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Equations in acylindrically hyperbolic groups and verbal closedness

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Abstract. Let H be an acylindrically hyperbolic group without nontrivial finite normal subgroups. We show that any finite system S of equations with constants from H is equivalent to a single equation. We also show that the algebraic set associated with S is, up to conjugacy, a projection of the algebraic set associated with a single splitted equation (such an equation has the form $w(x_1, \dots, x_n) = h$, where $w \in F(X)$, $h \in H$).

From this we deduce the following statement: *Let G be an arbitrary overgroup of the above group H. Then H is verbally closed in G if and only if it is algebraically closed in G.*

These statements have interesting implications; here we give only two of them: If H is a non-cyclic torsion-free hyperbolic group, then every (possibly infinite) system of equations with finitely many variables and with constants from H is equivalent to a single equation. We give a positive solution to Problem 5.2 from the paper [J. Group Theory 17 (2014), 29–40] of Myasnikov and Roman'kov: *Verbally closed subgroups of torsion-free hyperbolic groups are retracts.*

Moreover, we describe solutions of the equation $x^n y^m = a^n b^m$ in acylindrically hyperbolic groups (AH-groups), where a, b are non-commensurable jointly special loxodromic elements and n, m are integers with sufficiently large common divisor. We also prove the existence of special test words in AH-groups and give an application to endomorphisms of AH-groups.

Dedicated to my teacher Valerii Churkin on the occasion of his 75th birthday.

Contents

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1. Introduction

In 1943, Neumann [41] considered systems of equations over arbitrary groups and, motivated by field theory, introduced for groups such notions as adjoining of solutions and both algebraic and transcendent extensions. Inspired by this paper, Scott [58] introduced the notion of algebraically closed groups in the class of all groups. Since then the theory of equations over groups developed in two directions.

In the first direction, one studies which types of equations are solvable over groups from certain classes (e.g. over finite, residually finite, locally indicable or torsion-free groups). The branch which studies properties of algebraic sets in groups is called algebraic geometry over groups; see [3,39]. An extensive list of problems and results in this area can be found in the survey of Roman'kov [53] of 2012 and in the recent papers of Klyachko and Thom [29] and Nitsche and Thom [45].

In the second direction, one studies properties of *algebraically, existentially, and verbally closed groups* in certain overgroups or classes of groups (see Definition 2.2 below and a general definition of $\mathfrak{S}$-closedness suggested by Neumann in [43]). For problems and results in this area see the surveys of Leinen [33], Roman'kov [53], and the papers [2, 30, 31, 35, 40, 42, 43, 54–56, 58]. Note that this branch of group theory is closely related to logic in the form of model theory and recursive functions; see the book of Higman and Scott [22], Appendix A.4 in the book of Hodges [23], and the paper [26].

An important class of groups where both of these directions have a good chance for development is the class of *acylindrically hyperbolic groups*. These groups were implicitly studied in [5, 10, 14, 21, 63] before they were formally defined by Osin in [47]. In [47, Theorem 1.2], Osin proved that all definitions used in the above-mentioned papers are equivalent to his definition (AH_1); see Section 3.

For brevity, we also write *AH-groups* for acylindrically hyperbolic groups. We call a group *clean* if it does not contain nontrivial finite normal subgroups.

The class of AH-groups is large. It includes non-(virtually cyclic) groups that are hyperbolic relative to proper subgroups, many 3-manifold groups, groups of deficiency at least 2, many groups acting on trees, non-(virtually cyclic) groups acting properly on proper CAT(0)-spaces and containing rank-one elements, non-cyclic directly indecomposable right-angled Artin groups, all but finitely many mapping class groups, $\mathrm{Out}(F_n)$ for $n \geqslant 2$, and many other interesting groups; see the survey of Osin [50], where some interesting properties of AH-groups are listed as well.

However, almost nothing was known about solutions of equations and related problems in the class of AH-groups. In this paper, we describe solutions of certain equations of the form $x^n y^m = a^n b^m$ in AH-groups (see Proposition C and Corollary C1). Using this description, we construct certain test words for clean AH-groups (see Definition 11.1 and Corollaries D and E). We use these test words to study systems of equations over AH-groups and to establish relations between the verbal closedness, the algebraic closedness, and the retract property for AH-subgroups of groups. Below we briefly formulate some of the main results of the present paper (see Section 2 for full formulations).

– Theorem A says that *if H is a clean acylindrically hyperbolic group, then any finite system of equations with constants in H has the same set of solutions in H as a single equation*. Moreover, this set is a projection, up to conjugacy, of the set of solutions of a single *splitted* equation (see Definition 2.1).

Recall that a group H is called *equationally Noetherian* if every system of equations with constants from H and a finite number of variables is equivalent to a finite subsystem; see [3]. The equational noetherianity is important in the study of equations over groups, model theory of groups, and other questions; see [3,4,20,27,28,39,51,59–61]. Many interesting classes of groups enjoy this property (see [3,12,19,57,61]); in particular, hyperbolic groups are equationally Noetherian (see [65, Corollary 6.13] and [60, Theorem 1.22]).

– A simplified version of Corollary A1 says that if H is a clean non-elementary hyperbolic group, then every (possibly infinite) system of equations with constants in H and finitely many variables is equivalent to a single equation with coefficients in H; i.e., they have the same set of solutions in H.

– Theorem B says that *for any clean acylindrically hyperbolic group H and any overgroup G of H the notions of verbal and algebraic closedness of H in G are equivalent.*

Special cases of this theorem where H is a virtually free group or a free product of nontrivial groups were considered by Klyachko, Mazhuga, and Miroshnichenko in [31] and by Mazhuga in [38].

– Corollary B1 says that if H is a finitely generated clean acylindrically hyperbolic group and G is a finitely presented overgroup of H, then the notions of verbal and algebraic closedness of H in G are both equivalent to the assertion that H is a retract in G.

The same conclusion holds if H is an equationally Noetherian clean acylindrically hyperbolic group and G is an arbitrary overgroup which is finitely generated over H.

– Corollary B2 solves Problem 5.2 from the paper [40] of Myasnikov and Roman'kov: *Verbally closed subgroups of clean hyperbolic groups are retracts.*

In Section 2, we give exact formulations of all main results and describe the logical structure of the paper.

2. Main results

Let H be a group. An *equation* with variables $x_1, \dots, x_n$ and constants from H is an element of the free product $F_n * H$, where F_n is the free group with basis $x_1, \dots, x_n$.

Sometimes we write an equation f in the form $f(x_1,\dots,x_n;H)$ stressing that f involves the variables $x_1,\dots,x_n$ and constants from H. Sometimes, for convenience, we write an equation f_0f_1 in the form $f_0=f_1^{-1}$.

Let $S\subseteq F_n*H$ be a system of equations and let G be an overgroup of H. A tuple $(g_1,\dots,g_n)$ with components from G is called a *solution of the system S in G* if $f(g_1,\dots,g_n;H)=1$ in G for every equation $f(x_1,\dots,x_n;H)$ from S. Let $V_G(S)$ be the set of all solutions of the system S in G; i.e.,

$$V_G(S)=\big\{(g_1,\dots,g_n)\in G^n \mid f(g_1,\dots,g_n;H)=1 \text{ for all } f\in S\big\}.$$

2.1. Systems of equations versus a single equation

Definition 2.1. An equation $f\in F_n*G$ is called *splitted* if it has the form wg, where $w\in F_n$ and $g\in G$.

For $m\geqslant n$, let $\mathbf{pr}_n:G^m\to G^n$ be the projection to the first n coordinates; i.e., $\mathbf{pr}_n(g_1,\dots,g_m)=(g_1,\dots,g_n)$. For $g,u\in G$, we denote $g^u=u^{-1}gu$. For $(g_1,\dots,g_n)\in G^n$ and $u\in G$, we set $(g_1,\dots,g_n)^u=(g_1^u,\dots,g_n^u)$. The first main result of this paper is the following theorem.

Theorem A. *Let H be an acylindrically hyperbolic group without nontrivial finite normal subgroups. Let $S\subset F_n*H$ be a finite system of equations with constants from H. Then the following statements hold.*

(1) *There exists a single equation $f\in F_n*H$ such that*

$$V_H(f)=V_H(S).$$

(2) *There exists a natural number $k\geqslant n$ and a single* splitted *equation $f\in F_k*H$ of the form f_1f_0, where $f_1\in F_k$ and $f_0\in H$ such that the following two properties are satisfied:*

(a) *we have*

$$\mathbf{pr}_n\big(V_H(f)\big)=\bigcup_{\alpha\in\mathbb{Z}} V_H(S)^{f_0^\alpha},$$

(b) *for any overgroup G of the group H, we have*

$$\mathbf{pr}_n\big(V_G(f)\big)\supseteq\bigcup_{\alpha\in\mathbb{Z}} V_G(S)^{f_0^\alpha}.$$

(3) *There exist a natural number $k\geqslant n$ and two splitted equations $f,g\in F_k*H$ such that*

$$V_H(S)=\mathbf{pr}_n\big(V_H(f)\big)\cap\mathbf{pr}_n\big(V_H(g)\big).$$

In [19, Theorem D], Groves and Hull proved that any relatively hyperbolic group with respect to a finite collection of equationally Noetherian subgroups is equationally Noetherian. Using this fact we deduce the following corollary directly from Theorem A.

Corollary A1. *Suppose that H is a clean non-(virtually cyclic) relatively hyperbolic group with respect to a finite collection of proper equationally Noetherian subgroups. Then every (possibly infinite) system of equations with constants in H and finitely many variables is equivalent to a single equation with constants in H; i.e., they have the same set of solutions in H.*

In particular, this corollary is valid for all clean non-elementary hyperbolic groups.

2.2. Algebraic closedness, verbal closedness, and retracts

Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables and let $F(X)$ be the free group with basis X. We recall definitions of algebraically (verbally) closed subgroups and retracts.

Definition 2.2. Let H be a subgroup of a group G.

(a) The subgroup H is called *algebraically closed* in G if for any finite system of equations
$$S = \{W_i(x_1, \ldots, x_n; H) = 1 \mid i = 1, \ldots, m\}$$
with constants from H the following holds: if S has a solution in G, then it has a solution in H; see [40, 43].

(b) The subgroup H is called *verbally closed* in G if for any word $W \in F(X)$ and any element $h \in H$ the following holds: if the equation $W(x_1, \ldots, x_n) = h$ has a solution in G, then it has a solution in H; see [40, Definition 1.1].

(c) The subgroup H is called a *retract* of G if there is a homomorphism $\varphi : G \to H$ such that $\varphi|_H = \mathrm{id}$. The homomorphism φ is called a *retraction*.

Obviously, if H is a retract of G, then H is algebraically closed in G. Algebraic closedness implies verbal closedness, but the converse implication is not valid in general; see example in Remark 14.2.

The following proposition of Myasnikov and Roman'kov says that, under some general assumptions, the property of H to be algebraically closed in G is equivalent to the property of H to be a retract of G.

Recall that a group G is called *finitely generated over* a subgroup H if there exists a finite subset $X \subseteq G$ such that $G = \langle X, H \rangle$.

Proposition 2.3 ([40, Proposition 2.2]). *Let H be a subgroup of a group G. Suppose that at least one of the following holds:*

(a) *H is finitely generated and G is finitely presented,*

(b) *H is equationally Noetherian and G is finitely generated over H.*

Then H is algebraically closed in G if and only if H is a retract of G.

In [40], Myasnikov and Roman'kov initiated the study of verbal closedness. They proved (using nilpotent groups) that the algebraic and the verbal closedness and the property to be a retract are equivalent for subgroups of finitely generated free groups; see [40, Theorem 1.2]. Some other results on verbal closedness can be found in [30, 31, 36–38].

Our second main theorem establishes the equivalence of verbal and algebraic closedness for clean acylindrically hyperbolic subgroups of arbitrary groups.

Theorem B. *Let H be an acylindrically hyperbolic group without nontrivial finite normal subgroups and let G be an arbitrary overgroup of H. Then H is verbally closed in G if and only if H is algebraically closed in G.*

The assumption that H does not have nontrivial finite normal subgroups cannot be omitted (see example in Remark 14.2). Some special cases of this theorem were considered earlier in [31, 38]; see Remark 14.1.

The following corollary follows directly from Theorem B and Proposition 2.3.

Corollary B1 (see [40, Proposition 2.2] for the equivalence (1)⇔(3)). *Let H be a subgroup of a group G such that at least one of the following holds:*

(a) *H is finitely generated and G is finitely presented,*

(b) *H is equationally Noetherian and G is finitely generated over H.*

Suppose additionally that H is acylindrically hyperbolic and does not have nontrivial finite normal subgroups. Then the following three statements are equivalent:

(1) *H is algebraically closed in G,*

(2) *H is verbally closed in G,*

(3) *H is a retract of G.*

2.3. Solution of a problem of Myasnikov and Roman'kov on verbal closedness

In [40], Myasnikov and Roman'kov write that not much is known in general about verbally closed subgroups of a given group G and raise the following two problems.

Problem 5.1 in [40]. What are the verbally closed subgroups of a free nilpotent group of finite rank?

Problem 5.2 in [40]. Prove that verbally closed subgroups of a torsion-free hyperbolic group are retracts.

Problem 5.1 was solved by Roman'kov and Khisamiev in [54]. They proved the following. Let $\mathcal{N}_c$ be the variety of all nilpotent groups of class at most c and $N_{r,c}$ a free nilpotent group of finite rank r and nilpotency class c. A subgroup H of $N_{r,c}$ is verbally closed in $N_{r,c}$ if and only if H is a free factor of $N_{r,c}$ in the variety $\mathcal{N}_c$ (equivalently, an algebraically closed subgroup, or a retract of $N_{r,c}$).

Problem 5.2 (in a slightly general setting) is solved in this paper as follows.

Corollary B2 (Corollary 15.8; solution to Problem 5.2 in [40]). *Let G be a hyperbolic group and H a subgroup of G. Suppose that H does not have nontrivial finite normal subgroups. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Below we formulate more general corollaries about relatively hyperbolic (sub)groups. For a relevant terminology see the manuscript of Osin [49].

Corollary B3 (Corollary 15.6). *Let G be a group and let H be a subgroup of G such that G is finitely generated over H. Suppose that H is hyperbolic relative to a finite collection of equationally Noetherian proper subgroups and does not have nontrivial finite normal subgroups. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Corollary B4 (Corollary 15.7). *Let G be a relatively hyperbolic group with respect to a finite collection of finitely generated equationally Noetherian subgroups. Suppose that H is a non-parabolic subgroup of G such that H does not have nontrivial finite normal subgroups. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

2.4. Solutions of certain equations in acylindrically hyperbolic groups

In the course of the proof of Theorem A, we obtain a description of solutions of the equation $x^n y^m = a^n b^m$ in acylindrically hyperbolic groups for non-commensurable jointly special loxodromic elements a, b and numbers n, m with sufficiently large common divisor.

Suppose that G is an acylindrically hyperbolic group *with respect to a generating set X*; see Definition 3.2. Then any loxodromic, with respect to X, element $g \in G$ is contained in a unique maximal virtually cyclic subgroup $E_G(g)$ of G (see [14, Lemma 6.5]). This subgroup is called the *elementary subgroup associated with g*.

We call an element $g \in G$ *special with respect to X* if it is loxodromic with respect to X and $E_G(g) = \langle g \rangle$. Elements $g_1, \dots, g_k \in G$ are called *jointly special* if there exists a generating set X of G such that each g_i is special with respect to X (see precise definitions in Section 3).

Two elements $a, b \in G$ of infinite order are called *commensurable* if there exist $g \in G$ and $s, t \in \mathbb{Z} \setminus \{0\}$ such that $a^s = g^{-1} b^t g$.

Proposition C (Proposition 7.1). *Let G be an acylindrically hyperbolic group. Suppose that a and b are two non-commensurable jointly special elements of G. Then there exists a generating set Y of G containing $\mathcal{E} = \langle a \rangle \cup \langle b \rangle$ and there exists a number $N \in \mathbb{N}$ such that for all $n, m > N$ the following holds.*

If (c, d) is a solution of the equation $x^n y^m = a^n b^m$, then one of the following holds:

(1) *c and d are loxodromic with respect to Y and $E_G(d) = E_G(c)$;*

(2) *c is loxodromic with respect to Y, d is elliptic, and $d^m \in E_G(c)$;*

(3) *d is loxodromic with respect to Y, c is elliptic, and $c^n \in E_G(d)$;*

(4) *c and d are elliptic with respect to Y and one of the following holds:*

   (a) *c is conjugate to a and d is conjugate to b;*

   (b) *c is conjugate to b, d is conjugate to a, and $|n - m| \leqslant N$.*

The following corollary gives a simple description of solutions of this equation for certain n, m.

Corollary C1 (Corollary 9.5). *Let G be an acylindrically hyperbolic group. Suppose that $a, b \in G$ are two non-commensurable jointly special elements. Then there exists a number $\ell = \ell(a,b) \in \mathbb{N}$ such that for all $n, m \in \ell\mathbb{N}$, $n \neq m$, the equation $x^n y^m = a^n b^m$ is perfect; i.e., any solution of this equation in G is conjugate to (a, b) by a power of $a^n b^m$.*

The condition on $\gcd(n, m)$ in this corollary cannot be replaced by the condition that n, m are sufficiently large; see example in Remark 7.3.

2.5. Test words in acylindrically hyperbolic groups

Let G be a group. An element $g \in G$ is called a *test element* if any endomorphism $\varphi : G \to G$ for which $\varphi(g) = g$ is an automorphism (see [64, Definition 1], [46, Definition 1]).

Note that this concept was studied by Shpilrain in [62], before being made explicit in [46, 64]. It is well known due to Dehn and Nielsen that $[x_1, x_2]$ is a test word in F_2 (see [34, 44]). Other examples of test words in F_n were given by Zieschang [66, 67], Rips [52], Dold [15], and Shpilrain [62].

Turner [64] related test words in free groups with retracts; he proved that $w \in F_n$ is a test word if and only if w is not contained in a proper retract of F_n. Groves [18] extended this result to torsion-free hyperbolic groups.

In [25], Ivanov constructed the so-called C-test words in free groups and applied them to show that there exist two words $w_1, w_2 \in F_n$ such that any monomorphism $\varphi : F_n \to F_n$ is uniquely determined by $\varphi(w_1)$ and $\varphi(w_2)$. In [32], Lee constructed C-test words with some additional property.

In [40], Myasnikov and Roman'kov used Lee's test words to prove that verbally closed subgroups of F_n are retracts. We introduce the following variant of a test word, which helps us to prove Theorem A.

Definition 2.4. Let H be a group and let $a_1, \dots, a_k$ be some elements of H. A word $W(x_1, \dots, x_k)$ is called an $(a_1, \dots, a_k)$-*test word* if for every solution $(b_1, \dots, b_k)$ of the equation

$$W(a_1, \dots, a_k) = W(x_1, \dots, x_k)$$

in H, there exists a number $\alpha \in \mathbb{Z}$ such that $b_i = a_i^{U^\alpha}$ for $i = 1, \dots, k$, where $U = W(a_1, \dots, a_k)$.

In Section 12, we construct certain $(a_1, \dots, a_k)$-test words in clean acylindrically hyperbolic groups. In particular, we prove the following corollary.

Corollary D (Corollary 12.2). *Let H be an acylindrically hyperbolic group without non-trivial finite normal subgroups and let $a_1, \dots, a_k \in H$ (where $k \geqslant 3$) be jointly special and pairwise non-commensurable elements. Then there is an $(a_1, \dots, a_k)$-test word $\mathcal{U}_k(x_1, \dots, x_k)$ such that the elements $a_1, \dots, a_k$ together with $\mathcal{U}_k(a_1, \dots, a_k)$ are jointly special and pairwise non-commensurable.*

This corollary and a more general Proposition 12.1 are used to prove statements (1) and (2)–(3) of Theorem A, respectively.

The following corollary says that any clean finitely generated acylindrically hyperbolic group contains test elements satisfying a stronger condition than in [46, Definition 1]. This corollary follows directly from Proposition 10.7 and Corollary D.

Corollary E. *Let H be a finitely generated acylindrically hyperbolic group without non-trivial finite normal subgroups. Then there exists an element $w \in H$ such that for any endomorphism $\varphi : H \to H$ the equality $\varphi(w) = w$ implies that φ is a conjugation by a power of w.*

2.6. Uniform divergence of quasi-geodesics determined by loxodromic elements in acylindrically hyperbolic groups

Proposition C is proved with the help of the following two propositions which seem to be interesting for their own sake. The first one says that the quasi-geodesics determined by two loxodromic elements in acylindrically hyperbolic groups diverge uniformly.

Proposition F (Proposition 5.4). *Let G be a group and let X be a generating set of G. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and acylindrical. Then there exists a constant $N_0 > 0$ such that for any loxodromic (with respect to X) elements $c, d \in G$ with $E_G(c) \neq E_G(d)$ and for any $n, m \in \mathbb{N}$ we have that*

$$|c^n d^m|_X > \frac{\min\{n, m\}}{N_0}.$$

Proposition G (Proposition 5.6). *Let G be a group and let X be a generating set of G. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and acylindrical. Then there exists a constant $N_1 > 0$ such that for any loxodromic (with respect to X) element $c \in G$, any elliptic element $e \in G \setminus E_G(c)$, and any $n \in \mathbb{N}$, we have that*

$$|c^n e|_X > \frac{n}{N_1}.$$

We prove these propositions with the help of the *periodicity theorem* for acylindrically hyperbolic groups; see [6, Theorem 1.4]. This theorem and relevant notions are reproduced in Section 5.1 of the present paper. A special case of this theorem, where G is a free group and $r = 0$, can be found in the book of Adian [1] devoted to a solution of the Burnside problem (see statement 2.3 in Chapter I there).

Note that these propositions have another interesting application: in [8], we use them to describe homomorphisms from H to G, where H is a topological group which is either completely metrizable or locally compact Hausdorff, and G is an acylindrically hyperbolic group.

3. Acylindrically hyperbolic groups

We introduce general notation and recall some relevant definitions and statements from the papers [6, 14, 17, 47].

3.1. General notation

All generating sets considered in this paper are assumed to be symmetric, i.e., closed under taking inverse elements. Let G be a group generated by a subset X. For $g \in G$, let $|g|_X$ be the length of a shortest word in X representing g. The corresponding metric on G is denoted by d_X (or by d if X is clear from the context); thus, $d_X(a,b) = |a^{-1}b|_X$. The right Cayley graph of G with respect to X is denoted by $\Gamma(G, X)$. By a path p in the Cayley graph we mean a combinatorial path; the initial and the terminal vertices of p are denoted by p_- and p_+, respectively. The path inverse to p is denoted by $\bar{p}$. The length of p is denoted by $\ell(p)$. The label of p is denoted by $\mathbf{Lab}(p)$; we stress that the label is a formal word in the alphabet X. The canonical image of $\mathbf{Lab}(p)$ in G is denoted by $\mathbf{Lab}_G(p)$.

Given a real number $K \geqslant 0$, two paths p and q in $\Gamma(G, X)$ are called *K-similar* if $d(p_-, q_-) \leqslant K$ and $d(p_+, q_+) \leqslant K$.

Recall that a path p in $\Gamma(G, X)$ is called *$(\varkappa, \varepsilon)$-quasi-geodesic*, where $\varkappa \geqslant 1$, $\varepsilon \geqslant 0$, if $d(q_-, q_+) \geqslant \frac{1}{\varkappa}\ell(q) - \varepsilon$ for any subpath q of p.

The following remark is important. Suppose that $\{X_\lambda\}_{\lambda\in\Lambda}$ is a collection of subsets of a group G such that $\bigcup_{\lambda\in\Lambda} X_\lambda$ generates G. The alphabet $\mathcal{X} = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ determines the Cayley graph $\Gamma(G, \mathcal{X})$, where two vertices may be connected by many edges. This happens if some element $x \in G$ belongs to subsets X_λ and X_μ of G for different $\lambda, \mu \in \Lambda$. In this case, $\Gamma(G, \mathcal{X})$ contains two edges from g to gx for any vertex g. The labels of these edges are different since they belong to disjoint subsets of the alphabet $\mathcal{X}$; however these labels represent the same element x in G.

The following notation will shorten the forthcoming proofs. For $a, b, c \in \mathbb{R}$, we write $a \approx_c b$ if $|a - b| \leqslant c$. Note that $a \approx_c b$ and $b \approx_{c_1} d$ imply $a \approx_{c+c_1} d$.

For a group G and an element $a \in G$, we define a homomorphism $\hat{a} : G \to G$ by the rule $\hat{a}(g) = a^{-1}ga$. We also write g^a for $a^{-1}ga$.

3.2. Hyperbolic spaces

Let A, B, C be three points in a metric space $\mathfrak{X}$. Recall that the *Gromov product* of A, B with respect to C is the number

$$(A, B)_C := \frac{d(C, A) + d(C, B) - d(A, B)}{2}.$$

We use the following definition of a δ-hyperbolic space (see [11, Chapter III.H, Definition 1.16, and Proposition 1.17]).

For $\delta \geqslant 0$, we say that a geodesic triangle ABC in $\mathfrak{X}$ is *δ-thin* at the vertex C if for any two points A_1 and B_1 on the sides $[C, A]$ and $[C, B]$ with $d(C, A_1) = d(C, B_1) \leqslant (A, B)_C$, we have that $d(A_1, B_1) \leqslant \delta$.

We say that a metric space $\mathfrak{X}$ is *δ-hyperbolic* if it is geodesic and every geodesic triangle in $\mathfrak{X}$ is δ-thin at each of its vertices.

3.3. Two equivalent definitions of acylindrically hyperbolic groups

Definition 3.1 (see [10] and Introduction in [47]). An action of a group G on a metric space S is called *acylindrical* if for every $\varepsilon > 0$ there exist $R, N > 0$ such that for every two points x, y with $d(x, y) \geqslant R$, there are at most N elements $g \in G$ satisfying

$$d(x, gx) \leqslant \varepsilon \quad \text{and} \quad d(y, gy) \leqslant \varepsilon.$$

Given a generating set X of a group G, we say that the Cayley graph $\Gamma(G, X)$ is *acylindrical* if the left action of G on $\Gamma(G, X)$ is acylindrical. For Cayley graphs, the acylindricity condition can be rewritten as follows: for every $\varepsilon > 0$ there exist $R, N > 0$ such that for any $g \in G$ of length $|g|_X \geqslant R$ we have that

$$\big|\{f \in G \mid |f|_X \leqslant \varepsilon,\ |g^{-1}fg|_X \leqslant \varepsilon\}\big| \leqslant N.$$

Recall that an action of a group G on a hyperbolic space S is called *elementary* if the limit set of G on the Gromov boundary ∂S contains at most 2 points.

