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Curvature criterion for vanishing of group cohomology

Zohar Grinbaum-Reizis and Izhar Oppenheim

Abstract. We introduce a new geometric criterion for vanishing of cohomology for BN-pair groups. In particular, this new criterion yields a sharp vanishing of cohomology result for all BN-pair groups acting on non-thin affine buildings.

1. Introduction

In his seminal paper, Garland [13] developed a machinery to prove vanishing of cohomology with real coefficients for groups acting on Bruhat–Tits buildings. This machinery, known today as "Garland's method", was generalized by Ballmann and Świątkowski [3] to yield vanishing of cohomology with coefficients in unitary representations for groups acting properly and cocompactly on a simplicial complexes. These vanishing results had several applications (see [4], [20] and more recently [6]) and Garland's method also had several applications in combinatorics (see [14, Section 22.2] and references therein).

The main idea behind Garland's method is that vanishing of cohomology can be deduced for a group acting on a simplicial complex, given that the spectral gaps of the links of the simplicial complex are large enough. Consequently, it implies vanishing of cohomology up to rank for a group acting on a Bruhat–Tits building if the thickness of the building is large enough. However, it was already noted in Garland's original paper [13] that his method does not yield a sharp result in the case of affine buildings. Namely, it was known before Garland (by the work of Kazhdan [17]) that a group with a proper, cocompact action on a non-thin affine building has property (T) (i.e., its first cohomology vanishes), but the thickness condition given in Garland's work did not hold for every non-thin affine building. In other words, there are cases of non-thin affine buildings that are not covered by Garland's criterion for vanishing of cohomology. Later, Casselman [4] was able to remove this restriction and prove vanishing of cohomology for every group acting on a non-thin affine building, but his proof used entirely different methods.

In [9], Dymara and Januszkiewicz offered a different point of view on Garland's method: By assuming that the fundamental domain is a single simplex, they showed that the spectral gap can be replaced with the notion of the angle between subspaces. This change of perspective was very fruitful: Dymara and Januszkiewicz [9] used it to show

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vanishing of cohomology even when the stabilizers of vertices are not compact subgroups; Ershov, Jaikin and Kassabov pushed the idea of the angle between subgroups to prove several results regarding property (T) (see [10, 11, 15]); and the second-named author used these ideas to prove Banach versions of property (T) and vanishing of cohomology (see [18, 19]).

In this paper, we use the approach of Dymara and Januszkiewicz [9] and the ideas of Kassabov regarding the angle between subspaces [15] and get the following result for vanishing of cohomology for BN-pair groups:

Theorem 1.1. *Let G be a BN-pair group acting on a building X such that X is n-dimensional with $n \geq 2$ and all the 1-dimensional links of X are finite. Denote by C the cosine matrix of the Coxeter system associated with the Coxeter group that arises from the BN-pair of G, and by $\widetilde{\mu}$ the smallest eigenvalue of C. If X has thickness $\geq q+1$, where $q \geq 2$ and $\widetilde{\mu} > 1 - \frac{q+1}{2\sqrt{q}}$, then:*

(1) *For every continuous unitary representation π of G, it holds that $H^k(X, \pi) = 0$ for every $1 \leq k \leq n-1$.*
(2) *If $1 \leq k \leq n-1$ is a constant such that all the k-dimensional links of X are finite, then $H^i(G, \pi) = 0$ for every $1 \leq i \leq k$ and every continuous unitary representation π of G.*

As a corollary, we get another proof of the sharp vanishing result for groups acting on affine buildings of Casselman:

Corollary 1.2. *Let G be a BN-pair group such that the building X coming from the BN-pair of G is an n-dimensional, non-thin affine building, with $n \geq 2$. Then for every unitary continuous representation π of G, it holds that $H^k(G, \pi) = 0$ for every $1 \leq k \leq n-1$.*

Theorem 1.1 is a special case of a more general theorem that we will explain below after introducing the needed framework.

We start by introducing some terminology regarding simplicial complexes. Throughout, X will denote a simplicial complex and $X(i)$ will denote the set of the i-dimensional simplices of X (and we will use the convention that $X(-1) = \{\emptyset\}$). Below, we will use the following definitions:

- The simplicial complex X is called *pure n-dimensional* if the top-dimensional simplices in X are of dimension n and every simplex in X is contained in an n-dimensional simplex.
- A pure n-dimensional simplicial complex X is called *gallery connected* if for every $\sigma, \sigma' \in X(n)$, there is a sequence of n-dimensional simplices $\sigma = \sigma_1, \dots, \sigma_k = \sigma'$ such that for every i, $\sigma_i \cap \sigma_{i+1}$ is a simplex of dimension $n-1$.
- A pure n-dimensional simplicial complex X is called *$(n+1)$-partite* (or colorable) if there are disjoint sets of vertices $S_0, \dots, S_n$ of X called *the sides of X* such that every $\sigma \in X(n)$ has exactly one vertex in each side.

- For a $(n+1)$-partite simplicial complex X with sides $S_0, \dots, S_n$, we define *a type function*, denoted type : $X \to 2^{\{0,\dots,n\}}$, by $\text{type}(\tau) = \{i : \tau \cap S_i \neq \emptyset\}$.
- For a simplex $\sigma \in X$, the *link of* σ, denoted X_σ, is the simplicial complex defined by $X_\sigma = \{\tau \in X : \tau \cap \sigma = \emptyset, \ \tau \cup \sigma \in X\}$ (by this definition, $X_\emptyset = X$). Note that if X is pure n-dimensional and $(n+1)$-partite, then for every $\sigma \in X(i)$, X_σ is pure $(n-i-1)$-dimensional and $(n-i)$-partite.

Following Dymara and Januszkiewicz [9], we work in the following setup: let $n \geq 2$ and X be a pure n-dimensional, $(n+1)$-partite simplicial complex with sides $S_0, \dots, S_n$ and let G be a closed subgroup of $\text{Aut}(X)$ with respect to the compact-open topology. We consider the following properties for the couple (X, G):

($\mathcal{B}1$) All the 1-dimensional links are finite.

($\mathcal{B}2$) All the links of dimension ≥ 1 are gallery connected.

($\mathcal{B}3$) All the links are either finite or contractible (including X itself).

($\mathcal{B}4$) The group G acts simplicially on X, such that the action is transitive on $X(n)$ and type preserving, i.e., for every $\tau \in X$ and every $g \in G$, $\text{type}(\tau) = \text{type}(g.\tau)$.

Next, we define the cosine matrix of X that will play a central role in our criterion for vanishing of cohomology. In order to do so, we first recall some basic facts regarding simple random walks on graphs. For a finite graph (V, E), the simple random walk on (V, E) is an operator $M : \ell^2(V) \to \ell^2(V)$ defined by

$$M\phi(v) = \frac{1}{d(v)} \sum_{u, \{u,v\} \in E} \phi(u),$$

where $d(v)$ is the valency of v, i.e., the number of neighbors of v. We further recall that M is (similar to) a self-adjoint operator and as such has a spectral decomposition. Furthermore, all the eigenvalues of M are in the interval $[-1, 1]$ and if (V, E) is connected, then 1 is an eigenvalue of M with multiplicity 1 and all the other eigenvalues of M are strictly smaller than 1. Finally, we recall that if (V, E) is a connected bipartite graph, then the second largest eigenvalue of M is in the interval $[0, 1)$.

