Groups Geom. Dyn. 16 (2022), 705–723
DOI 10.4171/GGD/662

© 2022 European Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

The boundary at infinity of the curve complex and the relative Teichmüller space

Erica Klarreich

Abstract. In this paper we study the boundary at infinity of the curve complex $\mathcal{C}(S)$ of a surface S of finite type and the relative Teichmüller space $\mathcal{T}_{\mathrm{el}}(S)$ obtained from the Teichmüller space by collapsing each region where a simple closed curve is short to be a set of diameter 1. $\mathcal{C}(S)$ and $\mathcal{T}_{\mathrm{el}}(S)$ are quasi-isometric, and Masur–Minsky have shown that $\mathcal{C}(S)$ and $\mathcal{T}_{\mathrm{el}}(S)$ are hyperbolic in the sense of Gromov. We show that the boundary at infinity of $\mathcal{C}(S)$ and $\mathcal{T}_{\mathrm{el}}(S)$ is the space of topological equivalence classes of minimal foliations on S.

1. Introduction

There is a strong but limited analogy between the geometry of the Teichmüller space $\mathcal{T}(S)$ of a surface S and that of hyperbolic spaces. Teichmüller space has many of the large-scale qualities of hyperbolic space, and in fact the Teichmüller space of the torus is $\mathbb{H}^2$. At one point it was generally believed that the Teichmüller metric was negatively curved; however, Masur [13] showed that this is not so, apart from a few exceptional cases. Since then, Masur and Wolf [15] showed that $\mathcal{T}(S)$ is not even hyperbolic in the sense of Gromov.

One way in which $\mathcal{T}(S)$ differs from hyperbolic space is that it does not have a canonical compactification. A Gromov hyperbolic space has a boundary at infinity that is natural in the following two senses, among others: the boundary consists of all end-points of quasi-geodesic rays up to equivalence (two rays are equivalent if they stay a bounded distance from each other), and every isometry of the space extends continuously to a homeomorphism of the boundary. Teichmüller space cannot be equipped with such a compactification but rather gives rise to several compactifications, each with advantages and drawbacks.

Questions about the boundary of a hyperbolic space are interesting for many reasons; one is that they tie in to questions of rigidity of group actions by isometry on the space. For example, in the proof of Mostow's rigidity theorem, a key step in showing that two hyperbolic structures on the same compact 3-manifold are isometric is to show that a quasi-isometry between the two structures lifts to a map of $\mathbb{H}^3$ that extends continuously

2020 Mathematics Subject Classification. Primary 57M50; Secondary 20F67, 30F60, 32G15.
Keywords. Curve complex, boundary at infinity, relative Teichmüller space, measured foliation.

to $\partial_\infty \mathbb{H}^3$ (the Riemann sphere), and then to gain some control over the map on $\partial_\infty \mathbb{H}^3$. In another instance, Sullivan's rigidity theorem gives geometric information about a hyperbolic 3-manifold based on quasi-conformal information about its associated group action on $\partial_\infty \mathbb{H}^3$.

Although Teichmüller space is not hyperbolic, it is natural to be interested in boundaries of Teichmüller space, since they have a strong connection to deformation spaces of hyperbolic 3-manifolds. If M is a compact 3-manifold, there is a well-known parametrization of the space of geometrically finite hyperbolic structures on $\text{int}(M)$ by the Teichmüller space of the boundary of M; see [1]. One question is to understand the behavior of the hyperbolic structure on M as the Riemann surface structure on ∂M "degenerates", that is, goes to infinity in the Teichmüller space. More generally, an important problem in the theory is to describe all geometrically infinite hyperbolic structures on M; for this purpose Thurston has introduced an invariant called the *ending lamination* of ∂M, intended to play a similar role to that of the Teichmüller space of ∂M in the geometrically finite setting. Two important boundaries of $\mathcal{T}(S)$ by Teichmüller and Thurston involve compactifying $\mathcal{T}(S)$ by the measured foliation space, or equivalently the measured lamination space, which is related to but not the same as the space of possible ending laminations on S.

Masur and Minsky [14] have shown that although Teichmüller space is not Gromov hyperbolic, it is *relatively hyperbolic* with respect to a certain collection of closed subsets. In this paper we describe the boundary at infinity of the relative Teichmüller space and a closely related object, the curve complex. If α is a homotopy class of simple closed curves on S, a surface of finite type, let Thin_α denote the region of $\mathcal{T}(S)$ where the extremal length of α is less than or equal to ε, for some fixed small $\varepsilon > 0$. These regions play a role somewhat similar to that of horoballs in hyperbolic space; in fact for the torus, these regions are actual horoballs in $\mathbb{H}^2$. However, Minsky [19] has shown that in general the geometry of each region Thin_α is not hyperbolic, but rather has the large-scale geometry of a product space with the sup metric. On the other hand, these regions are in a sense the only obstacle to hyperbolicity: Masur and Minsky [14] have shown that Teichmüller space is relatively hyperbolic with respect to the family of regions $\{\text{Thin}_\alpha\}$. In other words, the *electric Teichmüller space* $\mathcal{T}_{\text{el}}(S)$ obtained from $\mathcal{T}(S)$ by collapsing each region Thin_α to diameter 1 is Gromov hyperbolic (this collapsing is done by adding a point for each set Thin_α that is distance $\frac{1}{2}$ from each point in Thin_α).

Since $\mathcal{T}_{\text{el}}(S)$ is Gromov hyperbolic, it can be equipped with a boundary at infinity $\partial_\infty \mathcal{T}_{\text{el}}(S)$. Our main result is the following:

Theorem 1.1. *The boundary at infinity of $\mathcal{T}_{\text{el}}(S)$ is homeomorphic to the space of minimal topological foliations on S.*

A foliation is minimal if no trajectory is a simple closed curve. This space of minimal foliations is exactly the space of possible ending laminations (or foliations) on a surface S that corresponds to a geometrically infinite end of a hyperbolic manifold that has no parabolics (see [23]). Here the topology on the space of minimal foliations is that obtained from the measured foliation space by forgetting the measures. This topology is Hausdorff (see Appendix A), unlike the topology on the full space of topological foliations; hence we may prove Theorem 1.1 using sequential arguments to establish continuity.

We will prove Theorem 1.1 by showing that the inclusion of $\mathcal{T}(S)$ in $\mathcal{T}_{\mathrm{el}}(S)$ extends continuously to a portion of the Teichmüller compactification of $\mathcal{T}(S)$ by the projective measured foliation space $\mathcal{PMF}(S)$:

Theorem 1.2. *The inclusion map from $\mathcal{T}(S)$ to $\mathcal{T}_{\mathrm{el}}(S)$ extends continuously to the portion $\mathcal{PMF}_{\min}(S)$ of $\mathcal{PMF}(S)$ consisting of minimal foliations, to give a map $\pi : \mathcal{PMF}_{\min}(S) \to \partial_\infty \mathcal{T}_{\mathrm{el}}(S)$. The map π is surjective, and $\pi(\mathcal{F}) = \pi(\mathcal{G})$ if and only if $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent. Moreover, any sequence $\{x_n\}$ in $\mathcal{T}(S)$ that converges to a point in $\mathcal{PMF}(S) \setminus \mathcal{PMF}_{\min}(S)$ cannot accumulate in the electric space onto any portion of $\partial_\infty \mathcal{T}_{\mathrm{el}}(S)$.*

If $\mathcal{F}_{\min}(S)$ is the space of minimal topological foliations on the surface S, then the map $\pi : \mathcal{PMF}_{\min}(S) \to \partial_\infty \mathcal{T}_{\mathrm{el}}(S)$ descends to a homeomorphism from $\mathcal{F}_{\min}(S)$ to $\partial_\infty \mathcal{T}_{\mathrm{el}}(S)$; hence Theorem 1.1 is a consequence of Theorem 1.2.

Another space that we can associate to S that has a close connection to the electric Teichmüller space is the *curve complex* $\mathcal{C}(S)$, originally described by Harvey in [10]. $\mathcal{C}(S)$ is a simplicial complex whose vertices are homotopy classes of non-peripheral simple closed curves on S. A collection of curves forms a simplex if all the curves may be simultaneously realized so that they are pairwise disjoint (when S is the torus, once-punctured torus or four-punctured sphere, it is appropriate to make a slightly different definition; see Section 4). $\mathcal{C}(S)$ can be given a metric structure by assigning to each simplex the geometry of a regular Euclidean simplex whose edges have length 1.

