{M}_g,$$

where $\pi^\gamma\colon \mathcal{T}_g \to \mathcal{M}_g^\gamma$ and $\pi\colon \mathcal{T}_g \to \mathcal{M}_g$ are the natural projections.

Remark 4.1. The function $X \mapsto (m_i\ell_X(\gamma_i))_{i=1}^k$ is well-defined for $X \in \mathcal{M}_g^\gamma$. The horosphere $\mathcal{S}^A_{R,\gamma} \subset \mathcal{M}_g^\gamma$ can be written as the pre-image of $R \cdot A$ under this function, where $R \cdot A$ is defined to be $\{(x_1, \dots, x_k) \in \mathbb{R}^k_{>0} : (x_1, \dots, x_k)/R \in A\}$.

4.1. Horospherical measures

We can choose $d-k$ simple closed curves $\alpha_{k+1},\dots,\alpha_d$ such that $\{\gamma_1,\dots,\gamma_k,\alpha_{k+1},\dots,\alpha_d\}$ is a pants decomposition. In the associated Fenchel–Nielsen coordinates, the horosphere $\widetilde{\mathcal{S}}^A_{R,\gamma}$ is an open subset of a simplex. Let μ_Δ denote the Weil–Petersson (Lebesgue) measure on this simplex. The *horospherical measure* $\mu^A_{R,\gamma}$ of an open subset $U\subset\mathcal{T}_g$ is defined to be

$$\mu^A_{R,\gamma}(U) := \mu_\Delta(U\cap\widetilde{\mathcal{S}}^A_{R,\gamma}).$$

The horospherical measure $\mu^A_{R,\gamma}$ is invariant under the action of the mapping class group, and hence descends to a measure on $\mathcal{M}^\gamma_g$ and a measure on $\mathcal{M}_g$; by abuse of notation we shall denote both by $\mu^A_{R,\gamma}$. Note that $\mathcal{M}^\gamma_g\to\mathcal{M}_g$ is a (ramified) covering map of infinite degree. However, its restriction on $\mathcal{S}^A_{R,\gamma}$ is of finite degree. Thus $\mu^A_{R,\gamma}$ on $\mathcal{M}_g$ is the push-forward measure of $\mu^A_{R,\gamma}$ by $\mathcal{M}^\gamma_g\to\mathcal{M}_g$. So for any open subset U of $\mathcal{M}_g$,

$$\mu^A_{R,\gamma}(\pi_\gamma^{-1}(U)) = [\mathrm{Stab}(\overline{\gamma}):\mathrm{Stab}(\gamma)]\cdot\mu^A_{R,\gamma}(U).$$

In particular, the total masses of $\mu^A_{R,\gamma}$ on $\mathcal{M}^\gamma_g$ and on $\mathcal{M}_g$ differ only by a multiplicative constant depending only on γ.

4.2. Total mass

The horospherical measure on $\mathcal{T}_g$ has infinite total mass. Nevertheless, its total mass is finite on $\mathcal{M}^\gamma_g$ and $\mathcal{M}_g$.

Proposition 4.2. *The total mass of $\mu^A_{R,\gamma}$ on $\mathcal{M}_g$ is*

$$M^A_{R,\gamma} = \frac{\kappa_\gamma}{[\mathrm{Stab}(\overline{\gamma}):\mathrm{Stab}(\gamma)]}\frac{1}{m_1\cdots m_k}\int_{R\cdot A} P_\gamma(x_1/m_k,\dots,x_k/m_k)\,\lambda(dx),$$

where $R\cdot A := \{(x_1,\dots,x_k)\in\mathbb{R}^k_{\geq 0} : (x_1,\dots,x_k)/R\in A\}$, λ is the Lebesgue measure on $\{(x_1,\dots,x_k)\in\mathbb{R}_{\geq 0} : x_1+\dots+x_k = R\}$, and P_γ is defined by the formula (2.2).

Proof. In the light of Remark 4.1, by taking f in Theorem 3.4 to be the indicator function

$$\mathbb{1}\big\{(x_1,\dots,x_k)\in\mathbb{R}^k_{>0} : R\leq m_1x_1+\dots+m_kx_k\leq R+\varepsilon,\\ (m_1x_1,\dots,m_kx_k)/(m_1x_1+\dots+m_kx_k)\in A\big\},$$

we obtain that $\mu^A_{R,\gamma}(\mathcal{M}_g)\cdot[\mathrm{Stab}(\overline{\gamma}):\mathrm{Stab}(\gamma)]$ is equal to

$$\begin{aligned}\mu^A_{R,\gamma}(\mathcal{M}^\gamma_g) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathcal{M}_g} f_\gamma(X)\,dX\\ &= \frac{\kappa_\gamma}{m_1\cdots m_k}\int_{R\cdot A} P_\gamma(x_1/m_1,\dots,x_k/m_k)\,\lambda(dx),\end{aligned}$$

the desired result. ∎

Corollary 4.3. *The total mass $M^A_{R,\gamma}$ (of $\mu^A_{R,\gamma}$ on $\mathcal{M}_g$) is a polynomial in R of degree $2d-1 = 6g-7$. Write C^A_γ for its leading coefficient. We have*

$$M^A_{R,\gamma} \sim C^A_\gamma \cdot R^{2d-1} \tag{4.1}$$

as $R \to \infty$, and C^A_γ can be calculated by

$$C^A_\gamma = \frac{\kappa_\gamma}{[\mathrm{Stab}(\overline{\gamma}) : \mathrm{Stab}(\gamma)]} \frac{1}{m_1 \cdots m_k} \int_A \overline{P}_\gamma(x_1/m_1, \ldots, x_k/m_k)\, \lambda(dx),$$

where λ is the Lebesgue measure on Δ^{k-1} and $\overline{P}_\gamma$ is the top-degree homogeneous part of the graph polynomial P_γ defined by (2.2).

