

On dual surjectivity and applications

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Abstract. We explore the dual version of Gottschalk’s conjecture recently introduced by Capobianco, Kari, and Taati, and the notion of dual surjectivity in general. We show that dual surjective groups satisfy Kaplansky’s direct finiteness conjecture for all fields of positive characteristic. By quantifying the notions of injectivity and post-surjectivity for cellular automata, we show that the image of the full topological shift under an injective cellular automaton is a subshift of finite type in a quantitative way. Moreover, we show that dual surjective groups are closed under ultraproducts, under elementary equivalence, and under certain semidirect products (using the ideas of Arzhantseva and Gal for the latter); they form a closed subset in the space of marked groups, fully residually dual surjective groups are dual surjective, etc. We also consider dual surjective systems for more general dynamical systems, namely for certain expansive algebraic actions, employing results of Chung and Li.

Introduction

In the beginning of the 1970s, W. Gottschalk introduced the following notion. Let G be a group and A a finite set, and let us consider the topological Bernoulli shift $G \curvearrowright A^G$. If any injective, continuous, and G -equivariant map $T: A^G \rightarrow A^G$ is also surjective, then G is called *surjective*. Gottschalk asked in [19] whether every group is surjective. The question reached its prominence after Gromov proved in [20] that groups that were later going to be called *sofic* (see [28]) are surjective. Sofic groups, originally introduced just because of Gottschalk’s question, now live a life on their own and are one of the most important classes of groups in geometric group theory, topological and measurable dynamics, graph theory, and beyond. Peculiarly, as of now, there is still no example of a non-sofic group, although the existence of such groups is rather generally expected.

A natural idea is to consider the *reverse* of the Gottschalk question. Perhaps the most direct attempt, that is, asking whether every surjective, continuous, and G -equivariant map $T: A^G \rightarrow A^G$ is injective, is not the right choice. Indeed, there are counterexamples even for $G = \mathbb{Z}$ (we refer to [7, Example 3.3.8, p. 61] for examples of such kind). It is however instructive to recall at this point the notion of *Garden of Eden*. Although surjectivity of the map $T: A^G \rightarrow A^G$ does not necessarily imply injectivity, it does imply, in some cases,

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a weaker notion called *pre-injectivity* (we refer further to Definition 1.5 of the paper below, or to the monograph [7, p. 112]). The theorems of Moore and Myhill [25, 26] established the equivalence of surjectivity and pre-injectivity for $G = \mathbb{Z}^d$. Later, the same equivalence was obtained for all finitely generated groups of subexponential growth in [24], and finally for all amenable groups in [10]. This line of research culminated rather recently when Bartholdi in the combination of the two papers [3] and [4] showed that the Garden of Eden equivalence characterizes the class of amenable groups. We remark that the Garden of Eden equivalence has been considered and proved for dynamical systems much more general than topological Bernoulli shifts. We refer to [23] and references therein for more information.

It was also very recently when Capobianco, Kari, and Taati found a proper reverse, or dual, of Gottschalk's question. In [5], they introduce the notion of *post-surjectivity*, which is strictly stronger than surjectivity, and ask for which groups, rightfully called *dual surjunctive*, post-surjectivity implies pre-injectivity. As Gromov did for surjunctive groups, they show that all sofic groups are dual surjunctive. Since, as they show for topological Bernoulli shifts, post-surjectivity together with pre-injectivity actually implies injectivity, one is led to a strong version of the Garden of Eden theorem which says that a continuous G -equivariant map $T: A^G \rightarrow A^G$ is injective if and only if it is post-surjective. All sofic groups therefore satisfy this strong version of Garden of Eden.

The aim of this note is to explore the notions of post-surjectivity and dual surjunctivity further. We simplify some arguments from [5] concerning post-surjectivity and pre-injectivity, and we investigate these notions in a quantitative way. We also introduce and investigate a notion of post-surjectivity for more general shifts and even more general expansive dynamical systems. Below is a selection of some of our results.

Results.

- (1) Dual surjunctive groups satisfy Kaplansky's direct finiteness conjecture for all fields of positive characteristic (see Theorem 4.1).
- (2) If $T: A^G \rightarrow A^G$ is an injective cellular automaton, then $T[A^G]$ is a subshift of finite type with memory set of the forbidden patterns precisely depending on the injectivity of T (see Theorem 2.2).
- (3) Dual surjunctive groups are closed under taking ultraproducts and under elementary equivalence, they form a closed subset in the space of marked groups, fully residually dual surjunctive groups are dual surjunctive (see Theorems 3.6 and 3.8, and Corollaries 3.9 and 3.13).
- (4) Algebraic expansive actions of any countable polycyclic-by-finite group (and under some additional conditions, of any amenable group) on compact metrizable abelian groups with completely positive entropy (with respect to the Haar measure on the compact group) are dual surjunctive (see Theorem 5.9).

1. Post-surjectivity and pre-injectivity

Throughout the paper, G is a group and A is a finite set having at least two elements. We topologize A^G with the product topology, where A is equipped with the discrete topology. When G is countably infinite, which will be the most interesting case, A^G is then obviously homeomorphic to the Cantor space. For this reason, unless explicitly stated otherwise, such as when working with ultraproducts of groups, G is assumed to be countable. Note that this requirement is only chosen to simplify the notation of the proofs, especially it allows us to work with sequences instead of more general nets. Most of the proofs have straightforward generalization to the uncountable case though.