Definition 3.2 (see [47, Definition 1.3]). A group G is called *acylindrically hyperbolic* if it satisfies one of the following equivalent conditions:

- (AH_1) there exists a generating set X of G such that the corresponding Cayley graph $\Gamma(G, X)$ is hyperbolic, $|\partial\Gamma(G, X)| > 2$, and the natural action of G on $\Gamma(G, X)$ is acylindrical;
- (AH_2) G admits a non-elementary acylindrical action on a hyperbolic space.

In the case (AH_1), we also write that G is *acylindrically hyperbolic with respect to X*.

Recall the following useful lemma.

Lemma 3.3 ([47, Lemma 5.1]). *For any group G and any generating sets X and Y of G such that*

$$\sup_{x\in X} |x|_Y < \infty \quad \textit{and} \quad \sup_{y\in Y} |y|_X < \infty,$$

the following hold:

- (a) *$\Gamma(G, X)$ is hyperbolic if and only if $\Gamma(G, Y)$ is hyperbolic,*
- (b) *$\Gamma(G, X)$ is acylindrical if and only if $\Gamma(G, Y)$ is acylindrical.*

3.4. Elliptic and loxodromic elements in acylindrically hyperbolic groups

Let G be a group acting on a metric space S. Recall that the *stable norm* of an element $g \in G$ for this action is defined as

$$\|g\| = \lim_{n\to\infty} \frac{1}{n}\, d(x, g^n x),$$

where x is an arbitrary point in S; see [13]. It is easy to check that this number is well defined, independent of x, that it is a conjugacy invariant, and that $\|g^k\| = |k| \cdot \|g\|$ for all $k \in \mathbb{Z}$. The following definition is standard.

Definition 3.4. Given a group G acting on a metric space S, an element $g \in G$ is called *elliptic* if some (equivalently, any) orbit of g is bounded, and *loxodromic* if the map $\mathbb{Z} \to S$ defined by $n \mapsto g^n x$ is a quasi-isometric embedding for some (equivalently, any) $x \in S$. That is, for $x \in S$, there exist $\varkappa \geqslant 1$ and $\varepsilon \geqslant 0$ such that for any $n, m \in \mathbb{Z}$ we have

$$d(g^n x, g^m x) \geqslant \frac{1}{\varkappa}|n - m| - \varepsilon.$$

Let X be a generating set of G. We say that $g \in G$ is *elliptic (respectively loxodromic) with respect to* X if g is elliptic (respectively loxodromic) for the canonical left action of G on the Cayley graph $\Gamma(G, X)$. If X is clear from a context, we omit the words "with respect to X."

The set of all elliptic (respectively loxodromic) elements of G with respect to X is denoted by $\mathrm{Ell}(G, X)$ (respectively by $\mathrm{Lox}(G, X)$).

Note that for groups acting on geodesic hyperbolic spaces, there is only one additional isometry type of an element: parabolic (see e.g. [13, Chapitre 9, Théorème 2.1]).

Bowditch [10, Lemma 2.2] proved that every element of a group acting acylindrically on a hyperbolic space is either elliptic or loxodromic (see a more general statement in [47, Theorem 1.1]). Moreover, he proved there that the infimum of the set of stable norms of all loxodromic elements for such an action is larger than zero (we assume that $\inf \emptyset = +\infty$).

From this fundamental result, we deduced in [6, Corollary 2.12] that, under certain assumptions, the quasi-geodesics associated with loxodromic elements have universal quasi-geodesic constants (see Definition 3.5 and Corollary 3.6 below).

Definition 3.5. Let G be a group and X a generating set of G. For any two elements $u, v \in G$, we choose a geodesic path $[u, v]$ in $\Gamma(G, X)$ from u to v so that $w[u, v] = [wu, wv]$ for any $w \in G$. With any element $x \in G$ and any loxodromic element $g \in G$, we associate the bi-infinite quasi-geodesic

$$L(x, g) = \bigcup_{i=-\infty}^{\infty} x[g^i, g^{i+1}].$$

We have that $L(x, g) = x\, L(1, g)$. The path $L(1, g)$ is called the *quasi-geodesic associated with* g.

Corollary 3.6 ([6, Corollary 2.12]). *Let G be a group and X a generating set of G. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and acylindrical. Then there exist $\varkappa \geqslant 1$ and $\varepsilon \geqslant 0$ such that the following holds:*

if an element $g \in G$ is loxodromic and shortest in its conjugacy class, then the quasi-geodesic $L(1, g)$ associated with g is a $(\varkappa, \varepsilon)$-quasi-geodesic.

Recall that any loxodromic element g in an acylindrically hyperbolic group G is contained in a unique maximal virtually cyclic subgroup [14, Lemma 6.5]. This subgroup, denoted by $E_G(g)$, is called the *elementary subgroup associated with* g; it can be described as follows (see equivalent definitions in [14, Corollary 6.6]):

$$\begin{aligned} E_G(g) &= \{f \in G \mid \exists n \in \mathbb{N} : f^{-1}g^n f = g^{\pm n}\} \\ &= \{f \in G \mid \exists k, m \in \mathbb{Z} \setminus \{0\} : f^{-1}g^k f = g^m\}. \end{aligned} \tag{3.1}$$

Lemma 3.7 (see [47, Lemma 6.8]). *Suppose that a group G acts acylindrically on a hyperbolic space S. Then there exists $L \in \mathbb{N}$ such that for every loxodromic element $g \in G$, $E_G(g)$ contains a normal infinite cyclic subgroup of index L.*

Definition 3.8. Suppose that G is an acylindrically hyperbolic group.

(a) An element $g \in G$ is called *special* if there exists a generating set X of G such that

- G is acylindrically hyperbolic with respect to X,
- g is loxodromic with respect to X, and
- $E_G(g) = \langle g \rangle$.

In this case, g is called *special with respect to X*.

(b) Elements $g_1, \dots, g_k \in G$ are called *jointly special* if there exists a generating set X of G such that each g_i is special with respect to X.

Note that point (a) of this definition was already used in the case of relatively hyperbolic groups (see comments in [48, Section 3]).

The following theorem helps to verify whether an acylindrical action of a group on a hyperbolic space is elementary or not.

Theorem 3.9 (see [47, Theorem 1.1]). *Let G be a group acting acylindrically on a hyperbolic space. Then G satisfies exactly one of the following conditions:*

(a) *G has bounded orbits;*

(b) *G is virtually cyclic and contains a loxodromic element;*

(c) *G contains infinitely many loxodromic elements whose limit sets are pairwise disjoint. In this case the action of G is non-elementary and G is acylindrically hyperbolic.*

3.5. Hyperbolically embedded subgroups

Let G be a group and $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G. A subset X of G is called a *relative generating set of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$* if G is generated by X together with the union of all H_λ. All relative generating sets are assumed to be symmetric. We define

$$\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda.$$

For the following two definitions, we assume that X is a relative generating set of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$.

Definition 3.10 (see [14, Definition 4.1]). The group G is called *weakly hyperbolic* relative to X and $\{H_\lambda\}_{\lambda\in\Lambda}$ if the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ is hyperbolic.

We consider the Cayley graph $\Gamma(H_\lambda, H_\lambda)$ as a complete subgraph of $\Gamma(G, X \sqcup \mathcal{H})$.

Definition 3.11 (see [14, Definition 4.2]). For every $\lambda \in \Lambda$, we introduce a *relative metric* $\hat{d}_\lambda : H_\lambda \times H_\lambda \to [0, +\infty]$ as follows:

let $a, b \in H_\lambda$. A path in $\Gamma(G, X \sqcup \mathcal{H})$ from a to b is called H_λ*-admissible* if it has no edges in the subgraph $\Gamma(H_\lambda, H_\lambda)$.

The distance $\hat{d}_\lambda(a, b)$ is defined to be the length of a shortest H_λ*-admissible* path connecting a to b if such exists. If no such path exists, we set $\hat{d}_\lambda(a, b) = \infty$.

Definition 3.12 (see [14, Definition 4.25]). Let G be a group and X a symmetric subset of G. A collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of G is called *hyperbolically embedded in G with respect to X* (we write $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$) if the following hold.

(a) The group G is generated by X together with the union of all H_λ and the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ is hyperbolic.

(b) For every $\lambda \in \Lambda$, the metric space $(H_\lambda, \hat{d}_\lambda)$ is proper. That is, any ball of finite radius in H_λ contains finitely many elements.

Further, we say that $\{H_\lambda\}_{\lambda\in\Lambda}$ is *hyperbolically embedded* in G and write $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$ if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$ for some $X \subseteq G$.

It was proved in [47, Theorem 1.2] that a group G is acylindrically hyperbolic if and only if it contains a proper infinite hyperbolically embedded subgroup.

Lemma 3.13 (see [14, Corollary 4.27]). *Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, and X, Y relative generating sets of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$. Suppose that $|X\Delta Y| < \infty$. Then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$ if and only if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, Y)$.*

There are examples which show that the condition $|X\Delta Y| < \infty$ cannot be replaced by the condition using supremum as in Lemma 3.3.

Lemma 3.14 (see [14, Proposition 4.33]). *Suppose that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$. Then, for each $\lambda \in \Lambda$, we have that $\{g \in G \mid |H_\lambda \cap g^{-1}H_\lambda g| = \infty\} \subseteq H_\lambda$.*

We use the following nontrivial theorem.

Theorem 3.15 (see [47, Theorem 5.4]). *Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a finite collection of subgroups of G and X a subset of G. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$. Then there exists $Y \subseteq G$ such that $X \subseteq Y$ and the following conditions hold.*

(a) *$\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, Y)$. In particular, the Cayley graph $\Gamma(G, Y \sqcup \mathcal{H})$ is hyperbolic.*

(b) *The action of G on $\Gamma(G, Y \sqcup \mathcal{H})$ is acylindrical.*

4. Preliminary statements

The main aim of this section is to prove Lemmas 4.7 and 4.8. These lemmas will be used in Section 5. The following lemma is an easy exercise and we leave it for the reader.

Lemma 4.1. *Let ABC be a geodesic triangle in a δ-hyperbolic space and let A_1 and B_1 be the middle points of the sides $[A, C]$ and $[B, C]$, respectively. Then $d(A_1, B_1) \leqslant d(A, B)/2 + 2\delta$.*

Lemma 4.2 (see [11, Chapter III.H, Theorem 1.7]). *For all $\delta \geqslant 0$, $\varkappa \geqslant 1$, $\epsilon \geqslant 0$, there exists a constant $\mu = \mu(\delta, \varkappa, \epsilon) > 0$ with the following property:*

if $\mathfrak{X}$ is a δ-hyperbolic space, p is a $(\varkappa, \epsilon)$-quasi-geodesic in $\mathfrak{X}$, and $[x, y]$ is a geodesic segment joining the endpoints of p, then the Hausdorff distance between $[x, y]$ and the image of p is at most μ.

The following lemma is an easy generalization of this statement.

Lemma 4.3 (see [6, Corollary 2.3]). *For any $\delta \geqslant 0$, $\varkappa \geqslant 1$, $\epsilon \geqslant 0$, $r \geqslant 0$, the following holds:*

if $\mathfrak{X}$ is a δ-hyperbolic space, p and q are $(\varkappa, \epsilon)$-quasi-geodesics in $\mathfrak{X}$ such that $\max\{d(p_-, q_-), d(p_+, q_+)\} \leqslant r$, then the Hausdorff distance between the images of p and q is at most $\eta(\delta, \varkappa, \epsilon, r) = r + 2\delta + 2\mu$, where $\mu = \mu(\delta, \varkappa, \epsilon)$ is the constant from Lemma 4.2.

The following lemma can be deduced straightforward from the definition of Gromov product.

Lemma 4.4. *Let ABC be a geodesic triangle in a metric space and let P and Q be points on its sides $[A, B]$ and $[B, C]$. Then $d(P, Q) \geqslant d(P, B) + d(B, Q) - 2(A, C)_B$.*

Lemma 4.5. *Let p, q be two $(\varkappa, \varepsilon)$-quasi-geodesics in a δ-hyperbolic space such that $p_+ = q_-$. Then their concatenation pq is a $(\varkappa, 2\alpha + \beta)$-quasi-geodesic, where α is the Gromov product of p_-, q_+ with respect to p_+ and $\beta \geqslant \varepsilon$ is a constant depending only on δ, $\varkappa$, ε.*

Proof. Let r be a subpath of pq. We shall estimate $d(r_-, r_+)$ from below by using $\ell(r)$. We consider only the case $r = p_1 q_1$, where p_1 is a terminal subpath of p and q_1 is an initial subpath of q. Denote $A = p_-$, $B = p_+$, $C = q_+$, $P_1 = r_-$, and $Q_1 = r_+$. By Lemma 4.2, there exist points $P \in [A, B]$ and $Q \in [B, C]$ such that $d(P_1, P) \leqslant \mu$, $d(Q_1, Q) \leqslant \mu$, where $\mu = \mu(\delta, \varkappa, \varepsilon)$. By Lemma 4.4, we have that $d(P, Q) \geqslant d(P, B) + d(B, Q) - 2\alpha$. Then

$$\begin{aligned} d(P_1, Q_1) &\geqslant d(P_1, B) + d(B, Q_1) - 2\alpha - 4\mu \\ &\geqslant \left(\frac{1}{\varkappa}\ell(p_1) - \varepsilon\right) + \left(\frac{1}{\varkappa}\ell(q_1) - \varepsilon\right) - 2\alpha - 4\mu = \frac{1}{\varkappa}\ell(r) - (2\alpha + 2\varepsilon + 4\mu). \end{aligned}$$

Therefore, the statement holds for $\beta = 2\varepsilon + 4\mu$. ∎

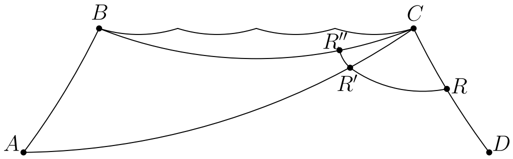

Figure 1. Illustration to Lemma 4.6.

The following lemma says that the concatenation $q_1q_2q_3$ of three $(\varkappa, \varepsilon)$-quasi-geodesic paths in a δ-hyperbolic space, where q_2 is sufficiently long, is a $(\varkappa, \varepsilon_0)$-quasi-geodesic for ε_0 depending only on δ, $\varkappa$, ε, and some Gromov products.

Lemma 4.6. *For any $\delta \geqslant 0$, $\varkappa \geqslant 1$, $\varepsilon \geqslant 0$, $\alpha \geqslant 0$, there exists $\varepsilon_0 \geqslant 0$ such that the following holds. Let $\mathfrak{X}$ be a δ-hyperbolic space, $q = q_0q_1q_2$ a path in $\mathfrak{X}$ such that q_0, q_1, q_2 are $(\varkappa, \varepsilon)$-quasi-geodesic paths satisfying the following two conditions:*

(1) $((q_0)_-, (q_1)_+)_{(q_0)_+} < \alpha$ *and* $((q_1)_-, (q_2)_+)_{(q_1)_+} < \alpha$;

(2) $d((q_1)_-, (q_1)_+) \geqslant 2(\alpha + \delta)$.

Then q is a $(\varkappa, \varepsilon_0)$-quasi-geodesic path.

Proof. By Lemma 4.5, the path (q_0q_1) is a $(\varkappa, \varepsilon_1)$-quasi-geodesic for $\varepsilon_1 = 2\alpha + \beta$, where $\beta = \beta(\delta, \varkappa, \varepsilon) \geqslant \varepsilon$. Since $\varepsilon_1 \geqslant \varepsilon$, the path q_2 is also a $(\varkappa, \varepsilon_1)$-quasi-geodesic. Now we apply Lemma 4.5 to (q_0q_1) and q_2. To complete the proof, it suffices to show that

$$\big((q_0q_1)_-, (q_2)_+\big)_{(q_0q_1)_+} < \alpha_1,$$

where $\alpha_1 := \alpha + \delta$. Denote $A = (q_0)_-$, $B = (q_1)_-$, $C = (q_1)_+$, and $D = (q_2)_+$.

Suppose the converse; i.e., $(A, D)_C \geqslant \alpha_1$. Then there exist two points $R \in [C, D]$ and $R' \in [C, A]$ such that $d(C, R) = d(C, R') = \alpha_1$ and $d(R, R') \leqslant \delta$; see Figure 1. From (1) and (2) we deduce that

$$(A, B)_C = d(B, C) - (A, C)_B > 2(\alpha + \delta) - \alpha = \alpha + 2\delta \geqslant \alpha_1.$$

Then there exists $R'' \in [C, B]$ such that $d(C, R'') = \alpha_1$ and $d(R'', R') \leqslant \delta$. Thus, $d(R, R'') \leqslant 2\delta$. On the other hand, using Lemma 4.4, we deduce that

$$d(R, R'') \geqslant d(C, R) + d(C, R'') - 2(B, D)_C > 2\alpha_1 - 2\alpha = 2\delta.$$

A contradiction. ∎

Lemma 4.7. *Let G be a group and X a generating set of G. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and acylindrical. Then there exist real numbers $\varkappa \geqslant 1$, $\varepsilon_0 \geqslant 0$, and a number $n_0 \in \mathbb{N}$ with the following property.*

Suppose that $n \geqslant n_0$ and $c \in G$ is a loxodromic element. Let $S(c)$ be the set of shortest elements in the conjugacy class of c and let $g \in G$ be a shortest element for which there exists $c_1 \in S(c)$ with $c = g^{-1}c_1g$. Then any path $p_0p_1 \cdots p_np_{n+1}$ in $\Gamma(G, X)$, where

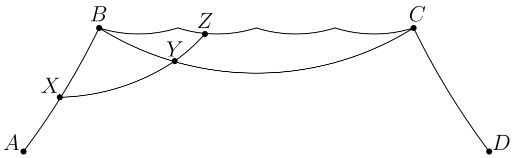

Figure 2. Illustration to Lemma 4.7.

$p_0, p_1, \dots, p_n, p_{n+1}$ are geodesics with labels representing $g^{-1}, c_1, \dots, c_1, g$, is a $(\varkappa, \varepsilon_0)$-quasi-geodesic. In particular,

$$|c^n|_X \geqslant \frac{1}{\varkappa}\big(n|c_1|_X + 2|g|_X\big) - \varepsilon_0 \geqslant \frac{1}{\varkappa} n - \varepsilon_0.$$

Proof. Let $\delta \geqslant 0$ be a constant such that $\Gamma(G, X)$ is δ-hyperbolic. Let n be an arbitrary positive integer. Suppose that $c, g, c_1 \in G$ are elements and $p_0, \dots, p_{n+1}$ are geodesics in $\Gamma(G, X)$ as above. We set $q_0 = p_0$, $q_1 = p_1 p_2 \cdots p_n$, and $q_2 = p_{n+1}$. Then q_1 is a $(\varkappa, \varepsilon)$-quasi-geodesic, where $\varkappa$ and ε are the constants from Corollary 3.6. According to Lemma 4.2, the Hausdorff distance between q_1 and $[(q_1)_-, (q_1)_+]$ is at most $\mu = \mu(\delta, \varkappa, \varepsilon)$. We set

$$n_0 := \lceil \varkappa(4\delta + 2\mu + \varepsilon + 2) \rceil.$$

Now we suppose that $n \geqslant n_0$. It suffices to show that the paths q_0, q_1, q_2 satisfy conditions (1) and (2) of Lemma 4.6 for $\alpha = \delta + \mu + 1$. Denote $A = (q_0)_-$, $B = (q_1)_-$, $C = (q_1)_+$, and $D = (q_2)_+$; see Figure 2. For (1), we shall check that

$$(A, C)_B < \delta + \mu + 1 \quad \text{and} \quad (B, D)_C < \delta + \mu + 1.$$

Because of symmetry, we check only the first inequality. To the contrary, suppose that $(A, C)_B \geqslant \delta + \mu + 1$. Then there exist vertices $X \in [B, A]$ and $Y \in [B, C]$ such that $d(B, X) = d(B, Y) = \lfloor \delta + \mu + 1 \rfloor$ and $d(X, Y) \leqslant \delta$. Since the Hausdorff distance between q_1 and $[B, C]$ is at most μ, there exists a vertex $Z \in q_1$ such that $d(Y, Z) \leqslant \mu$. Then

$$\begin{aligned} d(A, Z) &\leqslant d(A, X) + d(X, Y) + d(Y, Z) \\ &\leqslant d(A, X) + \delta + \mu < d(A, X) + d(B, X) = d(A, B). \end{aligned} \tag{4.1}$$

We have that $Z \in p_i = [Bc_1^i, Bc_1^{i+1}]$ for some $0 \leqslant i \leqslant n-1$. Let a and b be the elements of G representing the labels of $[Bc_1^i, Z]$ and $[Z, Bc_1^{i+1}]$, respectively. Then $c_1 = ab$. We set $c_2 = ba$ and $f = a^{-1}c_1^{-i}g$. Then c_2 is also shortest in the conjugacy class of c and we have that $c = g^{-1}c_1 g = f^{-1}c_2 f$. Since $Z = Bc_1^i a$ and $B = Ag^{-1}$, we have that $f = Z^{-1}A$. Then

$$|f|_X = d(Z, A) \overset{(4.1)}{<} d(B, A) = |g|_X$$

that contradicts the choice of g. Therefore, condition (1) is satisfied. Condition (2) follows from the above-mentioned fact that q_1 is a $(\varkappa, \varepsilon)$-quasi-geodesic:

$$d\big((q_1)_-, (q_1)_+\big) \geqslant \frac{1}{\varkappa}\ell(p_1 p_2 \cdots p_n) - \varepsilon \geqslant \frac{n}{\varkappa} - \varepsilon \geqslant \frac{n_0}{\varkappa} - \varepsilon \geqslant 2\alpha + 2\delta.$$

Thus, by Lemma 4.6, $q_0 q_1 q_2 = p_0 p_1 \cdots p_n p_{n+1}$ is a $(\varkappa, \varepsilon_0)$-quasi-geodesic for some universal constants $\varkappa$, ε_0. In particular,

$$|c^n|_X \geqslant \frac{1}{\varkappa}\ell(p_0 p_1 \cdots p_n p_{n+1}) - \varepsilon_0 = \frac{1}{\varkappa}\big(n|c_1|_X + 2|g|_X\big) - \varepsilon_0 \geqslant \frac{1}{\varkappa}n - \varepsilon_0. \qquad \blacksquare$$

Lemma 4.8. *For every $\delta \geqslant 0$, there exists $\varepsilon_1 = \varepsilon_1(\delta) \geqslant 0$ such that the following holds. Suppose that the Cayley graph of a group G with respect to a generating set X is δ-hyperbolic for some integer $\delta \geqslant 0$. Let $a, b \in G$ be conjugate elements satisfying $|a|_X \geqslant |b|_X + 4\delta + 2$. Then there exist $x, y \in G$ with the following properties:*

(a) *$a = x^{-1}yx$;*

(b) *$|y|_X \in \{|b|_X + 4\delta + 1, |b|_X + 4\delta + 2\}$;*

(c) *any path $q_0 q_1 q_2$ in $\Gamma(G, X)$, where q_0, q_1, q_2 are geodesics with labels representing x^{-1}, y, x, is a $(1, \varepsilon_1)$-quasi-geodesic.*

Proof. In the set $S := \{(y, x) \in G \times G \mid a = x^{-1}yx,\ |y|_X \leqslant |b|_X + 4\delta + 2\}$, we choose a pair $(y, x) \in S$ with minimal $|x|_X$. Clearly, (a) is valid. We claim that (b) is valid.

Suppose the contrary; i.e., $|y|_X \leqslant |b|_X + 4\delta$. Since $|a|_X \geqslant |b|_X + 4\delta + 2$, we have that $y \neq a$, hence $x \neq 1$. We write $x = x_1 x_2 \cdots x_n$ with $x_i \in X^{\pm}$, $i = 1, \dots, n$, and $n = |x|_X$. Then $(x_1^{-1} y x_1, x_2 x_3 \cdots x_n) \in S$. A contradiction to minimality of $|x|_X$.

Now we verify that (c) is valid for some ε_1 depending only on δ. Let $q = q_0 q_1 q_2$ be a path in $\Gamma(G, X)$ such that its subpaths q_0, q_1, q_2 are geodesics with labels representing x^{-1}, y, x. Without loss of generality, we may assume that $(q_0)_- = 1$. First, we show that conditions (1) and (2) of Lemma 4.6 are satisfied for $\alpha = \delta + 1$.

To the contrary, suppose that condition (1) of this lemma is not valid, say $((q_1)_-, (q_2)_+)_{(q_1)_+} \geqslant \delta + 1$; i.e., $(x^{-1}, x^{-1}yx)_{x^{-1}y} \geqslant \delta + 1$. Because of G-invariance of Gromov product, we have that

$$(1, yx)_y \geqslant \delta + 1.$$

Then there exist expressions $y = v_1 v_2$, $x = u_1 u_2$ such that $|y|_X = |v_1|_X + |v_2|_X$, $|x|_X = |u_1|_X + |u_2|_X$, $|v_2|_X = |u_1|_X = \delta + 1$, and $|v_2 u_1|_X \leqslant \delta$. Then the pair $(v_2 v_1, v_2 u_1 u_2)$ lies in S and

$$|v_2 u_1 u_2|_X \leqslant |v_2 u_1|_X + |u_2|_X \leqslant \delta + |u_2|_X < |u_1|_X + |u_2|_X = |x|_X.$$

A contradiction to minimality of $|x|_X$. Thus, condition (1) of Lemma 4.6 is valid. Condition (2) of this lemma is also valid, since

$$d\big((q_1)_-, (q_1)_+\big) = |y|_X \overset{\text{(b)}}{\geqslant} |b|_X + 4\delta + 1 \geqslant 4\delta + 2 = 2\alpha + 2\delta.$$

Recall that the paths q_0, q_1, q_2 are geodesic and hence $(1, 0)$-quasi-geodesic. Then, by Lemma 4.6, their concatenation $q_0q_1q_2$ is a $(1, \varepsilon_1)$-quasi-geodesic for some ε_1 depending only on δ. ∎

5. Uniform divergence of quasi-geodesics associated with loxodromic elements in acylindrically hyperbolic groups

The main aim of this section is to prove Propositions 5.4 and 5.6. We use Theorem 5.2 proved by the author in [6]; see Theorem 1.4 there. For convenience, we recall all necessary notions in the following subsection.