Definition 1.3 (The cosine matrix of X). Let $n \geq 2$ and X be a pure n-dimensional, $(n+1)$-partite simplicial complex with sides $S_0, \dots, S_n$ and let G be a closed subgroup of $\text{Aut}(X)$ with respect to the compact-open topology. Assume that (X, G) fulfill ($\mathcal{B}1$)–($\mathcal{B}4$) and define the cosine matrix of X, denoted $A = A(X)$, as follows: Let $\lambda_{i,j}$ be the second largest eigenvalue of the random walk of X_τ for $\tau \in X(n-2)$, $\text{type}(\tau) = \{0, \dots, n\} \setminus \{i, j\}$. Define A to be the $(n+1) \times (n+1)$ matrix indexed by $\{0, \dots, n\}$ as

$$A_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -\lambda_{i,j} & \text{if } i \neq j. \end{cases}$$

Remark 1.4. We note that if (X, G) fulfill ($\mathcal{B}1$)–($\mathcal{B}4$), then for every two simplices τ, τ', if $\text{type}(\tau) = \text{type}(\tau')$, then there is $g \in G$ such that $g.\tau = \tau'$ and, in particular, the links

X_τ and $X_{\tau'}$ are isomorphic as simplicial complexes, and thus the matrix A defined above is well-defined.

Remark 1.5. The reason we called A the "cosine matrix of X" will be further explained below. For now we just note that the definition above coincides with the definition of the cosine matrix of a Coxeter group G acting on a complex X defined in [5, Definition 6.8.11] (see Definition 4.3 below).

Using the definition of the cosine matrix of X, we can state our main vanishing result:

Theorem 1.6. *Let $n \geq 2$, let X be a pure n-dimensional, $(n+1)$-partite simplicial complex with sides $S_0, \dots, S_n$, and let G be a closed subgroup of $\operatorname{Aut}(X)$ with respect to the compact-open topology. If (X, G) fulfill $(\mathcal{B}1)$–$(\mathcal{B}4)$ and the cosine matrix of X is positive definite, then:*

(1) *For every continuous unitary representation π of G, it holds that $H^k(X, \pi) = 0$ for every $1 \leq k \leq n-1$.*
(2) *If $1 \leq k \leq n-1$ is a constant such that all the k-dimensional links of X are finite, then $H^i(G, \pi) = 0$ for every $1 \leq i \leq k$ and every continuous unitary representation π of G.*

In case X is a building, we will show in Section 4 that the smallest eigenvalue of the cosine matrix of X can be bounded from below by a function of the smallest eigenvalue of the Coxeter system and the thickness of the building. Using this fact, we will show that Theorem 1.6 implies Theorem 1.1.

Remark 1.7. The machinery developed in [9] allows also to compute the cohomology and not just prove vanishing. However, the statement of the computation includes introducing additional terminology and notation and therefore it is omitted from the introduction. More general statements of Theorems 1.1 and 1.6 that include computation of cohomology (even when it does not vanish) appear in the body of this paper; see Theorems 4.6 and 3.9.

While the theorems stated above concern vanishing of group cohomology, the ideas of Dymara and Januszkiewicz [9] reduce this problem to showing a decomposition theorem in Hilbert spaces. The main tool that they used to prove such a decomposition was the idea of the angle between subspaces. In the technical heart of this paper we use the results of Kassabov [15] regarding angles between subspaces to prove a general decomposition theorem in Hilbert spaces that is interesting by its own right. After doing this, we show how this decomposition can be applied to deducing vanishing of cohomology in the general framework of Dymara and Januszkiewicz and how to apply this result for BN-pair buildings.

Geometric interpretation of Theorem 1.6. Garland in his original paper used the term "p-adic curvature" for the eigenvalues of the random walks on the links. This term was used since the results were analogous to those of Matsushima who proved similar results for locally symmetric spaces. The condition for cohomology vanishing can be seen as a positive curvature condition as we will now explain. We start by recalling the following basic facts (see [2, Chapters 6, 7]): let $v_0, \dots, v_n$ be points in general position in the positive quadrant of the unit sphere of $\mathbb{R}^{n+1}$. These points can be thought as the vertices of a spherical simplex that is bounded by the subspaces $V_i = \text{span}\{v_0, \dots, \hat{v_i}, \dots, v_n\}$. The cosine matrix of these subspaces is defined by

$$A(V_0, \dots, V_n) = \begin{cases} 1 & \text{if } i = j, \\ -\cos(\angle(V_i, V_j)) & \text{if } i \neq j. \end{cases}$$

In particular, the volume of the spherical simplex can be bounded from below by a function on the smallest eigenvalue of $A(V_0, \dots, V_n)$ (see [2, Chapter 7, proof of Theorem 2.1]). Using the facts above as our geometric motivation, we note the following: the cosine matrix $A(X)$ of X being positive definite implies that there is a constant $\alpha > 0$ such that for every unitary representation $(\pi, \mathcal{H})$ and every equivariant "embedding" of our simplicial complex X in the unit sphere of $\mathcal{H}$, the spherical simplex spanned by the image of an n-simplex in X has a spherical volume of at least α. This statement is not precise, since our definition of an equivariant embedding is non-standard. A precise definition and an exact statement are given in Appendix A.

Structure of this paper. In Section 2, we prove a general decomposition theorem in Hilbert spaces. In Section 3, we show how this decomposition theorem implies vanishing of cohomology. In Section 4, we deduce a criterion for vanishing of cohomology for BN-pair groups and show that in the case of affine buildings this criterion gives a sharp vanishing result. In Appendix A, we give a further geometric interpretation for the vanishing criterion of Theorem 1.6.

2. Decomposition theorem in Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space and let $V_0, \dots, V_n \subseteq \mathcal{H}$ be closed subspaces.

Definition 2.1. For a set $\tau \subseteq \{0, \dots, n\}$ define the subspace $\mathcal{H}_\tau$ by

$$\mathcal{H}_\tau = \begin{cases} \bigcap_{i \in \{0,\dots,n\} \setminus \tau} V_i & \text{if } \tau \neq \{0, \dots, n\}, \\ \mathcal{H} & \text{if } \tau = \{0, \dots, n\}, \end{cases}$$

e.g., $\mathcal{H}_{\{0,\dots,n-1\}} = V_n$ and $\mathcal{H}_{\{0,\dots,n-2\}} = V_{n-1} \cap V_n$.

Note that for two sets $\tau, \eta \subseteq \{0, \dots, n\}$, $\mathcal{H}_{\eta \cap \tau} = \mathcal{H}_\eta \cap \mathcal{H}_\tau$ and in particular if $\eta \subseteq \tau$, then $\mathcal{H}_\eta \subseteq \mathcal{H}_\tau$. Also note that $\mathcal{H}_\emptyset = \bigcap_{i=0}^n V_i$.

Definition 2.2. For a set $\tau \subseteq \{0, \dots, n\}$ define the subspace $\mathcal{H}^\tau$ by

$$\mathcal{H}^\tau = \begin{cases} \mathcal{H}_\tau \cap \left(\bigcap_{\eta \subsetneq \tau} \mathcal{H}_\eta^\perp\right) & \text{if } \tau \neq \emptyset, \\ \mathcal{H}_\emptyset & \text{if } \tau = \emptyset, \end{cases}$$

and note that

$$\bigcap_{\eta \subsetneq \tau} \mathcal{H}_\eta^\perp = \Big(\sum_{\eta \subsetneq \tau} \mathcal{H}_\eta \Big)^\perp .$$

Definition 2.3 (Angle between subspaces; [15, Definition 3.2, Remark 3.19]). Let V_1 and V_2 be two closed subspaces in a Hilbert space. The cosine of $\angle(V_1, V_2)$ is defined by

$$\cos \angle(V_1, V_2) = \begin{cases} 0 & \text{if } V_1 \subseteq V_2 \text{ or } V_2 \subseteq V_1, \\ \sup\big\{ |\langle v_1, v_2 \rangle| : \|v_i\| = 1,\ v_i \in V_i,\ v_i \perp (V_1 \cap V_2) \big\} & \text{otherwise.} \end{cases}$$

Remark 2.4. There is an alternative definition of $\cos \angle(V_1, V_2)$ in the language of projections: denote by $P_{V_1}, P_{V_2}, P_{V_1 \cap V_2}$ the orthogonal projections on $V_1, V_2, V_1 \cap V_2$. Then

$$\cos \angle(V_1, V_2) = \| P_{V_1} P_{V_2} - P_{V_1 \cap V_2} \|.$$

The proof of the equivalence between this definition and the one given above is straightforward and can be found in [7, Lemma 9.5].