A group G acts (by homeomorphisms) on $A^G = \{f: G \rightarrow A\}$ by

$$g \cdot f(x) = f(g^{-1}x), \quad \text{for } g, x \in G, f \in A^G.$$

The corresponding dynamical system is called the *topological Bernoulli shift*, or just *topological (full)-shift*.

We need a few more definitions from the dynamics on topological shifts. We refer the reader to [7] for a detailed treatment.

Definition 1.1. Let G be a group and A a finite set. Any element $x \in A^G$ is called a configuration. Any map $p: D \rightarrow A$, where $D \subseteq G$ is (usually finite, but not always) a subset, is called a *pattern*. A pattern is called *finite* if its domain is finite.

Definition 1.2. Let G be a group and A a finite set. By a *subshift*, we mean any closed subset $X \subseteq A^G$ that is also closed under the shift by the elements of G .

If $P \subseteq G$ is a subset, by X_P we denote the set of patterns whose domain is P and which are restrictions of configurations from X . That is, $X_P := \{x \upharpoonright P: x \in X\}$.

The following two types of subshifts will be the most interesting for us.

Definition 1.3. Let $X \subseteq A^G$ be a subshift. We say that

- X is of *finite type* if there exists a finite set $\{p_1, \dots, p_n\}$ of finite patterns such that for $x \in A^G$ we have $x \in X$ if and only if for no $g \in G$ and $i \leq n$ we have $g \cdot x \upharpoonright \text{dom}(p_i) = p_i$.
- X is *strongly irreducible* if there exists a finite set $D \subseteq G$ such that for all finite patterns $p: P \rightarrow A$ and $p': P' \rightarrow A$, with $p \in X_P$, resp. $p' \in X_{P'}$ and $P \cdot D \cap P' = \emptyset$, there exists $x \in X$ such that $x \upharpoonright P = p$ and $x \upharpoonright P' = p'$.

Let $X, Y \subseteq A^G$ be subshifts. By $T: X \rightarrow Y$ we always mean a continuous G -equivariant map, that is

$$T(g \cdot f) = g \cdot T(f).$$

The map T is called a *cellular automaton* (further abbreviated CA). It is well known that every such T is induced by a map $\tau: A^F \rightarrow A$, where $F \subseteq G$ is a finite subset, called the

memory set for T , such that for all $f \in X, x \in G$, the following holds true:

$$T(f)(x) = \tau(f \upharpoonright x \cdot F),$$

where $f \upharpoonright x \cdot F$ is the pattern obtained by restricting f to $x \cdot F := \{xg : g \in F\}$.

Let us define an equivalence relation \sim of *almost equality* on A^G in the following way, for $c, d \in A^G$ we write

$$c \sim d \quad \text{if and only if} \quad \{g \in G : c(g) \neq d(g)\} \text{ is finite.}$$

We record the following fact, which immediately follows from [5, Proposition 1].

Fact 1.4. *Let $X \subseteq A^G$ be a strongly irreducible subshift. Then for every $x \in X$, the equivalence class $[x]_{\sim} \cap X$ is dense in X .*

In [5], Capobianco, Kari, and Taati introduced the following stronger version of surjectivity. We also recall below the by-now standard notion of pre-injectivity, a weaker version of injectivity.

Definition 1.5. (1) A map T is called *post-surjective* if whenever $T(g) \sim f'$, for any $g, f' \in A^G$, then there exists $g' \sim g$ such that $T(g') = f'$ (see [5, Definition 2]).

(2) A map T is called *pre-injective* if whenever $f \sim f'$ and $T(f) = T(f')$, then $f = f'$, for all $f, f' \in A^G$.

Lemma 1.6. *Let $X, Y \subseteq A^G$ be subshifts and Y be strongly irreducible. Let $T : X \rightarrow Y$ be a post-surjective CA. Then T is surjective.*

Proof. This is essentially proved in [5, Proposition 2]. Since we work in a slightly more general context, we re-prove it for the convenience of the reader. Pick some $y \in Y$ and let $x \in X$ be arbitrary. By strong irreducibility, the equivalence classes in \sim are dense in Y , so we can find a sequence $(y_n)_{n \in \mathbb{N}}$ such that $y_n \rightarrow y$ and $y_n \sim T(x)$, for every $n \in \mathbb{N}$. By post-surjectivity, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \sim x$ and $T(x_n) = y_n$, for every $n \in \mathbb{N}$. Let $x' \in X$ be an accumulation point of this sequence. It immediately follows that $T(x') = y$. ■

For $f_1, f_2 \in A^G$, we denote

$$\Delta(f_1, f_2) = \{g \in G : f_1(g) \neq f_2(g)\}.$$

We shall also need a stronger version of post-surjectivity, which is what is actually useful in applications. It turns out that for full shifts, the two notions of post-surjectivity are equivalent.

Definition 1.7. Let $X, Y \subseteq A^G$ be subshifts. A CA $T : X \rightarrow Y$ is *strongly post-surjective* if there exists a finite set $M \subseteq G$ such that for every $x \in X$ and $T(x) \sim z \in Y$ there exists $y \in X$ such that $y \sim x, T(y) = z$ and $\Delta(x, y) \subseteq \Delta(T(x), z) \cdot M$.

The set M will be called a *post-surjectivity set* for T .

Observation 1.8. (1) A CA $T: A^G \rightarrow A^G$ is post-surjective if and only if it is strongly post-surjective.