5.1. A periodicity theorem for acylindrically hyperbolic groups

Definition 5.1. Let Y be a generating set of G. Given a loxodromic element $a \in G$ and an element $x \in G$, consider the bi-infinite path $L(x, a)$ in $\Gamma(G, Y)$ obtained by connecting consequent points $\dots, xa^{-1}, x, xa, \dots$ by geodesic paths so that, for all $n \in \mathbb{Z}$, the path p_n connecting xa^n and xa^{n+1} has the same label as the path p_0 connecting x and xa. The paths p_n are called *a-periods* of $L(x, a)$, and the vertices xa^i, $i \in \mathbb{Z}$, are called the *phase vertices* of $L(x, a)$. For $k \in \mathbb{N}$, we say that a subpath $p \subset L(x, a)$ *contains k a-periods* if there exists $n \in \mathbb{Z}$ such that $p_n p_{n+1} \cdots p_{n+k-1}$ is a subpath of p.

Let G be a group and X a generating set of G. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and that G acts acylindrically on $\Gamma(G, X)$. In [10], Bowditch proved that the infimum of the set of stable norms (see Section 3.4 of the present paper) of all loxodromic elements of G with respect to X is a positive number. We denote this number by $\mathbf{inj}(G, X)$ and call it the *injectivity radius* of G with respect to X.

Theorem 5.2 ([6, Theorem 1.4]). *Let G be a group and X a generating set of G. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and that G acts acylindrically on $\Gamma(G, X)$. Then there exists a constant $\mathcal{C} > 0$ such that the following holds.*

Let $a, b \in G$ be two loxodromic elements which are shortest in their conjugacy classes and such that $|a|_X \geqslant |b|_X$. Let $x, y \in G$ be arbitrary elements and r an arbitrary non-negative integer. We set $f(r) = \frac{2r}{\mathbf{inj}(G,X)} + \mathcal{C}$.

If a subpath $p \subset L(x, a)$ contains at least $f(r)$ a-periods and lies in the r-neighborhood of $L(y, b)$, then there exist $s, t \neq 0$ such that $(x^{-1}y)b^s(y^{-1}x) = a^t$. In particular, a and b are commensurable.

Theorem 5.2 is illustrated by Figure 3.

Remark 5.3. (1) The main feature of Theorem 5.2 is that the function f is linear and does not depend on $|a|_X$ and $|b|_X$. Another point is that X is allowed to be infinite.

(2) In the conclusion of Theorem 5.2, we can write $z^{-1}b^s z = a^t$, where $z \in G$ is the element corresponding to the label of any path from any phase vertex of $L(y, b)$ to any phase vertex of $L(x, a)$.

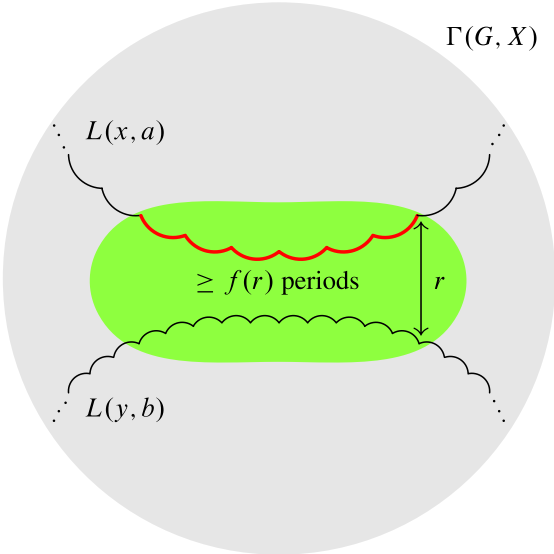

Figure 3. Illustration to Theorem 5.2.

5.2. Loxodromic-loxodromic case

Proposition 5.4. *Let G be a group and let X be a generating set of G. Suppose that the Cayley graph $\Gamma(G,X)$ is hyperbolic and acylindrical. Then there exists a constant $N_0 > 0$ such that for any loxodromic (with respect to X) elements $c, d \in G$ with $E_G(c) \neq E_G(d)$ and for any $n, m \in \mathbb{N}$ we have that*

$$|c^n d^m|_X > \frac{\min\{n,m\}}{N_0}.$$

Proof. Let $\delta \geqslant 0$ be a constant such that $\Gamma(G,X)$ is δ-hyperbolic. We use the following constants:

- $\varkappa \geqslant 1$, $\varepsilon_0 \geqslant 0$ and $n_0 \in \mathbb{N}$ are the constants from Lemma 4.7;
- $\mu = \mu(\delta, \varkappa, \varepsilon_0)$; see Lemma 4.2;
- $\mathscr{C}$ is the constant from Theorem 5.2.

We show that the proposition is valid for

$$N_0 = \max\left\{n_0, \frac{4000(\mu+\delta+1)}{\mathbf{inj}(G,X)} + 4\mathscr{C}\right\}. \tag{5.1}$$

Suppose to the contrary that there exist two loxodromic elements $c, d \in G$ satisfying $E_G(c) \neq E_G(d)$ and there exist $n, m \in \mathbb{N}$ such that

$$\min\{n,m\} \geqslant N_0 k, \quad \text{where } k = |c^n d^m|_X. \tag{5.2}$$

Clearly, $k \neq 0$.

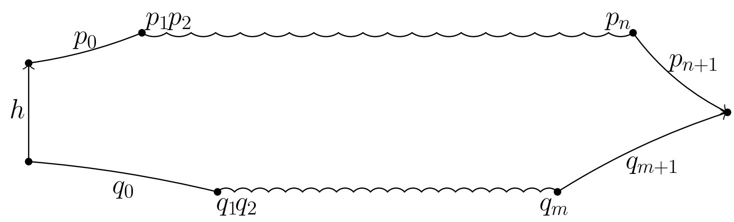

Figure 4

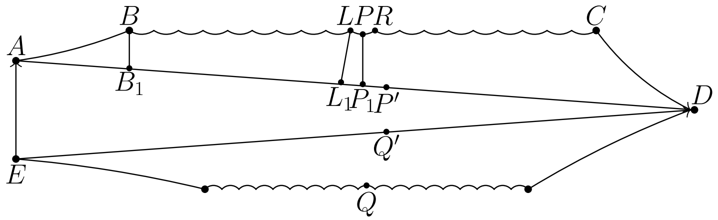

Figure 5

For any $g \in G$, let $S(g)$ be the set of shortest elements in the conjugacy class of g. Let $u \in G$ be a shortest element for which there exists $c_1 \in S(c)$ with the property $c = u^{-1}c_1u$. Let $v \in G$ be a shortest element for which there exists $d_1 \in S(d)$ with the property $d = v^{-1}d_1v$.

Without loss of generality, we assume that

$$|c_1|_X \geqslant |d_1|_X. \tag{5.3}$$

Denote $w = d^{-m}c^{-n}$. Then $|w|_X = k$ and we have the equation

$$u^{-1}\underbrace{c_1c_1\cdots c_1}_{n}uv^{-1}\underbrace{d_1\cdots d_1d_1}_{m}vw = 1. \tag{5.4}$$

Consider a geodesic $(n+m+5)$-gon $\mathcal{P} = p_0(p_1p_2\cdots p_n)p_{n+1}\bar{q}_{m+1}(\bar{q}_m\cdots\bar{q}_2\bar{q}_1)\bar{q}_0h$ in the Cayley graph $\Gamma(G,X)$ such that the labels of its sides correspond to the elements in the left side of (5.4) (see Figure 4). In particular, the labels of the paths $q_0, q_1, q_2, \dots, q_m, q_{m+1}$ correspond to the elements $v^{-1}, d_1^{-1}, d_1^{-1}, \dots, d_1^{-1}, v$.

Since $\min\{n,m\} \geqslant N_0k \geqslant N_0 \geqslant n_0$, we have by Lemma 4.7 that the paths $p := p_0p_1p_2\cdots p_np_{n+1}$ and $q = q_0q_1q_2\cdots q_mq_{m+1}$ are $(\varkappa, \varepsilon_0)$-quasi-geodesics. In the following claims we use the following constants:

$$K = k + 36\mu + 26\delta, \quad K_1 = K + 4\mu + 4\delta, \quad K_2 = 10K_1 + 2\mu + 2\delta.$$

Claim 1. The quasi-geodesic $p_1p_2\cdots p_n$ contains $n \geqslant 4f(K_2)$ c_1-periods. The quasi-geodesic $q_1q_2\cdots q_m$ contains $m \geqslant 4f(K_2)$ d_1-periods.

Proof. The claim follows straightforward from the definition of function $f(r)$ in Theorem 5.2 and from (5.1) and (5.2). ∎

We intend to apply Theorem 5.2 to these quasi-geodesics or to their parts. However, we cannot claim that the first quasi-geodesic is contained in the K_2-neighborhood of the second one.

Let P be the middle point of the quasi-geodesic $p_1 p_2 \cdots p_n$ and let Q be the middle point of the quasi-geodesic $q_1 q_2 \cdots q_m$.

Claim 2. We have that

$$d(P, Q) \leqslant K. \tag{5.5}$$

Proof. Denote $A = (p_0)_-$, $B = (p_1)_-$, $C = (p_{n+1})_-$, $D = (p_{n+1})_+$, and $E = (q_0)_-$; see Figure 5.

By Lemma 4.2, there exists a point $P_1 \in [A, D]$ such that

$$d(P, P_1) \leqslant \mu. \tag{5.6}$$

The point P divides the path p into two halves. In the following we define two points $L, R \in p$. If n is even, we set $L = P = R$. If n is odd, we set $L = (p_{\lceil \frac{n}{2} \rceil})_-$ and $R = (p_{\lceil \frac{n}{2} \rceil})_+$. By Lemma 4.3 applied to the first half of the quasi-geodesic p and the geodesic $[A, P_1]$, there exists a point $L_1 \in [A, P_1]$ such that

$$d(L, L_1) \leqslant d(P, P_1) + 2(\mu + \delta) \leqslant 3\mu + 2\delta. \tag{5.7}$$

Applying Lemma 4.3 once more, we obtain that there exists a point $B_1 \in [A, L_1]$ such that

$$d(B, B_1) \leqslant d(L, L_1) + 2(\mu + \delta) \leqslant 5\mu + 4\delta. \tag{5.8}$$

Using triangle inequality several times, we deduce that

$$\begin{aligned} d(A, P_1) &= d(A, B_1) + d(B_1, L_1) + d(L_1, P_1) \\ &\approx_T d(A, B) + d(B, L) + d(L, P), \end{aligned}$$

where $T = 2d(B, B_1) + 2d(L, L_1) + d(P, P_1)$. It follows from (5.6)–(5.8) that $T \leqslant 17\mu + 12\delta$. Hence

$$d(A, P_1) \approx_{17\mu+12\delta} d(A, B) + d(B, L) + d(L, P). \tag{5.9}$$

Analogously, we have that

$$d(D, P_1) \approx_{17\mu+12\delta} d(D, C) + d(C, R) + d(R, P). \tag{5.10}$$

Since the three summands in (5.9) are equal to the three summands in (5.10), we deduce that

$$d(A, P_1) \approx_{34\mu+24\delta} d(D, P_1).$$

Let P' be the middle point of $[A, D]$. Then $d(P_1, P') \leqslant 17\mu + 12\delta$. We have that

$$d(P, P') \leqslant d(P, P_1) + d(P_1, P') \leqslant 18\mu + 12\delta. \tag{5.11}$$

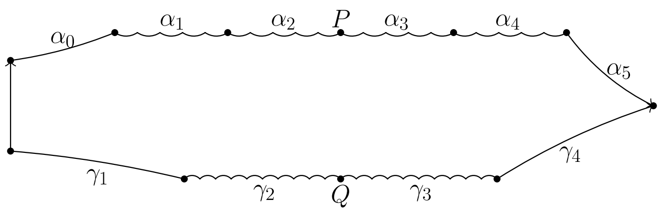

Figure 6

Analogously, if Q' is the middle point of $[E, D]$, then

$$d(Q, Q') \leqslant 18\mu + 12\delta. \tag{5.12}$$

By Lemma 4.1 applied to the geodesic triangle ADE, we have that

$$d(P', Q') \leqslant \frac{1}{2} d(A, E) + 2\delta = \frac{k}{2} + 2\delta. \tag{5.13}$$

Now the claim follows from (5.11)–(5.13). ■

We consider the decomposition $p_0 p_1 p_2 \cdots p_n p_{n+1} = \alpha_0\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$, where $\alpha_0 = p_0, \alpha_1 = p_1 \cdots p_{\lfloor \frac{n}{4} \rfloor}, \alpha_4 = p_{n-\lfloor \frac{n}{4} \rfloor+1} \cdots p_n, \alpha_5 = p_{n+1}$, and α_2 and α_3 are determined by the condition $(\alpha_2)_+ = P = (\alpha_3)_-$. We also consider the decomposition $q_0 q_1 \cdots q_m q_{m+1} = \gamma_1\gamma_2\gamma_3\gamma_4$, where $\gamma_1 = q_0$, $\gamma_4 = q_{m+1}$, and γ_2 and γ_3 are determined by the condition $(\gamma_2)_+ = Q = (\gamma_3)_-$; see Figure 6.

Claim 3. There are decompositions $\gamma_1\gamma_2 = \beta_0\beta_1\beta_2$ and $\gamma_3\gamma_4 = \beta_3\beta_4\beta_5$ such that α_i and β_i are K_1-similar for $i = 0, \dots, 5$. In particular, the Hausdorff distance between α_i and β_i is at most K_2 for $i = 0, \dots, 5$.

Proof. Because of symmetry, we show only that the first decomposition exists. We have that

$$d\big((\alpha_0\alpha_1\alpha_2)_-, (\gamma_1\gamma_2)_-\big) = k,$$
$$d\big((\alpha_0\alpha_1\alpha_2)_+, (\gamma_1\gamma_2)_+\big) \overset{(5.5)}{\leqslant} K.$$

By Lemma 4.3 applied to the $(\varkappa, \varepsilon_0)$-quasi-geodesics $\alpha_0\alpha_1\alpha_2$ and $\gamma_1\gamma_2$, there exists a point $U \in \gamma_1\gamma_2$ such that $d((\alpha_1)_+, U) \leqslant K + 2\mu + 2\delta$; see Figure 7.

Applying this lemma once more, we obtain a point V on the segment of $\gamma_1\gamma_2$ from E to U such that $d((\alpha_1)_-, V) \leqslant K + 4\mu + 4\delta = K_1$. The points V and U divide the path $\gamma_1\gamma_2$ into three consecutive subpaths. We denote them by $\beta_0, \beta_1, \beta_2$. By construction, the paths α_i and β_i are K_1-similar. Again by Lemma 4.3, the Hausdorff distance between α_i and β_i is at most $K_1 + 2\mu + 2\delta \leqslant K_2$. ■

It follows from $\gamma_1\gamma_2 = \beta_0\beta_1\beta_2$ that either β_2 is a subpath of γ_2, or $\beta_0\beta_1$ is an initial subpath of γ_1. Analogously, it follows from $\gamma_3\gamma_4 = \beta_3\beta_4\beta_5$ that either β_3 is a subpath of γ_3, or $\beta_4\beta_5$ is a terminal subpath of γ_4.

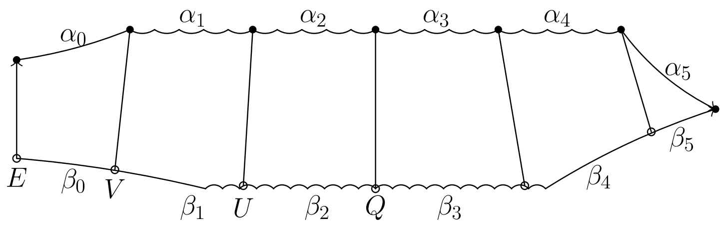

Figure 7

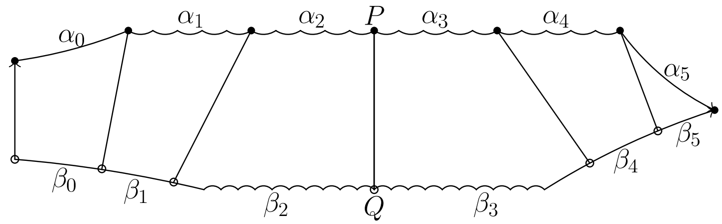

Figure 8

Case 1. Suppose that β_2 is a subpath of γ_2; see Figure 7.

Observe that α_2 and β_2 satisfy assumptions of Theorem 5.2. Indeed, in this case α_2 and β_2 are subpaths of the quasi-geodesics $L((p_n)_+, c_1)$ and $L((q_m)_+, d_1^{-1})$, respectively, $|c_1|_X \geqslant |d_1|_X$ by (5.3), the Hausdorff distance between α_2 and β_2 is at most K_2 by Claim 3, and α_2 contains at least $f(K_2)$ c_1-periods by Claim 1.

Hence, by Theorem 5.2 and Remark 5.3, there exist $s, t \neq 0$ such that

$$(uv^{-1})d_1^s(vu^{-1}) = c_1^t. \tag{5.14}$$

Indeed, vu^{-1} is the element which corresponds to the label of the path $q_{m+1}\overline{p_{n+1}}$ from the phase vertex $(q_m)_+$ to the phase vertex $(p_n)_+$. From (5.14) and from $c = u^{-1}c_1u$, $d = v^{-1}d_1v$, we deduce that $c^t = d^s$. Therefore, $E_G(c) = E_G(d)$. This contradicts the assumption of Proposition 5.4.

Case 2. Suppose that β_3 is a subpath of γ_3.

This case can be considered analogously. Thus, it remains to consider the following case.

Case 3. Suppose that $\beta_0\beta_1$ is an initial subpath of γ_1 and $\beta_4\beta_5$ is a terminal subpath of γ_4; see Figure 8.

Recall that the geodesics γ_1 and γ_4 have mutually inverse labels. Therefore, there exist $g \in G$ such that $\gamma_1 = g(\overline{\gamma_4})$. We denote

$$\alpha_0' = g(\overline{\alpha_5}), \quad \alpha_1' = g(\overline{\alpha_4}), \quad \beta_0' = g(\overline{\beta_5}), \quad \text{and} \quad \beta_1' = g(\overline{\beta_4});$$

see Figure 9. Then $\beta_0\beta_1$ and $\beta_0'\beta_1'$ are initial segments of γ_1.

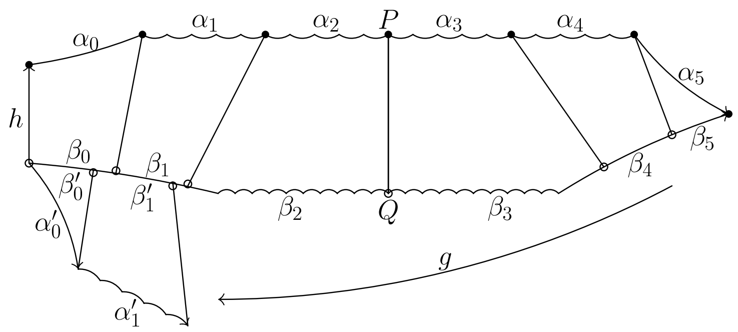

Figure 9

Claim 4. The Hausdorff distance between α_1 and α_1' is at most K_2.

Proof. Since β_i and α_i are K_1-similar for all i, we have that

$$\begin{aligned} d\big((\beta_j)_-,(\beta_j)_+\big) &\approx_{2K_1} d\big((\alpha_j)_-,(\alpha_j)_+\big),\\ d\big((\beta_j')_-,(\beta_j')_+\big) &\approx_{2K_1} d\big((\alpha_j')_-,(\alpha_j')_+\big) \end{aligned} \tag{5.15}$$

for $j=0,1$. Observe that α_0 and α_0' have the same labels and α_1 and α_1' have mutually inverse labels. Therefore, the numbers on the right sides of (5.15) are equal. This implies that

$$d\big((\beta_j)_-,(\beta_j)_+\big) \approx_{4K_1} d\big((\beta_j')_-,(\beta_j')_+\big).$$

Since $\beta_0\beta_1$ and $\beta_0'\beta_1'$ are initial segments of the geodesic γ_1, we deduce that

$$\begin{aligned} d\big((\beta_0)_+,(\beta_0')_+\big) &\leqslant 4K_1,\\ d\big((\beta_1)_+,(\beta_1')_+\big) &\leqslant 8K_1. \end{aligned}$$

From here, from the K_1-similarity of α_i and β_i, and from the K_1-similarity of α_i' and β_i' we deduce that

$$\begin{aligned} d\big((\alpha_1)_-,(\alpha_1')_-\big) &= d\big((\alpha_0)_+,(\alpha_0')_+\big)\\ &\leqslant d\big((\alpha_0)_+,(\beta_0)_+\big) + d\big((\beta_0)_+,(\beta_0')_+\big) + d\big((\beta_0')_+,(\alpha_0')_+\big)\\ &\leqslant K_1+4K_1+K_1 = 6K_1,\\ d\big((\alpha_1)_+,(\alpha_1')_+\big) &\leqslant d\big((\alpha_1)_+,(\beta_1)_+\big) + d\big((\beta_1)_+,(\beta_1')_+\big) + d\big((\beta_1')_+,(\alpha_1')_+\big)\\ &\leqslant K_1+8K_1+K_1 = 10K_1. \end{aligned}$$

By Lemma 4.3, the Hausdorff distance between α_1 and α_1' is at most K_2. ∎

Observe that α_1 and α_1' satisfy assumptions of Theorem 5.2. Indeed, α_1 and α_1' are subpaths of the quasi-geodesics $L((\alpha_1)_-, c_1)$ and $L((\alpha_1')_-, c_1^{-1})$, respectively, each of them contains at least $f(K_2)$ periods by Claim 1, and the Hausdorff distance between α_1 and α_1' is at most K_2 by Claim 4. By this theorem, there exist integers $s, t \neq 0$ such that $z^{-1}c_1^s z = c_1^t$, where $z \in G$ is the element representing the label of any path from $(\alpha_1')_-$ to $(\alpha_1)_-$. As such path we take $\overline{\alpha_0'}h\alpha_0$. Then $z = uwu^{-1}$, and we have $w^{-1}c^s w = c^t$. Hence

$$w \in E_G(c).$$

Since $w = d^{-m}c^{-n}$, we have $d^m = c^{-n}w^{-1} \in E_G(c)$ and hence $E_G(c) = E_G(d)$. This contradicts the assumption of Proposition 5.4.

Thus, the inequality (5.2) is impossible, and we are done. ■

5.3. Loxodromic-elliptic case

Remark 5.5. Suppose that G is a group and $X \subseteq G$ is a (possibly infinite) generating set of G such that $\Gamma(G, X)$ is δ-hyperbolic for some $\delta \geqslant 0$. A subgroup H of G is called *elliptic* (with respect to X) if, acting on $\Gamma(G, X)$, it has bounded orbits.

It is well known that any elliptic subgroup H of G can be conjugated into the ball of radius $4\delta + 1$ and center 1 in $\Gamma(G, X)$. The proof of this fact is given in [9] for the case where G is a hyperbolic group. It also works under the above assumptions. An alternative proof is given in [47, Corollary 6.7].

Proposition 5.6. *Let G be a group and let X be a generating set of G. Suppose that the Cayley graph $\Gamma(G, X)$ is hyperbolic and acylindrical. Then there exists a constant $N_1 > 0$ such that for any loxodromic (with respect to X) element $c \in G$, any elliptic element $e \in G \setminus E_G(c)$, and any $n \in \mathbb{N}$, we have that*

$$|c^n e|_X > \frac{n}{N_1}.$$

Proof. The proof is very similar to the proof of Lemma 5.4. However, some pieces are new and some are easier. Therefore, we decided to present a complete proof for clearness. Let $\delta \geqslant 0$ be a constant such that $\Gamma(G, X)$ is δ-hyperbolic.

We use the following constants:

- $\varkappa \geqslant 1$, $\varepsilon_0 \geqslant 0$, and $n_0 \in \mathbb{N}$ are the constants from Lemma 4.7;
- $\varepsilon_1 = \varepsilon_1(\delta) \geqslant 0$ is the constant from Lemma 4.8;
- $\varepsilon_2 := \max\{\varepsilon_0, \varepsilon_1\}$;
- $\mu = \mu(\delta, \varkappa, \varepsilon_2)$; see Lemma 4.2;
- $\mathscr{C}$ is the constant from Theorem 5.2.

We show that the proposition is valid for

$$N_1 = \max\left\{n_0,\ \frac{400(\mu + \delta + 1)}{\mathbf{inj}(G, X)} + 2\mathscr{C} + 1,\ \varkappa(\varepsilon_0 + 8\delta + 4)\right\}. \tag{5.16}$$

Suppose to the contrary that there exist a loxodromic (with respect to X) element $c \in G$, an elliptic element $e \in G \setminus E_G(c)$, and a number $n \in \mathbb{N}$ such that

$$n \geqslant N_1 k, \quad \text{where } k = |c^n e|_X. \tag{5.17}$$

Clearly, $k \neq 0$.

Claim 1. There exist $x, y \in G$ such that $e = x^{-1}yx$ and the following holds:

(1) $|y|_X \leqslant 8\delta + 3$,

(2) any path $q_0 q_1 q_2$ in $\Gamma(G, X)$, where q_0, q_1, q_2 are geodesics with labels representing x^{-1}, y, x, is a $(1, \varepsilon_1)$-quasi-geodesic path.

Proof. It follows from (5.16) and (5.17) that $n \geqslant N_1 k \geqslant N_1 \geqslant n_0$. Then, by Lemma 4.7, we have

$$|c^n|_X \geqslant \frac{1}{\varkappa} n - \varepsilon_0.$$

From this we deduce that

$$\begin{aligned} k = |c^n e|_X \geqslant |c^n|_X - |e|_X &\overset{(5.3)}{\geqslant} \frac{1}{\varkappa} n - \varepsilon_0 - |e|_X \geqslant \frac{1}{\varkappa} N_1 k - \varepsilon_0 - |e|_X \\ &\overset{(5.16)}{\geqslant} k(\varepsilon_0 + 8\delta + 4) - \varepsilon_0 - |e|_X \geqslant k + 8\delta + 3 - |e|_X. \end{aligned}$$

Hence $|e|_X \geqslant 8\delta + 3$. Since e is elliptic, e is conjugate to an element of G of length at most $4\delta + 1$ (see Remark 5.5). Thus, the assumption and, hence, the conclusion of Lemma 4.8 are satisfied. ∎

For any $g \in G$, let $S(g)$ be the set of shortest elements in the conjugacy class of g. Let $u \in G$ be a shortest element for which there exists $c_1 \in S(c)$ with the property $c = u^{-1} c_1 u$.