Definition 2.5. Let $V_0, \dots, V_n$ be closed subspaces in a Hilbert space. The cosine matrix $A = A(V_0, \dots, V_n)$ of $V_0, \dots, V_n$ is defined as follows: A is the $(n+1) \times (n+1)$ matrix with

$$A_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -\cos \angle(V_i, V_j) & \text{if } i \neq j. \end{cases}$$

Theorem 2.6 (Decomposition theorem). *Let $\mathcal{H}$ be a Hilbert space, let $V_0, \dots, V_n \subseteq \mathcal{H}$ be closed subspaces, and let A be the cosine matrix of $V_0, \dots, V_n$. If A is positive definite, then for every $\tau \subseteq \{0, 1, \dots, n\}$, it holds that $\mathcal{H}_\tau = \bigoplus_{\eta \subseteq \tau} \mathcal{H}^\eta$.*

The proof of this theorem will require some setup. We will start with defining an order relation between matrices that will be useful later on:

Definition 2.7. Let A, B be two square matrices of the same dimension. Write $A \preceq B$ if for every i, j, $A_{i,j} \leq B_{i,j}$.

The reason to define this order relation is the following:

Proposition 2.8. *Let A_1, A_2 be two square matrices of the same dimension such that they both have 1's along the main diagonal and all their other entries are non-positive. Let μ_i be the smallest eigenvalue of A_i for $i = 1, 2$. If $A_1 \preceq A_2$, then $\mu_1 \leq \mu_2$.*

Proof. Note that $I - A_1$ and $I - A_2$ are both non-negative matrices and as such, by the Perron–Frobenius theorem, their largest eigenvalues are achieved by a vector with non-negative entries. Thus, for $i = 1, 2$,

$$\mu_i = \max\big\{ \overline{v}^t A_i \overline{v} : \|\overline{v}\| = 1,\ \overline{v} \text{ has non-negative entries} \big\}.$$

The assumption $A_1 \preceq A_2$ implies that for every vector $\overline{v}$ with non-negative entries one has $\overline{v}^t A_1 \overline{v} \leq \overline{v}^t A_2 \overline{v}$ and therefore $\mu_1 \leq \mu_2$. ∎

The following result is proved in [15]:

Lemma 2.9 ([15, Lemma 4.2]). *The angles* $\angle(V_1 \cap V_3, V_2 \cap V_3)$ *satisfy the inequality*

$$\cos \angle(V_1 \cap V_3, V_2 \cap V_3) \leq \frac{\lambda_{12} + \lambda_{13}\lambda_{23}}{\sqrt{1-\lambda_{13}^2}\sqrt{1-\lambda_{23}^2}},$$

where $\lambda_{ij} = \cos \angle(V_i, V_j)$.

This lemma motivates the following definition:

Definition 2.10. Let $V_0, \dots, V_n$ be closed subspaces in a Hilbert space. Denote by A' the $n \times n$ matrix defined as follows:

$$A'_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -\delta_{ij} & \text{if } i \neq j, \end{cases}$$

where

$$\delta_{ij} = \frac{\lambda_{ij} + \lambda_{in}\lambda_{jn}}{\sqrt{1-\lambda_{in}^2}\sqrt{1-\lambda_{jn}^2}} \quad \text{and} \quad \lambda_{ij} = \cos \angle(V_i, V_j).$$

Additionally, we will need the following lemma:

Lemma 2.11. *For* $V_0, \dots, V_n, A, A'$ *as above, let* μ *be the smallest eigenvalue of* A *and* μ' *the smallest eigenvalue of* A'. *If* A *is positive definite, then* $\mu \leq \mu'$. *In particular, if* A *is positive definite, then* A' *is also positive definite.*

Proof. Let A'' be the $n \times n$ matrix defined by

$$A''_{i,j} = \begin{cases} 1 - \lambda_{in}^2 & \text{if } i = j, \\ -\lambda_{ij} - \lambda_{in}\lambda_{jn} & \text{if } i \neq j, \end{cases}$$

where $\lambda_{ij} = \cos \angle(V_i, V_j)$. As observed in the proof of [15, Theorem 5.1 (a)], the matrices A' and A'' have the relation $A' = DA''D$, where D is a diagonal matrix with entries

$$D_{i,i} = \frac{1}{\sqrt{1-\lambda_{in}^2}}.$$

Thus, it is enough to prove that if α is the smallest positive eigenvalue of A'' then $\mu \leq \alpha$.

Let $\overline{u} = (u_0\, u_1 \dots u_{n-1})^t$, $\|\overline{u}\| = 1$, be an eigenvector with the eigenvalue α. Let

$$B = \begin{pmatrix} \mathrm{Id} & 0 \\ -\lambda_n^t & 1 \end{pmatrix}, \quad \lambda_n = (\lambda_{0n} \dots \lambda_{n-1n})^t, \quad \text{and} \quad \overline{v} = B^{-1}\begin{pmatrix} \overline{u} \\ 0 \end{pmatrix}.$$

By the definition of $\overline{v}$, we have

$$\overline{v} = B^{-1}\begin{pmatrix} \overline{u} \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{u} \\ \lambda_{0n}u_0 + \dots + \lambda_{n-1n}u_{n-1} \end{pmatrix}.$$

Thus,

$$\|\overline{v}\|^2 = \|\overline{u}\|^2 + |\lambda_{0n} u_0 + \cdots + \lambda_{n-1n} u_{n-1}|^2 \geq \|\overline{u}\|^2 = 1. \tag{2.1}$$

As observed in the proof of [15, Theorem 5.1], A can be written as the product

$$A = \begin{pmatrix} \mathrm{Id}_{n\times n} & -\lambda_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A'' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{n\times n} & 0 \\ -\lambda_n^t & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \mu\|\overline{v}\|^2 \leq \langle A\overline{v}, \overline{v}\rangle &= \left\langle B^t \begin{pmatrix} A'' & 0 \\ 0 & 1 \end{pmatrix} B\overline{v}, \overline{v} \right\rangle = \left\langle \begin{pmatrix} A'' & 0 \\ 0 & 1 \end{pmatrix} B\overline{v}, B\overline{v} \right\rangle \\ &= \left\langle \begin{pmatrix} A'' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{u} \\ 0 \end{pmatrix}, \begin{pmatrix} \overline{u} \\ 0 \end{pmatrix} \right\rangle = \alpha\|\overline{u}\|^2 = \alpha. \end{aligned}$$

Together with (2.1), it follows that $\mu \leq \frac{1}{\|\overline{v}\|^2}\alpha \leq \alpha$ as needed. ■

Corollary 2.12. *Let $V_0, \ldots, V_n$ and $\mathcal{H}$ be as above. If $A(V_0, \ldots, V_n)$ is positive definite and the smallest eigenvalue of $A(V_0, \ldots, V_n)$ is greater than or equal to μ, then $A(V_0 \cap V_n, \ldots, V_{n-1} \cap V_n)$ is positive definite and the smallest eigenvalue of $A(V_0 \cap V_n, \ldots, V_{n-1} \cap V_n)$ is greater than or equal to μ.*

Proof. By Lemma 2.9, $A' \preceq A(V_0 \cap V_n, \ldots, V_{n-1} \cap V_n)$ and the corollary follows from Lemma 2.11 and Proposition 2.8. ■

Using this corollary, we can prove the decomposition theorem:

Proof of Theorem 2.6. Let $\mathcal{H}_\tau, \mathcal{H}^\tau$, and $\tau \subseteq \{0, \ldots, n\}$ be as above.