(2) Let $X, Y \subseteq A^G$ be subshifts and let $T: X \rightarrow Y$ be a strongly post-surjective and pre-injective CA with a finite post-surjectivity set M . Then for every $c, d \in X$, $c \sim d$, the following holds true:

$$\Delta(c, d) \subseteq \Delta(T(c), T(d)) \cdot M.$$

Proof. The non-trivial direction of (1) is the content of [5, Lemma 1]. For (2), notice that by strong post-surjectivity there is some $d' \sim c$ with $T(d') = T(d)$ such that $\Delta(c, d') \subseteq \Delta(T(c), T(d)) \cdot M$. By pre-injectivity, since $d' \sim d$ and $T(d) = T(d')$, we get that $d' = d$, and we are done. ■

Our aim is now to strengthen the main results from [5], saying that pre-injective and (strongly) post-surjective CA on full shift is reversible (see [5, Theorem 1]), and also to provide a simpler proof of it.

First, we need the following lemma. Suppose $X \subseteq A^G$ is a subshift. Let us provisionally call a CA $T: X \rightarrow A^G$ *uniformly injective* (till we show that it is equivalent to being injective) if there exists a finite set $N \subseteq G$ such that for all $x, y \in X$ we have $\Delta(x, y) \subseteq \Delta(T(x), T(y)) \cdot N^{-1}$. Call such a set N an *injectivity set* for T .

Lemma 1.9. Let $X \subseteq A^G$ be a subshift and let $T: X \rightarrow A^G$ be a CA. Then T is injective if and only if it is uniformly injective. Moreover, provided that T is injective, a set $N \subseteq G$ is an injectivity set for T if and only if it is a memory set for T^{-1} .

Proof. Clearly, if T is uniformly injective, then it is injective, so we prove the converse. Assume that T is injective and denote by Y the image $T[X]$. By compactness and G -equivariance, Y is a subshift and the inverse $T^{-1}: Y \rightarrow X$ is also continuous and G -equivariant, thus a CA. Let $N \subseteq G$ be a finite memory set for T^{-1} . By definition, for every $x, y \in Y$ we have $\Delta(T^{-1}(x), T^{-1}(y)) \subseteq \Delta(x, y) \cdot N^{-1}$, which can be translated to a statement that N is an injectivity set for T . It also shows the ‘Moreover’ part of the statement. ■

Theorem 1.10. Let $X, Y \subseteq A^G$ be subshifts and let T be a strongly post-surjective and pre-injective CA. If X is strongly irreducible, then T is injective. In particular, if both X and Y are strongly irreducible, then T is an isomorphism.

Proof. Suppose that X is strongly irreducible. By Lemma 1.9, it suffices to check that there is a finite injectivity set $N \subseteq G$ for T . Let $M \subseteq G$ be a post-surjectivity set for T . We claim that $M \cdot M^{-1}$ is an injectivity set for T . If not, then we can find $x, y \in X$ such that $x(1_G) \neq y(1_G)$, yet

$$T(x) \upharpoonright M \cdot M^{-1} = T(y) \upharpoonright M \cdot M^{-1}.$$

By strong irreducibility of X , we can assume that $x \sim y$. Indeed, by Fact 1.4 there is $y' \sim x$ which is sufficiently close to y so that $y'(1_G) = y(1_G)$ and $T(y) \upharpoonright M \cdot M^{-1} = T(y') \upharpoonright M \cdot M^{-1}$.

Since $\Delta(T(x), T(y)) \subseteq G \setminus M \cdot M^{-1}$, by Observation 1.8, we get that $\Delta(x, y) \subseteq G \setminus M$, which contradicts with $x(1_G) \neq y(1_G)$. The ‘in particular’ part follows by applying Lemma 1.6. ■

Corollary 1.11. *Let $T: A^G \rightarrow A^G$ be post-surjective and pre-injective. Then it is injective and also an isomorphism.*

Proof. It follows immediately from Theorem 1.10 and Observation 1.8. ■

2. More observations on injectivity and post-surjectivity

Let us have a closer look at the tight connection between injectivity and post-surjectivity.

The following lemma establishes the connection between the injectivity and post-surjectivity sets. Notice that reversible, i.e., bijective, cellular automata are post-surjective [5, Example 1].

Lemma 2.1. *Let $T: A^G \rightarrow A^G$ be a reversible cellular automaton.*

- *Let N be a symmetric injectivity set for T . Then it is also a post-surjectivity set for T .*
- *Let M be a symmetric post-surjectivity set for T . Then it is also an injectivity set for T .*

Proof. Let N be a symmetric injectivity set (note that if N is an arbitrary injectivity set, then $N \cup N^{-1}$ is symmetric and still an injectivity set). Suppose that it is not a post-surjectivity set. Then there exist $x, z \in A^G$ such that $\Delta(T(x), z) = \{1_G\}$, yet for every $y \sim x$ such that $T(y) = z$ we have $\Delta(x, y) \not\subseteq N$. Choose such $y \sim x$ using post-surjectivity of T . There exists $g \notin N$ such that $x(g) \neq y(g)$. However, since N is a symmetric injectivity set for T , we get that there is $h \in g \cdot N$ such that $T(x)(h) \neq T(y)(h)$. Since $h \neq 1_G$, this is a contradiction.