We denote $w = (c^n e)^{-1}$. Then $|w|_X = k$ and we have the equation

$$u^{-1} \underbrace{c_1 c_1 \cdots c_1}_{n} u x^{-1} y x w = 1. \tag{5.18}$$

Consider a geodesic $(n+6)$-gon $\mathcal{P} = p_0(p_1 p_2 \cdots p_n) p_{n+1} \bar{q}_2 \bar{q}_1 \bar{q}_0 h$ in the Cayley graph $\Gamma(G, X)$ such that the labels of its sides correspond to the elements in the left side of (5.18); see Figure 10.

In particular, the labels of the paths q_0, q_1, q_2 correspond to the elements x^{-1}, y^{-1}, and x. By Claim 1, the path $q := q_0 q_1 q_2$ is a $(1, \varepsilon_1)$-quasi-geodesic. Since $n \geqslant N_1 k \geqslant N_1 \geqslant n_0$, we have by Lemma 4.7 that the path $p := p_0 p_1 p_2 \cdots p_n p_{n+1}$ is a $(\varkappa, \varepsilon_0)$-quasi-geodesic. Then p and q are $(\varkappa, \varepsilon_2)$-quasi-geodesics. In the following claims, we use the following constants:

$$K = k + 36\mu + 26\delta, \quad K_1 = K + 2\mu + 6\delta + 2, \quad K_2 = K_1 + 2\mu + 2\delta.$$

Claim 2. The quasi-geodesic $p_1 p_2 \cdots p_n$ contains $n \geqslant 2f(2K_2) + 1$ c_1-periods.

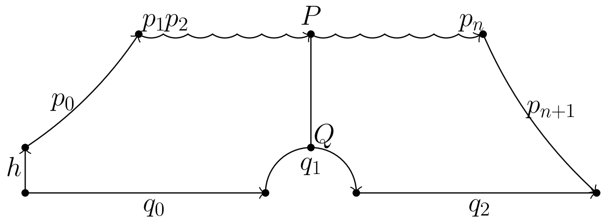

Figure 10

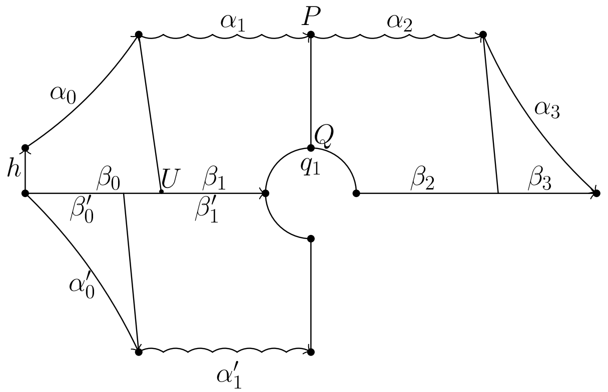

Figure 11

Proof. The claim follows straightforward from the definition of function $f(r)$ in Theorem 5.2 and from (5.16) and (5.17). ∎

Let P be the middle point of the quasi-geodesic $p_1 p_2 \cdots p_n$ and let Q be the middle point of the geodesic q_1. As in Claim 2 of the proof of Proposition 5.4, we have that $d(P, Q) \leqslant K$.

We consider the decomposition $p_0 p_1 p_2 \cdots p_n p_{n+1} = \alpha_0\alpha_1\alpha_2\alpha_3$, where $\alpha_0 = p_0$, $\alpha_3 = p_{n+1}$, and α_1 and α_2 are determined by the condition $(\alpha_1)_+ = P = (\alpha_2)_-$; see Figure 11.

Claim 3. There are decompositions $q_0 = \beta_0\beta_1$ and $q_2 = \beta_2\beta_3$ such that α_i and β_i are K_1-similar for $i = 0, \dots, 3$. In particular, the Hausdorff distance between α_i and β_i is at most K_2.

Proof. Because of symmetry, we show only that the first decomposition exists. By Claim 1, we have that $d((q_1)_-, (q_1)_+) = |y|_X \leqslant 8\delta + 3$. Hence

$$d\big((\alpha_0\alpha_1)_-, (q_0)_-\big) = k,$$

$$d\big((\alpha_0\alpha_1)_+, (q_0)_+\big) \leqslant d(P, Q) + \frac{1}{2} d\big((q_1)_-, (q_1)_+\big) < K + 4\delta + 2.$$

By Lemma 4.3 applied to the $(\varkappa, \varepsilon_2)$-quasi-geodesics $\alpha_0\alpha_1$ and q_0, there exists a point $U \in q_0$ such that $d((\alpha_1)_+, U) \leqslant K + 4\delta + 2 + (2\mu + 2\delta) = K_1$.

The point U divides the path q into two subpaths. We denote them by β_0, β_1. By construction, the paths α_i and β_i are K_1-similar. Again by Lemma 4.3, the Hausdorff distance between α_i and β_i is at most $K_1 + 2\mu + 2\delta = K_2$. ■

Recall that the geodesics q_0 and q_2 have mutually inverse labels. Therefore, there exists $g \in G$ such that $q_0 = g(\overline{q_2})$. We denote

$$\alpha_0' = g(\overline{\alpha_3}), \quad \alpha_1' = g(\overline{\alpha_2}), \quad \beta_0' = g(\overline{\beta_3}), \quad \text{and} \quad \beta_1' = g(\overline{\beta_2}).$$

Then $\beta_0\beta_1 = \beta_0'\beta_1' = q_0$.

Claim 4. One of α_1, α_1' lies in the $2K_2$-neighborhood of the other.

Proof. By Claim 3, the Hausdorff distance between α_1 and β_1 is at most K_2. Also the Hausdorff-distance between α_1' and β_1' is at most K_2. The claim follows from the fact that one of β_1, β_1' is a subsegment of the other. ■

Observe that α_1 and α_1' satisfy assumptions of Theorem 5.2. Indeed, α_1 and α_1' are subpaths of the quasi-geodesics $L((\alpha_1)_-, c_1)$ and $L((\alpha_1')_-, c_1^{-1})$, respectively, each of them contains at least $f(2K_2)$ periods by Claim 2, and one of them lies in the $2K_2$-neighborhood of the other by Claim 3. By this theorem, there exist integers $s, t \neq 0$ such that $z^{-1}c_1^s z = c_1^t$, where $z \in G$ is the element representing the label of any path from $(\alpha_1')_-$ to $(\alpha_1)_-$. As such path we take $\overline{\alpha_0'}h\alpha_0$. Then $z = uwu^{-1}$, and we have that $w^{-1}c^s w = c^t$. Hence

$$w \in E_G(c).$$

Since $w = (c^n e)^{-1}$, we have that $e \in E_G(c)$. This contradicts the assumption of Proposition 5.6. Thus, the inequality (5.17) is impossible, and we are done. ■

6. Extension of quasi-morphisms from hyperbolically embedded subgroups to the whole group

6.1. A sufficient condition for preserving the ellipticity under decreasing of a generating set

Lemma 6.1. *Let G be a group and X a generating set of G. Suppose that $\Gamma(G, X)$ is hyperbolic and acylindrical. Let $a_1, \dots, a_k \in G$ be finitely many loxodromic elements with respect to X. Let $g \in G$ be an element non-commensurable with elements of $A = \{a_1, \dots, a_k\}$. If g is elliptic with respect to $X' = X \cup E_G(a_1) \cup \dots \cup E_G(a_k)$, then g is elliptic with respect to X.*

Proof. Let B_i be a finite set of representatives of left cosets of $\langle a_i \rangle$ in $E_G(a_i)$, $i = 1, \dots, k$. Enlarging X by a finite set does not change the property of $\Gamma(G, X)$ to be

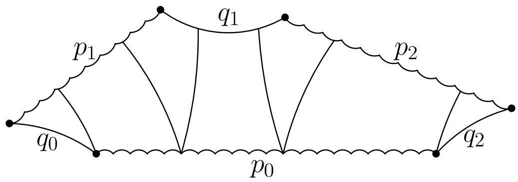

Figure 12. Case $s = 2$.

hyperbolic and acylindrical (see [47, Lemma 5.1]) and the property of elements of G to be elliptic or loxodromic. Therefore, we may assume that X contains $\bigcup_{i=1}^{k} B_i$. Let $\delta \geqslant 0$ be a constant such that $\Gamma(G, X)$ is δ-hyperbolic.

To the contrary, suppose that g is elliptic with respect to X' and not elliptic with respect to X.

Then the following holds.

(1) There exists $R > 0$ such that $|g^n|_{X'} \leqslant R$ for all $n \in \mathbb{Z}$.

(2) g is loxodromic with respect to X.

We fix an arbitrary $n \in \mathbb{N}$ and write $g^n = x_1 x_2 \cdots x_t$, where $x_i \in X'$ and t is minimal. In particular, $t \leqslant R$. Each element of $E_G(a_i)$ can be written in the form ba_i^k for some $b \in B_i \subset X$. Therefore, g^n can be written as

$$g^n = u_0 a_{i_1}^{k_1} u_1 a_{i_2}^{k_2} \cdots u_{s-1} a_{i_s}^{k_s} u_s,$$

where $u_0, \dots, u_s$ are words in X, $a_{i_1}, a_{i_2} \dots, a_{i_s} \in A$, $0 \leqslant s \leqslant t \leqslant R$, and

$$\sum_{j=0}^{s} |u_j|_X \leqslant t \leqslant R. \tag{6.1}$$

Let $f_0, h \in G$ be elements such that $g = f_0^{-1} h f_0$ and h is a shortest element (with respect to X) in the conjugacy class of g. For each $a_i \in A$, let $f_i, b_i \in G$ be elements such that $a_i = f_i^{-1} b_i f_i$ and b_i is a shortest element (with respect to X) in the conjugacy class of a_i. Let $F = \max\{|f_i|_X \mid i = 0, \dots, k\}$. Then

$$h^n = v_0 b_{i_1}^{k_1} v_1 b_{i_2}^{k_2} \cdots v_{s-1} b_{i_s}^{k_s} v_s,$$

where $v_0 = f_0 u_0 f_{i_1}^{-1}$, $v_j = f_{i_j} u_j f_{i_{j+1}}^{-1}$, $j = 1, \dots, s-1$, and $v_s = f_{i_s} u_s f_0^{-1}$. Using (6.1), we have that

$$\sum_{j=0}^{s} |v_j|_X \leqslant \sum_{j=0}^{s} |u_j|_X + 2|f_0|_X + 2\sum_{j=1}^{s} |f_{i_j}|_X \leqslant R + 2(R+1)F. \tag{6.2}$$

Let $\mathcal{P}$ be a geodesic $2(s+1)$-gon in $\Gamma(G, X)$ with sides $p_0, q_0, p_1, \dots, p_s, q_s$ such that $p_0, p_1, \dots, p_s$ are quasi-geodesics representing $h^{-n}, b_{i_1}^{k_1}, \dots, b_{i_s}^{k_s}$ and $q_0, \dots, q_s$ are geodesics representing $v_0, \dots, v_s$; see Figure 12.

Since the elements $h, b_{i_1}, \dots, b_{i_s}$ are loxodromic with respect to X and have minimal length in their conjugacy classes, the paths $p_0, p_1, \dots, p_s$ are $(\varkappa, \varepsilon)$-quasi-geodesics, where $\varkappa$ and ε are universal constants from Corollary 3.6. We set

$$\alpha = 2R\delta + 2\mu(\delta, \varkappa, \varepsilon),$$

where $\mu(\delta, \varkappa, \varepsilon)$ is the constant from Lemma 4.2.

Claim 1. The side p_0 lies in the α-neighborhood of the union of other sides of $\mathcal{P}$.

Proof. For each $i = 0, \dots, s$, we chose a geodesic segment $\widetilde{p_i}$ such that $(\widetilde{p_i})_- = (p_i)_-$ and $(\widetilde{p_i})_+ = (p_i)_+$. Consider the geodesic $2(s+1)$-gon $\widetilde{\mathcal{P}}$ with the sides $\widetilde{p_0}, q_0, \widetilde{p_1}, \dots, \widetilde{p_s}, q_s$. The side $\widetilde{p_0}$ lies in the $2s\delta$-neighborhood of the other sides of $\widetilde{\mathcal{P}}$. By Lemma 4.2, the Hausdorff distance between $\widetilde{p_i}$ and p_i is at most $\mu(\delta, \varkappa, \varepsilon)$ for every i. This completes the proof. ∎

For $i = 0, \dots, s$, we set

$$D_i = \varkappa\big(2\alpha + |v_i|_X + \varepsilon\big) + 1.$$

Claim 2. For any $i \in \{0, \dots, s\}$, the α-neighborhood of q_i contains at most D_i points of p_0.

Proof. Let Q_i be the set of points on p_0 which lie in the α-neighborhood of q_i. Suppose that $Q_i \neq \emptyset$ and let z_1 and z_2 be the leftmost and the rightmost points of Q_i on p_0. Then $d_X(z_1, z_2) \leqslant 2\alpha + \ell(q_i) = 2\alpha + |v_i|_X$. Therefore, the length of the subpath of p_0 connecting z_1 and z_2 is at most $\varkappa(2\alpha + |v_i|_X + \varepsilon)$, and the claim follows. ∎

We set

$$\beta = \eta(\delta, \varkappa, \varepsilon, \alpha),$$

where η is the function from Lemma 4.3. Also, for $j = 1, \dots, s$, we set

$$C_j := \max\big\{\varkappa\big(\big(f(\beta) + 2\big)|b_{i_j}|_X + 2\alpha + \varepsilon\big), \big(f(\beta) + 2\big)|h|_X\big\}, \qquad (6.3)$$

where f is the function from Theorem 5.2.

Claim 3. For any $j \in \{1, \dots, s\}$, the α-neighborhood of p_j contains at most C_j points of p_0.

Proof. Let c be the maximal subpath of p_0 for which there exists a subpath c' of p_j or $\overline{p_j}$ with the property

$$d_X(c_-, c'_-) \leqslant \alpha \quad \text{and} \quad d_X(c_+, c'_+) \leqslant \alpha. \qquad (6.4)$$

Suppose that the α-neighborhood of p_j contains more than C_j points of p_0. Then

$$\ell(c) \geqslant C_j.$$

We check the following statements:

(1) the Hausdorff distance between c and c' is at most β;

(2) the path c contains at least $f(\beta)$ h-periods;

(3) the path c' contains at least $f(\beta)$ $b_{i_j}^{\pm}$-periods.

Statement (1) follows from (6.4) and Lemma 4.3. Statement (2) follows from

$$\ell(c) \geqslant C_j \overset{(6.3)}{\geqslant} \big(f(\beta)+2\big)|h|_X.$$

Finally, statement (3) follows from

$$\begin{aligned}\ell(c') \geqslant d_X(c'_-, c'_+) \geqslant d_X(c_-, c_+) - d_X(c_-, c'_-) - d_X(c_+, c'_+) \geqslant d_X(c_-, c_+) - 2\alpha \\ \geqslant \frac{1}{\varkappa}\ell(c) - \varepsilon - 2\alpha \geqslant \frac{1}{\varkappa}C_j - \varepsilon - 2\alpha \overset{(6.3)}{\geqslant} \big(f(\beta)+2\big)|b_{i_j}|_X.\end{aligned}$$

It follows from statements (1)–(3) and Theorem 5.2 that h and b_{i_j} are commensurable. Then g and a_{i_j} are commensurable. A contradiction. ∎

It follows from Claims 1–3 that p_0 contains at most $\sum_{i=0}^{s} D_i + \sum_{j=1}^{s} C_j$ points. Using (6.2), we have that

$$\sum_{i=0}^{s} D_i \leqslant (R+1)\big(\varkappa(2\alpha+\varepsilon)+1\big) + \varkappa\big(R + 2(R+1)F\big).$$

We also have that

$$\sum_{j=1}^{s} C_j \leqslant R \max C_j \leqslant R \max\big\{\varkappa\big(\big(f(\beta)+2\big)\big(\max_{i=1,\dots,k}|a_i|_X\big) + 2\alpha + \varepsilon\big),\ \big(f(\beta)+2\big)|h|_X\big\}.$$

Therefore, $\ell_X(p_0)$ is bounded from above by a constant which does not depend on n. This contradicts the fact that $\ell_X(p_0) = n|h|_X \geqslant n$. ∎

6.2. Improving relative generating sets for hyperbolically embedded subgroups

Recall that in the situation $\{E_1, \dots, E_k\} \hookrightarrow_h G$, we use notation $\mathcal{E} = E_1 \sqcup \dots \sqcup E_k$. The following lemma follows directly from Definition 3.12.

Lemma 6.2. *Suppose that $\{E_1, \dots, E_k\} \hookrightarrow_h (G, X)$ and let Y be a subset of G such that $X \subseteq Y \subseteq \langle X\rangle$ and $\sup_{y\in Y} |y|_X < \infty$. Then $\{E_1, \dots, E_k\} \hookrightarrow_h (G, Y)$.*

Proof. The hyperbolicity of $\Gamma(G, Y \sqcup \mathcal{E})$ follows from the hyperbolicity of $\Gamma(G, X \sqcup \mathcal{E})$ by Lemma 3.3. The local finiteness of $(E_i, \widehat{d_i}^Y)$, where the relative metric $\widehat{d_i}^Y$ on E_i is defined using Y, follows from the local finiteness of $(E_i, \widehat{d_i}^X)$, where the relative metric $\widehat{d_i}^X$ on E_i is defined using X. ∎

Lemma 6.3. *Suppose that $\{E_1, \dots, E_k\} \hookrightarrow_h (G, X)$, where $E_1, \dots, E_k$ are infinite virtually cyclic subgroups of G. Then there exists a subset $Y \subseteq G$ containing X such that the following properties are satisfied:*

(1) *$\langle Y\rangle = G$,*

(2) *$\{E_1, \dots, E_k\} \hookrightarrow_h (G, Y)$,*

(3) *$\Gamma(G, Y)$ and $\Gamma(G, Y \sqcup \mathcal{E})$ are hyperbolic and acylindrical,*

(4) *if G is not virtually cyclic, then G is acylindrically hyperbolic with respect to Y,*

(5) *if $g \in G$ is an elliptic element with respect to $Y \cup \bigcup_{i=1}^{k} E_i$ such that g is non-commensurable with elements of $\bigcup_{i=1}^{k} E_i$ of infinite order, then g is elliptic with respect to Y. Moreover, there exists $u \in G$ such that $u^{-1} g^m u \in Y$ for all $m \in \mathbb{Z}$.*

Proof. By Theorem 3.15, there exists a subset $X_1 \subseteq G$ such that $X \subseteq X_1$ and the following conditions hold.

(i) $\{E_1, \ldots, E_k\} \hookrightarrow_h (G, X_1)$.

(ii) $\Gamma(G, X_1 \sqcup \mathcal{E})$ is hyperbolic and acylindrical.

We show that X_1 can be chosen so that, additionally, X_1 generates G. Let A_i be a finite generating set of E_i. We set $X_2 := X_1 \cup A_1 \cup \cdots \cup A_k$. Since $G = \langle X_1 \cup \bigcup_{i=1}^{k} E_i \rangle$, we have that $G = \langle X_2 \rangle$.

Claim 1. (a) $\{E_1, \ldots, E_k\} \hookrightarrow_h (G, X_2)$.

(b) $\Gamma(G, X_2 \sqcup \mathcal{E})$ is hyperbolic and acylindrical.

(c) $\Gamma(G, X_2)$ is hyperbolic and acylindrical.

Proof. Since $|X_1 \Delta X_2| < \infty$, we have that (i)$\Leftrightarrow$(a) by Lemma 3.13 and (ii)$\Leftrightarrow$(b) by Lemma 3.3. To prove (c), we first observe that $\{1\} \hookrightarrow_h (E_i, A_i)$ for $i = 1, \ldots, k$, and recall that $\{E_1, \ldots, E_k\} \hookrightarrow_h (G, X_1)$. By [14, Proposition 4.35], this implies that $\{1\} \hookrightarrow_h (G, X_2)$. In particular, by definition this means that $\Gamma(G, X_2)$ is hyperbolic. The acylindricity of $\Gamma(G, X_2)$ can be proved as in the part of the proof of [47, Theorem 1.4], starting from the words "let us show that $\Gamma(G, X)$ is acylindrical." ∎

Let $\delta > 0$ be a number such that $\Gamma(G, X_2)$ is δ-hyperbolic. We set

$$Y = \{g \in G \mid |g|_{X_2} \leqslant 4\delta + 1\}.$$

Since $G = \langle X_2 \rangle$, we have that $G = \langle Y \rangle$, i.e., (1). Statement (2) follows from Lemma 6.2 and Claim 1 (a).

Now we prove (3). The hyperbolicity and acylindricity of $\Gamma(G, Y \sqcup \mathcal{E})$ follows from the hyperbolicity and acylindricity of $\Gamma(G, X_2 \sqcup \mathcal{E})$ by Lemma 3.3. Analogously, the hyperbolicity and acylindricity of $\Gamma(G, Y)$ follows from the hyperbolicity and acylindricity of $\Gamma(G, X_2)$.

For (4) and (5), we first prove the following claim.

Claim 2. Let $a_i \in E_i$ be an arbitrary element of infinite order. Then a_i is loxodromic with respect to Y and $E_i = E_G(a_i)$.

Proof. By statement (3), $\Gamma(G, Y)$ is hyperbolic and acylindrical. Therefore any element of G is either elliptic or loxodromic with respect to Y. By statement (2), the space $(E_i, \widehat{d_i}^{Y})$ is locally finite, hence a_i cannot be elliptic with respect to Y.

Since $E_G(a_i)$ is the maximal virtually cyclic subgroup containing a_i, we have that $E_i \subseteq E_G(a_i)$. The inverse inclusion follows from Lemma 3.14 and the algebraic characterization of $E_G(a_i)$ in (3.1). ∎

We prove (4). Suppose that G is not virtually cyclic. Since $\Gamma(G, Y)$ is hyperbolic and acylindrical, it suffices to show that the action of G on $\Gamma(G, Y)$ is non-elementary. By Claim 2, this action has unbounded orbits. Thus, cases (a) and (b) of Theorem 3.9 are excluded. The remaining case (c) of this theorem says that G is acylindrically hyperbolic with respect to Y.

Finally, we prove (5). Suppose that $g \in G$ is an elliptic element with respect to $Y \cup \bigcup_{i=1}^{k} E_i$ and that g is non-commensurable with elements of $\bigcup_{i=1}^{k} E_i$ of infinite order. Assumptions of Lemma 6.1 are valid for $\Gamma(G, Y)$, the elements $a_1, \dots, a_k$ from Claim 2, and the element g. By this lemma, g is elliptic with respect to Y.

Since $\sup_{y \in Y} |y|_{X_2} \leqslant 4\delta + 1$, we conclude that g is elliptic with respect to X_2. By Remark 5.5, $\langle g \rangle$ is conjugated into Y. ∎

6.3. Quasi-morphisms

Let G be a group. Recall that a map $q : G \to \mathbb{R}$ is called a *quasi-morphism* if there exists a constant $\varepsilon > 0$ such that for every $f, g \in G$ we have that

$$\big|q(fg) - q(f) - q(g)\big| < \varepsilon.$$

For a quasi-morphism $q : G \to \mathbb{R}$, we define its *defect* $D(q)$ by

$$D(q) = \sup_{f,g \in G} \big|q(fg) - q(f) - q(g)\big|.$$

The quasi-morphism q is called *homogeneous* if $q(g^m) = m \cdot q(g)$ for all $g \in G$ and $m \in \mathbb{Z}$.

Remark 6.4. For every quasi-morphism $q : G \to \mathbb{R}$, there exists a unique *homogenous* quasi-morphism $\tilde{q} : G \to \mathbb{R}$ which lies at a bounded distance from q. This means that the following conditions are satisfied:

(1) $\tilde{q}(g^m) = m \cdot \tilde{q}(g)$ for all $g \in G$ and $m \in \mathbb{Z}$,

(2) there exists $C \geqslant 0$ such that $|q(g) - \tilde{q}(g)| \leqslant C$ for all $g \in G$.

The quasi-morphism $\tilde{q}$ is defined by the formula

$$\tilde{q}(g) = \lim_{m \to \infty} \frac{q(g^m)}{m}. \tag{6.5}$$

Lemma 6.5. *Suppose that $q : G \to \mathbb{R}$ is a homogeneous quasi-morphism. Then q is constant on each conjugacy class of elements of G. In particular, if $a, b \in G$ are two commensurable elements and $q(a) = 0$, then $q(b) = 0$.*

Proof. Let $u, g, h \in G$ be elements such that $u^{-1}gu = h$. Then

$$\big|q(g^m) - q(h^m)\big| \leqslant 2\big(q(u) + D(q)\big)$$

for all $m \in \mathbb{N}$. Hence

$$q(g) = \lim_{m \to \infty} \frac{q(g^m)}{m} = \lim_{m \to \infty} \frac{q(h^m)}{m} = q(h).$$ ∎

Statements (a) and (b) of the following corollary follow from [24, Theorem 4.2]. We show that statement (c) can be deduced from the proof of this theorem combined with Lemma 6.3.

Corollary 6.6. *Suppose that G is a group and $E_1, \ldots, E_k$ are infinite cyclic subgroups of G generated by elements $a_1, \ldots, a_k$, respectively. Suppose that $\{E_1, \ldots, E_k\} \hookrightarrow_h (G, X)$ and denote $E = \bigcup_{i=1}^{k} E_i$. Then for all $I \subseteq \{1, \ldots, k\}$, there exists a homogenous quasi-morphism $\tilde{q} : G \to \mathbb{R}$ such that the following hold:*

(a) $\tilde{q}(a_i) = 1$ *for all* $i \in I$,

(b) $\tilde{q}(a_i) = 0$ *for all* $i \notin I$,

(c) $\tilde{q}(g) = 0$ *for all* $g \in \mathrm{Ell}(G, X \cup E)$ *that are non-commensurable with elements of* E.

Proof. By Lemma 6.3, there exists a subset $Y \subseteq G$ such that $X \subseteq Y$ and the statements (1)–(5) of this lemma are satisfied. In particular, we have that $\{E_1, \ldots, E_k\} \hookrightarrow_h (G, Y)$. By [24, Theorem 4.2] applied to this hyperbolic embedding, there exists a quasi-morphism $q : G \to \mathbb{R}$ (possibly inhomogenous) such that the following hold:

(a′) $q(a_i^n) = n$ for all $i \in I$ and all $n \in \mathbb{Z}$,

(b′) $q(a_i^n) = 0$ for all $i \notin I$ and all $n \in \mathbb{Z}$.