Remark 4.4. It results from Theorem 2.2 that the polynomial $\overline{P}_\gamma$ can be expressed in terms of intersections numbers of ψ-classes on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$.

4.3. Horospherical measures on the unit sphere bundle

Define

$$\mathcal{P}\widetilde{\mathcal{S}}^A_{R,\gamma} := \big\{(X, \gamma/R) \in \mathcal{P}^1\mathcal{T}_g : \hat{\ell}_X(\gamma) \in A\big\}.$$

Note that $\mathcal{P}\widetilde{\mathcal{S}}^A_{R,\gamma}$ projects via $\mathcal{P}^1\mathcal{T}_g \to \mathcal{T}_g$ to $\widetilde{\mathcal{S}}^A_{R,\gamma}$, and is invariant under the earthquake flow.

Let ν_Δ denote the Lebesgue measure on $\mathcal{P}\widetilde{\mathcal{S}}^A_{R,\gamma}$. The *horospherical measure* $\nu^A_{R,\gamma}$ on $\mathcal{P}\mathcal{T}_g$ is defined by the formula

$$\nu^A_{R,\gamma}(U) := \nu_\Delta(U \cap \mathcal{P}\widetilde{\mathcal{S}}^A_{R,\gamma}),$$

where U is any open subset of $\mathcal{P}^1\mathcal{T}_g$. The measure $\nu^A_{R,\gamma}$ is Mod_g-invariant, and therefore descends to a measure on $\mathcal{P}^1\mathcal{M}_g$ which by abuse of notation we shall also denote by $\nu^A_{R,\gamma}$. Note that $\mu^A_{R,\gamma}$ is the push-forward of $\nu^A_{R,\gamma}$ via $\mathcal{P}^1\mathcal{M}_g \to \mathcal{M}_g$.

Notation. To simplify the notation, let us fix $X \in \mathcal{M}_g$, a multicurve $\gamma = (m_1\gamma_1, \ldots, m_k\gamma_k)$ on Σ_g, and an open subset A of the standard simplex $\Delta^{k-1} := \{(x_1, \ldots, x_k) \in \mathbb{R}_{\geq 0} : x_1 + \cdots + x_k = 1\}$. From now on we shall write μ_R for $\mu^A_{R,\gamma}$, ν_R for $\nu^A_{R,\gamma}$, and M_R for $M^A_{R,\gamma}$, unless otherwise stated.

5. Equidistribution

In this section, we establish the equidistribution of large horospheres. The proof is adapted from that of [10, Theorem 1.1].

Theorem 5.1. *We have weak convergence of probability measures on $\mathcal{P}^1\mathcal{M}_g$*

$$\frac{\nu_R}{M_R} \Rightarrow \frac{\nu_g}{b_g}$$

as $R \to \infty$.

The following immediate corollary is exceedingly useful late on.

Corollary 5.2. *We have weak convergence of probability measures on $\mathcal{M}_g$*

$$\frac{\mu_R}{M_R} \Rightarrow \frac{B(X)}{b_g}\mu_{\mathrm{WP}}$$

as $R \to \infty$.

Proof. This follows from the fact that μ_R is the push-forward of ν_R via $\mathcal{P}^1\mathcal{M}_g \to \mathcal{M}_g$ and Theorem 5.1. ∎

The proof of Theorem 5.1 rests on the following series of propositions.

Let ν be a weak limit of $(\nu_R/M_R)_{R>0}$.

Proposition 5.3. *The measure ν is invariant under the earthquake flow.*

Proposition 5.4. *The measure ν is absolutely continuous with respect to ν_g.*

Proposition 5.5. *The measure ν is a probability measure.*

Proof of Theorem 5.1. Proposition 5.3, Proposition 5.4, and Theorem 5.1 imply that ν and ν_g differ by a multiplicative constant, and it follows from Proposition 5.5 that this constant is 1. ∎

Proposition 5.3 is immediate.

Proof of Proposition 5.3. This follows from the fact that ν_R is invariant under the earthquake flow (since ν_g is). ∎

For the rest of this section we shall prove Propositions 5.4 and 5.5, which are more technical.

5.1. Escape to infinity?

In this subsection, we prove Proposition 5.5. The key ingredient is the following non-divergence result for the earthquake flow due to Y. Minsky and B. Weiss.