Conversely, suppose we are given a symmetric post-surjectivity set M . Suppose that it is not an injectivity set. Then there are $x, y \in A^G$ and $h \in G$ such that $x(h) \neq y(h)$, yet $T(x) \upharpoonright hM = T(y) \upharpoonright hM$. Without loss of generality, we may assume that $x \sim y$. Then $D := \Delta(T(x), T(y))$ is also finite and $D \cap hM = \emptyset$. However, by post-surjectivity, $\Delta(x, y) \subseteq D \cdot M$. Since $h \notin D \cdot M$, we reach a contradiction. ■

The finer analysis of injectivity and post-surjectivity will now have the following application. Suppose $T: A^G \rightarrow A^G$ is an injective CA and suppose we know some injectivity sets N and M , for T and T^{-1} respectively. We prove that in order to show that T is surjective, it suffices to verify that every pattern from A^{MN} is in the image of T . Second, we show that the image of every injective CA is a subshift of finite type, and additionally, we have some quantitative information about the size of forbidden patterns.

Let us recall a *GOE pattern* (*Garden of Eden pattern*) for $T: A^G \rightarrow A^G$ [7, Section 5.1]. It is a finite pattern $p: D \rightarrow A$ such that for no $f \in T[A^G]$, $f \upharpoonright D = p$. It is a basic application of compactness that if T is not surjective, then there is a non-trivial GOE pattern for T .

Theorem 2.2. *Let $T: A^G \rightarrow A^G$ be an injective CA. Let $N, M \subseteq G$ be finite injectivity sets for T and $T^{-1}: T[A^G] \rightarrow A^G$ respectively, both containing 1_G (which exist by Lemma 1.9).*

- (1) *Then $T[A^G] \subseteq A^G$ is a subshift of finite type, and moreover, the forbidden patterns are defined on MN .*
- (2) *More generally, denote by X^n the subshift $T^n[A^G]$ of A^G , where $n \in \mathbb{N}$. Then X^n is a subshift of finite type whose forbidden patterns are defined on MN^n .*
- (3) *If T is not surjective, then there exists a GOE pattern for T supported on MN .*

Proof. Set $X^n = T^n[A^G]$, for $n \in \mathbb{N}$ (so $X^1 = X$ and $X^0 = A^G$). By Lemma 1.9, N is a memory set for $T^{-1}: X \rightarrow A^G$, thus T^{-1} on X is defined by some $\tau: X_N \rightarrow A$, where $X_N = \{x \upharpoonright N: x \in X\}$. By extending τ to the whole A^N , we get a CA $S: A^G \rightarrow A^G$ whose restriction to X is equal to T^{-1} .

We start with (1). Set $D := MN$ and set $F = A^D \setminus X_D$. We claim that F is a finite set of forbidden patterns defining X . Let us denote by Y the subshift of A^G defined by forbidden patterns from F .

Let $x \in X$. It is clear that for every $g \in G$, $g^{-1} \cdot x \upharpoonright D \notin F$, so $X \subseteq Y$.

Conversely, let $y \in Y$ and let us show that $y \in X$. By definition, for every $g \in G$, $g^{-1} \cdot y \upharpoonright D \in X_D$, so there is $x \in X$ such that $y \upharpoonright gD = x \upharpoonright gD$. Since N , resp. M is a memory set for S , resp. T , we get

$$T \circ S(y)(g) = T \circ S(x)(g) = T \circ T^{-1}(x)(g) = x(g) = y(g).$$

It follows that $T \circ S(y) = y$, thus $y \in X$.

We continue with (2). We shall prove the statement by induction. For $n = 1$ this has been proved in (1). Suppose that $n > 1$ and the statement has been proved for $n - 1$. Set $D_n = MN^n$ and $F_n = A^{D_n} \setminus X_{D_n}^n$, and let Y_n be the subshift of A^G defined by forbidden patterns from F_n . We claim that $X^n = Y_n$. Again, it is clear that $X^n \subseteq Y_n$, so we show the other inclusion. Pick $y \in Y_n$. For every $g \in G$ there exists $x \in X^n$ such that $y \upharpoonright gD_n = x \upharpoonright gD_n$, so

$$S(y) \upharpoonright gD_{n-1} = S(x) \upharpoonright gD_{n-1},$$

and it follows by the induction hypothesis that $S(y) \in X^{n-1}$. Since the same argument as in (1) gives that $T \circ S(y) = y$, we get that $y \in X^n$.

We finish with (3). Retain the notation from (1). If T is not surjective, then $A_G \neq Y$ and thus F is non-empty. Any element of F is then a GOE pattern supported on MN . ■

3. Dual surjunctive groups and ultraproducts

Let us recall the following conjecture due to Gottschalk [19].

Conjecture 3.1. *Suppose that $T: A^G \rightarrow A^G$ is a G -equivariant continuous injective map, i.e., an injective cellular automaton. Then T is surjective and hence an isomorphism.*

A group G is called *surjunctive* if Conjecture 3.1 is true for G and any finite A . The class of surjunctive groups is closed under subgroups [28, Lemma 1.1] and ultraproducts [18, Theorem 3]. All sofic groups are surjunctive [20, 28].

A group G is *dual surjunctive* if every post-surjective cellular automaton $T: A^G \rightarrow A^G$ is pre-injective and hence is an isomorphism by Theorem 1.10. All sofic groups are dual surjunctive [5, Theorem 2].

Let us introduce several new classes of groups.