In the construction of the electric Teichmüller space, if the value of ε used to define the sets Thin_α is sufficiently small, then Thin_α and Thin_β intersect exactly when α and β have disjoint realizations on S, that is, when the elements α and β of $\mathcal{C}(S)$ are connected by an edge. Hence the 1-skeleton $\mathcal{C}_1(S)$ of $\mathcal{C}(S)$ describes the intersection pattern of the sets Thin_α; $\mathcal{C}_1(S)$ is the nerve of the collection $\{\mathrm{Thin}_\alpha\}$. The relationship between $\mathcal{T}_{\mathrm{el}}(S)$ and $\mathcal{C}(S)$ is not purely topological. Masur and Minsky [14] have shown that $\mathcal{T}_{\mathrm{el}}(S)$ is quasi-isometric to $\mathcal{C}_1(S)$ and $\mathcal{C}(S)$. This implies that $\mathcal{C}_1(S)$ and $\mathcal{C}(S)$ are also Gromov hyperbolic (although in the proof of Masur and Minsky, the implication goes in the other direction). Two Gromov hyperbolic spaces that are quasi-isometric have the same boundary at infinity, so a consequence of Theorem 1.1 is the following:

Theorem 1.3. *The boundary at infinity of the curve complex $\mathcal{C}(S)$ is the space of minimal foliations on S.*

As with Teichmüller space, the curve complex is important in the study of hyperbolic 3-manifolds. Let M be a compact 3-manifold whose interior admits a complete hyperbolic structure, and suppose S is a component of ∂M that corresponds to a geometrically infinite end e of M. Thurston [25], Bonahon [2], and Canary [3] have shown that there is a sequence of simple closed curves $\alpha_n \in \mathcal{C}_1(S)$ whose geodesic representatives in M "exit the end e", that is, are contained in smaller and smaller neighborhoods of S in M.

Further, they showed that every such sequence converges to a unique geodesic lamination (equivalently, foliation) on S. In the case when the hyperbolic structure on int(M) has a uniform lower bound on injectivity radius, Minsky [17, 18] has shown that the sequence $\{\alpha_n\}$ is a quasi-geodesic in $\mathcal{C}_1(S)$; a form of this was a key step in his proof of the *ending lamination conjecture* for such manifolds, giving quasi-isometric control of the ends of M.

Since the sequence $\{\alpha_n\}$ is a quasi-geodesic in $\mathcal{C}_1(S)$, it must converge to a point $\mathcal{F}$ in the boundary at infinity of $\mathcal{C}_1(S)$, which we have described as the space of minimal foliations (or laminations) on S. We will show that this description is natural, so that in particular when the sequence $\{\alpha_n\}$ in $\mathcal{C}_1(S)$ arises as described above in the context of hyperbolic 3-manifolds, the boundary point $\mathcal{F}$ is the ending lamination.

Theorem 1.4. *Let $\{\alpha_n\}$ be a sequence of elements of $\mathcal{C}_1(S)$ that converges to a foliation $\mathcal{F}$ in the boundary at infinity of $\mathcal{C}(S)$. Then regarding the curves α_n as elements of the projective measured foliation space $\mathcal{PMF}(S)$, every accumulation point of $\{\alpha_n\}$ in $\mathcal{PMF}(S)$ is topologically equivalent to $\mathcal{F}$.*

It is interesting to note that our description of the boundary of $\mathcal{C}(S)$ ultimately does not depend on our original choice of a Teichmüller compactification for $\mathcal{T}(S)$, even though the Teichmüller boundary of $\mathcal{T}(S)$ depends heavily on an initial choice of basepoint in $\mathcal{T}(S)$ (see Section 2 for more details). Kerckhoff [12] has shown that the action of the modular group by isometry on $\mathcal{T}(S)$ does not extend continuously to the Teichmüller boundary; on the other hand, the natural actions of the modular group on $\mathcal{T}_{\mathrm{el}}(S)$ and $\mathcal{C}(S)$ do extend to the boundary at infinity, since this is true of any action by isometry on a Gromov hyperbolic space. Hence the collapse used in the construction of $\mathcal{T}_{\mathrm{el}}(S)$ essentially "collapses" the discontinuity of the modular group action.

In Section 2 we will give an overview of some of the basic theory of Teichmüller space and quadratic differentials. Section 3 contains the essential ideas of Gromov hyperbolicity that we will need. In Section 4 we discuss in more detail Masur and Minsky's work on the electric Teichmüller space and the curve complex, and describe the quasi-isometry between them. In Section 5 we establish some facts about convergence properties of sequences of Teichmüller geodesics, which are used in Section 6 to prove the main theorems.

2. Quadratic differentials and the Teichmüller compactification of Teichmüller space

Let S be a surface of finite genus and finitely many punctures. The Teichmüller space $\mathcal{T}(S)$ is the space of all equivalence classes of conformal structures of finite type on S, where two conformal structures are equivalent if there is a conformal homeomorphism of one to the other that is isotopic to the identity on S. A conformal structure is of finite type if every puncture has a neighborhood that is conformally equivalent to a punctured disk. The Teichmüller distance between two points σ and $\tau \in \mathcal{T}(S)$ is defined by

$$d(\sigma, \tau) = \frac{1}{2} \log K(\sigma, \tau),$$

where $K(\sigma, \tau)$ is the minimal quasi-conformal dilatation of any homeomorphism from a representative of σ to a representative of τ in the correct homotopy class. The extremal map from σ to τ may be constructed explicitly using quadratic differentials.

A holomorphic quadratic differential q on a Riemann surface σ is a tensor of the form $q(z)dz^2$ in local coordinates, where $q(z)$ is holomorphic. We define

$$\|q\| = \iint_S |q(z)| dx dy.$$

Let $\mathcal{DQ}(\sigma)$ denote the open unit ball in the space $\mathcal{Q}(\sigma)$ of quadratic differentials on σ, and $\mathcal{SQ}(\sigma)$ the unit sphere.

Every $q \in \mathcal{DQ}(\sigma)$ determines a Beltrami differential $\|q\| \frac{\overline{q}}{|q|}$ on σ, which in turn determines a quasi-conformal map from σ to a new element τ_q of $\mathcal{T}(S)$; this map is the Teichmüller extremal map between σ and τ_q. The map that sends q to τ_q is a homeomorphism, giving an embedding of $\mathcal{T}(S)$ in $\mathcal{Q}(\sigma)$; $\mathcal{SQ}(\sigma)$ is the boundary of $\mathcal{T}(S)$ in $\mathcal{Q}(\sigma)$, and $\mathcal{T}(S) \cup \mathcal{SQ}(\sigma)$ gives a compactification of $\mathcal{T}(S)$ which we will denote $\overline{\mathcal{T}(S)}$, called the *Teichmüller compactification* of $\mathcal{T}(S)$.

Any $q \in \mathcal{Q}(\sigma)$ determines a pair $\mathcal{H}_q$ and $\mathcal{V}_q$ of measured foliations on S called the horizontal and vertical foliations. *Measured foliations* are equivalence classes of foliations of S with 3- or higher-pronged saddle singularities, equipped with transverse measures; the equivalence is by measure-preserving isotopy and Whitehead moves (that collapse singularities). We will denote the measured foliation space by $\mathcal{MF}(\mathcal{S})$ and the projectivized measured foliation space (obtained by scaling the measures) by $\mathcal{PMF}(S)$. The horizontal and vertical foliations associated to q give a metric on S in the conformal class of σ that is Euclidean away from the singularities. The map from $\mathcal{SQ}(\sigma)$ to $\mathcal{PMF}(S)$ defined by sending q to the projective class of its vertical foliation is a homeomorphism, so that we may think of $\mathcal{PMF}(S)$ as the boundary of $\mathcal{T}(S)$ (see [11]).

A unit-norm quadratic differential q on σ determines a directed geodesic line in $\mathcal{T}(S)$ as follows: for $0 \leq k < 1$, let σ_k denote the element of $\mathcal{T}(S)$ determined by the quasi-conformal homeomorphism given by the quadratic differential $k \cdot q$. Geometrically, the extremal map from σ to σ_k is obtained by contracting the transverse measure of $\mathcal{H}_q$ by a factor of $K^{-\frac{1}{2}}$ and expanding the transverse measure of $\mathcal{V}_q$ by $K^{\frac{1}{2}}$, where $K = \frac{1+k}{1-k}$; note that the extremal map is K-quasi-conformal. The family $\{\sigma_k : 0 \leq k < 1\}$, when parametrized by Teichmüller arclength, gives a Teichmüller geodesic ray; the family $\{\sigma_k : -1 < k < 1\}$ determines a complete geodesic line. Every ray and line through σ is so determined. We may think of the Teichmüller ray $\{\sigma_k : 0 \leq k < 1\}$ as terminating at the boundary point $q \in \mathcal{SQ}(\sigma)$, or equivalently, at the projective foliation $\mathcal{V}_q \in \mathcal{PMF}(S)$. Similarly, every pair of foliations in $\mathcal{PMF}(S)$ that fills up S (see the next section for the definition of filling up) determines a geodesic line in $\mathcal{T}(S)$, for which if $\tau \in L$ then the quadratic differential on τ that determines L has the two foliations as its horizontal and vertical foliations.

The compactification of $\overline{\mathcal{T}(S)}$ by endpoints of geodesic rays depends in a fundamental way on the choice of basepoint σ in $\mathcal{T}(S)$. Kerckhoff [12] has shown that there exist projective foliations $\mathcal{F} \in \mathcal{PMF}(S)$ such that there are choices of $\tau \in \mathcal{T}(S)$ for which the Teichmüller ray from τ determined by $\mathcal{F}$ does not converge in $\overline{\mathcal{T}(S)}$ to $\mathcal{F}$, but rather accumulates onto a portion of $\mathcal{PMF}(S)$ consisting of projective foliations that are topologically equivalent but not measure equivalent to $\mathcal{F}$.