Moreover, by construction of q in the proof of this theorem, we obtain

(c′) $q(y) = 0$ for all $y \in Y \setminus E$.

Let $\tilde{q}$ be the homogenous quasi-morphism obtained from q by (6.5). Then $\tilde{q}$ satisfies conditions (a), (b). We prove that $\tilde{q}$ satisfies condition (c).

Condition (c) is obviously valid for elements $g \in G$ of finite order. Suppose that $g \in G$ has infinite order and is elliptic with respect to $X \cup E$ and non-commensurable with elements of E. Then g is elliptic with respect to $Y \cup E$. By statement (5) of Lemma 6.3, there exists $u \in G$ such that $u^{-1}g^m u \in Y$ for all $m \in \mathbb{N}$. Since g is non-commensurable with elements of E, we have that $u^{-1}g^m u \in Y \setminus E$ for all $m \in \mathbb{N}$. By (c′), we obtain $q(u^{-1}g^m u) = 0$ for all $m \in \mathbb{N}$. Then $\tilde{q}(u^{-1}gu) = 0$. By Lemma 6.5, we have that $\tilde{q}(g) = 0$, i.e., (c). ∎

7. Equation $x^n y^m = a^n b^m$ in acylindrically hyperbolic groups

Proposition 7.1. *Let G be an acylindrically hyperbolic group. Suppose that a and b are two non-commensurable jointly special elements of G. Then there exists a generating set Y of G containing $\mathcal{E} = \langle a \rangle \cup \langle b \rangle$ and there exists a number $N \in \mathbb{N}$ such that for all $n, m > N$ the following holds:*

if (c, d) is a solution of the equation $x^n y^m = a^n b^m$, then one of the following holds:

(1) *c and d are loxodromic with respect to Y and $E_G(d) = E_G(c)$;*

(2) *c is loxodromic with respect to Y, d is elliptic, and $d^m \in E_G(c)$;*

(3) *d is loxodromic with respect to Y, c is elliptic, and $c^n \in E_G(d)$;*

(4) *c and d are elliptic with respect to Y and one of the following holds:*

(a) *c is conjugate to a and d is conjugate to b;*

(b) *c is conjugate to b, d is conjugate to a, and $|n-m| \leqslant N$.*

Proof. By [14, Theorem 6.8], there exists a subset $X_1 \subseteq G$ such that $\{\langle a\rangle, \langle b\rangle\} \hookrightarrow_h (G, X_1)$. Then, by [47, Theorem 5.4], there exists a subset $X \subseteq G$ such that $X_1 \subseteq X$ and the following conditions hold:

(1) $\{\langle a\rangle, \langle b\rangle\} \hookrightarrow_h (G, X)$,

(2) $\Gamma(G, X \sqcup \mathcal{E})$ is hyperbolic and acylindrical, where $\mathcal{E} = \langle a\rangle \cup \langle b\rangle$.

We set $Y = X \cup \mathcal{E}$. By Propositions 5.4 and 5.6 applied to (G, Y), there exist constants N_0 and N_1 satisfying conclusions of these propositions. Now we analyze the equation $c^n d^m = a^n b^m$.

Case 1. Suppose that c, d are loxodromic with respect to Y. Then, by Proposition 5.4, if $n, m \geqslant 2N_0$ and $E_G(c) \neq E_G(d)$, then $|c^n d^m|_Y > 2$. On the other hand, $|c^n d^m|_Y = |a^n b^m|_Y \leqslant 2$. Therefore, we have that $E_G(c) = E_G(d)$ if $n, m \geqslant 2N_0$.

Case 2. Suppose that c is loxodromic and d is elliptic with respect to Y. Then, by Proposition 5.6, the following holds: if $n \geqslant 2N_1$ and $d^m \notin E_G(c)$, then $|c^n d^m|_Y > 2$. On the other hand, $|c^n d^m|_Y = |a^n b^m|_Y \leqslant 2$. Therefore, we have that $d^m \in E_G(c)$ if $n \geqslant 2N_1$.

Case 3. Suppose that d is loxodromic and c is elliptic with respect to Y. Then, analogously to Case 2, we obtain $c^n \in E_G(d)$ if $m \geqslant 2N_1$.

Case 4. Suppose that c, d are elliptic with respect to Y.

In this case, we will use Corollary 6.6 applied to the hyperbolic embedding

$$\{\langle a\rangle, \langle b\rangle\} \hookrightarrow_h (G, X).$$

Let $q_a : G \to \mathbb{R}$ be a homogenous quasi-morphism such that $q_a(a) = 1$, $q_a(b) = 0$, and $q_a(g) = 0$ for all $g \in \mathrm{Ell}(G, Y)$ that are non-commensurable with a and b. By Lemma 6.5, if g is commensurable with b, then $q_a(g) = 0$. Thus, $q_a(g) = 0$ for all $g \in \mathrm{Ell}(G, Y)$ that are non-commensurable with a.

Analogously, let $q_b : G \to \mathbb{R}$ be a homogenous quasi-morphism such that $q_b(a) = 0$, $q_b(b) = 1$, and $q_b(g) = 0$ for all $g \in \mathrm{Ell}(G, Y)$ that are non-commensurable with b. We set

$$N_2 = 2\max\{D(q_a), D(q_b)\}.$$

and suppose that $n, m > N_2$. From the definition of a quasi-morphism, we have that

$$\big|q_a(a^n b^m) - q_a(a^n) - q_a(b^m)\big| \leqslant D(q_a).$$

Then

$$\big|q_a(a^n b^m) - n\big| \leqslant D(q_a). \tag{7.1}$$

Analogously,

$$\left|q_b(a^n b^m) - m\right| \leqslant D(q_b). \tag{7.2}$$

We also have that

$$\begin{aligned} \left|q_a(c^n d^m) - q_a(c^n) - q_a(d^m)\right| &\leqslant D(q_a), \\ \left|q_b(c^n d^m) - q_b(c^n) - q_b(d^m)\right| &\leqslant D(q_b). \end{aligned} \tag{7.3}$$

Subcase 4.1. Suppose that there exists $x \in \{a, b\}$ such that c and d are non-commensurable with x. Without loss of generality, we assume that $x = a$. Then (7.3) implies that

$$\left|q_a(c^n d^m)\right| \leqslant D(q_a). \tag{7.4}$$

It follows from (7.1) and (7.4) that $n \leqslant 2D(q_a) \leqslant N_2$. A contradiction.

Subcase 4.2. Suppose that c is commensurable with a and d is commensurable with b. The former means that there exist $u \in G$ and $s_1, s_2 \in \mathbb{Z} \setminus \{0\}$ such that $c^{s_1} = u^{-1}a^{s_2}u$. Then $c \in E_G(u^{-1}au)$. Since a is special, we have that $E_G(u^{-1}au) = \langle u^{-1}au\rangle$. Hence $c = u^{-1}a^s u$ for some $s \in \mathbb{Z} \setminus \{0\}$. Then (7.3) implies that

$$\left|q_a(c^n d^m) - sn\right| \leqslant D(q_a).$$

Using (7.1), we deduce that

$$|n - sn| \leqslant 2D(q_a) \leqslant N_2.$$

Since $n > N_2$, we have that $s = 1$; i.e., $c = u^{-1}au$. Analogously, d and b are conjugate.

Subcase 4.3. Suppose that c is commensurable with b and d is commensurable with a. The latter means that there exist $v \in G$ and $t_1, t_2 \in \mathbb{Z} \setminus \{0\}$ such that $d^{t_1} = v^{-1}a^{t_2}v$. Then $d \in E_G(v^{-1}av) = \langle v^{-1}av\rangle$. Hence $d = v^{-1}a^t v$ for some $t \in \mathbb{Z} \setminus \{0\}$. Then (7.3) implies that

$$\left|q_a(c^n d^m) - tm\right| \leqslant D(q_a).$$

Using (7.1), we deduce that

$$|n - tm| \leqslant 2D(q_a) \leqslant N_2. \tag{7.5}$$

Analogously, using the commensurability of c and b, we deduce that

$$|m - sn| \leqslant N_2 \tag{7.6}$$

for some $s \in \mathbb{Z} \setminus \{0\}$. It follows from $n, m > N_2$ and (7.5), (7.6) that $s, t \geqslant 1$. Without loss of generality, we assume that $m \geqslant n$. Then

$$m > N_2 \geqslant |n - tm| = tm - n \geqslant (t-1)m.$$

From this we deduce that $t = 1$ and $|n - m| \leqslant N_2$.

Taking into account all considered cases, we can set $N = \max\{2N_0, 2N_1, \lceil N_2\rceil\}$. ■

The following lemma says that if n, m in Proposition 7.1 have a certain common divisor, then only the subcase (a) in the conclusion of this proposition is possible; i.e., c is conjugate to a and d is conjugate to b. A description of these conjugates will be given in Corollary 9.5.

Lemma 7.2. *Let G be an acylindrically hyperbolic group. Suppose that $a, b \in G$ are two non-commensurable jointly special elements. Then there exists $\ell \in \mathbb{N}$ such that for all $n, m \in \ell\mathbb{N}$, $n \neq m$, the following holds:*

if $a^n b^m = c^n d^m$, where $c, d \in G$, then c is conjugate to a and d is conjugate to b.

Proof. We use the generating set Y as in the proof of Proposition 7.1. In particular, $\langle a\rangle \cup \langle b\rangle \subseteq Y$. We also use the homogenous quasi-morphisms $q_a : G \to \mathbb{R}$ and $q_b : G \to \mathbb{R}$ defined there. In particular, $q_a(a) = 1$, $q_a(b) = 0$, $q_b(a) = 0$, and $q_b(b) = 1$.

• By Lemma 3.7, there exists a number $L \in \mathbb{N}$ such that for every loxodromic element $g \in G$ (with respect to Y), the elementary subgroup $E_G(g)$ contains a normal cyclic subgroup of index L.

• By Lemma 4.7, there exist $\varkappa \geqslant 1$, $\varepsilon_0 \geqslant 0$, and $n_0 \in \mathbb{N}$ such that for any loxodromic (with respect to Y) element $g \in G$ and any $n \geqslant n_0$ we have that

$$|g^n|_Y \geqslant \frac{1}{\varkappa} n - \varepsilon_0.$$

• We set

$$\ell = LMNn_0,$$

where $M = \lceil (2+\varepsilon_0)\varkappa \rceil + 1$ and $N \in \mathbb{N}$ is the number from Proposition 7.1. Recall that in the proof of this proposition we defined N so that we have that

$$N \geqslant 2 \max\big\{D(q_a), D(q_b)\big\}.$$

Suppose that $a^n b^n = c^n d^m$, where $n, m \in \ell\mathbb{N}$, $n \neq m$. We analyze cases in the conclusion of Proposition 7.1.

(1) c and d are loxodromic with respect to Y and $E_G(c) = E_G(d)$.

By Lemma 3.7, there exists $z \in E_G(c)$ such that c^L and d^L are powers of z, say $c^L = z^s$ and $d^L = z^t$. Then $c^n d^m = z^{sn+mt/L}$. Observe that $sn + mt \neq 0$; otherwise $a^n b^m = c^n d^m = 1$ that is impossible by non-commensurability of a and b. Since $\ell = MLNn_0$ is a divisor of n and m, we have (using Lemma 4.7) that

$$2 \geqslant |a^n b^m|_Y = |z^{sn+mt/L}|_Y \geqslant \frac{1}{\varkappa}\frac{|sn+mt|}{L} - \varepsilon_0 \geqslant \frac{1}{\varkappa}\frac{\ell}{L} - \varepsilon_0 \geqslant \frac{1}{\varkappa} M - \varepsilon_0 > 2.$$

A contradiction.

(2) c is loxodromic with respect to Y, d is elliptic, and $d^m \in E_G(c)$.

By definition of L, the group $E_G(c)$ contains a normal infinite cyclic subgroup C of index L. Hence $c^L \in C$ and $d^{-m} c^L d^m = c^{\pm L}$. Since L is a divisor of n, this implies that

$$d^{-m} c^n d^m = c^{\pm n}.$$

The group C is generated by a loxodromic element, because it contains the loxodromic element c^L. Since $d^{mL} \in C$ and d is elliptic, we have that

$$d^{mL} = 1.$$

Subcase 1. Suppose that $d^{-m}c^n d^m = c^n$. Then

$$(a^n b^m)^L = (c^n d^m)^L = c^{nL} d^{mL} = c^{nL}.$$

It follows that

$$2L \geqslant \left|(a^n b^m)^L\right|_Y = |c^{nL}|_Y \geqslant \frac{1}{\varkappa} nL - \varepsilon_0 \geqslant \frac{1}{\varkappa} ML - \varepsilon_0 L > 2L.$$

A contradiction.

Subcase 2. Suppose that $d^{-m}c^n d^m = c^{-n}$. Then

$$(a^n b^m)^2 = (c^n d^m)^2 = d^{2m}.$$

Recalling that $d^{mL} = 1$, we deduce that $(a^n b^m)^{2L} = 1$. Since homogeneous quasi-morphisms vanish on periodic elements, we have that $q_a(a^n b^m) = 0$. In view of

$$\left|q_a(a^n b^m) - q_a(a^n) - q_a(b^m)\right| \leqslant D(q_a),$$

this implies that $n \leqslant D(q_a) \leqslant N < \ell$. A contradiction.

(3) d is loxodromic with respect to Y, c is elliptic, and $c^n \in E_G(d)$.
This case is impossible by the same reason as the previous one.
(4) c and d are elliptic with respect to Y and one of the following holds:

(a) c is conjugate to a and d is conjugate to b;

(b) c is conjugate to b, d is conjugate to a, and $|n - m| \leqslant N$.

The case (b) is impossible since $n, m \in \ell\mathbb{N}$, $n \neq m$, and N is a proper divisor of ℓ. Thus, only the case (a) is possible. ∎

Remark 7.3. The condition on $\gcd(n,m)$ in Lemma 7.2 cannot be replaced by the condition that n, m are sufficiently large. Indeed, if $\gcd(n,m) = 1$, then the equation $x^n y^m = a^n b^m$ in the free group $F(a,b)$ of rank 2 has infinitely many solutions

$$(x, y) = \big((a^n b^m)^s, (a^n b^m)^t\big),$$

where s, t are integers satisfying $ns + mt = 1$. None of the components of these solutions is conjugate to a or b.

8. Isolated components in geodesic polygons

In the following proof, we use Proposition 4.14 from [14]. Since this proposition and accompanied definitions are crucial in the following proof, we recall them here.

Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, and X a symmetrized subset of G. We assume that X together with $\{H_\lambda\}_{\lambda\in\Lambda}$ generates G. Let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$.

Definition 8.1 (see [14, Definition 4.5]). Let q be a path in the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$. A (non-trivial) subpath p of q is called an *H_λ-subpath* if the label of p is a word in the alphabet H_λ. An H_λ-subpath p of q is an *H_λ-component* if p is not contained in a longer subpath of q with this property. Two H_λ-components p_1, p_2 of a path q in $\Gamma(G, X \sqcup \mathcal{H})$ are called *connected* if there exists a path γ in $\Gamma(G, X \sqcup \mathcal{H})$ that connects some vertex of p_1 to some vertex of p_2, and $\mathbf{Lab}(\gamma)$ is a word consisting only of letters from H_λ.

Note that we can always assume that γ has length at most 1 as every element of H_λ is included in the set of generators. An H_λ-component p of a path q in $\Gamma(G, X \sqcup \mathcal{H})$ is *isolated* if it is not connected to any other component of q.

Recall that definitions of a weakly hyperbolic group and of a relative metric $\hat{d}_\lambda$ on H_λ were given in Section 3. Given a path p in $\Gamma(G, X \sqcup \mathcal{H})$, the canonical image of $\mathbf{Lab}(p)$ in G is denoted by $\mathbf{Lab}_G(p)$.

Definition 8.2 (see [14, Definition 4.13]). Let $\varkappa \geqslant 1$, $\varepsilon \geqslant 0$, and $n \geqslant 2$. Let $\mathcal{P} = p_1 \cdots p_n$ be an n-gon in $\Gamma(G, X \sqcup \mathcal{H})$ and let I be a subset of the set of its sides $\{p_1, \dots, p_n\}$ such that

(1) each side $p_i \in I$ is an isolated H_{λ_i}-component of $\mathcal{P}$ for some $\lambda_i \in \Lambda$,

(2) each side $p_i \notin I$ is a $(\varkappa, \varepsilon)$-quasi-geodesic.

We denote $s(\mathcal{P}, I) = \sum_{p_i \in I} \hat{d}_{\lambda_i}(1, \mathbf{Lab}_G(p_i))$.

Proposition 8.3 (see [14, Proposition 4.14]). *Suppose that G is weakly hyperbolic relative to X and $\{H_\lambda\}_{\lambda\in\Lambda}$. Then for any $\varkappa \geqslant 1$, $\varepsilon \geqslant 0$, there exists a constant $D(\varkappa, \varepsilon) > 0$ such that for any n-gon $\mathcal{P}$ in $\Gamma(G, X \sqcup \mathcal{H})$ and any subset I of the set of its sides satisfying conditions of Definition 8.2, we have that $s(\mathcal{P}, I) \leqslant D(\varkappa, \varepsilon)n$.*

Corollary 8.4. *Suppose that G is weakly hyperbolic relative to X and $\{H_\lambda\}_{\lambda\in\Lambda}$. Let $\mathcal{P} = p_1 p_2 p_3$ be a geodesic triangle in $\Gamma(G, X \sqcup \mathcal{H})$, where p_3 is an isolated component of $\mathcal{P}$ or a degenerate path. Suppose that q is an H_λ-component of $\mathcal{P}$ of the form $q = q_1 q_2$, where q_1 is a terminal subpath of p_1 and q_2 is an initial subpath of p_2. Then $\hat{d}_\lambda(1, \mathbf{Lab}_G(q)) \leqslant 4D(1, 0)$.*

Proof. Let q_1' and q_2' be paths such that $p_1 = q_1' q_1$ and $p_2 = q_2 q_2'$. Consider the 4-gon $\mathcal{P}' = q_1' q q_2' p_3$. The H_λ-component q of $\mathcal{P}'$ cannot be connected to p_3 by assumption and it cannot be connected to an H_λ-component of q_1' or q_2', since p_1 and p_2 are geodesics. Therefore, q is an isolated component in $\mathcal{P}'$, and we are done by Proposition 8.3. ∎

9. Perfect equations of kind $x^n y^m = a^n b^m$ in acylindrically hyperbolic groups

The first proposition in this section describes conjugators in the conclusion of Lemma 7.2. From this proposition and the lemma we deduce Corollary 9.5, which gives a clear description of solutions of the equation $x^n y^m = a^n b^m$ in acylindrically hyperbolic groups in the case where a, b are non-commensurable and jointly special, and n, m have a certain common divisor.

Proposition 9.1. *Let G be an acylindrically hyperbolic group. Suppose that $a, b \in G$ are two non-commensurable jointly special elements. Then there exists $N \in \mathbb{N}$ such that for any $n, m > N$ and any $u, v \in G$ satisfying*

$$(u^{-1}a^n u)(v^{-1}b^m v) = a^n b^m,$$

there exists $r \in \mathbb{Z}$ such that $u \in \langle a\rangle(a^n b^m)^r$ and $v \in \langle b\rangle(a^n b^m)^r$.

We give a proof of this proposition after introducing some auxiliary definitions and lemmas. These lemmas will be proved at the end of this section.

We set $H_a = \langle a\rangle$, $H_b = \langle b\rangle$, and $\mathcal{H} = H_a \sqcup H_b$. By [14, Theorem 6.8], there exists a symmetrized subset X of G such that $\{H_a, H_b\} \hookrightarrow_h (G, X)$. In particular, G is weakly hyperbolic relative to X and $\{H_a, H_b\}$. The associated relative metrics on H_a and on H_b are denoted by $\hat{d}_a$ and $\hat{d}_b$, respectively (see Definition 3.11).

A path p in $\Gamma(G, X \sqcup \mathcal{H})$ is called an *a-path* (respectively, a *b-path*) if the label of p is a word in letters from H_a (respectively, from H_b).

For an element $g = a^i \in H_a$, the number $|i|$ is called the *a-length* of g. Analogously we define the *b-length* of an element $g \in H_b$.

A word w in the alphabet $X \sqcup \mathcal{H}$ is called *geodesic* if it has minimal length among all words, representing the same element in G as w. In particular, w does not contain two consecutive letters which both lie in H_a or both lie in H_b.

Definition of the complexity of a word. Given a geodesic word w in the alphabet $X \sqcup \mathcal{H}$, we define its *complexity* $\mathrm{Compl}(w)$ as the pair $(|w|_{X\sqcup\mathcal{H}}, s)$, where s is the sum of a-lengths of its H_a-components plus the sum of b-lengths of its H_b-components. We order the pairs lexicographically: $(t_1', t_2') \prec (t_1, t_2)$ if $t_1' < t_1$ or $t_1 = t_1'$ and $t_2' < t_2$.

For an element $g \in G$, we define its complexity $\mathrm{Compl}(g)$ as the minimum of complexities of geodesic words w in the alphabet $X \sqcup \mathcal{H}$ representing g. Note that there is only finitely many elements in each descending chain of complexities.

For any pair (u, v) of elements of G, we define its complexity as follows:

$$\mathrm{Compl}(u, v) = \big(\mathrm{Compl}(u), \mathrm{Compl}(v)\big).$$

We write

$$\mathrm{Compl}(u_1, v_1) < \mathrm{Compl}(u, v)$$

if $\mathrm{Compl}(u_1) \prec \mathrm{Compl}(u)$ and $\mathrm{Compl}(v_1) \prec \mathrm{Compl}(v)$.

Definition of the number N. We have observed that G is weakly hyperbolic relative to X and $\{H_a, H_b\}$. Let $D = D(1, 0)$ be the constant from Proposition 8.3 for parameters $(\varkappa, \varepsilon) = (1, 0)$. We set

$$N_a = \max\big\{i \mid \hat{d}_a(1, a^i) \leqslant 9D\big\}, \quad N_b = \max\big\{i \mid \hat{d}_b(1, b^i) \leqslant 9D\big\}.$$

Since the spaces $(H_a, \hat{d}_a)$ and $(H_b, \hat{d}_b)$ are locally finite, the numbers N_a and N_b are finite. We set

$$N = 4 \cdot \max\{N_a, N_b\} + 1.$$

Then $N \in \mathbb{N}$. We will prove that this N satisfies Proposition 9.1. For the rest of the proof we assume that G, a, and b satisfy assumptions of this proposition and that $n, m > N$.

Consider the following equations in variables x, y:

$$(x^{-1}a^n x)(y^{-1}b^m y) = a^n b^m, \tag{9.1}$$

$$(x^{-1}a^n x)(y^{-1}b^m y) = b^m a^n. \tag{9.2}$$

Lemma 9.2. *Suppose that* (u, v) *is a solution of* (9.1) *such that* $u \notin \langle a\rangle$, $v \notin \langle b\rangle$. *We set* $(u_1, v_1) := (ua^n, va^n)$ *and* $(u_2, v_2) := (ub^{-m}, vb^{-m})$. *Then* (u_1, v_1) *and* (u_2, v_2) *are solutions of* (9.2), *and we have that*

$$\mathrm{Compl}(u_1, v_1) < \mathrm{Compl}(u, v) \quad \textit{or} \quad \mathrm{Compl}(u_2, v_2) < \mathrm{Compl}(u, v).$$

The following lemma is dual to Lemma 9.2.

Lemma 9.3. *Suppose that* (p, q) *is a solution of* (9.2) *such that* $p \notin \langle a\rangle$, $q \notin \langle b\rangle$. *We set* $(p_1, q_1) := (pa^{-n}, qa^{-n})$ *and* $(p_2, q_2) := (pb^m, qb^m)$. *Then* (p_1, q_1) *and* (p_2, q_2) *are solutions of* (9.1), *and we have that*

$$\mathrm{Compl}(p_1, q_1) < \mathrm{Compl}(p, q) \quad \textit{or} \quad \mathrm{Compl}(p_2, q_2) < \mathrm{Compl}(p, q).$$

Proofs of these lemmas will be given later.

Proof of Proposition 9.1. Suppose that (u, v) is a solution of (9.1). If $u \in \langle a\rangle$, then $v^{-1}b^m v = b^m$, hence $v \in E_G(b) = \langle b\rangle$, and we are done. Analogously, if $v \in \langle b\rangle$, then $u \in \langle a\rangle$, and we are done. Thus, we may assume that $u \notin \langle a\rangle$ and $v \notin \langle b\rangle$.

By Lemma 9.2, (ua^n, va^n) and (ub^{-m}, vb^{-m}) are solutions of (9.2) and we have that $\mathrm{Compl}(ua^n, va^n) < \mathrm{Compl}(u, v)$ or $\mathrm{Compl}(ub^{-m}, vb^{-m}) < \mathrm{Compl}(u, v)$. We consider only the first case

$$\mathrm{Compl}(ua^n, va^n) < \mathrm{Compl}(u, v), \tag{9.3}$$

since the second case can be considered analogously.

We may assume that (ua^n, va^n) satisfies assumption of Lemma 9.3. Indeed, the first assumption $ua^n \notin \langle a\rangle$ is satisfied since $u \notin \langle a\rangle$. Suppose that the second assumption is not satisfied; i.e., $va^n \in \langle b\rangle$. Then $v \in \langle b\rangle \cdot (a^n b^m)^{-1}$, and we deduce from (9.1) that $u^{-1}a^n u \cdot a^n b^m a^{-n} = a^n b^m$. It follows that $ua^n b^m \in E_G(a) = \langle a\rangle$. Hence $u \in \langle a\rangle(a^n b^m)^{-1}$, and we are done.

Thus, we assume that (ua^n, va^n) satisfies assumption of Lemma 9.3. By (9.3), the first case in the conclusion of this lemma cannot happen. Therefore, we have the second case; i.e., $(ua^n b^m, va^n b^m)$ satisfies (9.1) and

$$\mathrm{Compl}(ua^n b^m, va^n b^m) < \mathrm{Compl}(ua^n, va^n).$$

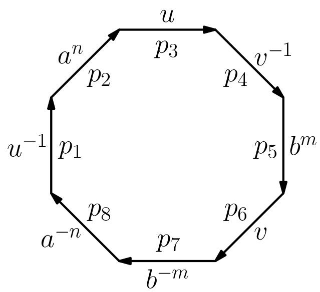

Figure 13

This formula and (9.3) imply that $\mathrm{Compl}(ua^n b^m, va^n b^m) < \mathrm{Compl}(u, v)$, and the statement of Proposition 9.1 follows by induction. ∎

Proof of Lemma 9.2. By an abuse of notation, for any element $w \in G$, we also denote by w a geodesic word of minimal complexity among all geodesic words in $X \sqcup \mathcal{H}$ representing the element w.