Definition 3.2. We call a group G *s-surjunctive* if for any finite set A and any strongly irreducible subshift of finite type $X \subseteq A^G$, every injective CA $T: X \rightarrow X$ is surjective.

Analogously, we call a group G *dual s-surjunctive* if for any finite set A and any strongly irreducible subshift of finite type $X \subseteq A^G$, every strongly post-surjective CA $T: X \rightarrow X$ is pre-injective (and hence an isomorphism by Theorem 1.10).

Moreover, we introduce the notions of *ss-surjunctivity* and *ss-dual surjunctivity*, which are defined as s-surjunctivity, resp. s-dual surjunctivity, in the following way: the subshift $X \subseteq A^G$ in the definition is required to be only strongly irreducible, not necessarily of finite type.

We recall here that the *Myhill property* of a subshift or of a more general dynamical system is the property that pre-injectivity of a continuous G -equivariant map implies its surjectivity, and the *Moore property* is the converse, i.e., surjectivity implies pre-injectivity.

Examples 3.3. (1) Every amenable group is ss-surjunctive (and therefore also s-surjunctive). This follows from [9], where the authors prove the Myhill property for amenable groups and strongly irreducible subshifts.

(2) Every amenable group is both s-surjunctive and s-dual surjunctive. This follows from [17], where the author shows the Garden of Eden theorem for amenable groups and strongly irreducible subshifts of finite type.

On the other hand, the existence of ss-dual surjunctive groups is more delicate. Clearly, every finite group is ss-dual surjunctive and so also every locally finite group is ss-dual surjunctive. We conjecture that every G that contains \mathbb{Z} as a subgroup is not ss-dual surjunctive. Fiorenzi in [16, Section 3] (see also [7, Exercise 5.49]) shows that the Moore property does not hold for \mathbb{Z} and strongly irreducible subshifts. Her example might be also a counterexample disproving ss-dual surjunctivity for \mathbb{Z} and groups containing \mathbb{Z} .

Problem 3.4. Are sofic groups s-surjunctive and s-dual surjunctive?

Now, we prove that the class of dual surjunctive groups is closed under subgroups and ultraproducts. The techniques used in the proof can be also applied to get a shorter proof of the result of Glebsky and Gordon from [18] that surjunctive groups are closed under ultraproducts.

Lemma 3.5. *A group is dual surjunctive if and only if all of its finitely generated subgroups are dual surjunctive. In particular, every subgroup of a dual surjunctive group is dual surjunctive.*

Proof. Let G be a group. First, suppose that G is dual surjunctive and let H be a (finitely generated) subgroup. Let $T: A^H \rightarrow A^H$ be a post-surjective CA. The map on a memory set for T also defines a CA $T': A^G \rightarrow A^G$ which is also post-surjective by [5, Proposition 4]. Therefore, since G is dual surjunctive, T' is pre-injective, and it easily follows that T is pre-injective as well.

Conversely, suppose that all finitely generated subgroups of G are dual surjunctive. Let $T: A^G \rightarrow A^G$ be a post-surjective CA defined using a map on a finite memory set T . This map defined a CA $T': A^H \rightarrow A^H$, where H is a subgroup generated by T . Again by [5, Proposition 4], T' is post-surjective, so it is pre-injective. It is again straightforward that T is then pre-injective as well. ■

Theorem 3.6. *Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of dual surjunctive groups and let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} . Then the ultraproduct $\prod_{\mathcal{U}} G_n$ is dual surjunctive as well.*

Proof. Denote the ultraproduct $\prod_{\mathcal{U}} G_n$ by \mathcal{G} . Let $T: A^{\mathcal{G}} \rightarrow A^{\mathcal{G}}$ be a post-surjective continuous \mathcal{G} -equivariant map. T is given by a map $\tau: A^F \rightarrow A$, where $F \subseteq \mathcal{G}$ is, without loss of generality, a finite symmetric memory set, which is also a post-surjectivity set for T . For each $f \in F$, choose a sequence $(f_n)_{n \in \mathbb{N}} \in \prod_n G_n$ representing f in \mathcal{G} . Then for each $n \in \mathbb{N}$, let $F_n \subseteq G_n$ consists of the n -th coordinates of those representatives, so for \mathcal{U} -many n we have a canonical bijection between F_n and F . For such n we define $\tau_n: A^{F_n} \rightarrow A$ using this canonical bijection and $\tau: A^F \rightarrow A$. Finally, we can then, for such n , define a continuous G_n -equivariant map $T_n: A^{G_n} \rightarrow A^{G_n}$ using τ_n . For other $n \in \mathbb{N}$, we may define τ_n and T_n arbitrarily.

For a sequence $(c_n)_{n \in \mathbb{N}}$, where $c_n \in A^{G_n}$, we denote by $(c_n)_{\mathcal{U}}$ the element $c = \prod_{\mathcal{U}} c_n$. That is, the element $c \in A^{\mathcal{G}}$ such that for each $g \in \mathcal{G}$, represented by a sequence $(g_n)_{n \in \mathbb{N}}$, where $g_n \in G_n$,

$$c(g) = a \quad \text{if and only if} \quad \forall^{\mathcal{U}} n (c_n(g_n) = a).$$

Denote by \mathcal{I} the subset of $A^{\mathcal{G}}$ consisting of elements of the form $(c_n)_{\mathcal{U}}$. It is straightforward to verify that $\mathcal{I} \subseteq A^{\mathcal{G}}$ is a dense subset which is invariant under the action of \mathcal{G} and under the relation \sim .