Intersection number. If α is a simple closed curve on S then α determines a foliation on S (which we will also call α) whose non-singular leaves are all freely homotopic to α. The non-singular leaves form a cylinder, and S is obtained by gluing the boundary curves in some preassigned manner. There is a one-to-one correspondence between transverse measures on α and positive real numbers: each measure corresponds to the height of the cylinder, that is, the minimal transverse measure of all arcs connecting the two boundary curves of the cylinder. If a measure has height c, we will denote the measured foliation by $c \cdot \alpha$. We define the *intersection number* of the foliations $c \cdot \alpha$ and $k \cdot \beta$ by

$$i(c \cdot \alpha, k \cdot \beta) = ck \cdot i(\alpha, \beta)$$

where the right-hand intersection number is just the geometric intersection number of the simple closed curves α and β (that is, the minimal number of crossings of any pair of representatives of α and β). Thurston has shown that the collection $\{c \cdot \alpha : \alpha \text{ a simple closed curve}, c \in \mathbb{R}^+\}$ is dense in $\mathcal{MF}(S)$, and that the intersection number extends continuously to a function $i : \mathcal{MF}(S) \times \mathcal{MF}(S) \to \mathbb{R}$ (see for instance [7]).

Note that although the intersection number of two *projective* measured foliations is not well-defined, it still makes sense to ask whether two projective measured foliations have zero or non-zero intersection number.

A foliation $\mathcal{F}$ is *minimal* if no leaves of $\mathcal{F}$ are simple closed curves. We say that two measured foliations are topologically equivalent if the topological foliations obtained by forgetting the measures are equivalent with respect to isotopy and Whitehead moves that collapse the singularities. Rees [24] has shown the following:

Proposition 2.1. *If $\mathcal{F}$ is minimal then $i(\mathcal{F}, \mathcal{G}) = 0$ if and only if $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent.*

We say that two foliations $\mathcal{F}$ and $\mathcal{G}$ *fill up* S if for every foliation $\mathcal{H} \in \mathcal{MF}(S)$, $\mathcal{H}$ has non-zero intersection number with at least one of $\mathcal{F}$ and $\mathcal{G}$. A consequence of Proposition 2.1 is that if $\mathcal{F}$ is minimal, then whenever $\mathcal{G}$ is not topologically equivalent to $\mathcal{F}$, $\mathcal{F}$ and $\mathcal{G}$ fill up S.

Let $\mathcal{F}_{\min}(S)$ denote the space of minimal topological foliations, with topology obtained from the space $\mathcal{PMF}_{\min}(S)$ of minimal projective measured foliations by forgetting the measures. Our goal is to show that $\mathcal{F}_{\min}(S)$ is homeomorphic to the boundary at infinity of the electric Teichmüller space. We will use sequential arguments to show that certain maps are continuous, so it is necessary to show the following proposition, whose proof can be found in Appendix A.

Proposition 2.2. *The space* $\mathcal{F}_{\min}(S)$ *is Hausdorff and first countable.*

The entire space $\mathcal{F}(S)$ of topological foliations on S is not Hausdorff. If α and β are two distinct homotopy classes of simple closed curves that can be realized disjointly on S, then regarded as topological foliations, α and β do not have disjoint neighborhoods; every neighborhood of α or β must contain the topological foliation containing both α and β, and whose non-singular leaves are all homotopic to α or β.

Extremal length. If γ is a free homotopy class of simple closed curves on S, an important conformal invariant is the extremal length of γ, which is defined as follows:

Definition 2.3. Let $\sigma \in \mathcal{T}(S)$, and let γ be a homotopy class of simple closed curves on S. The *extremal length* of γ on σ is defined by

$$\mathrm{ext}_\sigma(\gamma) = \sup_\rho \frac{(l_\rho(\gamma))^2}{A_\rho},$$

where ρ ranges over all metrics in the conformal class of σ, A_ρ denotes the area of S with respect to ρ, and $l_\rho(\gamma)$ is the infimum of the length of all representatives of γ with respect to ρ.

Extremal length may be extended to scalar multiples of simple closed curves by $\mathrm{ext}(k \cdot \gamma) = k^2\,\mathrm{ext}(\gamma)$, and extends continuously to the space of measured foliations.

Our goal is to describe the boundary of the relative Teichmüller space $\mathcal{T}_{\mathrm{el}}(S)$ obtained from $\mathcal{T}(S)$ by collapsing each of the regions Thin_γ of $\mathcal{T}(S)$ to be a set of bounded diameter, where Thin_γ is the region of $\mathcal{T}(S)$ where the simple closed curve γ has short extremal length. We will need the following lemma, which gives a connection between extremal length and intersection number (see for instance [21, Lemma 3.1–3.2] for a proof):

Proposition 2.4. *Let* q *be a quadratic differential with norm less than* 1 *on* $\sigma \in \mathcal{T}(S)$ *with horizontal and vertical foliations* $\mathcal{H}$ *and* $\mathcal{V}$, *and let* $\mathcal{F}$ *be a measured foliation on* S. *Then* $\mathrm{ext}_\sigma(\mathcal{F}) \geq (i(\mathcal{F},\mathcal{H}))^2$; *likewise* $\mathrm{ext}_\sigma(\mathcal{F}) \geq (i(\mathcal{F},\mathcal{V}))^2$.

3. Gromov-hyperbolic spaces

In this section we will present an overview of some of the basic theory of Gromov-hyperbolic spaces. References for the material in this section are [4, 5, 8, 9].

Let (Δ, d) be a metric space. If Δ is equipped with a basepoint 0, define the *Gromov product* $\langle x|y\rangle$ of the points x and y in Δ to be

$$\langle x|y\rangle = \langle x|y\rangle_0 = \frac{1}{2}(d(x,0) + d(y,0) - d(x,y)).$$

Definition 3.1. Let $\delta \geq 0$ be a real number. The metric space Δ is δ-*hyperbolic* if

$$\langle x|y\rangle \geq \min(\langle x|y\rangle, \langle y|z\rangle) - \delta$$

for every $x, y, z \in \Delta$ and for every choice of basepoint.

We say that Δ is hyperbolic in the sense of Gromov if Δ is δ-hyperbolic for some δ.

A metric space Δ is *geodesic* if any two points in Δ can be joined by a geodesic segment (not necessarily unique). If x and y are in Δ we write $[x, y]$, ambiguously, to denote some geodesic from x to y.

Heuristically, a δ-hyperbolic space is "tree-like"; more precisely, if we define an ε-*narrow* geodesic polygon to be one such that every point on each side of the polygon is at distance $\leq \varepsilon$ from a point in the union of the other sides, then we have the following:

Proposition 3.2. *In a geodesic δ-hyperbolic metric space, every n-sided polygon* $(n \geq 3)$ *is* $4(n-2)\delta$*-narrow.*

In a geodesic hyperbolic space, the Gromov product of two points x and y is roughly the distance from 0 to $[x, y]$; we have the following:

Proposition 3.3. *Let Δ be a geodesic, δ-hyperbolic space and let $x, y \in \Delta$. Then*

$$d(0, [x, y]) - 4\delta \leq \langle x|y\rangle \leq d(0, [x, y])$$

for every geodesic segment $[x, y]$.

The boundary at infinity of a hyperbolic space. If Δ is a hyperbolic space, Δ can be equipped with a boundary in a natural way. We say that a sequence $\{x_n\}$ of points in Δ *converges at infinity* if we have $\lim_{m,n\to\infty}\langle x_m|x_n\rangle = \infty$; note that this definition is independent of the choice of basepoint, by Proposition 3.3. Given two sequences $\{x_m\}$ and $\{y_n\}$ that converge at infinity, we say that $\{x_m\}$ and $\{y_n\}$ are *equivalent* if $\lim_{m,n\to\infty}\langle x_m|y_n\rangle = \infty$. Since Δ is hyperbolic, it is easily checked that this is an equivalence relation. Define the *boundary at infinity* $\partial_\infty\Delta$ of Δ to be the set of equivalence classes of sequences that converge at infinity. If $\xi \in \partial_\infty\Delta$ then we say that a sequence of points in Δ *converges to* ξ if the sequence belongs to the equivalence class ξ. Write $\overline{\Delta} = \Delta \cup \partial_\infty\Delta$. When the space Δ is a proper metric space, the boundary at infinity may also be described as the set of equivalence classes of quasi-geodesic rays, where two rays are equivalent if they are a bounded Hausdorff distance from each other.

Quasi-isometries and quasi-geodesics. Let Δ_0 and Δ be two metric spaces. Let $k \geq 1$ and $\mu \geq 0$ be real numbers. A quasi-isometry from Δ_0 to Δ is a relation R between elements of Δ_0 and Δ that has the coarse behavior of an isometry. Specifically, let R relate every element of Δ_0 to some subset of Δ (so that we allow a given point in Δ_0 to be related to multiple points in Δ). We say that R is a (k, μ)*-quasi-isometry* if for all x_1 and $x_2 \in \Delta_0$,

$$\frac{1}{k}d(x_1, x_2) - \mu \leq d(y_1, y_2) \leq kd(x_1, x_2) + \mu$$

whenever $x_1 R y_1$ and $x_2 R y_2$. Note that for a quasi-isometry, given $x \in \Delta_0$ there is an upper bound to the diameter of the set $\{y \in \Delta : xRy\}$, that is independent of x.