It follows from the definition that $\mathcal{H}_\tau = \sum_{\eta\subseteq\tau} \mathcal{H}^\eta$ for every $\tau \subseteq \{0, 1, \ldots, n\}$. Thus, we are left to prove that this is a direct sum. Also, without loss of generality, it is enough to prove that $\mathcal{H} = \bigoplus_{\eta\subseteq\{0,1,\ldots,n\}} \mathcal{H}^\eta$. We will prove this decomposition by induction on n.

For $n = 0$, the condition on A holds vacuously. By definition, $\mathcal{H}^\emptyset = \mathcal{H}_\emptyset = V_0$ and $\mathcal{H}_{\{0\}} = \mathcal{H}$. Thus, $\mathcal{H}^{\{0\}} = \mathcal{H} \cap V_0^\perp = V_0^\perp$ and obviously $\mathcal{H} = V_0 \oplus V_0^\perp = \mathcal{H}^{\{0\}} \oplus \mathcal{H}^\emptyset$ as needed.

Assume that $n > 0$ and that the decomposition holds for $n - 1$. Let $V_0, \ldots, V_n$ be spaces of a Hilbert space $\mathcal{H}$ such that $A(V_0, \ldots, V_n)$ is positive definite and denote its smallest eigenvalue by μ.

We will first show that it follows from the induction assumption that

$$V_i = \bigoplus_{\eta\subseteq\{0,\ldots,\hat{i},\ldots,n\}} \mathcal{H}^\eta \tag{2.2}$$

for every $0 \leq i \leq n$.

Without loss of generality, it is enough to show this for $i = n$. Denote $\mathcal{H}' = V_n$ and $V_0' = V_0 \cap V_n, \ldots, V_{n-1}' = V_{n-1} \cap V_n$. By Corollary 2.12, $A(V_0', \ldots, V_{n-1}')$ is positive definite. Note that, by definition, $\mathcal{H}'_{\{0,\dots,n-1\}} = V_n = \mathcal{H}_{\{0,\dots,n-1\}}$. Also note that for every $\eta \subsetneq \{0,\dots,n-1\}$,

$$\mathcal{H}'_\eta = \bigcap_{i\in\{0,\dots,n-1\}\setminus\eta} V'_i = \bigcap_{i\in\{0,\dots,n-1\}\setminus\eta} V_i \cap V_n = \bigcap_{i\in\{0,\dots,n\}\setminus\eta} V_i = \mathcal{H}_\eta.$$

Thus $(\mathcal{H}')^\eta = \mathcal{H}^\eta$ for every $\eta \subseteq \{0,\dots,n-1\}$. By the induction assumption,

$$V_n = \mathcal{H}' = \bigoplus_{\eta\subseteq\{0,\dots,n-1\}} (\mathcal{H}')^\eta = \bigoplus_{\eta\subseteq\{0,\dots,n-1\}} \mathcal{H}^\eta,$$

as needed.

Next, we will prove that given $\{v_\eta \in \mathcal{H}^\eta\}_{\eta\subseteq\{0,\dots,n\}}$, if

$$\sum_{\eta\subseteq\{0,\dots,n\}} v_\eta = 0,$$

then $v_\eta = 0$ for every η.

Fix $\{v_\eta \in \mathcal{H}^\eta\}_{\eta\subseteq\{0,\dots,n\}}$ as above such that

$$\sum_{\eta\subseteq\{0,\dots,n\}} v_\eta = 0.$$

By definition $v_{\{0,1,\dots,n\}} \perp v_\eta$ for every $\eta \subseteq \{0,1,\dots,n\}$ and therefore $v_{\{0,1,\dots,n\}} = 0$. Thus,

$$\sum_{\eta\subsetneq\{0,1,\dots,n\}} v_\eta = 0.$$

We rewrite this sum as

$$\sum_{\eta\subsetneq\{0,1,\dots,n\}} v_\eta = \sum_{i=0}^{n} u_i,$$

where

$$\begin{aligned} u_n &= \sum_{\emptyset\neq\eta\subseteq\{0,1,\dots,n-1\}} v_\eta, \\ u_i &= \sum_{\eta\subseteq\{0,1,\dots,i-1\}} v_{\eta\cup\{i+1,\dots,n\}} \quad \text{for } 0 < i < n, \\ u_0 &= v_{\{1,\dots,n\}}. \end{aligned}$$

We observe that for every i, $u_i \in V_i$ and by (2.2) it follows that if $u_i = 0$ then all the summands in the sum that define u_i are 0. Therefore it is enough to prove that for every i, $u_i = 0$.

Notice that $u_n \in V_n$ and for every $0 \le i < n$,

$$u_i \in V_i \cap \bigcap_{j>i}^{n} (V_i \cap V_j)^\perp.$$

As a consequence, for every $0 \le i \ne j \le n$ we have $|\langle u_i, u_j\rangle| \le \cos\angle(V_i, V_j)\|u_i\|\,\|u_j\|$. Thus, if we denote $\lambda_{i,j} = \cos\angle(V_i, V_j)$, it follows that

$$\begin{aligned} 0 = \Big\|\sum_{i=0}^{n} u_i\Big\|^2 &\ge \sum_{i=0}^{n}\|u_i\|^2 - 2\sum_{0\le i<j\le n}|\langle u_i, u_j\rangle| \\ &\ge \sum_{i=0}^{n}\|u_i\|^2 - \sum_{0\le i<j\le n} 2\lambda_{ij}\|u_i\|\,\|u_j\| \\ &= \big(\|u_0\|\,\|u_1\|\ldots\|u_n\|\big)A\big(\|u_0\|\,\|u_1\|\ldots\|u_n\|\big)^t \\ &\ge \mu\Big(\sum_{i=0}^{n}\|u_i\|^2\Big). \end{aligned}$$

Therefore $u_i = 0$ for every $0 \le i \le n$ as needed. ∎

3. Vanishing of cohomology for groups acting on simplicial complexes

The aim of this section is to prove Theorem 1.6 that gives a criterion for cohomology vanishing for groups acting on simplicial complexes (under the assumptions ($\mathcal{B}1$)–($\mathcal{B}4$)).

Let $n \ge 2$, let X be a pure n-dimensional, $(n+1)$-partite simplicial complex with sides $S_0, \ldots, S_n$, and let G be a closed subgroup of $\mathrm{Aut}(X)$ with respect to the compact-open topology. Assume that (X, G) fulfill ($\mathcal{B}1$)–($\mathcal{B}4$) and fix $\triangle \in X(n)$. For a simplex $\tau \subseteq \triangle$, we denote by G_τ the subgroup of G stabilizing τ (and use the convention $G_\emptyset = G$). For $0 \le i \le n$, let $\sigma_i \subseteq \triangle$ be the $(n-1)$-dimensional face of $\triangle$ such that $\mathrm{type}(\sigma_i) = \{0, \ldots, \hat{i}, \ldots, n\}$.

Given a continuous unitary representation π of G on a Hilbert space $\mathcal{H}$, define $V_i = V_i(\pi)$ by

$$V_i = \mathcal{H}^{\pi(G_{\sigma_i})} = \big\{v \in \mathcal{H} : \forall g \in G_{\sigma_i},\ \pi(g).v = v\big\}.$$

Also for every $\tau \subseteq \triangle$ define

$$\mathcal{H}_\tau = \mathcal{H}_{\mathrm{type}(\tau)}, \quad \mathcal{H}^\tau = \mathcal{H}^{\mathrm{type}(\tau)},$$

where $\mathcal{H}_{\mathrm{type}(\tau)}$ and $\mathcal{H}^{\mathrm{type}(\tau)}$ are defined as in Definitions 2.1 and 2.2.