Claim 3.7. *For \mathcal{U} -many n , F_n is a post-surjectivity set for T_n . In particular, for \mathcal{U} -many n , T_n is post-surjective.*

Indeed, otherwise, we get for \mathcal{U} -many n elements c_n and $e_n \sim d_n := T_n(c_n)$ such that there is no $c'_n \sim c_n$ satisfying $T_n(c'_n) = e_n$ and $\Delta(c_n, c'_n) \subseteq \Delta(d_n, e_n) \cdot F_n$. Clearly, without loss of generality, we can assume that for \mathcal{U} -many n we have $\Delta(d_n, e_n) = \{1_{G_n}\}$.

Then we have

- $T((c_n)\mathcal{U}) = (d_n)\mathcal{U}$,
- $\Delta((d_n)\mathcal{U}, (e_n)\mathcal{U}) = \{1_{\mathcal{G}}\}$.

Since T has F as a post-surjectivity set, we can find $c' \in A^{\mathcal{G}}$ such that $T(c') = (e_n)\mathcal{U}$ and $\Delta(c', (c_n)\mathcal{U}) \subseteq F$. Since the set \mathcal{I} is invariant under the relation \sim , we have $c' \in \mathcal{I}$ and we can find elements $c'_n \in A^{G_n}$ for each n , so that $c' = (c'_n)\mathcal{U}$. It easily follows that for \mathcal{U} -many n , $\Delta(c'_n, c_n) \subseteq F_n$ and $T_n(c'_n) = e_n$, a contradiction. This finishes the proof of the claim.

By our assumption that the groups G_n are dual surjunctive, it follows that for \mathcal{U} -many n , $T_n: A^{G_n} \rightarrow A^{G_n}$ is pre-injective.

Now, suppose that T is not pre-injective. This means that there are elements $c \sim d \in A^{\mathcal{G}}$ such that $c \neq d$ and $T(c) = T(d)$. Denote by D the non-empty finite set $\Delta(c, d)$. As in the beginning of the proof for F , we can find non-empty finite sets $D_n \subseteq G_n$, with $|D_n| = |D|$, consisting of the n -th coordinates of representatives for elements of D .

Now, since \mathcal{I} is dense in $A^{\mathcal{G}}$, we can find nets of sequences $\{(c_n^\alpha)_n: \alpha \in S\}$ and $\{(d_n^\alpha)_n: \alpha \in S\}$, where S is some index set and we have

- $(c_n^\alpha)\mathcal{U} \rightarrow c$ and $(d_n^\alpha)\mathcal{U} \rightarrow d$,
- for every $\alpha \in S$ and for \mathcal{U} -many n , $\Delta(c_n^\alpha, d_n^\alpha) = D_n$.

It follows that for every $\alpha \in S$ and \mathcal{U} -many n , since T_n is pre-injective with memory set F_n , that we have

$$\emptyset \neq \Delta(T_n(c_n^\alpha), T_n(d_n^\alpha)) \subseteq D_n \cdot F_n.$$

Consequently, we get that for every $\alpha \in S$,

$$\emptyset \neq \Delta(T((c_n^\alpha)\mathcal{U}), T((d_n^\alpha)\mathcal{U})) \subseteq D \cdot F.$$

By compactness, passing to a subnet if necessary, we can without loss of generality assume that there exists a non-empty finite set $E \subseteq D \cdot F$ such that for every $\alpha \in S$,

$$\emptyset \neq \Delta(T((c_n^\alpha)\mathcal{U}), T((d_n^\alpha)\mathcal{U})) = E.$$

Since $(c_n^\alpha)\mathcal{U} \rightarrow c$ and $(d_n^\alpha)\mathcal{U} \rightarrow d$, and also $T((c_n^\alpha)\mathcal{U}) \rightarrow T(c)$ and $T((d_n^\alpha)\mathcal{U}) \rightarrow T(d)$, we obtain that

$$\emptyset \neq \Delta(T(c), T(d)) = E,$$

in particular $T(c) \neq T(d)$. This contradiction finishes the proof of the theorem. ■

The previous result has as a corollary a topological description of the dual surjunctive groups in the space of marked groups. Let us define the background.

Let S be a fixed finite set. One can topologize the set of (isomorphism classes of) groups with S as a generating set as follows. First, we identify each such group G with F_S/N , where F_S is a free group on generators from S and $N \triangleleft F_S$ is a normal subgroup. Then, it suffices to notice that the set of normal subgroups of F_S is a closed subset of 2^{F_S} , therefore, it is a compact metrizable space (homeomorphic to the Cantor space). Let us denote this space by X_S (see [11, 13] and the references therein).

It is known that for a fixed finite set S , the set of surjective groups is closed in X_S (see [7, Section 3.7], [8, Corollary 1.3] or [18, 20]). We prove an analogous result for the set of dual surjective groups.