We say that R is a cobounded quasi-isometry if in addition, there is some constant L such that if $y \in \Delta$, y is within L of some point that is related by R to a point in Δ_0. If R is a cobounded quasi-isometry then R has a quasi-inverse, that is, a relation R' that relates each element of Δ to some subset of Δ_0, with the following property: there is some constant K for which if x and x' are elements of Δ_0 such that for some $y \in \Delta$, xRy and $yR'x'$, then $d(x, x') \leq K$.

A quasi-isometry between two δ-hyperbolic spaces extends continuously to the boundary, in the following sense:

Theorem 3.4. *Let Δ_0 and Δ be Gromov-hyperbolic, and let $h : \Delta_0 \to \Delta$ be a quasi-isometry. For every sequence $\{x_n\}$ of points in Δ_0 that converges to a point ξ in $\partial_\infty \Delta_0$, the sequence $\{h(x_n)\}$ converges to a point in $\partial_\infty \Delta$ that depends only on ξ, so that h defines a continuous map from $\partial_\infty \Delta_0$ to $\partial_\infty \Delta$. The map $h : \partial_\infty \Delta_0 \to \partial_\infty \Delta$ is injective.*

Theorem 3.4 is, among other things, a key step in the proof of Mostow's rigidity theorem.

If the metric on Δ is a path metric, a (k, μ)-*quasi-geodesic* is a rectifiable path $p : I \to \Delta$, where I is an interval in $\mathbb{R}$, such that for all s and t in I,

$$\frac{1}{k} l(p|_{[s,t]}) - \mu \leq d(p(s), p(t)) \leq k \cdot l(p|_{[s,t]}) + \mu.$$

Note that if a path $p : I \to \Delta$ is parametrized by arc length then it is a quasi-geodesic if and only if it is a quasi-isometry.

The behavior of quasi-geodesics in the large is like that of actual geodesics. In particular, we have the following analogue of Proposition 3.3:

Proposition 3.5. *Let $s : I \to \Delta$ be a quasi-geodesic with endpoints x and y. Then there are constants K and C that only depend on the quasi-geodesic constants of s and the hyperbolicity constant of Δ, such that*

$$\frac{1}{K} d(0, s(I)) - C \leq \langle x|y \rangle \leq K d(0, s(I)) + C.$$

4. The curve complex and the relative hyperbolic space

The curve complex. If S is an oriented surface of finite type, an important related object is a simplicial complex called the *curve complex*. Except in the cases when S is the torus, the once-punctured torus or a sphere with 4 or fewer punctures, we define the curve complex $\mathcal{C}(S)$ in the following way: the vertices of $\mathcal{C}(S)$ are homotopy classes of non-peripheral simple closed curves on S. Two curves are connected by an edge if they may be realized disjointly on S, and in general a collection of curves spans a simplex if the curves may be realized disjointly on S.

When S is a sphere with 3 or fewer punctures, there are no non-peripheral curves on S, so $\mathcal{C}(S)$ is empty. When S is the 4-punctured sphere, the torus, or the once-punctured torus, there are non-peripheral simple closed curves on S, but every pair of curves must intersect, so $\mathcal{C}(S)$ has no edges. For these three surfaces, a more interesting space to consider is the complex in which two curves are connected by an edge if they can be realized with the smallest intersection number possible on S (one for the tori; two for the sphere); we alter the definition of $\mathcal{C}(S)$ in this way. In these cases, $\mathcal{C}(S)$ is the Farey graph, which is well-understood (see for example [20, 22]).

We give $\mathcal{C}(S)$ a metric structure by making every simplex a regular Euclidean simplex whose edges have length 1. The main result of [14] is the following:

Theorem 4.1 (Masur–Minsky). *$\mathcal{C}(S)$ is a δ-hyperbolic space, where δ depends on S.*

Note that $\mathcal{C}(S)$ is clearly quasi-isometric to its 1-skeleton $\mathcal{C}_1(S)$, so that in particular $\mathcal{C}_1(S)$ is also Gromov-hyperbolic.

The relative Teichmüller space. For a fixed $\varepsilon > 0$, for each curve $\alpha \in \mathcal{C}_0(S)$ denote

$$\mathrm{Thin}_\alpha = \left\{\sigma \in \mathcal{T}(S) : \mathrm{ext}_\sigma(\alpha) \leq \varepsilon\right\}.$$

We will assume that ε has been chosen sufficiently small that the collar lemma holds; in that case, a collection of sets $\mathrm{Thin}_{\alpha_1}, \dots, \mathrm{Thin}_{\alpha_n}$ has non-empty intersection if and only if $\alpha_1, \dots, \alpha_n$ can be realized disjointly on S, that is, if $\alpha_1, \dots, \alpha_n$ form a simplex in $\mathcal{C}(S)$.

We form the *relative* or *electric Teichmüller space* $\mathcal{T}_{\mathrm{el}}(S)$ (following terminology of Farb [6]) by attaching a new point P_α for each set Thin_α and an interval of length $\frac{1}{2}$ from P_α to each point in Thin_α. We give $\mathcal{T}_{\mathrm{el}}(S)$ the *electric metric d_{el}* obtained from path length.

Masur and Minsky have shown the following:

Theorem 4.2 ([14]). *$\mathcal{T}_{\mathrm{el}}(S)$ is quasi-isometric to $\mathcal{C}_1(S)$.*

The quasi-isometry R between $\mathcal{C}_1(S)$ and $\mathcal{T}_{\mathrm{el}}(S)$ is defined as follows: if α is a curve in $\mathcal{C}_0(S)$, α is related to the set Thin_α (or equally well, to the "added-on" point P_α). It is not difficult to see that the relation R between $\mathcal{C}_0(S)$ and $\mathcal{T}_{\mathrm{el}}(S)$ is a quasi-isometry (see [14] for a proof). $\mathcal{C}_0(S)$ is $\frac{1}{2}$-dense in $\mathcal{C}_1(S)$ (that is, every point in $\mathcal{C}_1(S)$ is within $\frac{1}{2}$ of a point in $\mathcal{C}_0(S)$) and the collection $\{\mathrm{Thin}_\alpha\}$ is D-dense in $\mathcal{T}_{\mathrm{el}}(S)$ for some D, so the relation R may easily be extended to be a cobounded quasi-isometry from $\mathcal{C}_1(S)$ to $\mathcal{T}_{\mathrm{el}}(S)$, making $\mathcal{C}_1(S)$ and $\mathcal{T}_{\mathrm{el}}(S)$ quasi-isometric.

An immediate corollary of Theorem 4.2 and Theorem 4.1 is the following:

Theorem 4.3 ([14]). *The electric Teichmüller space $\mathcal{T}_{\mathrm{el}}(S)$ is hyperbolic in the sense of Gromov.*

We will use $\langle\cdot|\cdot\rangle_{\mathrm{el}}$ to denote the Gromov product on $\mathcal{T}_{\mathrm{el}}(S)$.

Quasi-geodesics in $\mathcal{T}_{\mathrm{el}}(S)$. Since $\mathcal{T}(S)$ is contained in $\mathcal{T}_{\mathrm{el}}(S)$, each Teichmüller geodesic is a path in $\mathcal{T}_{\mathrm{el}}(S)$. Because certain portions of $\mathcal{T}(S)$ are collapsed to sets of bounded diameter in $\mathcal{T}_{\text{el}}(S)$, a path whose Teichmüller length is very large may be contained in a subset of $\mathcal{T}_{\text{el}}(S)$ whose diameter is small. So to understand the geometry of these paths in $\mathcal{T}_{\text{el}}(S)$, we introduce the notion of *arclength on the scale* c, after Masur–Minsky: if $c > 0$ and $p : [a,b] \to \mathcal{T}_{\text{el}}(S)$ is a path, we define $l_c(p[a,b]) = c \cdot n$ where n is the smallest number for which $[a,b]$ can be subdivided into n closed subintervals $J_1, \dots, J_n$ such that $\operatorname{diam}_{\mathcal{T}_{\text{el}}(S)}(p(J_i)) \leq c$.

We will say that a path $p : [a,b] \to \mathcal{T}_{\text{el}}(S)$ in $\mathcal{T}_{\text{el}}(S)$ is an electric quasi-geodesic if for some $c > 0$, $k \geq 1$ and $u > 0$ we have

$$\frac{1}{k} l_c(p[s,t]) - \mu \leq d_{\text{el}}(p(s), p(t)) \leq k \cdot l_c(p[s,t]) + \mu$$

for all s and t in $[a,b]$ (note that the right-hand side of the inequality is automatic).