Theorem 5.6 ([9, Theorem E2], [10, Corollary 5.12]). *For any $c > 0$, there exists $\varepsilon > 0$, depending only on c, such that for any $x \in \mathcal{T}_g$ and any $\lambda \in \mathcal{ML}_g$, the following dichotomy holds:*

(1) *There exists a simple closed curve α disjoint from λ, and $\ell_x(\alpha) < \varepsilon$.*

(2) *We have*

$$\liminf_{T\to\infty} \frac{|\{t \in [0,T] : \pi(\mathrm{tw}_\lambda^t(x)) \in \mathcal{M}_g^{\geq\varepsilon}\}|}{T} > 1 - c,$$

where $\pi\colon \mathcal{T}_g \to \mathcal{M}_g$ is the natural projection, and $\mathcal{M}_g^{\geq\varepsilon}$ is the compact subset of $\mathcal{M}_g$ consisting of all surfaces whose shortest closed geodesic has length at least ε.

Proof of Proposition 5.5. It is enough to prove that for any $\delta > 0$, we can find a compact subset K_δ of $\mathcal{P}^1\mathcal{M}_g$ such that

$$\liminf_{R\to\infty} \frac{\nu_R(K_\delta)}{M_R} \geq 1 - \delta.$$

The strategy is to show that there exists $\varepsilon > 0$, depending only on δ, such that the pre-image of $\mathcal{M}_g^{\geq\varepsilon}$ under $\mathcal{P}^1\mathcal{M}_g \to \mathcal{M}_g$ possess the desired property. In other words,

$$\liminf_{R\to\infty} \frac{\mu_R(\mathcal{M}_g^{\geq\varepsilon})}{M_R} \geq 1 - \delta.$$

Taking $c = \delta/2$, Theorem 5.6 allows us to write $\widetilde{\mathcal{S}}_R \subset \mathcal{T}_g$ as the disjoint union of $\widetilde{\mathcal{S}}_1$ and $\widetilde{\mathcal{S}}_2$ corresponding to the two possibilities. For convenience, we shall adapt the convention that $\overline{\mathcal{S}}_*$ (resp. $\mathcal{S}_*$) denotes the image of $\widetilde{\mathcal{S}}_*$ under $\mathcal{T}_g \to \mathcal{M}_g$ (resp. $\mathcal{T}_g \to \mathcal{M}_g^\gamma$), where $*$ is a certain index.

First, we show that $\mu_R(\overline{\mathcal{S}}_1) \leq \mu_R(\mathcal{S}_1) = \mathrm{o}(M_R)$ as $R \to \infty$ even when $A = \Delta^{k-1}$ (the subset of the simplex that we choose to define μ_R is the whole simplex). For any point in $\widetilde{\mathcal{S}}_1$, at least one of the following holds:

(1) α is freely homotopic to γ_i for some $1 \leq i \leq k$.

(2) α is disjoint from $\gamma_1, \dots, \gamma_k$.

Thus $\widetilde{\mathcal{S}}_1$ can be written as the union of $\widetilde{\mathcal{S}}_{1,1}$ and $\widetilde{\mathcal{S}}_{1,2}$ corresponding to the two cases above. To simplify the notation, in the following estimates of $\mu_R(\mathcal{S}_{1,1})$ and $\mu_R(\mathcal{S}_{1,2})$ we assume that γ is primitive, i.e. $m_1 = \cdots = m_k = 1$ (the calculation differs from the general case only by a multiplicative constant).

For each i, the corresponding horospherical volume of $\mathcal{S}_{1,1}$ can be estimated by taking f in Theorem 3.4 to be the indicator function

$$\mathbb{1}\big\{(x_1, \dots, x_k) \in \mathbb{R}^k_{\geq 0} : R \leq x_1 + \cdots + x_k \leq R + h,\ x_i < \varepsilon\big\},$$

and we obtain

$$\mu_R(\mathcal{S}_{1,1}) \leq \sum_{i=1}^{k} \lim_{h\to 0} \frac{\kappa_\gamma}{h} \cdot \int_0^\varepsilon dx_i \int_{\Delta^{k-1}_{[R-x_i, R+h-x_i]}} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_k \, P_\gamma(x_1, \dots, x_k),$$

where

$$\Delta^{k-1}_{[R-x_i, R+h-x_i]} := \big\{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \in \mathbb{R}^{k-1}_{\geq 0} : R \leq x_1 + \cdots + x_k \leq R + h\big\}.$$

Since P_γ is a polynomial of degree $2d - k$ and $x_1 \cdots x_k$ is a factor of P_γ, we have $\mu_R(\mathcal{S}_{1,1}) = \mathrm{O}(\varepsilon^2 R^{2d-3})$.

We now suppose that α is disjoint from $\gamma_1, \ldots, \gamma_k$. Denote by (γ, α) the ordered multi-curve $(\gamma_1, \ldots, \gamma_k, \alpha)$. Again, by applying Theorem 3.4, f being the indicator function

$$\mathbb{1}\big\{(x_1, \ldots, x_k, y) \in \mathbb{R}^{k+1}_{>0} : R \leq x_1 + \cdots + x_k \leq R + h,\ y < \varepsilon\big\},$$

we obtain the corresponding horospherical volume

$$\lim_{h \to 0} \frac{\kappa_\gamma}{h} \int_0^\varepsilon dy \int_{\Delta^k_{[R,R+h]}} dx_1 \cdots dx_k\, P_{(\gamma,\alpha)}(x_1, \ldots, x_k, y),$$

where

$$\Delta^k_{[R,R+h]} := \big\{(x_1, \ldots, x_k) \in \mathbb{R}^k_{\geq 0} : R \leq x_1 + \cdots + x_k \leq R + h\big\}.$$

Since $P_{(\gamma,\alpha)}$ is a polynomial of degree $2d - k - 1$ of which y is a factor, and there are only finitely many topological types of (γ, α), we have $\mu_R(\mathcal{S}_{1,2}) = \mathrm{O}(\varepsilon^2 R^{2d-3})$.