Theorem 3.8. *For a fixed finite set S , the set of dual surjective groups is closed in the space of S -marked groups X_S .*

Proof. Let $(N_i)_{i \in \mathbb{N}}$ be a sequence of normal subgroups of F_S converging to a normal subgroup $N \triangleleft F_S$ such that for each $i \in \mathbb{N}$, F_S/N_i is dual surjective. We prove that $G := F_S/N$ is dual surjective. Pick a non-principal ultrafilter \mathcal{U} and let \mathcal{G} be the corresponding ultraproduct of $(F_S/N_i)_{i \in \mathbb{N}}$. By Theorem 3.6, \mathcal{G} is dual surjective. The map from G to \mathcal{G} defined on the generating set S by the diagonal map $s \rightarrow (s)_{\mathcal{U}}$ is clearly a monomorphism. So G embeds as a subgroup of \mathcal{G} , and therefore it is dual surjective itself. ■

Corollary 3.9. *Fully residually dual surjective groups are dual surjective.*

Proof. Let G be fully residually dual surjective. We may suppose it is finitely generated by a finite generating set S . It is then easy to see that G is a limit, in the space of marked groups X_S , of dual surjective groups. So it is dual surjective itself by Theorem 3.8. ■

Using a result of Arzhantseva and Gal [2], we can now obtain the same closure property that they have originally obtained for the class of surjective groups.

Corollary 3.10. *Let G be a semidirect product $H \rtimes F$, where H is dual surjective and F is a finitely generated residually finite group. Then G is dual surjective.*

Proof. By [2, Theorem 1], it suffices to check that

- (1) fully residually dual surjective groups are dual surjective,
- (2) semidirect extensions of dual surjective groups by finite groups are dual surjective.

Item (1) has been proved in Corollary 3.9, and in order to prove (2), it suffices to show that virtually dual surjective groups are dual surjective. We proceed as in [2, Lemma 6]. Let $H \leq G$ be such that $|G:H| < \infty$ and H is dual surjective. Let $T: A^G \rightarrow A^G$ be post-surjective. A^G as an H -shift is isomorphic to the shift $(A^{H \setminus G})^H$, and it is easy to check that the induced H -equivariant map $T': (A^{H \setminus G})^H \rightarrow (A^{H \setminus G})^H$ is post-surjective. Therefore it is injective and so is T . ■

Lemma 3.11. *Let G be a group and $N \triangleleft G$ be a normal subgroup which is finitely generated and residually finite. Assume that G/N is (dual) surjunctive. Then $G/Z(N)$ is (dual) surjunctive as well.*

Proof. The group G acts by conjugation $G \ni g \mapsto i_g \in \text{Aut}(N)$ on N . Consider

$$F: G \rightarrow G/N \times \text{Aut}(N), \quad F(g) = (g/N, i_g).$$

That gives an embedding of $G/Z(N)$ into $G/N \times \text{Aut}(N)$ which is (dual) surjunctive, as if N is a finitely generated residually finite group, then $\text{Aut}(N)$ is residually finite. ■

Conjecture 3.12. *If G is (dual) surjunctive then free product $\mathbb{Z} * G$ is (dual) surjunctive as well.*

One of the basic notions in model theory is that of an *elementary equivalence*. Two structures A and B in a language L are *elementary equivalent* if they satisfy the same first order L -sentences, that is a theorem expressible in L is true in A if and only if it is true in B .

Corollary 3.13. *Suppose that A and B are elementary equivalent groups in a group theory language $\{\cdot\}$ and A is dual surjunctive. Then B is also dual surjunctive.*

Proof. If A and B have the same universal theory, then B embeds into some ultrapower of A and the conclusion follows from Theorem 3.6. ■

4. Direct finiteness conjecture

In the late 1960s, I. Kaplansky proved [21] that if K is a field of characteristic 0 and G is any group, then the group ring $K[G]$ is *directly finite*. Let us recall that a ring R with 1 is called *directly finite* if for any $x, y \in R$, the condition $xy = 1$ implies $yx = 1$. *Kaplansky’s direct finiteness conjecture* says that $K[G]$ is directly finite for any field K and any group G .

This conjecture attracted a lot of attention recently in a more general case when K is a division ring. See [1] for the proof when G is a residually amenable group and also [14] for the computational approach in characteristic 0. The most general result on Kaplansky conjecture was established in [15] (see also [6, Corollary 1.4]), when G is a sofic group. All sofic groups are surjunctive and dual surjunctive. The main idea of the proof in [15] is to construct an embedding of $K[G]$ into simple continuous von Neumann regular ring.

We give below an elementary proof of this conjecture when G is surjunctive or dual surjunctive group and K is an arbitrary field of positive characteristic. Our proof covers the case of sofic groups.

Theorem 4.1. *Surjunctive and dual surjunctive groups satisfy Kaplansky’s direct finiteness conjecture for fields of positive characteristic.*

Proof. Let us first assume that K is a finite field. In this special case, the argument below for surjunctive groups was given in [15] (right after the statement of Conjecture 2). Observe that the group ring $K[G]$ is dense in K^G . Every element $a \in K[G]$ induces a continuous linear map T_a

$$T_a: K^G \rightarrow K^G, \quad T_a(f) = f * a, \quad \text{where } (f * a)(x) = \sum_{y \in G} f(y)a(y^{-1}x),$$

which is G -equivariant, that is, $T_a(g \cdot f)(x) = g \cdot T_a(f)(x)$. Moreover, $T_b \circ T_a = T_{ab}$ for $a, b \in K[G]$.

Suppose now $ab = 1$ for some $a, b \in K[G]$. Then $T_b \circ T_a = T_{ab} = \text{id}$ is the identity on $K[G]$, which is dense in K^G . Therefore, it is the identity on K^G .