Masur and Minsky have shown the following, which will be important for understanding the boundary at infinity of $\mathcal{T}_{\text{el}}(S)$:

Theorem 4.4 ([14]). *Teichmüller geodesics in $\mathcal{T}(S)$ are electric quasi-geodesics in $\mathcal{T}_{\text{el}}(S)$, with uniform quasi-geodesic constants.*

5. Convergence of sequences of Teichmüller geodesics

For the remainder of the paper we will assume that we have chosen a basepoint $0 \in \mathcal{T}(S)$, giving an identification of $\mathcal{T}(S)$ with the open unit ball of quadratic differentials on 0, and a compactification $\overline{\mathcal{T}(S)}$ of $\mathcal{T}(S)$ by endpoints of Teichmüller geodesic rays from 0 (that is, by unit norm quadratic differentials or equivalently, by projective measured foliations).

In view of Proposition 3.5 and the fact that Teichmüller geodesics are electric quasi-geodesics, we can get some control over the behavior of sequences going to infinity in the electric space if we know the behavior of the Teichmüller geodesic segments between elements of the sequences. The main fact we will need is the following:

Proposition 5.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be minimal foliations in $\mathcal{PMF}(S)$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{T}(S)$ that converge to $\mathcal{F}$ and $\mathcal{G}$, respectively, and let s_n denote the geodesic segment with endpoints x_n and y_n. Then as $n \to \infty$, the sequence $\{s_n\}$ accumulates onto a set s in $\overline{\mathcal{T}(S)}$ with the following properties:*

(1) *$s \cap \mathcal{T}(S)$ is a collection of geodesic lines whose horizontal and vertical foliations are topologically equivalent to $\mathcal{F}$ and $\mathcal{G}$; this collection is non-empty exactly when $\mathcal{F}$ and $\mathcal{G}$ fill up S (that is, when $\mathcal{F}$ and $\mathcal{G}$ are not topologically equivalent).*

(2) *$s \cap \partial\mathcal{T}(S)$ consists of foliations in $\mathcal{PMF}(S)$ that are topologically equivalent to $\mathcal{F}$ or $\mathcal{G}$.*

Proof. We will begin by showing property (2). Let $\{z_n\}$ be a sequence of points lying on the segments s_n such that $z_n \to \mathcal{Z} \in \mathcal{PMF}(S)$; we will show that $\mathcal{Z}$ is topologically equivalent to either $\mathcal{F}$ or $\mathcal{G}$.

Suppose first that the z_n lie over a compact region of moduli space. Then we claim that after dropping to a subsequence there is a sequence $\{\alpha_n\}$ of distinct simple closed curves on S such that $\text{ext}_{z_n}(\alpha_n)$ is bounded. Since the z_n lie over a compact region of moduli space, there are elements f_n of the mapping class group that move the z_n to some fixed compact region of Teichmüller space; since the z_n are not contained in a compact region of Teichmüller space, we can drop to a subsequence so that the maps f_n are all distinct. So, after dropping to a further subsequence, there is some curve α on S for which the curves $\alpha_n = f_n^{-1}(\alpha)$ are all distinct; these curves will have bounded extremal length on the surfaces z_n, establishing the claim. Now since $\mathcal{PMF}(S)$ is compact, after dropping to a further subsequence, the sequence $\{\alpha_n\}$ converges in $\mathcal{PMF}(S)$; hence there exist constants r_n such that the sequence $\{r_n\alpha_n\}$ converges in $\mathcal{MF}(\mathcal{S})$ to a foliation $\mathcal{Z}'$, and since the curves α_n are all distinct, we have $r_n \to 0$. If instead the z_n do not lie over a compact region of moduli space then after dropping to a subsequence there is a sequence $\{\alpha_n\}$ of (possibly non-distinct) simple closed curves such that $\text{ext}_{z_n}(\alpha_n) \to 0$, and a sequence of bounded constants r_n such that $r_n\alpha_n$ converges to some $\mathcal{Z}' \in \mathcal{MF}(\mathcal{S})$.

Let q_n denote the quadratic differential on the basepoint 0 that is associated to z_n by the identification of $\mathcal{T}(S)$ with $\mathcal{DQ}(0)$, so that after dropping to a subsequence, $q_n \to q \in \mathcal{SQ}(0)$ whose vertical foliation is $\mathcal{Z}$. Let $\mathcal{F}_n$ denote the vertical foliation of q_n. If we pull back q_n by the Teichmüller extremal map between 0 and z_n to get a quadratic differential $\tilde{q}_n$ on z_n, the vertical foliation of $\tilde{q}_n$ is $K_n^{1/2}\mathcal{F}_n$, where K_n is the quasi-conformality constant of the extremal map. By Lemma 2.4, we have $\text{ext}_{z_n}(r_n\alpha_n) \geq (i(r_n\alpha_n, K_n^{1/2}\mathcal{F}_n))^2$, so $i(r_n\alpha_n, \mathcal{F}_n) \to 0$ as $n \to \infty$. So we have $i(\mathcal{Z}', \mathcal{Z}) = 0$.

On z_n, let ϕ_n denote the quadratic differential determining the segment s_n, and let $\mathcal{H}_n$ and $\mathcal{V}_n$ denote the horizontal and vertical foliations associated to ϕ_n (so that as we move along s_n in the direction from x_n to y_n, the transverse measure of $\mathcal{H}_n$ contracts and the transverse measure of $\mathcal{V}_n$ grows). Since $\text{ext}_{z_n}(\alpha_n) \geq (i(\alpha_n, \mathcal{H}_n))^2$, we have $i(r_n\alpha_n, \mathcal{H}_n) \to 0$; likewise $i(r_n\alpha_n, \mathcal{V}_n) \to 0$. Let a_n and b_n be constants such that after dropping to subsequences, $a_n\mathcal{H}_n$ and $b_n\mathcal{V}_n$ converge to some $\mathcal{H}$ and $\mathcal{V} \in MF(S)$, respectively. $\|\phi_n\| = i(\mathcal{H}_n, \mathcal{V}_n) = 1$, so since $i(\mathcal{H}, \mathcal{V})$ must be finite, the product $a_n b_n$ is bounded. So we must have at least one of the sequences $\{a_n\}$ and $\{b_n\}$ bounded (say $\{a_n\}$). Then $i(r_n\alpha_n, a_n\mathcal{H}_n) \to 0$ as $n \to \infty$, so $i(\mathcal{Z}', \mathcal{H}) = 0$.

Let $\widetilde{\phi_n}$ denote the quadratic differential on x_n obtained by pulling back ϕ_n by the Teichmüller extremal map from x_n to z_n. Let $\widetilde{\mathcal{H}_n}$ denote the horizontal foliation of $\widetilde{\phi_n}$. As we move along s_n from z_n back to x_n horizontal measure grows, so we have $k_n\widetilde{\mathcal{H}_n} = \mathcal{H}_n$ where the constants k_n are less than 1. Following the argument of the first paragraph of the proof, there is a sequence $\{\beta_n\}$ of simple closed curves on S and a sequence of bounded positive constants t_n such that $\text{ext}_{x_n}(t_n\beta_n) \to 0$ and $t_n\beta_n \to \mathcal{F}'$ where $i(\mathcal{F}, \mathcal{F}') = 0$ (so that $\mathcal{F}'$ is topologically equivalent to $\mathcal{F}$, by minimality of $\mathcal{F}$). This implies that $i(t_n\beta_n, \widetilde{\mathcal{H}_n}) \to 0$, so $i(t_n\beta_n, \mathcal{H}_n) \to 0$. Taking limits, $i(\mathcal{F}', \mathcal{H}) = 0$ so $\mathcal{H}$ is also topologically equivalent to $\mathcal{F}$. But we have already shown that $i(\mathcal{H}, \mathcal{Z}') = i(\mathcal{Z}', \mathcal{Z}) = 0$, so by minimality $\mathcal{Z}$ is topologically equivalent to $\mathcal{F}$, establishing property (2).

To show that $s \cap \mathcal{T}(S)$ consists of geodesic lines determined by horizontal and vertical foliations topologically equivalent to $\mathcal{F}$ and $\mathcal{G}$, suppose now that $\{z_n\}$ is a sequence of points in the segments s_n such that $z_n \to z \in \mathcal{T}(S)$. Again, let ϕ_n denote the quadratic differential on z_n that determines the segment s_n, and let $\mathcal{H}_n$ and $\mathcal{V}_n$ denote the associated horizontal and vertical foliations. After descending to a subsequence, we can assume that $q_n \to q$, a quadratic differential on z; $\mathcal{H}_n$ and $\mathcal{V}_n$ will converge respectively to the horizontal foliation $\mathcal{H}$ and vertical foliation $\mathcal{V}$ of q. By arguments similar to those of the preceding paragraphs, $\mathcal{H}$ and $\mathcal{V}$ are topologically equivalent to $\mathcal{F}$ and $\mathcal{G}$, respectively. Now the segments s_n all intersect a compact neighborhood of z, so since they form an equicontinuous family of maps, a subsequence must converge uniformly on compact sets to the complete geodesic line containing z determined by q.

When $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent, it is impossible for any point in $\mathcal{T}(S)$ to support a quadratic differential whose horizontal and vertical foliations are topologically equivalent to $\mathcal{F}$ and $\mathcal{G}$; hence when $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent, $s \cap \mathcal{T}(S)$ must be empty.