The following definitions of the core complex and the Davis chamber appear in the paper of Dymara and Januszkiewicz [9]. The inspiration to it is attributed in [9] to a similar construction of M. W. Davis in the setting of Coxeter complexes.

Definition 3.1 ([9, Definition 1.3]). Let X be a simplicial complex. Take the first barycentric subdivision X' of X. The core X_D of X is the subcomplex of X', consisting of the simplices, spanned by barycenters of simplices of X with compact links.

Definition 3.2 ([9, Definition 1.5]). Assume that (X, G) fulfill ($\mathcal{B}1$)–($\mathcal{B}4$). Let $\triangle$ be a chamber of X and let $\triangle'$ be the first barycentric subdivision of $\triangle$. The *Davis chamber* D is the subcomplex of $\triangle'$ consisting of simplices whose vertices are barycenters of simplices of $\triangle$ with finite links in X.

For any $\sigma \subset \Delta$, denote by Δ_σ the union of faces of Δ not containing σ, and put $D_\sigma = D \cap \Delta_\sigma$.

Dymara and Januszkiewicz proved the following condition for vanishing of $H^*(G,\pi)$:

Theorem 3.3 ([9, Theorems 5.1, 5.2]). *Assume that (X,G) fulfill $(\mathcal{B}1)$–$(\mathcal{B}4)$ and let $\triangle$ be a chamber in X. Given a continuous unitary representation π of G on a Hilbert space $\mathcal{H}$, if for every $\tau \subsetneq \triangle$, it holds that $\mathcal{H}_\tau = \bigoplus_{\eta \subseteq \tau} \mathcal{H}^\eta$, then:*

(1) $H^i(X,\pi) = 0$ *for every* $1 \leq i \leq n-1$.

(2) *If* $1 \leq k \leq n-1$ *is a constant such that all the* k*-dimensional links of* X *are finite, then* $H^i(G,\pi) = 0$ *for every* $1 \leq i \leq k$.

Moreover, Dymara and Januszkiewicz also generalized their result and gave a formula for computation of the group cohomology:

Theorem 3.4 ([9, Theorem 7.1]). *Assume that (X,G) fulfill $(\mathcal{B}1)$–$(\mathcal{B}4)$ and let $\triangle$ be a chamber in X. Given a continuous unitary representation π of G on a Hilbert space $\mathcal{H}$, if for every $\tau \subsetneq \triangle$, it holds that $\mathcal{H}_\tau = \bigoplus_{\eta \subseteq \tau} \mathcal{H}^\eta$, then*

$$H^*(G,\pi) = \bigoplus_{\sigma \subseteq \Delta} \widetilde{H}^{*-1}(D_\sigma; \mathcal{H}^\sigma),$$

where D_σ are the subcomplexes of the Davis chamber defined above. Moreover, these cohomology spaces are Hausdorff.

Remark 3.5. The theorems stated above are not exactly formulated as it appears in [9]. First, in [9], the subspaces $\mathcal{H}_\tau$ are defined a little differently, namely, for $\tau \subseteq \triangle$, $\mathcal{H}_\tau = \mathcal{H}^{\pi(G_\tau)}$. The discrepancy between the definitions is resolved by [9, Proposition 4.1] that states that for every $\tau \subsetneq \triangle$, G_τ is generated by $\{G_\sigma : \sigma \subseteq \triangle, \sigma \in X(n-1), \tau \subseteq \sigma\}$, and thus for every $\tau \subsetneq \triangle$,

$$\mathcal{H}_\tau = \bigcap_{\substack{\sigma \subseteq \triangle, \sigma \in X(n-1), \\ \tau \subseteq \sigma}} \mathcal{H}_\sigma.$$

Thus,

$$\mathcal{H}_{\mathrm{type}(\tau)} = \bigcap_{\substack{\sigma \subseteq \triangle, \sigma \in X(n-1), \\ \mathrm{type}(\tau) \subseteq \mathrm{type}(\sigma)}} \mathcal{H}_\sigma = \bigcap_{i \in \{0,\dots,n\} \setminus \tau} V_i,$$

as needed. Second, in [9], the condition for Theorem 3.3 is given as a bound on the eigenvalues of the Laplacian on the 1-dimensional links, but this bound is only used to prove the existence of the decomposition $\mathcal{H}_\tau = \bigoplus_{\eta \subseteq \tau} \mathcal{H}^\eta$ and the vanishing of cohomology results [9, Theorems 5.1, 5.2, 7.1] follow from that decomposition.

Combining Theorems 3.3 and 3.4 with Theorem 2.6 leads to the following:

Theorem 3.6. *Let $X, G, \triangle$ be as above. Given a continuous unitary representation π of G on a Hilbert space $\mathcal{H}$, if $A(V_0(\pi), \dots, V_n(\pi))$ is positive definite, then:*

(1) $H^i(X, \pi) = 0$ *for every* $1 \leq i \leq n - 1$.

(2) *If* $1 \leq k \leq n - 1$ *is a constant such that all the k-dimensional links of X are finite, then* $H^i(G, \pi) = 0$ *for every* $1 \leq i \leq k$.

(3) $H^*(G, \pi) = \bigoplus_{\sigma \subseteq \Delta} \widetilde{H}^{*-1}(D_\sigma; \mathcal{H}^\sigma)$ *and these cohomology spaces are Hausdorff.*

Proof. By Theorem 2.6, if $A(V_0(\pi), \dots, V_n(\pi))$ is positive definite, then for every $\tau \subsetneq \triangle$, it holds that $\mathcal{H}_\tau = \bigoplus_{\eta \subseteq \tau} \mathcal{H}^\eta$ and the assertions regarding vanishing of cohomology follow from Theorem 3.3. ∎

Thus, in order to prove vanishing of cohomology, it is enough to prove that for every unitary representation π, the matrix $A(V_0(\pi), \dots, V_n(\pi))$ is positive definite. It was already observed in [9] that the angles between $V_i(\pi)$'s can be bounded by the second largest eigenvalues of the simple random walk on the 1-dimensional links of X:

Lemma 3.7 ([9, Lemma 4.6, step 1]; see also [16, Theorem 1.7] and [18, Corollary 4.20]). *Let $X, G, \triangle$ be as above. For $0 \leq i, j \leq n$, where $i \neq j$, denote by $\lambda_{i,j}$ the second largest eigenvalue on the simple random walk on X_τ, where $\tau \in X(n-2)$ with* $\operatorname{type}(\tau) = \{0, \dots, n\} \setminus \{i, j\}$. *Then for every unitary representation π on a Hilbert space $\mathcal{H}$, if $V_i = V_i(\pi)$ and $V_j = V_j(\pi)$ are defined as above, then* $\cos \angle(V_i, V_j) \leq \lambda_{i,j}$.