Using Corollary 4.3, we deduce

$$\frac{\mu_R(\overline{\mathcal{S}}_1)}{M_R} \leq \frac{\mu_R(\overline{\mathcal{S}}_{1,1}) + \mu_R(\overline{\mathcal{S}}_{1,2})}{M_R} \leq \frac{\mu_R(\mathcal{S}_{1,1}) + \mu_R(\mathcal{S}_{1,2})}{M_R} = \mathrm{O}(\varepsilon^2 R^{-2}) = \mathrm{o}(1)$$

as $R \to \infty$.

Let us now consider $\overline{\mathcal{S}}_2$. One observes that every $p \in \overline{\mathcal{S}}_2$ lies in a unique 1-periodic earthquake flow orbit along the direction $\gamma_p := \gamma_1/\ell_p(\gamma_1) + \cdots + \gamma_k/\ell_p(\gamma_k)$, and $\overline{\mathcal{S}}_2$ can be written as the disjoint union of such orbits. (If one completes γ to a pants decomposition by adding $d - k$ simple curves, such orbits are parallel straight lines in the $(\tau_{\gamma_1}, \ldots, \tau_{\gamma_k})$-coordinates plane.) By Theorem 5.6,

$$\Big|\big\{t \in [0,1] : \pi(\mathrm{tw}^t_{\gamma_p}(p)) \in \mathcal{M}_g^{\geq\varepsilon}\big\}\Big| > 1 - \delta/2,$$

for all $p \in \mathcal{S}_2$. Thus,

$$\frac{\mu_R(\overline{\mathcal{S}}_2 \cap \mathcal{M}_g^{\geq\varepsilon})}{\mu_R(\overline{\mathcal{S}}_2)} \geq 1 - \delta/2.$$

Therefore,

$$\begin{aligned}\frac{\mu_R(\mathcal{M}_g^{\geq\varepsilon})}{M_R} &= \frac{\mu_R(\mathcal{M}_g^{\geq\varepsilon} \cap \overline{\mathcal{S}}_1) + \mu_R(\mathcal{M}_g^{\geq\varepsilon} \cap \overline{\mathcal{S}}_2)}{M_R} \\ &= 0 + \frac{\mu_R(\mathcal{M}_g^{\geq\varepsilon} \cap \overline{\mathcal{S}}_2)}{\mu_R(\overline{\mathcal{S}}_2)} \frac{M_R - \mu_R(\overline{\mathcal{S}}_1)}{M_R} \\ &\geq (1 - \delta/2)(1 - \mathrm{o}(1))\end{aligned}$$

as $R \to \infty$. The $\mathrm{o}(1)$ term can be made, e.g., smaller than $\delta/2$, by increasing R. The proof is thus complete. ∎

5.2. Absolute continuity

In this subsection, we prove Proposition 5.4.

We use the following notation throughout the subsection. Let d denote $3g-3$. We write $f = \mathrm{O}_K(g)$ if there exists $C > 0$, depending only on K, such that $f \le Cg$, and we write $f = \Theta_K(g)$ if there exists C, depending only on K, such that $(1/C)g \le f \le Cg$.

The key to the proof are the following estimates.

Theorem 5.7 ([1, 10, 14]). *Let $\varepsilon \in (0,1)$ and let K be a compact subset of $\mathcal{M}_g$. Then:*

(1) *For any $x \in K$, $\mu_{\mathrm{WP}}(\mathbb{B}_x(\varepsilon)) = \Theta_K(\varepsilon^{2d})$.*

(2) *For any $x \in \pi^{-1}(K) \subset \mathcal{T}_g$, $\mu_R(\mathbb{B}_x(\varepsilon)) = \mathrm{O}_K(\varepsilon^{2d-1}/R)$.*

Remark 5.8. The first part of the preceding theorem is [10, Theorem 5.5.a]. Mirzakhani proved the second part in the case when γ is a simple closed curve [10, Theorem 5.5.b], and claimed a more general version [14, Proposition 2.1.b] without proof. The proof of [10, Theorem 5.5.b] is concise and not easy to follow. See also the footnote on p. 390 in [16]. A much stronger estimate is obtained by Arana-Herrera in a different approach [1, Proposition 1.5].

The rest of the proof of Proposition 5.4 can be adapted from Mirzakhani's original proof in the case when γ is simple. Let us sketch her arguments for the sake of self-containedness.

Corollary 5.9. *Let $U \subset \mathbb{P}(\mathcal{ML}_g)$ be open, $K \subset \mathcal{T}_g$ be compact, $x \in K$, and $p\colon \mathcal{T}_g \times \mathbb{P}(\mathcal{ML}_g) \to \mathcal{P}^1\mathcal{M}_g$ be the natural projection. For $\varepsilon \in (0,1)$, we have*

$$\frac{\nu_R(p(\mathbb{B}_x(\varepsilon) \times U))}{M_R} = \mathrm{O}_K(\nu_g(\mathbb{B}_x(\varepsilon) \times U_x)),$$

where $U_x := \{\lambda \in \mathcal{ML}_g : \ell_x(\lambda) \le 1,\ [\lambda] \in U\}$.