It remains to show that when $\mathcal{F}$ and $\mathcal{G}$ fill S, the intersection $s \cap \mathcal{T}(S)$ is non-empty. We have shown that $s \cap \partial\mathcal{T}(S)$ consists of foliations in $\mathcal{PMF}(S)$ that are topologically equivalent to $\mathcal{F}$ or $\mathcal{G}$. The set of foliations in $\mathcal{PMF}(S)$ topologically equivalent to $\mathcal{F}$ is closed (likewise for $\mathcal{G}$), since if $\mathcal{F}_n$ is a sequence of foliations topologically equivalent to $\mathcal{F}$ and $\mathcal{F}_n \to \mathcal{H} \in \mathcal{PMF}(S)$ then we have $i(\mathcal{F}, \mathcal{H}) = 0$, so that $\mathcal{H}$ is topologically equivalent to $\mathcal{F}$. So since $\mathcal{F}$ and $\mathcal{G}$ are not topologically equivalent, $s \cap \partial\mathcal{T}(S)$ consists of at least two connected components. The segments s_n are connected, so their accumulation set s must be connected; hence $s \cap \mathcal{T}(S)$ cannot be empty. ■

Note that in the course of the proof we have also shown the following about sequences of segments whose endpoints converge to foliations that are not minimal:

Proposition 5.2. *Let x_n and y_n be sequences in $\mathcal{T}(S)$ converging to $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{PMF}(S)$, let s_n be the geodesic segment with endpoints x_n and y_n, and let s be the set of accumulation points in $\overline{\mathcal{T}(S)}$ of the segments s_n. Then the only possible minimal foliations in $s \cap \mathcal{PMF}(S)$ are those (if any) that are topologically equivalent to $\mathcal{F}$ or $\mathcal{G}$.*

Using similar arguments, we can prove the following about convergence of Teichmüller rays emanating from a common point (not necessarily the chosen basepoint 0 in $\mathcal{T}(S)$):

Proposition 5.3. *Let z be a fixed point in $\mathcal{T}(S)$, let z_n be a sequence of points in $\mathcal{T}(S)$ that converges to $\mathcal{Z} \in \mathcal{PMF}(S)$, and let r_n be the geodesic segment from z to z_n. After descending to a subsequence, the segments r_n converge uniformly on compact sets to a geodesic ray r with vertical foliation $\mathcal{V}$, such that $i(\mathcal{Z}, \mathcal{V}) = 0$.*

Proof. Assume that the segments r_n are paths parametrized by arclength, and extend the r_n to maps $r_n : \mathbb{R} \to \mathcal{T}(S)$ by setting $r_n(t) = z_n$ for all $t \geq d(z, z_n)$. The family $\{r_n\}$ is equicontinuous, so by Ascoli's theorem, after dropping to a subsequence the maps r_n converge uniformly on compact sets to a map $r : \mathbb{R} \to \mathcal{T}(S)$, which is necessarily a geodesic ray emanating from z.

Let $\mathcal{V}$ be the vertical foliation of r. We wish to show that $i(\mathcal{Z}, \mathcal{V}) = 0$. Let ϕ_n be the quadratic differential on z determining the segment r_n, and let $\mathcal{V}_n$ be the vertical foliation of ϕ_n, so that $\mathcal{V}_n \to \mathcal{V}$. Then $\mathrm{ext}_z \mathcal{V}_n \to \mathrm{ext}_z \mathcal{V}$, so $\mathrm{ext}_z \mathcal{V}_n$ is bounded. Now (see [12])

$$d(z, z_n) = \frac{1}{2} \log\Big(\frac{\mathrm{ext}_z \mathcal{V}_n}{\mathrm{ext}_{z_n} \mathcal{V}_n}\Big),$$

so since $d(z, z_n) \to \infty$, we have $\mathrm{ext}_{z_n} \mathcal{V}_n \to 0$ as $n \to \infty$. Now the argument of the third paragraph of the proof of Proposition 5.1 (changing the $r_n \alpha_n$ to $\mathcal{V}_n$) shows that $i(\mathcal{Z}, \mathcal{V}) = 0$. ■

6. The boundary of the relative Teichmüller space

As a start to proving Theorem 1.2 we will prove the following, which shows that minimal foliations in $\mathcal{PMF}(S)$ are an infinite electric distance from any point in $\mathcal{T}(S)$.

Proposition 6.1. *Let $\mathcal{F} \in \mathcal{PMF}(S)$ be minimal and let $\{z_n\}$ be a sequence of points in $\mathcal{T}(S)$ that converges to $\mathcal{F}$. Then $d_{\mathrm{el}}(0, z_n) \to \infty$ as $n \to \infty$.*

Proof. Suppose that $d_{\mathrm{el}}(0, z_n)$ does not go to infinity. Then after dropping to a subsequence we may assume that the z_n lie in a bounded electric neighborhood of 0. As in the proof of Proposition 5.1, we can construct a sequence of curves α_n such that the values $\mathrm{ext}_{z_n}(\alpha_n)$ are bounded, and such that for some bounded constants r_n, the sequence $r_n \alpha_n$ converges in $\mathcal{MF}(S)$ to a foliation $\mathcal{F}_0$ such that $i(\mathcal{F}, \mathcal{F}_0) = 0$. Now since α_n has bounded extremal length on z_n, we have that z_n lies in a bounded neighborhood of Thin_{α_n}, so the values $d_{\mathrm{el}}(0, \mathrm{Thin}_{\alpha_n})$ are bounded. So the curves α_n, regarded as elements of the curve complex, are a bounded distance (say M) from some fixed curve α. Now for each α_n we can construct a chain of curves $\alpha_{n,0}, \ldots, \alpha_{n,M}$ such that $\alpha_{n,0} = \alpha_n$ and $\alpha_{n,M} = \alpha$, and for all i, $d(\alpha_{n,i}, \alpha_{n+1,i}) = 1$. So $\alpha_{n,i}$ and $\alpha_{n,i+1}$ are disjoint, or in other words, $i(\alpha_{n,i}, \alpha_{n,i+1}) = 0$. After dropping to subsequences, for each fixed i, the sequence $\alpha_{n,i}$ converges (after bounded rescaling) to a measured foliation $\mathcal{F}_i$, and for all i we have $i(\mathcal{F}_i, \mathcal{F}_{i+1}) = 0$. Since $\mathcal{F}$ is minimal, this implies that all the foliations $\mathcal{F}_i$ are topologically equivalent to $\mathcal{F}$. But $\mathcal{F}_M = \alpha$, which gives a contradiction. ■

The proof of Theorem 1.2 will be divided into the next three propositions. We begin by showing that we have a well-defined, continuous map from $\mathcal{PMF}_{\min}(S)$ to $\partial_\infty \mathcal{T}_{\mathrm{el}}(S)$.

Proposition 6.2. *The inclusion map from $\mathcal{T}(S)$ to $\mathcal{T}_{\mathrm{el}}(S)$ extends continuously to the portion $\mathcal{PMF}_{\min}(S)$ of $\mathcal{PMF}(S)$ consisting of minimal foliations, to give a map $\pi : \mathcal{PMF}_{\min}(S) \to \partial_\infty \mathcal{T}_{\mathrm{el}}(S)$.*

Proof. Let $\mathcal{F} \in \mathcal{PMF}_{\min}(S)$. We must show that every sequence $\{z_n\}$ in $\mathcal{T}(S)$ converging to $\mathcal{F}$, considered as a sequence in $\mathcal{T}_{\text{el}}(S)$, converges to a unique point in $\partial_\infty \mathcal{T}_{\text{el}}(S)$. So suppose that there is a sequence $\{z_n\} \to \mathcal{F}$ that does not converge to a point in $\partial_\infty \mathcal{T}_{\text{el}}(S)$. Then there are subsequences $\{x_n\}$ and $\{y_n\}$ of $\{z_n\}$ such that $\langle x_n | y_n \rangle_{\text{el}}$ is bounded. Let s_n denote the Teichmüller geodesic segment between x_n and y_n. Since the segments s_n are electric quasi-geodesics with uniform quasi-geodesic constants, by Proposition 3.5 there is a point p_n on each s_n that is a bounded electric distance from 0. By Proposition 5.1, the points p_n converge to a foliation in $\mathcal{PMF}_{\min}(S)$ that is topologically equivalent to $\mathcal{F}$. But then according to Proposition 6.1, $d_{\text{el}}(0, p_n)$ must go to infinity as $n \to \infty$. This gives a contradiction. ■

We now show that the non-injectivity of the map $\pi : \mathcal{PMF}_{\min}(S) \to \partial_\infty \mathcal{T}_{\text{el}}(S)$ is limited to identifying foliations that are topologically equivalent but not measure equivalent.

Proposition 6.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be minimal foliations in $\mathcal{PMF}(S)$. Then $\pi(\mathcal{F}) = \pi(\mathcal{G})$ if and only if $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent.*

Proof. Suppose first that $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent, and suppose that $\pi(\mathcal{F}) \neq \pi(\mathcal{G})$. Then the same argument as in the proof of Proposition 6.2 would give a sequence of points $\{p_n\}$ that are a bounded electric distance from 0 and that converge to a minimal foliation in $\mathcal{PMF}(S)$; but this is impossible by Proposition 6.1. Hence when $\mathcal{F}$ and $\mathcal{G}$ are topologically equivalent, $\pi(\mathcal{F}) = \pi(\mathcal{G})$.