Corollary 3.8. *If $A(X)$ is the cosine matrix of X defined in Definition 1.3, then for every unitary representation π, $A(V_0(\pi), \dots, V_n(\pi)) \succeq A(X)$. In particular, if $A(X)$ is positive definite, then by Proposition 2.8, for every unitary representation π, $A(V_0(\pi), \dots, V_n(\pi))$ is positive definite.*

Using Corollary 3.8 and Theorems 2.6, 3.3, 3.4, we can prove the following more general form of Theorem 1.6:

Theorem 3.9. *Let $n \geq 2$, let X be a pure n-dimensional, $(n+1)$-partite simplicial complex with sides $S_0, \dots, S_n$, and let G be a closed subgroup of* $\operatorname{Aut}(X)$ *with respect to the compact-open topology. If (X, G) fulfill $(\mathcal{B}1)$–$(\mathcal{B}4)$ and the cosine matrix of X is positive definite, then:*

(1) *For every continuous unitary representation π of G, it holds that $H^k(X, \pi) = 0$ for every $1 \leq k \leq n - 1$.*

(2) *If* $1 \leq k \leq n - 1$ *is a constant such that all the k-dimensional links of X are finite, then $H^i(G, \pi) = 0$ for every $1 \leq i \leq k$ and every continuous unitary representation π of G.*

(3) $H^*(G, \pi) = \bigoplus_{\sigma \subseteq \Delta} \widetilde{H}^{*-1}(D_\sigma; \mathcal{H}^\sigma)$ *and these cohomology spaces are Hausdorff.*

Proof. Let $X, G, \triangle$ be as above and let π be some unitary representation of G on a Hilbert space $\mathcal{H}$. Assume that the cosine matrix of X defined in Definition 1.3 is positive definite. Thus, by Corollary 3.8, the matrix $A(V_0(\pi), \dots, V_n(\pi))$ is also positive definite and by Theorem 2.6, for every $\tau \subsetneq \triangle$, $\mathcal{H}_\tau = \bigoplus_{\eta \subseteq \tau} \mathcal{H}^\eta$. Thus the three assertions stated above follow directly by Theorem 3.3. ∎

4. Vanishing of cohomology for groups acting on buildings

The aim of this section is to prove Theorem 1.1 and to show that it can be used to prove Theorem 1.2 regarding vanishing of cohomologies for groups acting on affine buildings. We start by recalling some definitions regarding Coxeter systems:

Definition 4.1 (Coxeter matrix, Coxeter system). A Coxeter matrix $M = (m_{s,t})$ on a finite set S is an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ for all $s \in S$, and $m_{s,t} \geq 2$ for all $s, t \in S$, $s \neq t$.

A Coxeter matrix M defines a Coxeter system (W, S), where $W = \langle S | \mathcal{R} \rangle$ is a group generated by S with relations $\mathcal{R} = \{(st)^{m_{st}} : s, t \in S\}$ ($m_{st} = \infty$ means that no relation of the form $(st)^m$ is imposed).

Remark 4.2. A standard fact regarding Coxeter systems is that every Coxeter system acts by type preserving automorphisms on a partite, pure $(|S| - 1)$-dimensional simplicial complex $\Sigma(W, S)$ called the Coxeter complex (see [1, Chapter 3]), such that $(\Sigma(W, S), W)$ fulfill $(\mathcal{B}2)$–$(\mathcal{B}4)$ and if $m_{s,t} < \infty$ for every $s, t \in S$, then $(\Sigma(W, S), W)$ also fulfill $(\mathcal{B}1)$.

Definition 4.3 ([5, Definition 6.8.11]). The cosine matrix associated to a Coxeter matrix M is the $S \times S$ matrix $C = (c_{ij})$ defined by

$$c_{i,j} = -\cos\Big(\frac{\pi}{m_{s_i, s_j}}\Big).$$

When $m_{i,j} = \infty$ we define $c_{i,j} = -1$.

Observation 4.4. *Assume that (W, S) is a Coxeter system such that $m_{s,t} < \infty$ for every $s, t \in S$. Denote $S = \{s_0, \dots, s_n\}$ and abbreviate $m_{s_i, s_j} = m_{i,j}$. Then C defined above is exactly the cosine matrix of $\Sigma(W, S)$ defined in Definition 1.3. Indeed, by* [1, Corollary 3.20]*, for every $0 \leq i, j \leq n$, $i \neq j$, the link of type $\{i, j\}$ is a $2m_{i,j}$-gon and it is easy to verify that the second largest eigenvalue of the simple random walk on a $2m_{i,j}$-gon is $\cos(\frac{\pi}{m_{i,j}})$ and thus for every i, j,*

$$c_{i,j} = -\cos\Big(\frac{\pi}{m_{i,j}}\Big)$$

(for $i = j$, $m_{i,i} = 1$ and therefore $c_{i,i} = 1$).

Lemma 4.5. *Let X be a building of dimension ≥ 2 and thickness $\geq q+1$ where $q \geq 2$ such that all the 1-dimensional links of X are compact (i.e., finite) and let C be the cosine matrix of the Coxeter complex (i.e., the apartment) of the building X. Then for $A = A(X)$ being the cosine matrix of X it holds that*

$$A \succeq 2\frac{\sqrt{q}}{q+1}C + \Big(1 - 2\frac{\sqrt{q}}{q+1}\Big)I. \tag{4.1}$$

In particular, if $\widetilde{\mu}$ denotes the smallest eigenvalue of C, then $\widetilde{\mu} > 1 - \frac{q+1}{2\sqrt{q}}$ implies that A is positive definite.

Proof. Let X be as above. As a simplicial complex, X is a partite and we fix a type function on the vertices. For every $0 \leq i, j \leq n$, $i \neq j$, let $X_{i,j}$ denote the 1-dimensional link X_τ, where $\tau \in X(n-2)$ and $\text{type}(\tau) = \{0, \dots, n\} \setminus \{i, j\}$. Moreover, let $\lambda_{i,j}$ denote the second largest eigenvalue of the simple random walk on $X_{i,j}$. By our assumption $X_{i,j}$ is a finite graph and since X is a building of thickness $\geq q+1$, $X_{i,j}$ is a 1-dimensional spherical building of minimal degree $\geq q+1$. Let M be the Coxeter matrix of the Coxeter system associated with X such that M is indexed according to our type function, i.e., the entries of M are indexed by $\{0, \dots, n\}$ and for every i, j, $i \neq j$, $m_{i,j}$ is the diameter of $X_{i,j}$.

We recall that by a classical result of Feit and Higman [12], for every $i \neq j$, if $X_{i,j}$ has a minimal degree > 2, then $m_{i,j} \in \{2, 3, 4, 6, 8\}$ and in [13, Theorem 7.10], $\lambda_{i,j}$ was computed for every such choice of $m_{i,j}$. The computations of [13, Theorem 7.10] allow $X_{i,j}$ to be a bi-regular graph and $\lambda_{i,j}$ is computed according to the degrees of $X_{i,j}$, but here we will only state the results assuming that the degrees are greater than or equal to $q+1$: Let i, j, $i \neq j$.

- If $m_{i,j} = 2$, then $\lambda_{i,j} = 0$.
- If $m_{i,j} = 3$, then $\lambda_{i,j} \leq \frac{\sqrt{q}}{q+1}$.
- If $m_{i,j} = 4$, then $\lambda_{i,j} \leq \sqrt{2}\frac{\sqrt{q}}{q+1}$.
- If $m_{i,j} = 6$, then $\lambda_{i,j} \leq \sqrt{3}\frac{\sqrt{q}}{q+1}$.
- If $m_{i,j} = 8$, then $\lambda_{i,j} \leq \sqrt{2+\sqrt{2}}\frac{\sqrt{q}}{q+1}$.

We observe that in all the inequalities above,

$$\lambda_{i,j} \leq \cos\Big(\frac{\pi}{m_{i,j}}\Big)\frac{2\sqrt{q}}{q+1}.$$

Thus, we conclude (4.1). ∎

This lemma readily applies the following more general form of Theorem 1.1 that appeared in the introduction.