Let (u, v) be a solution of (9.1) satisfying assumption of Lemma 9.2. Obviously, $(u_1, v_1) := (ua^n, va^n)$ and $(u_2, v_2) := (ub^{-m}, vb^{-m})$ are solutions of (9.2). Thus, it suffices to prove that one of the following holds:

(a) both words u and v end with a power of a which is smaller than $-n/2$;

(b) both words u and v end with a power of b which is larger than $m/2$.

Since $n, m > N$, we have that

$$n > 4N_a = 4 \max \{ i \mid \hat{d}_a(1, a^i) \leqslant 9D \}, \tag{9.4}$$

$$m > 4N_b = 4 \max \{ i \mid \hat{d}_b(1, b^i) \leqslant 9D \}. \tag{9.5}$$

Let $\mathcal{P} = p_1 p_2 \cdots p_8$ be a geodesic 8-gon in $\Gamma(G, X \sqcup \mathcal{H})$ with sides p_i labeled by consecutive syllables of the word

$$u^{-1} a^n u\, v^{-1} b^m v b^{-m} a^{-n}.$$

More precisely, $\mathbf{Lab}(p_1) = u^{-1}$, $\mathbf{Lab}(p_2) = a^n, \ldots,$ and $\mathbf{Lab}(p_8) = a^{-n}$. Observe that the sides p_2, p_5, p_7, p_8 of $\mathcal{P}$ are edges labeled by powers of a and b; see Figure 13.

By assumption of lemma, the words u and v are nonempty. Write $u = a^i u'$ and $v = b^j v'$, where $i, j \in \mathbb{Z}$, the first letter of u' does not lie in H_a, and the first letter of v' does not lie in H_b. As (u, v), the pair (u', v') is also a solution of (9.1) satisfying assumption of lemma. Obviously, if we prove it for (u', v'), then we prove it for (u, v) too.

Thus, we may assume that the first letter of u is not a nontrivial power of a, and the first letter of v is not a nontrivial power of b.

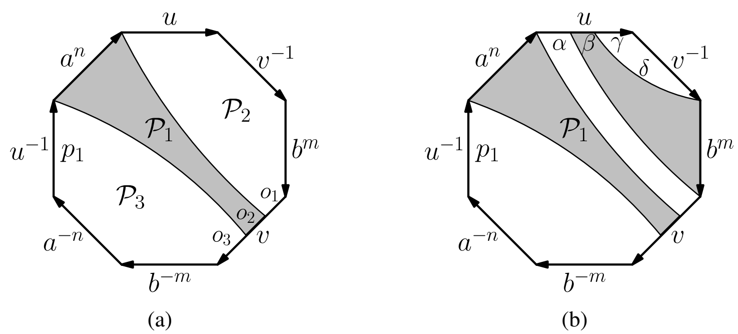

Figure 14

Then p_2 is an H_a-component of $\mathcal{P}$ and p_5 is an H_b-component of $\mathcal{P}$. It follows from (9.4) and (9.5) that

$$\hat{d}_a\big(1, \mathbf{Lab}_G(p_2)\big) = \hat{d}_a(1, a^n) > 9D,$$
$$\hat{d}_b\big(1, \mathbf{Lab}_G(p_5)\big) = \hat{d}_b(1, b^{-m}) = \hat{d}_b(1, b^m) > 9D.$$

These inequalities and Proposition 8.3 imply that the components p_2 and p_5 are not isolated in $\mathcal{P}$.

Then p_2 is connected to a component of p_4 or p_6 and p_5 is connected to a component of p_1 or p_3.

Case 1. Suppose that p_2 is connected to an H_a-component of p_6.

Then there exists a geodesic rectangle $\mathcal{P}_1 = p_2 r_1 o_2 r_2$, where o_2 is an H_a-component of p_6 (see Figure 14(a)). Let o_1, o_3 be subpaths of p_6 such that $p_6 = o_1 o_2 o_3$. We consider two complementary geodesic 5-gons $\mathcal{P}_2 = p_3 p_4 p_5 o_1 \overline{r_1}$ and $\mathcal{P}_3 = p_1 \overline{r_2} o_3 p_7 p_8$.

The path p_5 is not an isolated H_b-component of $\mathcal{P}_2$ (by Proposition 8.3 and using the inequality $\hat{d}_b(1, \mathbf{Lab}_G(p_5)) > 9D$). Therefore, p_5 is connected to some H_b-component β of p_3 (see Figure 14(b)). Then $p_3 = \alpha\beta\gamma$ for some subpaths α, γ of p_3. Let δ be a geodesic b-path from $(p_5)_-$ to β_+. We consider the triangle $\Delta_1 = \gamma p_4 \delta$. The path δ is an isolated H_b-component of Δ_1 or a degenerate path.

In the rest of the proof we use the following notation. For any nontrivial path p in the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$, let p° and $p^\bullet$ denote the first and the last edges of p, respectively.

Let t be the H_b-component of $\mathcal{P}$ containing the edge p_7. Then t is contained in $o_3 p_7$ (see Figure 15).

Subcase 1.1. Suppose that t is not connected to a component of p_1.

Then t is isolated in the 5-gon $\mathcal{P}_3$.

By Proposition 8.3 applied to $\mathcal{P}_3$, we obtain

$$\hat{d}_b\big(1, \mathbf{Lab}_G(t)\big) \leqslant 5D. \tag{9.6}$$

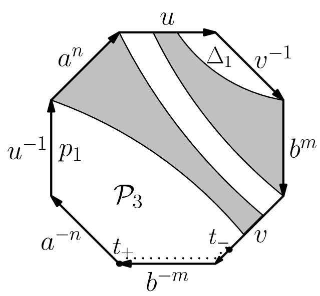

Figure 15

By (9.5), we have that

$$\hat{d}_b\big(1, \mathbf{Lab}_G(p_7)\big) = \hat{d}_b(1, b^{-m}) > 9D.$$

Therefore, p_7 is a proper subpath of t, and hence $t = p_6^\bullet p_7$ with

$$\hat{d}_b\big(1, \mathbf{Lab}_G(p_6^\bullet)\big) > 4D.$$

Since $\mathbf{Lab}_G(p_6) = \mathbf{Lab}_G(\overline{p_4})$, we have that $\mathbf{Lab}_G(p_6^\bullet) = (\mathbf{Lab}_G(p_4^\circ))^{-1}$. Hence

$$\hat{d}_b\big(1, \mathbf{Lab}_G(p_4^\circ)\big) > 4D. \tag{9.7}$$

We claim that the b-path p_4° cannot be a component of Δ_1. Indeed, if it were, we could apply Corollary 8.4 to the triangle Δ_1, its side δ (which is an isolated H_b-component of Δ_1 or a degenerate path), and to the component p_4°, and get a contradiction to (9.7).

Hence $p_3^\bullet p_4^\circ$ is a component of Δ_1 and, by Corollary 8.4, we have that

$$\hat{d}_b\big(1, \mathbf{Lab}_G(p_3^\bullet p_4^\circ)\big) \leqslant 4D. \tag{9.8}$$

Now we estimate $y, z \in \mathbb{Z}$ such that $\mathbf{Lab}_G(p_3^\bullet) = b^y$ and $\mathbf{Lab}_G(p_6^\bullet) = b^z$. Since

$$\mathbf{Lab}_G(t) = \mathbf{Lab}_G(p_6^\bullet p_7) = b^{z-m},$$

$$\mathbf{Lab}_G(p_3^\bullet p_4^\circ) = \mathbf{Lab}_G(p_3^\bullet)\big(\mathbf{Lab}_G(p_6^\bullet)\big)^{-1} = b^{y-z},$$

we deduce from (9.6) and (9.8) that

$$|z - m| \leqslant N_b \quad \text{and} \quad |y - z| \leqslant N_b.$$

Since $m > 4N_b$, we have that

$$z \geqslant m - N_b > \frac{3}{4}m \quad \text{and} \quad y \geqslant z - N_b > \frac{1}{2}m.$$

Observe that the word $u = \mathbf{Lab}(p_3)$ ends with $\mathbf{Lab}(p_3^\bullet) = b^y$ and the word $v = \mathbf{Lab}(p_6)$ ends with $\mathbf{Lab}(p_6^\bullet) = b^z$. Thus, by previous estimations, both u and v end with a power of b which is larger than $m/2$, and we are done.

Subcase 1.2. Suppose that t is connected to some component of p_1.

Let τ be a geodesic b-path from the initial point of this component to t_+ (see Figure 16). We consider the triangle $\Delta_2 = p_8\sigma\tau$, where σ is the initial path of p_1 with the

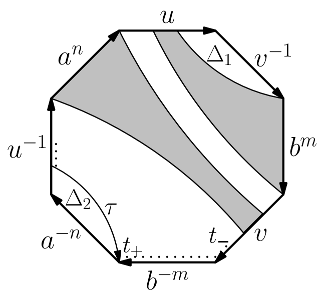

Figure 16

endpoint $\sigma_+ = \tau_-$. The path τ is either an isolated H_b-component of Δ_2 or a degenerate path. Let q be an H_a-component of Δ_2 containing p_8. Then q is also isolated in Δ_2.

By Proposition 8.3, we have that

$$\hat{d}_a\big(1, \mathbf{Lab}_G(q)\big) \leqslant 3D. \tag{9.9}$$

By (9.4), we have that

$$\hat{d}_a\big(1, \mathbf{Lab}_G(p_8)\big) = \hat{d}_a(1, a^{-n}) > 9D.$$

Therefore, p_8 is a proper subpath of q, and hence $q = p_8 p_1^\circ$ with

$$\hat{d}_a\big(1, \mathbf{Lab}_G(p_1^\circ)\big) > 6D.$$

Since $\mathbf{Lab}_G(p_1) = \mathbf{Lab}_G(\overline{p_3})$, we have that $\mathbf{Lab}_G(p_1^\circ) = (\mathbf{Lab}_G(p_3^\bullet))^{-1}$. Hence

$$\hat{d}_a\big(1, \mathbf{Lab}_G(p_3^\bullet)\big) > 6D.$$

By Corollary 8.4 applied to the triangle Δ_1, the a-path $p_3^\bullet$ cannot be a component of Δ_1. Hence $p_3^\bullet p_4^\circ$ is a component of Δ_1 and, by this corollary, we have that

$$\hat{d}_a\big(1, \mathbf{Lab}_G(p_3^\bullet p_4^\circ)\big) \leqslant 4D. \tag{9.10}$$

Now we estimate $z, y \in \mathbb{Z}$ such that $\mathbf{Lab}_G(p_1^\circ) = a^z$ and $\mathbf{Lab}_G(p_4^\circ) = a^y$. Since

$$\mathbf{Lab}_G(q) = \mathbf{Lab}_G(p_8 p_1^\circ) = a^{-n+z},$$
$$\mathbf{Lab}_G(p_3^\bullet p_4^\circ) = \big(\mathbf{Lab}_G(p_1^\circ)\big)^{-1}\mathbf{Lab}_G(p_4^\circ) = a^{-z+y},$$

we deduce from (9.9) and (9.10) that

$$|-n+z| \leqslant N_a \quad \text{and} \quad |-z+y| \leqslant N_a.$$

Since $n > 4N_a$, we have that

$$z \geqslant n - N_a > \frac{3}{4}n \quad \text{and} \quad y \geqslant z - N_a > \frac{1}{2}n.$$

Observe that the word $u = \mathbf{Lab}(\overline{p_1})$ ends with $(\mathbf{Lab}(p_1^\circ))^{-1} = a^{-z}$ and the word $v = \mathbf{Lab}(\overline{p_4})$ ends with $(\mathbf{Lab}(p_4^\circ))^{-1} = a^{-y}$. Thus, by previous estimations, both u and v end with a power of a which is smaller than $-n/2$, and we are done.

Case 2. Suppose that p_2 is connected to an H_a-component of p_4.

Arguing as in Case 1, we can prove that p_5 is connected to a component of p_1. After that, renaming a, b, u, v, n, m by b^{-1}, a^{-1}, v, u, m, n, respectively, we reduce to Case 1. ∎

To simplify formulations, we introduce the following definition.

Definition 9.4. Let $g \in G$ and $n, m \in \mathbb{Z}$. The equation $x^n y^m = g$ in variables x, y is called *perfect* if it has a solution (x_0, y_0) in G and any solution of this equation has the form $(x_0^{g^\alpha}, y_0^{g^\alpha})$ for some $\alpha \in \mathbb{Z}$.

The following corollary directly follows from Lemma 7.2 and Proposition 9.1.

Corollary 9.5. *Let G be an acylindrically hyperbolic group. Suppose that $a, b \in G$ are two non-commensurable jointly special elements. Then there exists a number $\ell = \ell(a, b) \in \mathbb{N}$ such that for all $n, m \in \ell\mathbb{N}$, $n \neq m$, the equation $x^n y^m = a^n b^m$ is perfect.*

10. Special generating sets for finitely generated acylindrically hyperbolic groups

The main purpose of this section is Proposition 10.6, which is essentially used in the proof of Theorem A. Proposition 10.7 is only used in the proof of Corollary E. Other statements of this section will be used in Sections 11–13.

The following lemma proven in [14] is crucial in many proofs. Therefore, we reproduce it here.

Lemma 10.1 (see [14, Lemma 4.21]). *Let G be a group weakly hyperbolic relative to X and $\{H_\lambda\}_{\lambda \in \Lambda}$ and let $\mathcal{W}$ be the set consisting of all words U in $X \sqcup \mathcal{H}$ such that*

- (W$_1$) *U contains no subwords of type xy, where $x, y \in X$,*
- (W$_2$) *if U contains a letter $h \in H_\lambda$ for some $\lambda \in \Lambda$, then $\hat{d}_\lambda(1, h) > 50D$, where $D = D(1, 0)$ is given by Proposition 8.3,*
- (W$_3$) *if $h_1 x h_2$ (respectively, $h_1 h_2$) is a subword of U, where $x \in X$, $h_1 \in H_\lambda$, $h_2 \in H_\mu$, then either $\lambda \neq \mu$ or the element represented by x in G does not belong to H_λ (respectively, $\lambda \neq \mu$).*

Then the following hold.

- (a) *Every path in $\Gamma(G, X \sqcup \mathcal{H})$ labeled by a word from $\mathcal{W}$ is $(4, 1)$-quasi-geodesic.*
- (b) *For every $\varepsilon > 0$ and every integer $K > 0$, there exist $R = R(\varepsilon, K) > 0$ satisfying the following condition. Let p, q be two paths in $\Gamma(G, X \sqcup \mathcal{H})$ such that $\ell(p) \geqslant R$, $\mathbf{Lab}(p), \mathbf{Lab}(q) \in \mathcal{W}$, and p, q are oriented ε-close; i.e.,*

$$\max\left\{d(p_-, q_-), d(p_+, q_+)\right\} \leqslant \varepsilon.$$

Then there exist K consecutive components of p which are connected to K consecutive components of q.

Corollary 10.2. *Suppose that G is a group, $Y \subset G$ a subset, and $E_1, \dots, E_k$ are subgroups of G such that*

$$\{E_1, \dots, E_k\} \hookrightarrow_h (G, Y).$$

Suppose that $a_1, \dots, a_k$ are elements of infinite order from $E_1, \dots, E_k$, respectively. Then there exists $N \in \mathbb{N}$ such that if $n_1, \dots, n_k \geqslant N$, then every cyclically reduced word W of syllable length at least 2 in the alphabet $\{a_1^{n_1}, \dots, a_k^{n_k}\}^{\pm}$ represents a loxodromic element of G with respect to $Y \sqcup \mathcal{E}$, where $\mathcal{E} = E_1 \sqcup \dots \sqcup E_k$. In particular, $\langle a_1^{n_1}, \dots, a_k^{n_k}\rangle$ is a free group of rank k.

Moreover, if $\langle Y \rangle = G$, then each word W as above represents a loxodromic element of G with respect to Y.

Proof. For $i = 1, \dots, k$, let $\hat{d}_i$ be the metric on E_i associated with the embedding

$$\{E_1, \dots, E_k\} \hookrightarrow_h (G, Y).$$

Let

$$N_i := \max \left\{n \in \mathbb{N} \mid \hat{d}_i(1, a_i^n) \leqslant 50D\right\},$$

where $D = D(1, 0)$ is given by Proposition 8.3. We claim that

$$N = \max\{N_i \mid i = 1, \dots, k\} + 1$$

satisfies the corollary. Let $n_1, \dots, n_k \geqslant N$ and let W be a cyclically reduced word in the alphabet $\{a_1^{n_1}, \dots, a_k^{n_k}\}^{\pm}$ such that the syllable length of W is at least 2. Using conjugation, we may assume that the first and the last letters of W do not coincide and are not inverse to each other.

Let U be the word in the alphabet $\langle a_1^{n_1}\rangle \sqcup \dots \sqcup \langle a_k^{n_k}\rangle$ obtained from W by replacing each syllable of W of kind $\underbrace{a_i^{\pm n_i} \cdots a_i^{\pm n_i}}_{s}$ by the unique letter $a_i^{\pm s n_i}$. Then U^m satisfies conditions (W_1)–(W_3) of Lemma 10.1 for any $m \in \mathbb{N}$. Let p_m be the path in $\Gamma(G, Y \sqcup \mathcal{E})$ labeled by U^m, such that $(p_m)_- = 1$. Since, by this lemma, the path p_m is $(4, 1)$-quasi-geodesic, we have that $d(1, (p_m)_+) \geqslant \ell(p_m)/4 - 1 \geqslant m/2 - 1$. Then U, and hence W, represents a loxodromic element of G with respect to $Y \sqcup \mathcal{E}$. In particular, $W \neq 1$ in G. If W is of syllable length 1, i.e., $W = a_i^{n_i m}$ for some $i \in \{1, \dots, k\}$ and $m \neq 0$, then, obviously, $W \neq 1$ in G. Therefore, $\langle a_1^{n_1}, \dots, a_k^{n_k}\rangle$ is a free group of rank k.

The last statement of corollary obviously follows from the first one. ∎

The following lemma is closely related to [14, Corollary 6.12].

Lemma 10.3. *Let G be a group, $X \subseteq G$, and $H \hookrightarrow_h (G, X)$ a finitely generated infinite subgroup. Then for any finite collection of elements $a_1, \dots, a_s \in G \setminus H$ and any infinite subset $\widetilde{H} \subseteq H$, there exist elements $h_1, \dots, h_s \in \widetilde{H}$ such that $a_1 h_1, \dots, a_s h_s$ are pairwise non-commensurable loxodromic elements with respect to the action of G on $\Gamma(G, X \sqcup H)$.*

Proof. First, we show that conditions (a′) and (b) of [14, Theorem 6.11] are satisfied for some extended relative generating set X_1. Though the conclusion of this theorem is not sufficient for our aims, the proof is sufficient and can be easily adopted to obtain the desired statement.

Definition of X_1. Let B be a finite generating set of H. We set

$$X_1 = X \cup B \cup \{a_1, \ldots, a_s\}^{\pm}.$$

Since $|X \triangle X_1| < \infty$, we have that $H \hookrightarrow_h (G, X_1)$ by Lemma 3.13. Let $\hat{d}$ be the relative metric on H associated with this embedding.

Verification of condition (a′). We shall show that

- $\hat{d}(1, h) < \infty$ for any $h \in H$ and
- H is unbounded with respect to $\hat{d}$.

The former follows from the fact that the relative generating set X_1 contains a finite generating set of H, namely B. The latter follows from the fact that the metric space $(H, \hat{d})$ is locally finite (since $H \hookrightarrow_h (G, X_1)$) and the assumption that H is infinite.

Verification of condition (b) for the elements $a_1, \ldots, a_s$. We shall check that

- these elements lie in X_1 and
- $|H^{a_i} \cap H| < \infty$ for $i = 1, \ldots, s$.

The former is valid by definition of X_1, the latter follows from the assumption that $a_i \in G \setminus H$ (see Lemma 3.14).

Thus, conditions (a′) and (b) of [14, Theorem 6.11] are satisfied for the extended relative generating set X_1. Now we look into the proof of this theorem. Condition (a′) and the local finiteness of $(H, \hat{d})$ enable to choose $h_1, \ldots, h_s$ in $\widetilde{H}$ such that

$$\hat{d}(1, h_1) > 50D,$$
$$\hat{d}(1, h_{i+1}) > \hat{d}(1, h_i) + 8D, \quad i = 1, \ldots, s-1,$$

where $D = D(1, 0)$ is provided by Proposition 8.3. We set $f_i = a_i h_i$. Then the proof that $f_1, \ldots, f_s$ are non-commensurable and loxodromic with respect to the action of G on $\Gamma(G, X_1 \sqcup H)$ is the same as in [14, Theorem 6.11]. Since $X \subseteq X_1$, these elements remain loxodromic with respect to the action of G on $\Gamma(G, X \sqcup H)$. ∎

Lemma 10.4. *Let G be an acylindrically hyperbolic group with respect to a generating set Z and let $a, b \in G$ be two non-commensurable loxodromic with respect to Z elements, where, additionally, a is special. Then there exists a positive integer n_0 such that for any $n, m \geqslant n_0$ the element $g = a^n b^m$ is special with respect to some generating set, in particular $E_G(g) = \langle g \rangle$.*

Proof. Since G acts acylindrically on the hyperbolic space $\Gamma(G, Z)$ and a, b are non-commensurable and loxodromic with respect to Z, we can apply [14, Theorem 6.8] which says that in this situation there exists a subset $X \subset G$ such that $\{E_G(a), E_G(b)\} \hookrightarrow_h (G, X)$. Denote $\mathcal{E} = E_G(a) \sqcup E_G(b)$. By Theorem 3.15, there exists $Y \subseteq G$ such that $X \subseteq Y$ and the following conditions hold.

(a) $\{E_G(a), E_G(b)\} \hookrightarrow_h (G, Y)$. In particular, the Cayley graph $\Gamma(G, Y \sqcup \mathcal{E})$ is hyperbolic.

(b) The action of G on $\Gamma(G, Y \sqcup \mathcal{E})$ is acylindrical.

Let $\hat{d}_1$ and $\hat{d}_2$ be the relative metrics on $E_G(a)$ and on $E_G(b)$, respectively, associated with the embedding $\{E_G(a), E_G(b)\} \hookrightarrow_h (G, Y)$. Then there exists n_0 such that for any $l \geqslant n_0$ we have that $\hat{d}_1(1, a^l) > 50D$ and $\hat{d}_2(1, b^l) > 50D$, where $D = D(1, 0)$ is the constant from Proposition 8.3. Let $n, m \geqslant n_0$ and $g = a^n b^m$. Observe that conditions (W_1)–(W_3) of Lemma 10.1 are satisfied for g considered as a word of length 2 in the alphabet $\mathcal{E} \subseteq Y \sqcup \mathcal{E}$. By part (a) of Lemma 10.1, the element g is loxodromic with respect to $Y \sqcup \mathcal{E}$. Since G acts acylindrically on the hyperbolic space $\Gamma(G, Y \sqcup \mathcal{E})$ and contains a loxodromic element with respect to this action and G is not virtually cyclic, we conclude that this action is non-elementary (see Theorem 3.9). Therefore, G is acylindrically hyperbolic with respect to $Y \sqcup \mathcal{E}$ and the subgroup $E_G(g)$ is well defined.

The rest of the proof is very similar to the second part of the proof of [14, Lemma 6.18]. Let $t \in E_G(g)$. Then $tg^{\sigma k} = g^k t$ for some $k \in \mathbb{N}$ and $\sigma \in \{-1, 1\}$. Consider the paths p and q in $\Gamma(G, Y \sqcup \mathcal{E})$ labeled by $(a^n b^m)^k$ and $(a^n b^m)^{\sigma k}$, respectively, such that $p_- = 1$, $q_- = t$. We have that $d_{Y \sqcup \mathcal{E}}(p_-, q_-) = d_{Y \sqcup \mathcal{E}}(p_+, q_+) = \varepsilon$, where $\varepsilon = |t|_{Y \sqcup \mathcal{E}}$.

Let $R = R(\varepsilon, 3)$ be as in the part (b) of Lemma 10.1. Passing to a multiple of k if necessary, we may assume that $\ell(p) \geqslant R$. Then by statement (b) of Lemma 10.1, there exist 3 consecutive components p_1, p_2, p_3 of p that are connected to 3 consecutive components q_1, q_2, q_3 of q.

Without loss of generality, we may assume that p_1, p_3, q_1, q_3 are $E_G(a)$-components while p_2, q_2 are $E_G(b)$-components. Let e_j be a path connecting $(p_j)_+$ to $(q_j)_+$ in $\Gamma(G, Y \sqcup \mathcal{E})$ and let z_j be the element of G represented by $\mathbf{Lab}(e_j)$, $j = 1, 2$. Then $z_j \in E_G(a) \cap E_G(b)$. But $E_G(a) \cap E_G(b) = 1$ since a, b are non-commensurable and $E_G(a) = \langle a \rangle$ by assumption. Thus, $z_j = 1$. Reading the label of the closed path $e_1 q_2 \overline{e_2}\, \overline{p_2}$, we obtain $b^{\sigma m} b^{-m} = 1$. Therefore, $\sigma = 1$ and the label of q is $(a^n b^m)^k$. Reading the labels of the segment of p from 1 to $(p_1)_+$, e_1, and the segment of $\bar{q}$ from $(q_1)_+$ to t, we obtain $t = g^l$ for some $l \in \mathbb{Z}$. Therefore, $E_G(g) \leqslant \langle g \rangle$. Hence $E_G(g) = \langle g \rangle$ and g is special with respect to $Y \sqcup \mathcal{E}$. ∎

Lemma 10.5. *Suppose that G is an acylindrically hyperbolic group without nontrivial finite normal subgroups. Then there exist an element $g \in G$ and a generating set Y of G such that g is special with respect to Y and $\langle g \rangle \hookrightarrow_h (G, Y)$.*

Proof. By [14, Lemma 6.18], there exist a subset $X \subseteq G$, a subgroup $E \hookrightarrow_h (G, X)$, and an element $g \in G$ such that $E = \langle g \rangle \times K(G)$, where $K(G)$ is the maximal finite normal subgroup of G. By assumption, $K(G) = 1$. Then $\langle g \rangle \hookrightarrow_h (G, X)$.