Now suppose that $\mathcal{F}$ and $\mathcal{G}$ are not topologically equivalent, and let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathcal{T}(S)$ converging to $\mathcal{F}$ and $\mathcal{G}$, respectively. We will show that $\{x_n\}$ and $\{y_n\}$ do not converge to the same point in $\partial_\infty \mathcal{T}_{\text{el}}(S)$, by showing that we can drop to subsequences so that $\langle x_n | y_n \rangle_{\text{el}}$ is bounded as $n \to \infty$. Let s_n denote the Teichmüller geodesic segment with endpoints x_n and y_n. Since $\mathcal{F}$ and $\mathcal{G}$ are not topologically equivalent, by Proposition 5.1 we can drop to a subsequence so that the s_n converge uniformly on compact sets to a Teichmüller geodesic line L. Choose a point $p \in L$, and a sequence $p_n \in s_n$ converging to p. Then as $n \to \infty$, $d(0, p_n)$ is bounded, hence $d_{\text{el}}(0, \pi(p_n))$ is also bounded. So by Proposition 3.5, $\langle x_n | y_n \rangle_{\text{el}}$ is bounded as $n \to \infty$. Thus $\pi(\mathcal{F}) \neq \pi(\mathcal{G})$. ■

The following proposition completes the proof of Theorem 1.2.

Proposition 6.4. *The map $\pi : \mathcal{PMF}_{\min}(S) \to \partial_\infty \mathcal{T}_{\text{el}}(S)$ is surjective. Moreover, if $\{x_n\}$ is a sequence in $\mathcal{T}(S)$ that converges to a non-minimal foliation in $\mathcal{PMF}(S)$ then no subsequence of $\{x_n\}$ converges in the electric space $\mathcal{T}_{\text{el}}(S)$ to a point in $\partial_\infty \mathcal{T}_{\text{el}}(S)$.*

Proof. Let $\mathcal{X} \in \partial_\infty \mathcal{T}_{\text{el}}(S)$, and let x_n be a sequence in $\mathcal{T}_{\text{el}}(S)$ that converges to $\mathcal{X}$; without loss of generality we may assume that each x_n lies in $\mathcal{T}(S)$, since if x_n is one of the added-on points in the construction of $\mathcal{T}_{\text{el}}(S)$ then we may replace x_n by a point in $\mathcal{T}(S)$ that is distance $\frac{1}{2}$ from x_n, without changing the convergence properties of the sequence $\{x_n\}$. We will show that a subsequence of $\{x_n\}$ converges to a minimal foliation $\mathcal{F} \in \mathcal{PMF}(S)$; then $\pi(\mathcal{F}) = \mathcal{X}$.

Since $\overline{\mathcal{T}(S)}$ is compact, after dropping to a subsequence, $\{x_n\}$ converges to some $\mathcal{F} \in \mathcal{PMF}(S)$. Suppose $\mathcal{F}$ is not minimal. We will show that for some $B < \infty$, for each x_n there are infinitely many x_m such that $\langle x_n | x_m \rangle_{\text{el}} < B$; this would contradict convergence in $\overline{\mathcal{T}_{\text{el}}(S)}$ of the sequence $\{x_n\}$. Fix x_n, and let r_{mn} denote the geodesic segment with endpoints x_n and x_m. By Proposition 5.3, a subsequence of the r_{mn} (which we will again call r_{mn}) converges uniformly on compact sets to a geodesic ray r_n. Let $\mathcal{H}_n$ denote the horizontal foliation of r_n; by Proposition 5.3 we have $i(\mathcal{F}, \mathcal{H}_n) = 0$. The foliations $\mathcal{F}$ and $\mathcal{H}_n$ are not minimal, so each one contains a simple closed curve, which we will denote α and γ_n, respectively. Now we have $i(\alpha, \gamma_n) = 0$, so that in the curve complex, the distance from α to γ_n is at most 1; hence the electric distance from Thin_α to Thin_{γ_n} is bounded independent of n, since the curve complex is quasi-isometric to $\mathcal{T}_{\text{el}}(S)$.

The simple closed curve γ_n contained in $\mathcal{H}_n$ may be chosen so that as $t \to \infty$, $\text{ext}_{r_n(t)}\, \gamma_n \to 0$ (see [16, Lemma 8.3]). So for all sufficiently large t, $r_n(t)$ belongs to Thin_{γ_n}. Since the rays r_{mn} converge to r_n uniformly on compact sets, for all m sufficiently large there is a point p_{mn} on r_{mn} that lies in Thin_{γ_n}. Now we have

$$d_{\text{el}}(0, p_{mn}) \leq d_{\text{el}}(0, \text{Thin}_{\gamma_n}) + 1 \leq d_{\text{el}}(0, \text{Thin}_\alpha) + d_{\text{el}}(\text{Thin}_\alpha, \text{Thin}\, \gamma_n) + 2$$

since each thin set has diameter 1 (here the electric distance between two sets S_1 and S_2 means the smallest distance between any pair of points in S_1 and S_2, respectively). Note that the right-hand side of the inequality does not depend on n or m since the value $d_{\text{el}}(\text{Thin}_\alpha, \text{Thin}_{\gamma_n})$ is bounded independent of n. Now by Proposition 3.5 we have that for all m sufficiently large, $\langle x_n | x_m \rangle_{\text{el}}$ is bounded, and the bound does not depend on n or m; this contradicts the fact that the sequence $\{x_n\}$ converges to a point in the boundary at infinity of $\mathcal{T}_{\text{el}}(S)$, so our assumption that $\mathcal{F}$ is not minimal must be false. Hence $\mathcal{F}$ is minimal, and we have $\pi(\mathcal{F}) = \mathcal{X}$. ■

Note that given a non-minimal foliation $\mathcal{F}$, there are sequences in $\mathcal{T}(S)$ converging to $\mathcal{F}$ whose electric distance from 0 goes to infinity; however, no subsequences of these will converge to a point in $\partial_\infty \mathcal{T}_{\text{el}}(S)$, so that in particular $\mathcal{T}_{\text{el}}(S) \cup \partial_\infty \mathcal{T}_{\text{el}}(S)$ is not compact. It is simple to construct such sequences: the minimal foliations are dense in $\mathcal{PMF}(S)$ (see for instance [7]), so there is a sequence $\{\mathcal{F}_n\}$ of minimal foliations that converges to $\mathcal{F}$. By Proposition 6.1, for every $M > 0$, each $\mathcal{F}_n$ has a neighborhood whose points are all at least M from 0 in the electric metric; hence we may easily choose a sequence $\{p_n\}$ of points contained in small neighborhoods of the foliations $\mathcal{F}_n$, such that $\{p_n\}$ converges to $\mathcal{F}$ and $d_{\text{el}}(0, p_n) \to \infty$.

If $\mathcal{F}$ is a foliation in $\mathcal{PMF}(S)$, let $\tau(\mathcal{F})$ denote the equivalence class of foliations in $\mathcal{PMF}(S)$ that are topologically equivalent to $\mathcal{F}$. We have shown that the boundary at infinity of $\mathcal{T}_{\text{el}}(S)$ and $C(S)$ can be identified with topological equivalence classes of minimal foliations. In spite of the fact that the Teichmüller compactification of $\mathcal{T}(S)$ by $\mathcal{PMF}(S)$ depends heavily on the choice of basepoint, the arguments we have given show that the description we have obtained of the boundary of $C(S)$ is natural:

Theorem 1.4. *Let $\{\alpha_n\}$ be a sequence of elements of $\mathcal{C}_1(S)$ that converges to a foliation $\mathcal{F}$ in the boundary at infinity of $\mathcal{C}(S)$. Then regarding the curves α_n as elements of the projective measured foliation space $\mathcal{PMF}(S)$, every accumulation point of $\{\alpha_n\}$ in $\mathcal{PMF}(S)$ is topologically equivalent to $\mathcal{F}$.*

A. Appendix

In order to use sequential arguments to prove the continuity results of the main theorems, it is necessary to understand the point-set topology of $\mathcal{F}_{\min}(S)$, the space of minimal topological foliations on S. This is particularly important in light of the fact that the entire space $\mathcal{F}(S)$ of topological foliations, with the topology induced from $\mathcal{PMF}(S)$ by forgetting the measures, is not Hausdorff. We will begin with the following:

Proposition A.1. *The measure-forgetting quotient map $p : \mathcal{PMF}_{\min}(S) \to \mathcal{F}_{\min}(S)$ is a closed map, and the pre-image of any point of $\mathcal{F}_{\min}(S)$ is compact.*

Proof. To show that p is a closed map, let $K \subset \mathcal{PMF}_{\min}(S)$ be a closed set. Then we claim that the set $p^{-1}(p(K))$ is closed; this will imply that $p(K)$ is closed. So, let $\{x_n\}$ be s sequence in $p^{-1}(p(K))$ that converges to a point x in $\mathcal{PMF}_{\min}(S)$; we must show that $x \in p^{-1}(p(K))$. There is a sequence of $y_n \in K$ such that $p(x_n) = p(y_n)$. Since $\mathcal{PMF}(S)$ is compact, after dropping to a subsequence we may assume that $y_n \to y \in \mathcal{PMF}(S)$. Now since $p(x_n) = p(y_n)$, we have that x_n and y_n are topologically equivalent, which implies that $i(x_n, y_n) = 0$. Hence $i(x, y) = 0$, so since x is minimal, x and y are topologically equivalent by Proposition 2.1, so that $p(x) = p(y)$. We now know y to be in $\mathcal{PMF}_{\min}(S)$, so since K is closed in $\mathcal{PMF}_{\min}(S)$, we have $y \in K$. This in turn implies that $x \in p^{-1}(p(K))$, so $p^{-1}(p(K))$ is closed.