Theorem 4.6. *Let G be a BN-pair group acting on a building X such that X is n-dimensional with $n \geq 2$ and all the 1-dimensional links of X are finite. Denote by C the cosine matrix of the Coxeter system associated with the Coxeter group that arises from the BN-pair of G and by $\widetilde{\mu}$ the smallest eigenvalue of C. If X has thickness $\geq q+1$, where $q \geq 2$ and $\widetilde{\mu} > 1 - \frac{q+1}{2\sqrt{q}}$, then:*

(1) *For every continuous unitary representation π of G, it holds that $H^k(X, \pi) = 0$ for every $1 \leq k \leq n-1$.*

(2) *If $1 \leq k \leq n-1$ is a constant such that all the k-dimensional links of X are finite, then $H^i(G, \pi) = 0$ for every $1 \leq i \leq k$ and every continuous unitary representation π of G.*

(3) *$H^*(G, \pi) = \bigoplus_{\sigma \subseteq \Delta} \widetilde{H}^{*-1}(D_\sigma; \mathcal{H}^\sigma)$ and these cohomology spaces are Hausdorff.*

Proof of Theorem 1.1. Let X, G be as in Theorem 1.1 and let C be the cosine matrix of the Coxeter complex (i.e., the apartment) of X. Denote by μ the smallest eigenvalue of $A(X)$ and by $\widetilde{\mu}$ the smallest eigenvalue of C. By our assumption, X has thickness $\geq q+1$ and thus by Lemma 4.5 and Proposition 2.8,

$$\mu \geq 2\frac{\sqrt{q}}{q+1}\widetilde{\mu} + 1 - 2\frac{\sqrt{q}}{q+1}.$$

It follows that if $\widetilde{\mu} > 1 - \frac{q+1}{2\sqrt{q}}$, then $\mu > 0$, i.e., $A(X)$ is positive definite and the assertions above follow from Theorem 3.9. ∎

The sharp vanishing result for affine buildings stated in Theorem 1.2 is a consequence of Theorem 1.1 and of a well-establish fact regarding the cosine matrix of affine Coxeter complexes:

Proof of Theorem 1.2. Let G be a BN-pair group such that the building X coming from the BN-pair of G is an n-dimensional, non-thin affine building, with $n \geq 2$ and let C be the cosine matrix of the Coxeter complex (i.e., the apartment) of X. Since X is non-thin, it has thickness ≥ 3 and therefore by Theorem 1.1, it is enough to prove that $\widetilde{\mu} > 1 - \frac{3}{2\sqrt{2}}$, where $\widetilde{\mu}$ is the smallest eigenvalue of C. By [5, Theorem 6.8.12], the cosine matrix of an affine Coxeter complex is positive semidefinite (with co-rank 1), thus $\widetilde{\mu} = 0 > 1 - \frac{3}{2\sqrt{2}}$ as needed.

Note that all the links of X that are not X itself are compact and therefore by Theorem 1.6, $H^i(G, \pi) = 0$ for every $1 \leq i \leq n-1$ and every continuous unitary representation π. ∎

Another consequence of Theorem 1.1 is vanishing of cohomology for Kac–Moody groups acting on Kac–Moody buildings given that the thickness is large enough and all the 1-dimensional links are finite. This was already proved by Dymara and Januszkiewicz in [9] where the condition on the thickness was that it should be greater than or equal to

$\frac{1}{25}(1764)^n$, where n is the dimension of the building. Theorem 1.1 shows that the same vanishing result holds under a much weaker condition on the thickness. In order to illustrate this point, we perform an exact calculation of a specific example of a (hyperbolic) Kac–Moody group (this example was chosen rather arbitrarily and one can perform similar computations for any specific example of a BN-pair group):

Example 4.7. Consider the Coxeter group whose Coxeter diagram is a square such that one of the edges is labelled 4 and the remaining ones are labelled 3. By the work of Tits [21], for every finite field $\mathbb{F}_q$, there is a BN-pair group $G(q)$ acting on a building X with thickness $q+1$ and the Coxeter group as above. Note that according to the Coxeter diagram the links of the vertices are spherical building (see [8, Section 4]) and in particular the links of all the vertices are finite and we can apply Theorem 1.1. The cosine matrix of this Coxeter group is

$$\begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

and its smallest eigenvalue is $\frac{-\sqrt{2}+1}{2}$. Note that for every $q \geq 4$, it holds that

$$\frac{-\sqrt{2}+1}{2} > 1 - \frac{q+1}{2\sqrt{q}},$$

and the conditions of Theorem 1.1 are satisfied. Thus, in this example, for every $q \geq 4$ and every unitary representation of $G(q)$, it holds that $H^1(G(q),\pi) = H^2(G(q),\pi) = 0$.

A. Equivariant embedding interpretation of our criterion

The aim of this appendix is to give a geometric interpretation to our vanishing criterion, i.e., to give a geometric meaning to the condition that the cosine matrix of a complex X is positive definite. Let X be an n-dimensional simplicial complex and G a group acting simplicially on X such that (X,G) fulfill $(\mathcal{B}1)$–$(\mathcal{B}4)$.

Let $(\pi,\mathcal{H})$ be a unitary representation of G. Below, we fix $\triangle = \{x_0,\dots,x_n\} \in X(n)$ and use the notations of Section 3, i.e.:

(1) For $\tau \subseteq \triangle$, G_τ is the stabilizer subgroup of τ in G.

(2) For $i \in \{0,\dots,n\}$, $V_i(\pi) = \mathcal{H}^{\pi(G_{\{x_k : k\in\{0,\dots,n\}\setminus\{i\}\}})}$.

(3) For $\tau \subseteq \triangle$, $\mathcal{H}_\tau = \bigcap_{i\notin\tau} V_i(\pi) = \mathcal{H}^{\pi(G_\tau)}$.

Lemma A.1. *Let (X,G), $\triangle = \{x_0,\dots,x_n\} \in X(n)$ and $\{G_\tau\}_{\tau\subseteq\triangle}$ be as above. For every $\tau,\tau' \subseteq \triangle$, $\langle G_\tau, G_{\tau'}\rangle = G_{\tau\cap\tau'}$.*

Proof. If $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$, there is nothing to prove, thus we will assume that this is not the case and, in particular, that $|\tau\cap\tau'| \leq n-1$.

Observe that for every $|\tau \cap \tau'| \leq n-1$, the couple $(X_{\{x_j : j \in \tau \cap \tau'\}}, G_{\tau \cap \tau'})$ also fulfills $(\mathcal{B}1)$–$(\mathcal{B}4)$, and if we fix $\{x_j : j \in \{0, \dots, n\} \setminus \tau \cap \tau'\}$ to be a fundamental domain of $X_{\{x_j : j \in \tau \cap \tau'\}}$, we will get exactly the subgroups $\{G_{\tau \cap \tau' \cup \eta}\}_{\eta \subseteq \triangle \setminus \tau \cap \tau'}$. Thus, it is enough to prove that if $\tau \cap \tau' = \emptyset$, it follows that $\langle G_\tau, G_{\tau'} \rangle = G$.

By definition, $G_\tau, G_{\tau'} \subseteq G$, thus $\langle G_\tau, G_{\tau'} \rangle \subseteq G$. Next, we will show that for every $g \in G$, g can be written as a product of elements in $G_\tau \cup G_{\tau'}$.

Fix some $g \in G$. By $(\mathcal{B}2)$, $\triangle$ and $g.\triangle$ are connected by a gallery, i.e., there are n-dimensional simplices $\sigma_1, \dots, \sigma_k$ such that $\sigma_j \cap \sigma_{j+1} \in X(n-1)$ and $\triangle = \sigma_1, \dots, \sigma_k = g.\triangle$. By induction on k we will prove that if k is the length of the connecting gallery, then $g \in (G_\tau \cup G_{\tau'})^{k+1}$.

If $k = 0$, then $g.\triangle = \triangle$ and thus $g \in G_\triangle \subseteq G_\tau$.