Proof. It is enough to prove this for $A = \Delta^{k-1}$. By (2.1), for any $y \in \mathbb{B}_x(\varepsilon)$, we have

$$(1-\varepsilon)^{2d} \cdot \mu_{\mathrm{Th}}(U_x) \le \mu_{\mathrm{Th}}(U_y) \le (1+\varepsilon)^{2d} \cdot \mu_{\mathrm{Th}}(U_x), \tag{5.1}$$

and so

$$\begin{aligned}
&\#\big\{\alpha \in \mathrm{Mod}_g \cdot \gamma : [\alpha] \in U,\ \ell_y(\alpha) = R \text{ for some } y \in \mathbb{B}_x(\varepsilon)\big\} \\
&\le \#\big\{\alpha \in \mathrm{Mod}_g \cdot \gamma : [\alpha] \in U,\ (1-\varepsilon)R \le \ell_x(\alpha) \le (1+\varepsilon)R\big\} \\
&= \mathrm{O}_K(\varepsilon R^{2d} \mu_{\mathrm{Th}}(U_x)).
\end{aligned}$$

Hence Theorem 5.7 (2) implies that $\nu_R(p(\mathbb{B}_x(\varepsilon) \times U)) = \mathrm{O}_K(\varepsilon^{2d} R^{2d-1} \mu_{\mathrm{Th}}(U_x))$. The result now follows from Theorem 5.7 (1) and Corollary 4.3. ∎

We need one further technical lemma.

Lemma 5.10. *Let K be a compact subset of $\mathcal{P}^1\mathcal{T}_g$. For any $N \subset K$ with $\nu_g(N) = 0$, and any $\varepsilon > 0$, there exists an open cover $\{\mathbb{B}_{X_i}(r_i) \times U_i : i \in \mathbb{Z}_{\geq 1}\}$ of N, where for all i, $X_i \in \mathcal{T}_g$, $r_i \in (0,1)$, and $U_i \subset \mathbb{P}(\mathcal{ML}_g)$ is open, such that*

$$\sum_{i \geq 1} \nu_g(\mathbb{B}_{X_i}(r_i) \times U_i) \leq \varepsilon.$$

Proof. Fix a choice of Fenchel–Nielsen coordinates. There exists an open cover $\{B_{X_i}(r_i) \times U_i : i \in \mathbb{Z}_{\geq 1}\}$ of N, where $B_{X_i}(r_i)$ is the Euclidean ball of radius r_i centered at X_i, such that $\sum_{i\geq 1} \nu_g(B_{X_i}(r_i) \times U_i) \leq \varepsilon$, and $\sup_{i \geq 1} r_i$ can be made as small as we please (since ν_g is a Lebesgue class measure). It follows from the compactness of $K \times [0,1]$ that there exists a constant s depending only on K such that $B_x(r) \subset \mathbb{B}_x(s \cdot r)$ for any $x \in K$ and any $r \in [0,1]$. By Theorem 5.7 (1), there exists a constant s' depending only on K such that $\mu_{\mathrm{WP}}(\mathbb{B}_x(s \cdot r)) \leq s' \cdot \mu_{\mathrm{WP}}(B_x(r))$ for any $x \in K$, and any $r < 1/2s$. Therefore, by (5.1),

$$\sum_{i \geq 1} \nu_g(\mathbb{B}_{X_i}(s \cdot r_i) \times U_i) \leq 3^{2d} s' \sum_{i \geq 1} \nu_g(B_{X_i}(r_i) \times U_i) \leq 3^{2d} s' \varepsilon = \mathrm{O}_K(\varepsilon)$$

and the lemma follows. ■

Proof of Proposition 5.4. It is sufficient, as before, to consider the case in which A is the whole simplex Δ^{k-1}. Let $N \subset \mathcal{P}^1\mathcal{M}_g$ with $\nu_g(N) = 0$. By Proposition 5.5, we may assume that N is contained in a compact set $K \subset \mathcal{P}^1\mathcal{M}_g$. Lemma 5.10 implies that for any $\varepsilon > 0$, there exists an open cover $\{\mathbb{B}_{X_i}(r_i) \times U_i : i \in \mathbb{Z}_{\geq 1}\}$ of N, such that

$$\sum_{i \geq 1} \nu_g(\mathbb{B}_{X_i}(r_i) \times U_i) \leq \varepsilon.$$

Hence, it follows from Corollary 5.9 that

$$\sum_{i \geq 1} \frac{\nu_R(\mathbb{B}_{X_i}(r_i) \times U_i)}{M_R} = \sum_{i \geq 1} \mathrm{O}_K(\nu_g(\mathbb{B}_{X_i}(r_i) \times U_{X_i})) \leq \mathrm{O}_K(\varepsilon).$$

The proof is thus complete. ■

6. Counting

The main result of this section is the following theorem which is a refined version of [13, Theorem 1.1].