To show that the pre-image of any point is compact, let z be a point in $\mathcal{F}_{\min}(S)$ and let $Z = p^{-1}(z)$. Let $\{x_n\}$ be a sequence of points in Z; since $\mathcal{PMF}(S)$ is compact, after dropping to a subsequence we may assume that x_n converges to some $x \in \mathcal{PMF}(S)$. Let y be a fixed point in Z. Then the set Z is the set of all foliations in $\mathcal{PMF}(S)$ that are topologically equivalent to y. Hence $i(y, x_n) = 0$ for all n, so $i(y, x) = 0$. Thus by minimality, x is topologically equivalent to y, so $x \in Z$. So Z is compact. ∎

The space $\mathcal{PMF}(S)$ is metrizable and normal, since it is a topological sphere; hence so is $\mathcal{PMF}_{\min}(S) \subset \mathcal{PMF}(S)$. The following proposition will establish in particular that $\mathcal{F}_{\min}(S)$ is first countable and Hausdorff, which are exactly the properties needed in order for sequential arguments to prove continuity:

Proposition A.2. *Let X be a metric space that is normal, and let $p : X \to \hat{X}$ be a quotient map that is a closed map, and such that the pre-image of any point of $\hat{X}$ is compact. Then the quotient topology on $\hat{X}$ is first countable and normal.*

Proof. We will show first that $\hat{X}$ is normal. Let S and T be disjoint closed sets in $\hat{X}$; we must show that S and T have disjoint neighborhoods. The sets $p^{-1}(S)$ and $p^{-1}(T)$ are closed and disjoint in X, so since X is normal, there are disjoint open sets U and V such that $p^{-1}(S) \subset U$ and $p^{-1}(T) \subset V$. Then $X - U$ and $X - V$ are closed, so $p(X - U)$ and $p(X - V)$ are closed since p is a closed map. Now S has empty intersection with $p(X - U)$, so $\hat{X} - p(X - U)$ is a neighborhood of S; likewise $\hat{X} - p(X - V)$ is a neighborhood of T. It is easily checked that the sets $\hat{X} - p(X - U)$ and $\hat{X} - p(X - V)$ are disjoint, which establishes normality.

To show that $\hat{X}$ is first countable, let $z \in \hat{X}$; we must define a countable neighborhood basis around z. Let $Z = p^{-1}(z)$, and let U_n be the open neighborhood around Z of radius $\frac{1}{n}$. Let $V_n = p(U_n)$. If V is any neighborhood of z, then $p^{-1}(V)$ is a neighborhood of the set Z, so since by assumption Z is compact, $p^{-1}(V)$ must contain one of the sets U_n; hence V must contain one of the sets V_n. So we will be done if we can show that every V_n contains a neighborhood of z. In X, let $W_n = p^{-1}(V_n)$; note that $U_n \subset \operatorname{int}(W_n)$. Let $S_n = X - \operatorname{int}(W_n)$, so that $S_n \cap U_n = \emptyset$. The set $p(S_n)$ is closed in $\hat{X}$ since p is a closed map, so by normality of $\hat{X}$, there is some neighborhood T_n of x disjoint from $p(S_n)$. Now $p^{-1}(T_n) \subset W_n$, so $T_n \subset p(W_n) = V_n$. Hence the sets T_n form a local basis of neighborhoods of z. ■

Acknowledgments. I would like to thank Dick Canary and Yair Minsky for interesting and helpful conversations, and Howard Masur for suggesting a portion of the argument for Proposition 5.1. I would like to thank Matthew Tucker-Simmons for encouraging me, nearly 20 years after this paper was written, to post it on the arXiv, and for updating the TEX-file.

References

[1] L. Bers, Spaces of Kleinian groups. In *Several complex variables, I* (Proc. Conf., Univ. of Maryland, College Park, Md., 1970), pp. 9–34, Springer, Berlin, 1970 MR 0271333

[2] F. Bonahon, Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)* **124** (1986), no. 1, 71–158 Zbl 0671.57008 MR 847953

[3] R. D. Canary, Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **6** (1993), no. 1, 1–35 Zbl 0810.57006 MR 1166330

[4] J. W. Cannon, The theory of negatively curved spaces and groups. In *Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989)*, pp. 315–369, Oxford University Press, New York, 1991 Zbl 0764.57002 MR 1130181

[5] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes*. Lecture Notes in Math. 1441, Springer, Berlin, 1990 Zbl 0727.20018 MR 1075994

[6] B. S. Farb, *Relatively hyperbolic and automatic groups with applications to negatively curved manifolds*. ProQuest LLC, Ann Arbor, MI, 1994 MR 2690989

[7] A. Fathi, F. Laudenbach, and V. Poenaru, *Travaux de Thurston sur les Surfaces*. Astérisque 66–67, Société Mathématique de France, Paris, 1979 MR 568308

[8] É. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'aprés Mikhael Gromov*. Progr. Math. 83, Birkhaüser, Basel, 1990. MR 1086648.

[9] M. Gromov, Hyperbolic groups. In *Essays in group theory*, pp. 75–263, Math. Sci. Res. Inst. Publ. 8, Springer, New York, 1987 MR 919829

[10] W. J. Harvey, Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference* (State Univ. New York, Stony Brook, N.Y., 1978), pp. 245–251, Ann. of Math. Stud. 97, Princeton University Press, Princeton, 1981 Zbl 0461.30036 MR 624817

[11] J. Hubbard and H. Masur, Quadratic differentials and foliations. *Acta Math.* **142** (1979), no. 3-4, 221–274 Zbl 0415.30038 MR 523212

[12] S. P. Kerckhoff, The asymptotic geometry of Teichmüller space. *Topology* **19** (1980), no. 1, 23–41 Zbl 0439.30012 MR 559474

[13] H. Masur, On a class of geodesics in Teichmüller space. *Ann. of Math. (2)* **102** (1975), no. 2, 205–221 Zbl 0322.32010 MR 385173

[14] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.* **138** (1999), no. 1, 103–149 Zbl 0941.32012 MR 1714338

[15] H. A. Masur and M. Wolf, Teichmüller space is not Gromov hyperbolic. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **20** (1995), no. 2, 259–267 Zbl 0878.32015 MR 1346811

[16] Y. N. Minsky, Harmonic maps, length, and energy in Teichmüller space. *J. Differential Geom.* **35** (1992), no. 1, 151–217 Zbl 0763.53042 MR 1152229

[17] Y. N. Minsky, Teichmüller geodesics and ends of hyperbolic 3-manifolds. *Topology* **32** (1993), no. 3, 625–647 Zbl 0793.58010 MR 1231968

[18] Y. N. Minsky, On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **7** (1994), no. 3, 539–588 Zbl 0808.30027 MR 1257060

[19] Y. N. Minsky, Extremal length estimates and product regions in Teichmüller space. *Duke Math. J.* **83** (1996), no. 2, 249–286 Zbl 0861.32015 MR 1390649

[20] Y. N. Minsky, A geometric approach to the complex of curves on a surface. In *Topology and Teichmüller spaces* (Katinkulta, 1995), pp. 149–158, World Scientific, River Edge, 1996 Zbl 0937.30027 MR 1659683

[21] Y. N. Minsky, Quasi-projections in Teichmüller space. *J. Reine Angew. Math.* **473** (1996), 121–136 Zbl 0848.30031 MR 1390685

[22] Y. N. Minsky, The classification of punctured-torus groups. *Ann. of Math. (2)* **149** (1999), no. 2, 559–626 Zbl 0939.30034 MR 1689341

[23] K. Ohshika, Ending laminations and boundaries for deformation spaces of Kleinian groups. *J. London Math. Soc. (2)* **42** (1990), no. 1, 111–121 Zbl 0715.30032 MR 1078179

[24] M. Rees, An alternative approach to the ergodic theory of measured foliations on surfaces. *Ergodic Theory Dynam. Systems* **1** (1981), no. 4, 461–488 (1982) Zbl 0539.58018 MR 662738

[25] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1*. Princeton Math. Ser. 35, Princeton University Press, Princeton, 1997 Zbl 0873.57001 MR 1435975

Received 24 August 2020.

Erica Klarreich
1627 Blake St., Berkeley, CA 94703, USA; klarreic@gmail.com