Assume that our claim is true for $k-1$ and let $\triangle = \sigma_1, \dots, \sigma_k = g.\triangle$. Note that since $\sigma_1 \cap \sigma_2 \in X(n-1)$, there is some i_0 such that $\sigma_1 \cap \sigma_2 = \{x_i : i \in \triangle \setminus \{i_0\}\}$. By $(\mathcal{B}4)$, it follows that there is some $g' \in G_{\triangle \setminus \{i_0\}}$ such that $\sigma_1 = g'.\sigma_2$. Note that $G_{\triangle \setminus \{i_0\}} \subseteq G_\tau \cup G_{\tau'}$ and thus $g' \in G_\tau \cup G_{\tau'}$. Thus, $\sigma_2 = g'.\sigma_1, \dots, \sigma_k = g.\triangle$ is a gallery of length $k-1$ and also $\sigma_1, (g')^{-1}.\sigma_3, \dots, (g')^{-1}.\sigma_k = (g')^{-1} g.\triangle$ is a gallery of length $k-1$. It follows that $(g')^{-1} g \in (G_\tau \cup G_{\tau'})^k$ and therefore $g \in g'(G_\tau \cup G_{\tau'})^k \subseteq (G_\tau \cup G_{\tau'})^{k+1}$ as needed. ∎

Let $\phi : X(0) \to \mathcal{H}$. We recall that the map ϕ is called *equivariant* (with respect to π) if for every $g \in G$ and every vertex x of X it holds that $\phi(g.x) = \pi(g)\phi(x)$.

Observation A.2. *Fix $\triangle = \{x_0, \dots, x_n\} \in X(n)$ and let $(\pi, \mathcal{H})$ be a unitary representation of G. We observe that by $(\mathcal{B}4)$, an equivariant map $\phi : X(0) \to \mathcal{H}$ is uniquely determined by the choices of $\phi(x_i)$, $i = 0, \dots, n$, and that $\phi(x_i) \in \mathcal{H}^{\pi(G_{\{x_i\}})}$ for every $i = 0, \dots, n$. Vice-versa, every choice $v_i \in \mathcal{H}^{\pi(G_{\{x_i\}})}$, $i = 0, \dots, n$, defines an equivariant map ϕ via $\phi(g.x_i) = \pi(g)v_i$ (note that this indeed defines a well-defined equivariant map $\phi : X(0) \to \mathcal{H}$).*

Proposition A.3. *Let $(\pi, \mathcal{H})$ be a unitary representation of G without any non-trivial invariant vectors and let $\phi : X(0) \to \mathcal{H}$ be an equivariant map such that $\phi(x) \neq 0$ for every $x \in X(0)$. Then for every $\tau \subsetneq \triangle$ and every $x_i \notin \tau$, $P_{\mathcal{H}_\tau}\phi(x_i) \neq \phi(x_i)$.*

Proof. Assume toward contradiction that $P_{\mathcal{H}_\tau}\phi(x_i) = \phi(x_i)$. Then by Observation A.2, $\phi(x_i) \in \mathcal{H}_\tau \cap \mathcal{H}_{\{x_i\}}$ and thus it is stabilized by both G_τ and $G_{\{x_i\}}$. By our assumption $x_i \notin \tau$ and thus $\phi(x_i)$ is stabilized by $\langle G_\tau, G_{\{x_i\}} \rangle = G_\emptyset = G$ (here we use Lemma A.1). Therefore $\phi(x_i)$ is a non-trivial invariant vector and this contradicts our assumption. ∎

We define $\phi : X(0) \to \mathcal{H}$ to be an *equivariant embedding of X into the unit sphere of $\mathcal{H}$* if the following holds:

(1) For every $x \in X(0)$, $\|\phi(x)\| = 1$.

(2) The map ϕ is equivariant.

(3) For every $i \neq j$, $P_{\mathcal{H}_{\triangle_{i,j}}}\phi(x_i) \in \operatorname{span}\{\phi(x) : x \in \triangle_{i,j}\}$, where $\triangle_{i,j} = \{x_k : k \in \{0, \dots, n\} \setminus \{i, j\}\}$.

The last condition may be thought of being in general position with respect to the subspaces $V_i(\pi)$.

The following theorem gives a geometric interpretation of our criterion that the cosine matrix $A = A(X)$ is positive definite. Basically it states that for any representation $(\pi, \mathcal{H})$ with no non-trivial invariant vectors, an n-simplex of X is mapped to a spherical simplex that cannot be "too small" (there is a lower bound on the spherical n-volume of the image).

Theorem A.4. *Assume that the cosine matrix $A = A(X)$ of X is positive definite. Then there is a constant $\alpha > 0$ that depends on the smallest positive eigenvalue of A such that for every unitary representation $(\pi, \mathcal{H})$ without non-trivial invariant vectors and any $\phi : X(0) \to \mathcal{H}$ equivariant embedding of X into the unit sphere of $\mathcal{H}$, $\{\phi(x_0), \dots, \phi(x_n)\}$ are vertices of an n-dimensional spherical simplex with a spherical n-volume of at least α.*

Proof. Let $(\pi, \mathcal{H})$ be some unitary representation and $\phi : X(0) \to \mathcal{H}$ an equivariant embedding of X into the unit sphere of $\mathcal{H}$.

By Proposition A.3, $\phi(x_0), \dots, \phi(x_n)$ are in general position in $\mathcal{H}$ and thus span an n-dimensional spherical simplex in the $(n+1)$-dimensional subspace $V' = \operatorname{span}\{\phi(x_k) : k \in \{0, \dots, n\}\}$. Restricting our attention to V', the spherical simplex at hand is bounded by the subspaces $V_i' = \operatorname{span}\{\phi(x_k) : k \in \{0, \dots, n\} \setminus \{i\}\}$. The cosine matrix of these subspaces is defined as above:

$$A(V_0', \dots, V_n') = \begin{cases} 1 & \text{if } i = j, \\ -\cos(\angle(V_i', V_j')) & \text{if } i \neq j. \end{cases}$$

In particular, the volume of the spherical simplex can be bounded from below by a positive increasing function on the smallest eigenvalue of $A(V_0', \dots, V_n')$ (see [2, Chapter 7, proof of Theorem 2.1]). Therefore in order to prove Theorem A.4 it is sufficient to show that the smallest eigenvalue of $A(X)$ is a lower bound for the smallest eigenvalue of $A(V_0', \dots, V_n')$.

We note that for every i, j, $V_i' \cap V_j' = \operatorname{span}\{\phi(x_k) : k \in \{0, \dots, n\} \setminus \{i, j\}\}$ and $V_i' \cap (V_i' \cap V_j')^\perp$ is the 1-dimensional space that can be written as

$$V_i' \cap (V_i' \cap V_j')^\perp = \operatorname{span}\big\{\phi(x_i) - P_{V_i' \cap V_j'}\phi(x_i)\big\}.$$

By our definition of an equivariant embedding, $P_{V_i' \cap V_j'}\phi(x_i) = P_{V_i(\pi) \cap V_j(\pi)}\phi(x_i)$ and thus for every i, j, $V_i' \cap (V_i' \cap V_j')^\perp \subseteq V_i(\pi) \cap (V_i(\pi) \cap V_j(\pi))^\perp$. It follows that

$$\cos(\angle(V_i', V_j')) \leq \cos(\angle(V_i(\pi), V_j(\pi))).$$

By Corollary 3.8, $A(V_0', \dots, V_n') \succeq A(X)$ and thus by Proposition 2.8 the smallest eigenvalue of $A(V_0', \dots, V_n')$ is bounded from below by the smallest eigenvalue of $A(X)$ as needed. ∎

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Zohar Grinbaum-Reizis
Department of Mathematics, Ben-Gurion University of the Negev, Be'er Sheva 84105, Israel;
reiziszo@post.bgu.ac.il

Izhar Oppenheim
Department of Mathematics, Ben-Gurion University of the Negev, Be'er Sheva 84105, Israel;
izharo@bgu.ac.il