Theorem 6.1. *Let $X \in \mathcal{M}_g$, $\gamma = (m_1\gamma_1, \ldots, m_k\gamma_k)$ be an ordered multicurve, and $A \subset \Delta^{k-1}$ be open. We have*

$$\#\big\{\alpha \in \mathrm{Mod}_g \cdot \gamma : \ell_X(\alpha) \leq R,\ \hat{\ell}_X(\alpha) \in A\big\} \sim C_\gamma^A \, \frac{[\mathrm{Stab}(\overline{\gamma}) : \mathrm{Stab}(\gamma)]}{2d} \, \frac{B(X)}{b_g} \, R^{2d}$$

as $R \to \infty$.

By virtue of Lemma 3.3, this multicurves counting problem can be transformed to a counting problem on $\mathcal{M}_g^\gamma$. Let us begin by introducing some definitions that we need to state our counting result on $\mathcal{M}_g^\gamma$.

The *horoball* (on $\mathcal{M}_g^\gamma$) is defined by

$$\mathcal{B}_R := \{X \in \mathcal{M}_g^\gamma : \ell_X(\gamma) \leq R,\ \hat{\ell}_X(\gamma) \in A\} = \bigcup_{0<r\leq R} \mathcal{S}_r$$

and its associated measure $\mu_{\leq R}$ is defined by the formula

$$\mu_{\leq R}(U) := \int_0^R \mu_r(U)\,dr = \mu_{\mathrm{WP}}(U \cap \mathcal{B}_R),$$

where U is any open subset of $\mathcal{M}_g^\gamma$. By abuse of notation, we shall also use $\mu_{\leq R}$ to denote the measure on $\mathcal{M}_g$ defined by the formula

$$\nu_{\leq R}(U) := \int_0^R \mu_r(U)\,dr = \mu_{\mathrm{WP}}(U \cap \pi_\gamma(\mathcal{B}_R)),$$

for any open subset U of $\mathcal{M}_g$. Let $X \in \mathcal{M}_g$ and let $N(R)$ denote the number of pre-images of X under $\pi_\gamma\colon \mathcal{M}_g^\gamma \to \mathcal{M}_g$ which lie within the horoball $\mathcal{B}_R \subset \mathcal{M}_g^\gamma$, i.e.,

$$N(R) := \#\{\pi_\gamma^{-1}(X) \cap \mathcal{B}_R\}.$$

We have the following counting result on $\mathcal{M}_g^\gamma$.

Theorem 6.2. *Let $X \in \mathcal{M}_g$, $\gamma = (m_1\gamma_1, \dots, m_k\gamma_k)$ be an ordered multicurve, and $A \subset \Delta^{k-1}$ be open. Then we have*

$$N(R) \sim C_\gamma^A \frac{[\mathrm{Stab}(\overline{\gamma}) : \mathrm{Stab}(\gamma)]}{2d} \frac{B(X)}{b_g} R^{2d}$$

as $R \to \infty$.

As an immediate corollary, we get the main result of this section:

Proof of Theorem 6.1. This follows at once from Theorem 6.2 and Lemma 3.3. ∎

We introduce a family of subsets $A_{a,b}$ of Δ^{k-1}, indexed by $a = (a_1, \dots, a_{k-1}) \in [0,1]^{k-1}$ and $b = (b_1, \dots, b_{k-1}) \in [0,1]^{k-1}$ such that $a_i < b_i$ for all $1 \leq i \leq k-1$, and defined by

$$A_{a,b} := \{(x_1, \dots, x_k) \in \Delta^{k-1} : a_i \leq x_i \leq b_i,\ \forall 1 \leq i \leq k-1\}.$$

To prove Theorem 6.2, it is enough to check the case when $A = A_{a,b}$ for all a, b. In order to abbreviate our formulas, for the rest of this section we write

$$A := A_{a,b}, \quad A_+ := A_{\frac{1-\varepsilon}{1+\varepsilon}a, \frac{1+\varepsilon}{1-\varepsilon}b}, \quad A_- := A_{\frac{1+\varepsilon}{1-\varepsilon}a, \frac{1-\varepsilon}{1+\varepsilon}b},$$

where we adopt the convention that $\frac{1+\varepsilon}{1-\varepsilon}b_i = 1$ if $\frac{1+\varepsilon}{1-\varepsilon}b_i > 1$, and we write $\mathcal{B}_R^+ := \mathcal{B}_R^{A_+}$, $\mu_{\leq R}^+ := \mu_{\leq R}^+$, etc. The reason for the choice of A_+ and A_- is the following elementary lemma.