It was shown in the proof of this lemma that g can be chosen to be loxodromic with respect to some generating set. In particular, we may assume that g has infinite order. Note that G is not virtually cyclic, since G is acylindrically hyperbolic.

By Lemma 6.3, there exists a generating set Y of G such that G is acylindrically hyperbolic with respect to Y and $\langle g\rangle \hookrightarrow_h (G,Y)$. Let $\hat{d}$ be the relative metric on $\langle g\rangle$ associated with this embedding. Since the metric space $(\langle g\rangle, \hat{d}\,)$ is locally finite, g cannot be elliptic with respect to Y. Then g is loxodromic with respect to Y. Since $\langle g\rangle \hookrightarrow_h (G,Y)$, we deduce from Lemma 3.14 and (3.1) that $E_G(g) = \langle g\rangle$. Thus, g is special with respect to Y. ■

Proposition 10.6. *Suppose that G is an acylindrically hyperbolic group without nontrivial finite normal subgroups. Then there are special loxodromic elements $a, g \in G$ such that for any integer $k > 0$ the coset $a\langle g\rangle$ contains k pairwise non-commensurable and jointly special elements.*

Proof. By Lemma 10.5, there exist an element $g \in G$ and a generating set Y of G such that g is special with respect to Y and $H \hookrightarrow_h (G,Y)$, where $H = \langle g\rangle$. It follows from Definition 3.8 that G is acylindrically hyperbolic with respect to Y.

Let $b \in G \setminus H$ be an arbitrary element. By Lemma 10.3, there exist two non-commensurable loxodromic elements bg^s, bg^t with respect to $Y \sqcup H$. It follows that they are loxodromic with respect to Y. At least one of them, say $c := bg^s$, is non-commensurable with g. In particular,

$$\langle c\rangle \cap \langle g\rangle = 1.$$

By Lemma 10.4, there exists a positive integer n_0 such that for any $n, m \geqslant n_0$ the element $c^n g^m$ is special with respect to some generating set; in particular,

$$E_G(c^n g^m) = \langle c^n g^m\rangle. \tag{10.1}$$

By Lemma 10.3, for any $k \in \mathbb{N}$, there exist natural numbers $m_1 < m_2 < \cdots < m_k$ such that $m_1 \geqslant n_0$ and the elements $c^{n_0} g^{m_1}, \ldots, c^{n_0} g^{m_k}$ are pairwise non-commensurable and loxodromic with respect to $Y \sqcup H$. Then they are loxodromic with respect to Y. Moreover, by (10.1), we have that $E_G(c^{n_0} g^{m_i}) = \langle c^{n_0} g^{m_i}\rangle$ for $i = 1, \ldots, k$. Thus, these elements are pairwise non-commensurable and jointly special with respect to Y.

We set $a = c^{n_0} g^{n_0}$. Then a, g, and k elements $ag^{m_1-n_0}, \ldots, ag^{m_k-n_0}$ satisfy the conclusion of proposition. ■

Remark. One can prove a stronger version of this lemma, saying that for any infinite subset $I \subseteq a\langle g\rangle$, there exists an infinite subset of I consisting of pairwise non-commensurable and jointly special elements.

As we mentioned above, the following proposition is only used for the proof of Corollary E. The proof of this proposition is very similar to the proof of Proposition 10.6.

Proposition 10.7. *Suppose that G is a finitely generated acylindrically hyperbolic group without nontrivial finite normal subgroups. Then, for any $n \in \mathbb{N}$, G can be generated by a finite set A such that $|A| \geqslant n$ and the elements of A are pairwise non-commensurable and jointly special.*

Proof. By Lemma 10.5, there exist an element $g \in G$ and a generating set Y of G such that g is special with respect to Y and $H \hookrightarrow_h (G, Y)$, where $H = \langle g \rangle$. It follows from Definition 3.8 that G is acylindrically hyperbolic with respect to Y.

Suppose that $G = \langle a_1, \dots, a_l \rangle$. Removing those a_i that are powers of g, we may assume that $G = \langle g, a_1, \dots, a_k \rangle$ for some $1 \leqslant k \leqslant l$, where $a_i \notin H$ for $i = 1, \dots, k$.

Step 1. We show how to find a finite generating set B of G such that $g \in B$ and the elements of B are pairwise non-commensurable and loxodromic with respect to Y.

We set $G_0 = \langle g \rangle$ and $G_i = \langle g, a_1, \dots, a_i \rangle$ for $i = 1, \dots, k$. Note that $G = G_k$. Arguing inductively, we fix $i \in \{0, \dots, k-1\}$ and suppose that we have found a finite generating set B_i of G_i such that $g \in B_i$ and the elements of B_i are pairwise non-commensurable and loxodromic with respect to Y.

We set $s = |B_i| + 1$. By Lemma 10.3, there exist positive integers $n_1 < n_2 < \cdots < n_s$ such that the elements of the set $\{a_{i+1} g^{n_1}, a_{i+1} g^{n_2}, \dots, a_{i+1} g^{n_s}\}$ are pairwise non-commensurable and loxodromic with respect to $Y \sqcup H$. It follows that they are loxodromic with respect to Y. Since the number of these elements is $|B_i| + 1$, there exists $j \in \{1, \dots, s\}$ such that the elements of $B_{i+1} := B_i \cup \{a_{i+1} g^{n_j}\}$ are pairwise non-commensurable. Since $g \in B_i$, we deduce that $G_{i+1} = \langle B_{i+1} \rangle$. Finally, we set $B = B_k$.

Thus, we may assume from the beginning that $G = \langle g, a_1, \dots, a_k \rangle$, where $g, a_1, \dots, a_k$ are pairwise non-commensurable and loxodromic with respect to Y. Recall that g is special with respect to Y.

Step 2. We show how to find a finite generating set of G consisting of pairwise non-commensurable and special elements with respect to Y.

We set $A_0 = \{g\}$. Arguing inductively, we fix $i \in \{0, \dots, k-1\}$ and suppose that we have found a finite subset $A_i \subset G$ such that $g, a_1, \dots, a_i \in \langle A_i \rangle$ and the elements of A_i are pairwise non-commensurable and special with respect to Y. We set $s = 2|A_i| + 2$ and construct a finite set $A_{i+1} \subset G$ with analogous properties.

By Lemma 10.4, there exists $n_0 \in \mathbb{N}$ such that for any $n, m \geqslant n_0$ the element $a_{i+1}^n g^m$ is special with respect to some generating set of G. In particular, we have that

$$E_G(a_{i+1}^n g^m) = \langle a_{i+1}^n g^m \rangle \tag{10.2}$$

for any $n, m \geqslant n_0$. By Lemma 10.3, there exist s integers $m_1, m_2, \dots, m_s \geqslant n_0$ such that the elements $a_{i+1}^{n_0+1} g^{m_1}, a_{i+1}^{n_0+2} g^{m_2}, \dots, a_{i+1}^{n_0+s} g^{m_s}$ are pairwise non-commensurable and loxodromic with respect to $Y \sqcup H$ (and hence with respect to Y). By (10.2) they are special with respect to Y. Since the number of these elements is $2|A_i| + 2$, there exists an odd $j \in \{1, \dots, s-1\}$ such that the elements of

$$A_{i+1} := A_i \cup \{a_{i+1}^{n_0+j} g^{m_j}, a_{i+1}^{n_0+j+1} g^{m_{j+1}}\}$$

are pairwise non-commensurable. By construction, all elements of A_{i+1} are special with respect to Y. Since $g \in A_i$, we have $a_{i+1} \in \langle A_{i+1} \rangle$.

Observe that $\langle A_k \rangle = G$ and $|A_k| = 2k+1$, where $k \geqslant 1$ is fixed before Step 2. Repeating the construction of Step 2 several times, we can obtain a finite generating set A of G with desired properties and of arbitrary large finite cardinality. ■

11. Test words in acylindrically hyperbolic groups: a special case

A background on test words and our definition of an $(a_1, \dots, a_k)$-test word are given in Section 2.5. For convenience, we recall this definition here. In this section, we construct certain $(a_1, \dots, a_k)$-test words in acylindrically hyperbolic groups.

Definition 11.1. Let H be a group and let $a_1, \dots, a_k$ be some elements of H. A word $W(x_1, \dots, x_k)$ is called an $(a_1, \dots, a_k)$-*test word* if for every solution $(b_1, \dots, b_k)$ of the equation

$$W(a_1, \dots, a_k) = W(x_1, \dots, x_k)$$

in H, there exists a number $\alpha \in \mathbb{Z}$ such that $b_i = a_i^{U^\alpha}$ for $i = 1, \dots, k$, where $U = W(a_1, \dots, a_k)$.

Remark 11.2. Let H be an acylindrically hyperbolic group without nontrivial normal finite subgroup. The following lemma says, in particular, that if $a_1, a_2, a_3 \in H$ are jointly special and pairwise non-commensurable elements, then there exist an $(a_1, a_2, a_3, 1)$-test word. The reason for the general formulation of this lemma (i.e., for $k \geqslant 3$ elements) is that it serves as the basis of induction for Proposition 12.1.

Lemma 11.3. *Let H be an acylindrically hyperbolic group without nontrivial normal finite subgroups and let $a_1, \dots, a_k \in H$ ($k \geqslant 3$) be jointly special and pairwise non-commensurable elements. Then there exist* 10 *positive integers k_1, l_1, m_1, k_2, l_2, m_2, s, p, q, t such that the following holds.*

Let U be the left side of the equation

$$\big((a_1^{k_1} a_3^{l_1})^{m_1} (a_2^{k_2} a_3^{l_2})^{m_2}\big)^s (a_2^p a_3^q)^t = \big((x_1^{k_1} x_3^{l_1})^{m_1} (x_2^{k_2} x_3^{l_2})^{m_2}\big)^s \big(x_2^p (x_3 y_3)^q\big)^t.$$

Then, for any solution $(x_1, x_2, x_3, y_3) = (b_1, b_2, b_3, c_3)$ of this equation in H, there exists an integer number α such that

$$b_1 = a_1^{U^\alpha}, \quad b_2 = a_2^{U^\alpha}, \quad b_3 = a_3^{U^\alpha}, \quad c_3 = 1.$$

Moreover, the 10 *exponents can be chosen so that, additionally to the above statement, the elements $U, a_1, \dots, a_k$, will be jointly special and pairwise non-commensurable.*

Proof. First we find an appropriate generating set of H.

Claim 1. There exists a generating set Y of H such that the following properties are satisfied.

(i) The group H is acylindrically hyperbolic with respect to Y.

(ii) The elements $a_1, \dots, a_k$ are special with respect to Y.

(iii) $\{\langle a_1 \rangle, \dots, \langle a_k \rangle\} \hookrightarrow_h (H, Y)$.

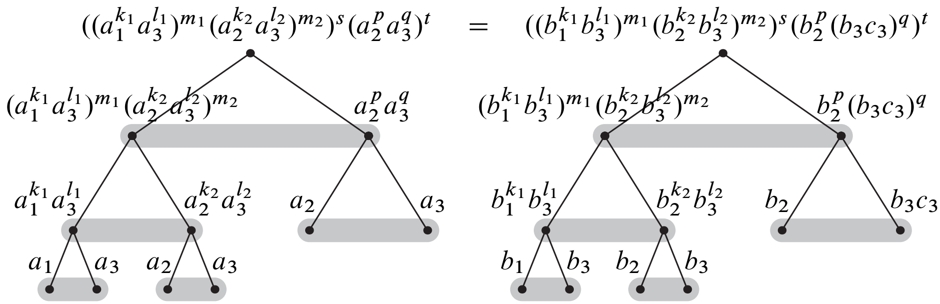

Figure 17

Proof. Conditions of Lemma 11.3 imply that $E_H(a_j) = \langle a_j \rangle$, $j = 1, \dots, k$, and

$$\{\langle a_1\rangle, \dots, \langle a_k\rangle\} \hookrightarrow_h H \tag{11.1}$$

(see [14, Theorem 6.8]). Applying Lemma 6.3 to this hyperbolic embedding, we obtain a generating set Y of H such that H is acylindrically hyperbolic with respect to Y and $\{\langle a_1\rangle, \dots, \langle a_k\rangle\} \hookrightarrow_h (H, Y)$. Thus, the properties (i) and (iii) are satisfied.

We prove (ii). Property (i) implies that any element of H is either elliptic or loxodromic with respect to Y. For $j = 1, \dots, k$, let $\widehat{d_j}^Y$ be the relative metric on $\langle a_j\rangle$ associated with the hyperbolic embedding (11.1). By definition, the space $(\langle a_j\rangle, \widehat{d_j}^Y)$ is locally finite. Therefore, a_j cannot be elliptic with respect to Y. Thus, a_j is loxodromic with respect to Y and satisfies $E_G(a_j) = \langle a_j\rangle$. Hence a_j is special with respect to Y. ■

We use the following notation.

Notation. Given $a, b, c, d \in H$, we say that the pair (a, b) is *conjugate* to the pair (c, d) if there exists $g \in H$ such that $g^{-1}ag = c$ and $g^{-1}bg = d$. In this case, we write $(a, b) \sim (c, d)$.

Let $k_1, l_1, m_1, k_2, l_2, m_2, s, p, q, t$ be arbitrary 10 positive integers (we call them exponents) and let (b_1, b_2, b_3, c_3) be a solution of the equation in Lemma 11.3:

$$\big((a_1^{k_1}a_3^{l_1})^{m_1}(a_2^{k_2}a_3^{l_2})^{m_2}\big)^s(a_2^pa_3^q)^t = \big((b_1^{k_1}b_3^{l_1})^{m_1}(b_2^{k_2}b_3^{l_2})^{m_2}\big)^s\big(b_2^p(b_3c_3)^q\big)^t. \tag{11.2}$$

The diagrams on Figure 17 reflect the nested structure of the left and the right sides of this equation.

We explain the structure of forthcoming proof.

• In Step 1, we will choose 10 exponents so that assumptions of Corollary 9.5 became applicable to 5 pairs of labels of the left diagram (we put them in 5 shadowed regions).

• In Step 2, we will start from the root equation and deduce from Corollary 9.5 consequently the following formulas:

(1) $((a_1^{k_1}a_3^{l_1})^{m_1}(a_2^{k_2}a_3^{l_2})^{m_2}, a_2^pa_3^q) \sim ((b_1^{k_1}b_3^{l_1})^{m_1}(b_2^{k_2}b_3^{l_2})^{m_2}, b_2^p(b_3c_3)^q)$;

(2) $(a_1^{k_1}a_3^{l_1}, a_2^{k_2}a_3^{l_2}) \sim (b_1^{k_1}b_3^{l_1}, b_2^{k_2}b_3^{l_2})$ and $(a_2, a_3) \sim (b_2, b_3c_3)$;

(3) $(a_1, a_3) \sim (b_1, b_3)$ and $(a_2, a_3) \sim (b_2, b_3)$.

• In Step 3, we will analyze these formulas and deduce the statement of lemma.

We fix $m \in \mathbb{N}$ such that $\langle a_1^m, a_2^m, a_3^m\rangle$ is a free group of rank 3 (see Corollary 10.2). Let Y be the generating set of H from Claim 1.

Step 1. In the following, we will use

- Corollary 9.5 (to provide perfectness of equations),
- Corollary 10.2 (to construct many loxodromic elements with respect to Y),
- Lemma 10.3 (to construct many non-commensurable elements), and
- Lemma 10.4 (to provide $E_G(g) = \langle g\rangle$ for each constructed element g).

We will also use the principle that if $u, v \in H$ are non-commensurable, then any element $g \in H$ is non-commensurable with at least one of u, v.

(a) We choose $k_1, l_1 \in m\mathbb{N}$ so that

(1) the equation $a_1^{k_1}a_3^{l_1} = x^{k_1}y^{l_1}$ is perfect,

(2) the element $a_1^{k_1}a_3^{l_1}$ is special with respect to Y.

In details: by Corollary 10.2, the elements $a_1^ia_3^j$ are loxodromic with respect to Y for all sufficiently large i, j. By Lemma 10.4, $E_G(a_1^ia_3^j) = \langle a_1^ia_3^j\rangle$ for all sufficiently large i, j. Thus, $a_1^ia_3^j$ is special with respect to Y for all sufficiently large i, j. Then we apply Corollary 9.5 to provide the perfectness.

(b) We choose $k_2, l_2 \in m\mathbb{N}$ so that

(1) the equation $a_2^{k_2}a_3^{l_2} = x^{k_2}y^{l_2}$ is perfect;

(2) the element $a_2^{k_2}a_3^{l_2}$ is special with respect to Y and non-commensurable with $a_1^{k_1}a_3^{l_1}$.

(c) We choose $m_1, m_2 \in \mathbb{N}$ so that

(1) the equation $(a_1^{k_1}a_3^{l_1})^{m_1}(a_2^{k_2}a_3^{l_2})^{m_2} = x^{m_1}y^{m_2}$ is perfect;

(2) the element $(a_1^{k_1}a_3^{l_1})^{m_1}(a_2^{k_2}a_3^{l_2})^{m_2}$ is special with respect to Y.

(d) We choose $p, q \in m\mathbb{N}$ so that

(1) the equation $a_2^pa_3^q = x^py^q$ is perfect;

(2) the element $a_2^pa_3^q$ is special with respect to Y and non-commensurable with $(a_1^{k_1}a_3^{l_1})^{m_1}(a_2^{k_2}a_3^{l_2})^{m_2}$.

(e) We choose $s, t \in \mathbb{N}$ so that

(1) the following equation is perfect:

$$\big((a_1^{k_1}a_3^{l_1})^{m_1}(a_2^{k_2}a_3^{l_2})^{m_2}\big)^s(a_2^pa_3^q)^t = x^sy^t;$$

(2) the element on the left side of this equation is special with respect to Y and non-commensurable with elements $a_1, \dots, a_k$.

Notation. Let A, B, C, D, E denote the left sides of equations in (a), (b), (c), (d), (e), respectively.

Step 2. By (11.2) and (e), there exists $\varepsilon \in \mathbb{Z}$ such that

$$(a_1^{k_1}a_3^{l_1})^{m_1}(a_2^{k_2}a_3^{l_2})^{m_2} = \big((b_1^{k_1}b_3^{l_1})^{m_1}(b_2^{k_2}b_3^{l_2})^{m_2}\big)^{E^\varepsilon}, \tag{11.3}$$

$$a_2^p a_3^q = \big(b_2^p(b_3c_3)^q\big)^{E^\varepsilon}. \tag{11.4}$$

By (11.3) and (c), there exists $\gamma \in \mathbb{Z}$ such that

$$a_1^{k_1}a_3^{l_1} = (b_1^{k_1}b_3^{l_1})^{E^\varepsilon C^\gamma}, \tag{11.5}$$

$$a_2^{k_2}a_3^{l_2} = (b_2^{k_2}b_3^{l_2})^{E^\varepsilon C^\gamma}. \tag{11.6}$$

By (11.4) and (d), there exists $\delta \in \mathbb{Z}$ such that

$$a_2 = b_2^{E^\varepsilon D^\delta}, \quad a_3 = (b_3c_3)^{E^\varepsilon D^\delta}. \tag{11.7}$$

By (11.5) and (a), there exists $\alpha \in \mathbb{Z}$ such that

$$a_1 = b_1^{E^\varepsilon C^\gamma A^\alpha}, \quad a_3 = b_3^{E^\varepsilon C^\gamma A^\alpha}. \tag{11.8}$$

By (11.6) and (b), there exists $\beta \in \mathbb{Z}$ such that

$$a_2 = b_2^{E^\varepsilon C^\gamma B^\beta}, \quad a_3 = b_3^{E^\varepsilon C^\gamma B^\beta}. \tag{11.9}$$

Step 3. From the last equations in (11.8) and (11.9), we deduce that $A^{-\alpha}B^\beta$ centralizes a_3. We claim that $\alpha = \beta = 0$.

Indeed, let H_1 be the subgroup of H generated by a_1^m, a_2^m, a_3^m. By the choice of m, H_1 is free of rank 3. Since $A^{-\alpha}B^\beta$ lies in H_1 and centralizes a_3^m (which is primitive in H_1), the element $A^{-\alpha}B^\beta$ is a power of a_3^m. Consider the homomorphism

$$\varphi : H_1 \to \mathbb{Z}\times\mathbb{Z}, \quad a_1^m \mapsto (1,0), \quad a_2^m \mapsto (0,1), \quad a_3^m \mapsto (0,0).$$

Then

$$\varphi(A^{-\alpha}B^\beta) = \left(-\frac{k_1}{m}\alpha, \frac{k_2}{m}\beta\right) = (0,0).$$

Hence $\alpha = \beta = 0$.

Using this, we deduce from the first equations in (11.7) and (11.9) that $C^{-\gamma}D^\delta$ centralizes a_2. We claim that $\gamma = \delta = 0$. Indeed, as above we deduce that $C^{-\gamma}D^\delta$ is a power of a_2^m. Consider the homomorphism

$$\psi : H_1 \to \mathbb{Z}\times\mathbb{Z}, \quad a_1^m \mapsto (1,0), \quad a_2^m \mapsto (0,0), \quad a_3^m \mapsto (0,1).$$

Then

$$\psi(C^{-\gamma}D^\delta) = \left(-\frac{k_1m_1}{m}\gamma, \frac{q}{m}\delta - \left(\frac{l_1m_1}{m} + \frac{l_2m_2}{m}\right)\gamma\right) = (0,0).$$

Hence $\gamma = \delta = 0$. Then the last equations in (11.7) and (11.8) imply that $c_3 = 1$. Moreover, the equations (11.7)–(11.8) imply that $a_1 = b_1^{E^e}$, $a_2 = b_2^{E^e}$, $a_3 = b_3^{E^e}$.

Finally note that the elements $E, a_1, \dots, a_k$ are pairwise non-commensurable and jointly special by (e) and Claim 1 (ii). ■

12. Test words in acylindrically hyperbolic groups: the general case

Notation. We write $\mathbf{1}^k$ for the tuple $(\underbrace{1,\dots,1}_{k})$.

The aim of this section is to prove the following proposition, which will be used in Section 13.

However, to prove this proposition, we need the following generalization, which is simultaneously a generalization of Lemma 11.3.

Proposition 12.1. *Let H be an acylindrically hyperbolic group without nontrivial normal finite subgroups and let $a_1,\dots,a_k \in H$ (where $k \geqslant 3$) be jointly special and pairwise non-commensurable elements. Then there exists an $(a_1,\dots,a_k,\mathbf{1}^{k-2})$-test word $W_k(x_1,\dots,x_k,y_3,\dots,y_k)$ such that the elements $a_1,\dots,a_k$ together with the element $W_k(a_1,\dots,a_k,\mathbf{1}^{k-2})$ are jointly special and pairwise non-commensurable.*

Proof. It suffices to prove, by induction on n, the following claim.

Claim. For any $n = 3,\dots,k$, there exists an $(a_1,\dots,a_n,\mathbf{1}^{n-2})$-test word $W_n(x_1,\dots,x_n, y_3,\dots,y_n)$ such that the elements $a_1,\dots,a_k$ together with the element $W_n(a_1,\dots,a_n, \mathbf{1}^{n-2})$ are jointly special and pairwise non-commensurable.

For $n = 3$, this statement is valid by Lemma 11.3. Suppose that for some $3 \leqslant n < k$, we have constructed the desired word $W_n = W_n(x_1,\dots,x_n,y_3,\dots,y_n)$. We show how to construct W_{n+1}.

Denote $A = W_n(a_1,\dots,a_n,\mathbf{1}^{n-2})$. Since the elements $A, a_1,\dots,a_k$ are jointly special and pairwise non-commensurable, they satisfy the assumption of Lemma 11.3. By this lemma, there exist positive integers $k_1, l_1, m_1, k_2, l_2, m_2, s, p, q, t$ such that the following holds.

(a) The word

$$\mathfrak{M}(X,x_n,x_{n+1},y_{n+1}) = \big((X^{k_1}x_{n+1}^{l_1})^{m_1}(x_n^{k_2}x_{n+1}^{l_2})^{m_2}\big)^s\big(x_n^p(x_{n+1}y_{n+1})^q\big)^t$$

in variables (X,x_n,x_{n+1},y_{n+1}) is an $(A,a_n,a_{n+1},1)$-test word.

(b) The elements $\mathfrak{M}(A,a_n,a_{n+1},1)$, A, $a_1,\dots,a_k$ are jointly special and pairwise non-commensurable.

We define

$$W_{n+1}(x_1,\dots,x_{n+1},y_3,\dots,y_{n+1}) = \mathfrak{M}(W_n,x_n,x_{n+1},y_{n+1}).$$

First, we prove that W_{n+1} is an $(a_1,\dots,a_{n+1},\mathbf{1}^{n-1})$-test word.

Suppose that for some elements $b_1,\dots,b_{n+1},c_3,\dots,c_{n+1}$ in H we have that

$$\begin{aligned}&\big((W_n^{k_1}(a_1,\dots,a_n,1,\dots,1)a_{n+1}^{l_1}\big)^{m_1}(a_n^{k_2}a_{n+1}^{l_2})^{m_2}\big)^s\big(a_n^p(a_{n+1}\cdot 1)^q\big)^t\\ &\quad = \big((W_n^{k_1}(b_1,\dots,b_n,c_3,\dots,c_n)b_{n+1}^{l_1}\big)^{m_1}(b_n^{k_2}b_{n+1}^{l_2})^{m_2}\big)^s\big(b_n^p(b_{n+1}c_{n+1})^q\big)^t.\end{aligned}$$

Denote $B := W_n(b_1, \dots, b_n, c_3, \dots, c_n)$ and write this equation shorter:

$$\begin{aligned}&\big((A^{k_1} a_{n+1}^{l_1})^{m_1} (a_n^{k_2} a_{n+1}^{l_2})^{m_2}\big)^s \big(a_n^p (a_{n+1} \cdot 1)^q\big)^t \\ &\quad = \big((B^{k_1} b_{n+1}^{l_1})^{m_1} (b_n^{k_2} b_{n+1}^{l_2})^{m_2}\big)^s \big(b_n^p (b_{n+1} c_{n+1})^q\big)^t.\end{aligned}$$

Let U be the left side of this equation.