Lemma 6.3. *Choose $\varepsilon \in (0,1)$ small enough to ensure that A_- and A_+ are well-defined, and let $x, y \in \mathcal{M}_g^\gamma$ with $d_{\mathrm{Th}}(x,y) \leq \varepsilon$. We have:*

(1) *If $x \in \mathcal{B}_{(1-\varepsilon)R}^-$, then $y \in \mathcal{B}_R$,*

(2) *If $x \in \mathcal{B}_R$, then $y \in \mathcal{B}_{(1+\varepsilon)R}^+$.*

Proof. Suppose that $x \in \mathcal{B}_R$. It follows from inequality (2.1) that

$$\ell_y(\gamma) \leq (1+\varepsilon)\ell_x(\gamma) \leq (1+\varepsilon)R$$

and

$$\frac{1-\varepsilon}{1+\varepsilon} a_i \leq \frac{(1-\varepsilon)\ell_x(m_i\gamma_i)}{(1+\varepsilon)\ell_x(\gamma)} \leq \frac{\ell_y(m_i\gamma_i)}{\ell_y(\gamma)} \leq \frac{(1+\varepsilon)\ell_x(m_i\gamma_i)}{(1-\varepsilon)\ell_x(\gamma)} \leq \frac{1+\varepsilon}{1-\varepsilon} b_i,$$

which shows that $y \in \mathcal{B}_{(1+\varepsilon)R}^+$. Part (1) can be proved in a similar manner. ∎

Proof of Theorem 6.2. We can choose $\varepsilon \in (0,1)$ such that $\mathbb{B}_{Y_1}(\varepsilon) \cap \mathbb{B}_{Y_2}(\varepsilon) = \varnothing$ for any distinct pre-images Y_1, Y_2 of X under $\pi_\gamma \colon \mathcal{M}_g^\gamma \to \mathcal{M}_g$. Let us write

$$N_-(R) := \#\{Y \in \pi_\gamma^{-1}(X) \subset \mathcal{M}_g^\gamma : \mathbb{B}_Y(\varepsilon) \subset \mathcal{B}_R\},$$

for the set of all $Y \in \mathcal{M}_g^\gamma$ such that Y projects to X and the Thurston distance ball of radius ε centered at Y is entirely included within the horoball $\mathcal{B}_R \subset \mathcal{M}_g^\gamma$. Furthermore, we write

$$N_+(R) := \#\{Y \in \pi_\gamma^{-1}(X) \subset \mathcal{M}_g^\gamma : \mathbb{B}_Y(\varepsilon) \cap \mathcal{B}_R \neq \varnothing\}$$

for the set of all $Y \in \mathcal{M}_g^\gamma$ that project to X such that $\mathbb{B}_Y(\varepsilon)$ intersects $\mathcal{B}_R$. By definition,

$$N_-(R) \leq N(R) \leq N_+(R).$$

It follows from Lemma 6.3 that

$$\begin{aligned} N_+(R) \cdot \mu_{\mathrm{WP}}(\mathbb{B}_X(\varepsilon)) &\leq \mu_{\mathrm{WP}}(\pi_\gamma^{-1}(\mathbb{B}_X(\varepsilon)) \cap \mathcal{B}_{(1+\varepsilon)R}^+) \\ &= \mu_{\leq(1+\varepsilon)R}^+(\pi_\gamma^{-1}(\mathbb{B}_X(\varepsilon))) \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \mu_{\leq(1-\varepsilon)R}^-(\pi_\gamma^{-1}(\mathbb{B}_X(\varepsilon))) &= \mu_{\mathrm{WP}}(\pi_\gamma^{-1}(\mathbb{B}_X(\varepsilon)) \cap \mathcal{B}_{(1-\varepsilon)R}^-) \\ &\leq N_-(R) \cdot \mu_{\mathrm{WP}}(\mathbb{B}_X(\varepsilon)). \end{aligned} \tag{6.2}$$

For any open subset $U \subset \mathcal{M}_g$,

$$\mu_{\leq R}(\pi_\gamma^{-1}(U)) = [\mathrm{Stab}(\overline{\gamma}) : \mathrm{Stab}(\gamma)] \cdot \mu_{\leq R}(U). \tag{6.3}$$

We deduce from (6.1), (6.2), and (6.3) that

$$\mu^-_{\le(1-\varepsilon)R}(\mathbb{B}_X(\varepsilon)) \le \frac{N(R)\cdot\mu_{\mathrm{WP}}(\mathbb{B}_X(\varepsilon))}{[\mathrm{Stab}(\overline{\gamma}):\mathrm{Stab}(\gamma)]} \le \mu^+_{\le(1+\varepsilon)R}(\mathbb{B}_X(\varepsilon)),$$

where $\mathbb{B}_X(\varepsilon)\subset\mathcal{M}_g$. Hence,

$$\begin{aligned}\lim_{R\to\infty}\frac{\mu^+_{(1+\varepsilon)R}(\mathbb{B}_X(\varepsilon))}{R^{2d}} &= \lim_{R\to\infty}\frac{1}{R}\int_0^{(1+\varepsilon)R}\frac{\mu^+_t(\mathbb{B}_X(\varepsilon))}{R^{2d-1}}\,dt\\ &= C^+\cdot\lim_{R\to\infty}\frac{1}{R^{2d}}\int_0^{(1+\varepsilon)L} t^{2d-1}\frac{\mu^+_t(\mathbb{B}_X(\varepsilon))}{C^+\cdot t^{2d-1}}\,dt,\end{aligned}\tag{6.4}$$

where $C^+ := C_\gamma^{A+}$ is given by (4.1). By Corollaries 5.2 and 4.3,

$$\frac{\mu^+_t(\mathbb{B}_X(\varepsilon))}{C^+\cdot t^{2d-1}} = \frac{1}{b_g}\int_{\mathbb{B}_X(\varepsilon)} B(Y)\,dY + \mathrm{o}(1)$$

as $t\to\infty$, where $\mathrm{o}(1)$ is bounded and the constant depends only on γ and A^+. Thus (6.4) is equal to

$$\frac{(1+\varepsilon)^{2d}\,C^+}{2d\,b_g}\int_{\mathbb{B}_X(\varepsilon)} B(Y)\,dY.$$

Therefore,

$$\frac{\mu_{\mathrm{WP}}(\mathbb{B}_X(\varepsilon))}{[\mathrm{Stab}(\overline{\gamma}):\mathrm{Stab}(\gamma)]}\limsup_{R\to\infty}\frac{N(R)}{R^{2d}} \le \frac{(1+\varepsilon)^{2d}\,C^+}{2d\,b_g}\int_{\mathbb{B}_X(\varepsilon)} B(Y)\,dY,$$

and similarly,

$$\frac{(1-\varepsilon)^{2d}\,C^-}{2d\,b_g}\int_{\mathbb{B}_X(\varepsilon)} B(Y)\,dY \le \frac{\mu_{\mathrm{WP}}(\mathbb{B}_X(\varepsilon))}{[\mathrm{Stab}(\overline{\gamma}):\mathrm{Stab}(\gamma)]}\liminf_{R\to\infty}\frac{N(R)}{R^{2d}},$$

where $C^- := C_\gamma^{A-}$. Taking $\varepsilon\to 0$, we obtain

$$\lim_{R\to\infty}\frac{N(R)}{R^{2d}} = C_\gamma^A\,\frac{[\mathrm{Stab}(\overline{\gamma}):\mathrm{Stab}(\gamma)]}{2d}\,\frac{B(X)}{b_g}.$$

This established the theorem. ∎

7. Statistics

Proof of Theorem 1.2. Theorem 6.1 implies

$$\lim_{R\to\infty}\mathbb{P}(\hat{\ell}_{X,R,\gamma}\in A) = \frac{C_\gamma^A}{C_\gamma^{\Delta^{k-1}}}.$$

The assertion now follows from Corollary 4.3. ∎

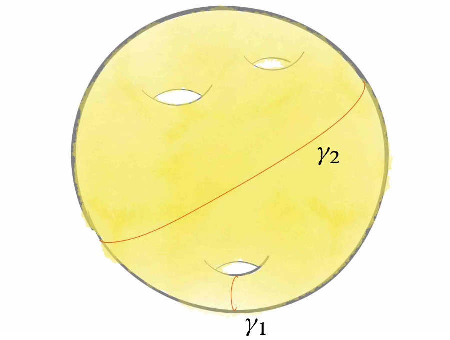

Figure 2. Example 7.2

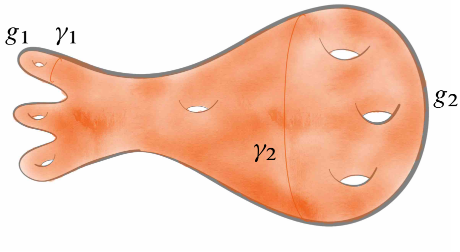

Figure 3. Example 7.3

Example 7.1. If $\gamma = (\gamma_1, \dots, \gamma_{3g-3})$ is a pants decomposition, then $g(v) = 0$, $n(v) = 3$, and $V_{g(v),n(v)} = 1$ for all $v \in V$. Thus $P_\gamma(x_1, \dots, x_{3g-3}) = x_1 \cdots x_{3g-3}$, and Theorem 1.2 reduces to Theorem 1.1.

Example 7.2. Let $\gamma = (\gamma_1, \gamma_2)$, where γ_2 is separating and separates Σ_g into a torus with a hole and a surface of type $(g-1, 1)$, and γ_1 sits on the torus with a hole is non-separating as in Figure 2. Then its associated graph polynomial $\overline{P}_\gamma$ is equal to

$$x_1 x_2 \cdot \overline{V}_{0,3}(x_1, x_1, x_2) \cdot \overline{V}_{g-1,1}(x_2) = \text{constant} \cdot x_1 x_2^{6g-9}.$$

This implies that in a random multi-geodesic of topological type (γ_1, γ_2) on a hyperbolic surface of genus $g \gg 2$, the separating component is very likely to be much longer than the non-separating component.

Example 7.3. Let (γ_1, γ_2) be an ordered multicurve such that, for $i = 1, 2$, γ_i is separating, and γ_i bounds two surfaces of type $(g_i, 1)$ (genus g_i with 1 boundary component) and $(g - g_1 - g_2, 2)$, respectively, as shown in Figure 3. Then $\overline{P}_\gamma$ is

$$\begin{aligned} &x_1 x_2 \cdot \overline{V}_{g_1,1}(x_1)\, \overline{V}_{g_2,1}(x_2)\, \overline{V}_{g-g_1-g_2,2}(x_1, x_2) \\ &\quad = \text{constant} \cdot x_1^{6g_1-3} x_2^{6g_2-3} \cdot \overline{V}_{g-g_1-g_2,2}(x_1, x_2), \end{aligned}$$

where $\overline{V}_{g-g_1-g_2,2}$ is a symmetric polynomial. So in a typical multi-geodesic of type (γ_1, γ_2), the first component is shorter than the second if $g_1 < g_2$.

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