Since, by statement (a), $\mathfrak{M}(X, x_n, x_{n+1}, y_{n+1})$ is an $(A, a_n, a_{n+1}, 1)$-test word, there exists $\alpha \in \mathbb{Z}$ such that

$$B = A^{U^\alpha}, \tag{12.1}$$

$$b_n = a_n^{U^\alpha}, \quad b_{n+1} = a_{n+1}^{U^\alpha}, \quad c_{n+1} = 1. \tag{12.2}$$

From (12.1) we deduce that

$$W_n(b_1^{U^{-\alpha}}, \dots, b_n^{U^{-\alpha}}, c_3^{U^{-\alpha}}, \dots, c_n^{U^{-\alpha}}) = W_n(a_1, \dots, a_n, \mathbf{1}^{n-2}) = A.$$

Since W_n is an $(a_1, \dots, a_n, \mathbf{1}^{n-2})$-test word, there exists $\beta \in \mathbb{Z}$ such that

$$\begin{aligned} b_1^{U^{-\alpha}} &= a_1^{A^\beta}, \\ &\vdots \\ b_n^{U^{-\alpha}} &= a_n^{A^\beta}, \end{aligned} \tag{12.3}$$

and

$$c_3 = \dots = c_n = 1. \tag{12.4}$$

From the first equation in (12.2) and the last equation in (12.3), we deduce that $a_n^{A^\beta} = a_n$. Since A and a_n are jointly special and non-commensurable, we have that $\beta = 0$. Then (12.2)–(12.4) imply that

$$(b_1, \dots, b_{n+1}, c_3, \dots, c_{n+1}) = (a_1^{U^\alpha}, \dots, a_{n+1}^{U^\alpha}, \mathbf{1}^{n-1});$$

i.e., the word W_{n+1} is an $(a_1, \dots, a_{n+1}, \mathbf{1}^{n-1})$-test word.

It remains to show that the elements $W_{n+1}(a_1, \dots, a_{n+1}, \mathbf{1}^{n-1}), a_1, \dots, a_k$ are jointly special and pairwise non-commensurable. This follows from statement (b) and the fact that $W_{n+1}(a_1, \dots, a_{n+1}, \mathbf{1}^{n-1}) = \mathfrak{M}(A, a_n, a_{n+1}, 1)$. ∎

In some cases it suffices to use the following corollary, which is a weaker version of Proposition 12.1. This corollary follows from Proposition 12.1 if we set there

$$\mathcal{U}_k(x_1, \dots, x_k) := W_k(x_1, \dots, x_k, \mathbf{1}^{k-2}).$$

Corollary 12.2. *Let H be an acylindrically hyperbolic group without nontrivial finite normal subgroups and let $a_1, \dots, a_k \in H$ (where $k \geqslant 3$) be jointly special and pairwise non-commensurable elements. Then there is an $(a_1, \dots, a_k)$-test word $\mathcal{U}_k(x_1, \dots, x_k)$ such that the elements $a_1, \dots, a_k$ together with $\mathcal{U}_k(a_1, \dots, a_k)$ are jointly special and pairwise non-commensurable.*

13. Proof of Theorem A

Lemma 13.1. *Let H be a group. For any finite system of equations $S \subset F_n * H$ there exists an integer $k \geqslant 0$ and a finite system $S' \subset F_{n+k} * H$ consisting of only splitted equations such that $|S'| \geqslant |S|$ and*

$$V_H(S') = V_H(S) \times \{g_1\} \times \cdots \times \{g_k\}$$

for some elements $g_1, \ldots, g_k \in H$.

Proof. To define S', one should replace each constant h in S by a new variable x_h and add the equation $x_h h^{-1}$. ■

Notation. To shorten notation, we write $\underline{x}$ instead of the tuple $(x_1, \ldots, x_n)$.

Proof of Theorem A. (1) Let $S = \{s_1, \ldots, s_k\}$, where $s_i \in F_n * H$. We take arbitrary $(k + 2)$ jointly special and pairwise non-commensurable elements $a_1, \ldots, a_{k+2} \in H$. The existence of such elements is guaranteed by Proposition 10.6. By Proposition 12.1, there exists an $(a_1, \ldots, a_{k+2}, \mathbf{1}^k)$-test word $W_{k+2}(z_1, \ldots, z_{k+2}, y_1, \ldots, y_k)$. Then the desired equation is

$$f : W_{k+2}(a_1, \ldots, a_{k+2}, \mathbf{1}^k) = W_{k+2}(a_1, \ldots, a_{k+2}, s_1, \ldots, s_k).$$

(2) By Lemma 13.1, we may assume that S consists of splitted equations:

$$S = \big\{w_i(x_1, \ldots, x_n) = h_i \mid i = 1, \ldots, m\big\}. \tag{13.1}$$

By Proposition 10.6, there exist special elements a, b such that the coset $a\langle b\rangle$ contains $(2m + 2)$ pairwise non-commensurable jointly special elements, say

$$ab^s, ab^t, ab^{k_1}, \ldots, ab^{k_m}, ab^{l_1}, \ldots, ab^{l_m}.$$

Then the elements of the tuple

$$T = (ab^s, ab^t,\ h_i^{-1}ab^{k_i}h_i,\ h_i^{-1}ab^{l_i}h_i;\ i = 1, \ldots, m)$$

are also pairwise non-commensurable and jointly special. Let $\mathcal{U}_{2m+2}$ be the T-test word from Corollary 12.2. We set

$$f_0 = \mathcal{U}_{2m+2}(ab^s, ab^t,\ h_i^{-1}ab^{k_i}h_i,\ h_i^{-1}ab^{l_i}h_i;\ i = 1, \ldots, m).$$

Now we introduce two new variables y, z and set

$$f_1^{-1} = \mathcal{U}_{2m+2}\big(yz^s, yz^t,\ w_i(\underline{x})^{-1}yz^{k_i}w_i(\underline{x}),\ w_i(\underline{x})^{-1}yz^{l_i}w_i(\underline{x});\ i = 1, \ldots, m\big).$$

We show that the splitted equation f written in the form

$$f_1^{-1} = f_0. \tag{13.2}$$

satisfies the statements (a) and (b) of Theorem A.

(a) Suppose that f has a solution in H, say

$$(x_1, \dots, x_n, y, z) = (C_1, \dots, C_n, A, B).$$

We shall show that there exists $\alpha \in \mathbb{Z}$ such that $(C_1, \dots, C_n)^{f_0^{-\alpha}}$ is a solution of the system S.

Set $H_i = w_i(\underline{C})$. Then we have that

$$\begin{aligned} &\mathcal{U}_{2m+2}(ab^s, ab^t,\ h_i^{-1}ab^{k_i}h_i,\ h_i^{-1}ab^{l_i}h_i;\ i = 1, \dots, m) \\ &\quad = \mathcal{U}_{2m+2}(AB^s, AB^t,\ H_i^{-1}AB^{k_i}H_i,\ H_i^{-1}AB^{l_i}H_i;\ i = 1, \dots, m). \end{aligned} \tag{13.3}$$

By definition of the test word (see Definition 11.1) applied to $\mathcal{U}_{2m+2}$ and (13.3), there exists $\alpha \in \mathbb{Z}$ such that the formulas (13.4)–(13.7) are valid:

$$(ab^s)^{f_0^\alpha} = AB^s, \tag{13.4}$$

$$(ab^t)^{f_0^\alpha} = AB^t, \tag{13.5}$$

$$(h_i^{-1}ab^{k_i}h_i)^{f_0^\alpha} = H_i^{-1}AB^{k_i}H_i, \tag{13.6}$$

$$(h_i^{-1}ab^{l_i}h_i)^{f_0^\alpha} = H_i^{-1}AB^{l_i}H_i. \tag{13.7}$$

It follows from (13.4) and (13.5) that $(b^{s-t})^{f_0^\alpha} = B^{s-t}$. Since b is special, we have that

$$B = b^{f_0^\alpha}. \tag{13.8}$$

From (13.4) and (13.8), we obtain

$$A = a^{f_0^\alpha}. \tag{13.9}$$

Substituting (13.8) and (13.9) in (13.6) and (13.7), we deduce that

$$\begin{aligned} h_i^{f_0^\alpha} H_i^{-1} &\in C_H\big((ab^{k_i})^{f_0^\alpha}\big) \cap C_H\big((ab^{l_i})^{f_0^\alpha}\big) \\ &\overset{(3.1)}{\subseteq} E_H\big((ab^{k_i})^{f_0^\alpha}\big) \cap E_H\big((ab^{l_i})^{f_0^\alpha}\big) \\ &= \big\langle (ab^{k_i})^{f_0^\alpha} \big\rangle \cap \big\langle (ab^{l_i})^{f_0^\alpha} \big\rangle. \end{aligned}$$

The latter equation holds since the elements ab^{k_i} and ab^{l_i} are special. This intersection is trivial since ab^{k_i} and ab^{l_i} are non-commensurable. Therefore,

$$w_i(\underline{C}) = h_i^{f_0^\alpha}$$

for $i = 1, \dots, m$. Hence $\underline{C}^{f_0^{-\alpha}}$ is a solution of S.

(b) Since $V_G(f)$ is invariant under conjugation by the element f_0, it suffices to check that

$$\mathbf{pr}_n\big(V_G(f)\big) \supseteq V_G(S).$$

The latter is trivial: if $(x_1, \dots, x_n) = (c_1, \dots, c_n)$ is a solution of the system S in G, then

$$(x_1, \dots, x_n, y, z) = (c_1, \dots, c_n, a, b)$$

is a solution of (13.2) in G.

(3) We may again assume that S has the form (13.1). We may additionally assume that the set $\{h_1, \dots, h_m\}$ of right sides of equations from S contains two non-commensurable special elements from H; otherwise we could take two non-commensurable special elements $u, v \in H$ and add two equations $x_{n+1} = u$ and $x_{n+2} = v$ to S. Obviously, the set of solutions of the old system S is a projection of the set of solutions of the new system S.

In the following, we will use the tuple T, the element f_0, and the equation f defined in (a). Thus, we have that

$$\mathbf{pr}_n\big(V_H(f)\big) = \bigcup_{\alpha\in\mathbb{Z}} V_H(S)^{f_0^\alpha}. \tag{13.10}$$

By Corollary 12.2, all components of T together with f_0 are pairwise non-commensurable and jointly special. Let T' be the tuple obtained from T by adding the component f_0:

$$T' = (f_0,\ ab^s,\ ab^t,\ h_i^{-1}ab^{k_i}h_i,\ h_i^{-1}ab^{l_i}h_i;\ i = 1,\dots,m).$$

Let $\mathcal{U}_{2m+3}$ be the T'-test word from Corollary 12.1. We set

$$g_0 = \mathcal{U}_{2m+3}(f_0,\ ab^s,\ ab^t,\ h_i^{-1}ab^{k_i}h_i,\ h_i^{-1}ab^{l_i}h_i;\ i = 1,\dots,m).$$

Now we introduce new variables t, y, z and define the word

$$g_1^{-1} = \mathcal{U}_{2m+3}\big(t,\ yz^s,\ yz^t,\ w_i(\underline{x})^{-1}yz^{k_i}w_i(\underline{x}),\ w_i(\underline{x})^{-1}yz^{l_i}w_i(\underline{x});\ i = 1,\dots,m\big).$$

Let g be the equation g_1g_0. Using the same arguments as in the proof of (a), we obtain

$$\mathbf{pr}_n\big(V_H(g)\big) = \bigcup_{\alpha\in\mathbb{Z}} V_H(S)^{g_0^\alpha}. \tag{13.11}$$

Claim 1. Suppose that for some $\alpha, \beta \in \mathbb{Z}$ we have that

$$V_H(S)^{f_0^\alpha} \cap V_H(S)^{g_0^\beta} \neq \emptyset.$$

Then $\alpha = \beta = 0$.

Proof. By assumption, the set $\{h_1, \dots, h_m\}$ of right sides of equations from S contains two non-commensurable special elements, say u, v. Then

$$u^{f_0^\alpha} = u^{g_0^\beta} \quad \text{and} \quad v^{f_0^\alpha} = v^{g_0^\beta},$$

and we deduce that

$$f_0^\alpha g_0^{-\beta} \in E_H(u) \cap E_H(v) = \langle u\rangle \cap \langle v\rangle = 1.$$

The penultimate equation holds since u and b are special, and the latter equation holds since u and v are non-commensurable.

By Corollary 12.2, all components of T' together with g_0 are pairwise non-commensurable and jointly special. In particular, g_0 and f_0 are non-commensurable and have infinite orders. From this and $f_0^\alpha = g_0^\beta$, we obtain $\alpha = \beta = 0$. ■

This claim and equations (13.10) and (13.11) imply that

$$V_H(S) = \mathbf{pr}_n\big(V_H(f)\big) \cap \mathbf{pr}_n\big(V_H(g)\big). \qquad \blacksquare$$

Remark 13.2. In the general case, one splitted equation in statement (3) of Theorem A is not sufficient. Indeed, let H be an arbitrary nontrivial group and let $S = \{w_i(x_1, \dots, x_n) = h_i \mid i = 1, \dots, m\}$, $m \geqslant 2$, be a finite system of splitted equations with constants h_i from H such that $C_H(h_1) \cap C_H(h_2) = 1$ and $V_H(S) \neq \emptyset$. Then for any splitted equation $f \in F_k * H$, where $k \geqslant n$, we have that

$$V_H(S) \neq \mathbf{pr}_n\big(V_H(f)\big).$$

This follows from the following observations.

(a) If f is a splitted equation of the form $f_1 f_0$, where $f_1 \in F_k$ and $f_0 \in H$, then $(V_H(f))^{f_0} = V_H(f)$. Moreover, we have that $(V_H(f))^g = V_H(f)$ for any $g \in H$ if $f_0 = 1$.

(b) $V_H(S)^g \cap V_H(S) = \emptyset$ for every nontrivial $g \in H$. This can be proved similarly to the proof of Claim 1.

14. Proof of Theorem B

Proof of Theorem B. Suppose that H is verbally closed in G. We show that H is algebraically closed in G. Let S be a finite system of equations with constants from H such that $V_G(S) \neq \emptyset$. We shall show that $V_H(S) \neq \emptyset$.

Let f be a splitted equation as in statement (2) of Theorem A. By part (b) of this statement, we have that $V_G(S) \subseteq \mathbf{pr}_n(V_G(f))$, hence $V_G(f) \neq \emptyset$. Since H is verbally closed in G, we have that $V_H(f) \neq \emptyset$. By part (a) of statement (2) of Theorem A, we have that $V_H(S) \neq \emptyset$. Thus, H is algebraically closed in G. The converse implication is obvious. $\blacksquare$

Remark 14.1. Consider the free product

$$H = \mathop{*}_{\alpha \in \mathfrak{A}} H_\alpha,$$

where $\mathfrak{A}$ is an arbitrary set of cardinal larger than 1 and each H_α is nontrivial. In [38], Mazhuga showed that if H is a verbally closed subgroup of a group G, then H is algebraically closed in G. This result (except the very special case $H = \mathbb{Z}_2 * \mathbb{Z}_2$, which was first considered in [31]) follows from our Theorem B.

Indeed, H can be splitted as $H = A * B$, where A and B are nontrivial; hence H is relatively hyperbolic with respect to $\{A, B\}$. It is well known that if a non-(virtually cyclic) group is relatively hyperbolic with respect to a collection of proper subgroups, then it is acylindrically hyperbolic. Therefore, if $H \ncong \mathbb{Z}_2 * \mathbb{Z}_2$, then H is acylindrically hyperbolic, and we can apply Theorem B.

Remark 14.2. In Theorem B and Corollary B1, the assumption that H does not have nontrivial finite normal subgroups cannot be omitted. Indeed, consider two copies of the dihedral group D_4:

$$A = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ b^{-1}ab = a^{-1} \rangle,$$
$$B = \langle c, d \mid c^4 = 1,\ d^2 = 1,\ d^{-1}cd = c^{-1} \rangle.$$

Let $\varphi : B \to A$ be the isomorphism sending c to a and d to b. We write $A \underset{a^2=c^2}{\times} B$ for the quotient of the direct product $A \times B$ by the cyclic subgroup $\langle (a^2, c^2) \rangle$. We identify A and B with their canonical isomorphic images in this quotient. Then the elements of $A \underset{a^2=c^2}{\times} B$ can be written as pq, where $p \in A$, $q \in B$. If $p, p_1 \in A$ and $q, q_1 \in B$, then $pq = p_1q_1$ in this quotient if and only if $p = p_1$ and $q = q_1$, or $p_1 = pa^2$ and $q_1 = qc^2$. Let F be the free group of rank 2. We set

$$G = F \times (A \underset{a^2=c^2}{\times} B) = (F \times A) \underset{a^2=c^2}{\times} B$$

and consider $H = F \times A$ as a subgroup of G. Clearly, H is hyperbolic. Since H is not virtually cyclic, it is acylindrically hyperbolic.

Claim. The following statements hold.

(a) A is verbally closed in $A \underset{a^2=c^2}{\times} B$.

(b) H is verbally closed in G.

(c) H is not a retract of G.

(d) H is not algebraically closed in G.

Proof. (a) Suppose that an equation $W(x_1, \dots, x_n) = v1$, where $v \in A$, has a solution $x_1 = p_1q_1, \dots, x_n = p_nq_n$ in $A \underset{a^2=c^2}{\times} B$. We shall find a solution in A. Using commutativity, we deduce that $W(p_1, \dots, p_n)W(q_1, \dots, q_n) = v1$. Then we have two cases.

Case 1. $W(p_1, \dots, p_n) = v$ and $W(q_1, \dots, q_n) = 1$.

Then $(p_1, \dots, p_n)$ is the desired solution.

Case 2. $W(p_1, \dots, p_n) = va^2$ and $W(q_1, \dots, q_n) = c^2$.

If the exponent sum of some letter x_i in W, say of the letter x_1, is odd, then $(p_1a^2, \dots, p_n)$ is the desired solution. Suppose that the exponent sum of any x_i in W is even. Then $W(p_1, \dots, p_n) \in A^2 = \{1, a^2\}$, hence $v = 1$ or $v = a^2$. If $v = 1$, then $(1, \dots, 1)$ is the desired solution, and if $v = a^2$, then $(\varphi(q_1), \dots, \varphi(q_n))$ is one.

Statement (b) follows from (a) by using the fact that F is a complementary direct summand to $A \underset{a^2=c^2}{\times} B$ in G.

(c) Suppose that $\psi : G \to H$ a retraction. Then

$$\big[\psi(B), H\big] = \big[\psi(B), \psi(H)\big] = \psi\big([B, H]\big) = 1.$$

Therefore, $\psi(B) \subseteq \langle a^2 \rangle$. Since $\langle a^2 \rangle = H \cap B$, we have that $\psi(B) = \langle a^2 \rangle$. We obtain that $\langle c^2 \rangle$ is a retract of B. A contradiction.

Statement (d) follows from (c) and Proposition 2.3. ■

15. Solution to a problem of Myasnikov and Roman'kov

As it was mentioned in the introduction, the following corollary follows directly from Theorem B and Proposition 2.3.

Corollary B1. *Let H be a subgroup of a group G such that at least one of the following holds:*

(a) *H is finitely generated and G is finitely presented;*

(b) *H is equationally Noetherian and G is finitely generated over H.*

Suppose additionally that H is acylindrically hyperbolic and does not have nontrivial finite normal subgroups. Then the following three statements are equivalent:

(1) *H is algebraically closed in G,*

(2) *H is verbally closed in G,*

(3) *H is a retract of G.*

In this section, we deduce three further corollaries from Corollary B1. The last one, Corollary 15.8, solves Problem 5.2 from the paper [40].

Note that these corollaries allow H to be a virtually cyclic group, and Corollary B1 not. Therefore, we first prove Proposition 15.3, which deals with the case where H is virtually cyclic.

Though this proposition follows from statement (1) in [31, Theorem 1] about virtually free subgroups, we prefer to indicate a proof which uses only [31, Theorem 2] about dihedral subgroups. The first lemma is simple; we extracted its proof from the proof of [40, Lemma 3.1].

Lemma 15.1. *Let G be a group such that its abelianization G^{ab} is finitely generated. Let H be an infinite cyclic subgroup of G. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Proof. It suffices to prove that if H is verbally closed in G, then H is a retract of G. Thus, suppose that $H = \langle h \rangle$ is an infinite cyclic verbally closed subgroup of G. Let $\mathrm{Tor}(G^{ab})$ be the subgroup of G^{ab} consisting of all elements of finite order and let $\varphi : G \to G^{ab}/\mathrm{Tor}(G^{ab})$ be the canonical homomorphism. We claim that $H \cap [G, G] = 1$. Indeed, suppose that $h_1 = [g_1, g_2] \cdots [g_{2k-1}, g_{2k}]$ for some $h_1 \in H$ and $g_i \in G$, $i = 1, \dots, 2k$. Consider the equation $h_1 = [x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$. Since this equation has a solution in G, it has a solution in H. Since H is abelian, we have that $h_1 = 1$.

Thus, φ embeds H into $G^{ab}/\mathrm{Tor}(G^{ab})$. We claim that $\varphi(h)$ is primitive in the free abelian group $G^{ab}/\mathrm{Tor}(G^{ab})$. Indeed, otherwise we would have that $\varphi(h) = \varphi(g)^t$ for some $g \in G$ and $t \geqslant 2$. Let s be the order of $\mathrm{Tor}(G^{ab})$. Then $h^s = g^{st}[g_1, g_2] \cdots [g_{2k-1}, g_{2k}]$

for some $g_i \in G$, $i = 1, \dots, 2k$. Consider the equation $h^s = x^{st}[x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$. Since this equation has a solution in G, it has a solution in H. Then $h^s = h_1^{st}$ for some $h_1 \in H$. Hence $h = h_1^t$ and we have that $t = \pm 1$. A contradiction.

Thus, H is embedded into $G^{ab}/\operatorname{Tor}(G^{ab})$ as a direct summand. Hence H is a retract of G. ∎

Lemma 15.2 (see [31, Theorem 2]). *Suppose that H is an infinite dihedral subgroup of a finitely generated group G. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Proposition 15.3 (see also [31, Theorem 1]). *Let H be a virtually cyclic subgroup of a finitely generated group G. Suppose that H does not have nontrivial finite normal subgroups. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Proof. We may assume that H is nontrivial. It is well known (see, for example, [16, Lemma 2.5]) that every virtually cyclic group has a finite-by-cyclic subgroup of index at most 2. Thus, there exists a subgroup $H_0 \leqslant H$ of index at most 2 and a finite normal subgroup $K \leqslant H_0$ such that H_0/K is cyclic. By assumptions, H cannot be finite. Therefore, $H_0/K \cong \mathbb{Z}$, which implies that K is the largest finite normal subgroup of H_0. Hence K is normal in H. Since H does not have nontrivial finite normal subgroups, we obtain that $K = 1$. Then H is either infinite cyclic or infinite dihedral, and the statement follows from Lemmas 15.1 and 15.2. ∎

The following lemma says that, in some sense, generic subgroups of relatively hyperbolic groups are acylindrically hyperbolic. For terminology concerning relatively hyperbolic groups we refer to [49].

Lemma 15.4. *Suppose that G is a group that is relatively hyperbolic with respect to a collection of subgroups $\{P_\lambda\}_{\lambda \in \Lambda}$. Suppose that H is a non-(virtually cyclic) non-parabolic subgroup of G. Then H is acylindrically hyperbolic.*

In particular, the following groups are acylindrically hyperbolic:

(1) *non-(virtually cyclic) groups that are hyperbolic relative to a collection of proper subgroups (see also [50]);*

(2) *non-(virtually cyclic) subgroups of hyperbolic groups.*

Proof. Let X be a finite relative generating set of G and let $\mathcal{P} = \bigsqcup_{\lambda \in \Lambda} P_\lambda$. Then the Cayley graph $\Gamma(G, X \sqcup \mathcal{P})$ is hyperbolic by [49, Corollary 2.54] and the action of G on $\Gamma(G, X \sqcup \mathcal{P})$ is acylindrical by [47, Proposition 5.2]. In particular, H acts acylindrically on $\Gamma(G, X \sqcup \mathcal{P})$.

By [7, Lemma 2.9], a subgroup of a relatively hyperbolic group contains a loxodromic element if and only if it is infinite and non-parabolic. Thus, H contains a loxodromic element. Hence H has unbounded orbits acting on $\Gamma(G, X \sqcup \mathcal{P})$. Then the statement follows from Theorem 3.9. ∎

The first two corollaries are about relatively hyperbolic groups, and in their proofs we use the following remarkable result of Groves and Hull.

Theorem 15.5 (see [19, Theorem D]). *Suppose that G is a relatively hyperbolic group with respect to a finite collection of subgroups $\{H_1, \dots, H_n\}$. Then G is equationally Noetherian if and only if each H_i is equationally Noetherian.*

Corollary 15.6. *Let G be a group and let H be a subgroup of G such that G is finitely generated over H. Suppose that H is hyperbolic relative to a finite collection of equationally Noetherian proper subgroups and does not have nontrivial finite normal subgroups. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Proof. By Proposition 15.3, we may assume that H is non-(virtually cyclic). Then, by Lemma 15.4, H is acylindrically hyperbolic. Moreover, H is equationally Noetherian by the result of Groves and Hull [19, Theorem D]. Then the statement follows from Corollary B1. ∎

The proof of the following corollary is similar to that of the previous one; we give it for completeness.

Corollary 15.7. *Let G be a relatively hyperbolic group with respect to a finite collection of finitely generated equationally Noetherian subgroups. Suppose that H is a non-parabolic subgroup of G such that H does not have nontrivial finite normal subgroups. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Proof. It follows from the assumptions that G is finitely generated. By Proposition 15.3, we may assume that H is non-(virtually cyclic). Then, by Lemma 15.4, H is acylindrically hyperbolic. By the result of Groves and Hull [19, Theorem D], G is equationally Noetherian. Any subgroup of an equationally Noetherian group is equationally Noetherian. Therefore, H is equationally Noetherian. Then the statement follows from Corollary B1. ∎

The following corollary follows directly from the previous one. Indeed, every hyperbolic group is relatively hyperbolic with respect to the trivial subgroup.

Corollary 15.8 (Solution to Problem 5.2 in [40]). *Let G be a hyperbolic group and H a subgroup of G. Suppose that H does not have nontrivial finite normal subgroups. Then the conditions that H is algebraically closed in G, H is verbally closed in G, and H is a retract of G are equivalent.*

Remark 15.9. This corollary can be proved without using Corollary 15.7. A direct proof can be obtained from the above proof if, instead of the result of Groves and Hull, we will use the result of Reinfeldt and Weidmann [65, Corollary 6.13] that all hyperbolic groups are equationally Noetherian.

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