Claim 4.2. *The map T_b is post-surjective and T_a is injective.*

Proof. The injectivity of T_a is clear since $T_b \circ T_a = T_{ab} = \text{id}$. We prove that T_b is post-surjective. Suppose $c = T_b(e)$ and $c \sim d$. Then $T_a(c) \sim T_a(d)$, so $e \sim e + T_a(d) - T_a(c) =: e'$ and

$$\begin{aligned} T_b(e') &= T_b \circ T_a(d) + T_b(e - T_a(c)) = T_{ab}(d) + T_b(e) - T_{ab}(c) \\ &= T_{ab}(d) + T_b(e) - T_{bab}(e) = d + T_b(e) - T_b(e) = d. \end{aligned}$$

Hence T_b is post-surjective. ■

Since G is dual surjunctive (surjunctive respectively), T_b is a bijective (T_a is a bijective respectively) continuous map. Thus T_a is the inverse of T_b , so $T_a \circ T_b = T_{ba}$ is the identity on K^G as well. Hence $ba = 1$.

Suppose K is an arbitrary field of positive characteristic and $ab = 1$ but $ba \neq 1$ for some $a, b \in K[G]$. Let $R = \langle k_i, l_i : 1 \leq i \leq n \rangle$ be a subring of K generated as a subring by the coefficients of a and b . Hence, R is a finitely generated domain. Since $ba \neq 1$, we may write

$$ba = \sum_{j=1}^m p_j g_j,$$

where $0 \neq p_j \in K$ and elements $g_j \in G$ are pairwise distinct. We may assume that either $g_1 \neq 1_G$ or $ba = p_1 1_G$, where $p_1 \notin \{0, 1\}$. In the former case, there exists a maximal ideal $I \triangleleft R$ such that $p_1 \notin I$, while in the latter case there exists a maximal ideal $I \triangleleft R$ such that $p_1 - 1 \notin I$ (see, e.g., [27, Lemma 3.2 (iv)]). Then $F = R/I$ is a field, which is finitely generated as a ring. Therefore F is a finite field (see, e.g., [27, Lemma 3.2 (iii)]). Let $f: R[G] \rightarrow F[G]$ be the quotient homomorphism. Then $1 = f(ab) = f(a)f(b)$, but $f(ba) = f(b)f(a) \neq 1$. Indeed, in the former case, $f(p_1 g_1) = f(p_1)g_1 \neq 0$, while in the latter case, $f(p_1 1_G) = f(p_1) \neq 1$. This finishes the proof of Theorem 4.1. ■

We prove that the class of groups satisfying Kaplansky conjecture is closed under taking ultraproducts.

Proposition 4.3. *Suppose R is a ring and $(G_i)_{i \in I}$ is a family of groups such that $R[G_i]$ is directly finite for $i \in I$. Then $R[G]$ is directly finite, where $G = \prod_{i \in I} G_i / \mathcal{U}$ is an ultraproduct of $(G_i)_{i \in I}$. In particular, the class of groups satisfying Kaplansky’s direct finiteness conjecture is closed under taking ultraproducts.*

Proof. Pick some $x, y \in R[G]$. That is $x = \sum_{j=1}^n c_j g_j, y = \sum_{k=1}^m c'_k g'_k$, where $c_j, c'_k \in R, g_j = (g_{i,j})_{i \in I} / \mathcal{U} \in G, g'_k = (g'_{i,k})_{i \in I} / \mathcal{U} \in G$ for some $g_{i,j}, g'_{i,k} \in G_i$ and $g_s \neq g_t, g'_p \neq g'_q$ for $1 \leq s \neq t \leq n$ and $1 \leq p \neq q \leq m$.

Consider $x_i = \sum_{j=1}^n c_j g_{i,j}$ and $y_i = \sum_{k=1}^m c'_k g'_{i,k}$ ($x_i, y_i \in R[G_i]$). In order to finish the proof, it is enough to prove the following claim.

Claim 4.4. $xy = 1$ if and only if $\{i \in I : x_i y_i = 1\} \in \mathcal{U}$.

Proof. Let

$$I_1 = \{i \in I : \forall_{1 \leq j, j' \leq n} \forall_{1 \leq k, k' \leq m} g_j \cdot g'_k = g_{j'} \cdot g'_{k'} \Leftrightarrow g_{i,j} \cdot g'_{i,k} = g_{i,j'} \cdot g'_{i,k'}\},$$

$$I_2 = \{i \in I : \forall_{1 \leq j \leq n} \forall_{1 \leq k \leq m} g_j \cdot g'_k = e_G \Leftrightarrow g_{i,j} \cdot g'_{i,k} = e_{G_i}\}.$$

Clearly $I_1, I_2 \in \mathcal{U}$. Moreover, for $i \in I_1 \cap I_2$, the canonical form of $x_i y_i$ in $R[G_i]$ has the same coefficients as the canonical form of xy in $R[G]$ (because $x \cdot y = \sum \sum c_j c'_k g_j \cdot g'_k$). Therefore for $i \in I_1 \cap I_2, xy = 1$ if and only if $x_i y_i = 1$. ■

5. Expansive dynamical systems

In the last section, we consider dual surjectivity for more general dynamical systems than subshifts. Here we follow and apply mainly the seminal results of Chung and Li [12], and Li [23] on expansive algebraic actions.

Let X be a compact metrizable space with some compatible metric d which we may assume, without loss of generality, to be bounded by 1. An action $\alpha : G \curvearrowright X$ of a group G on X by homeomorphisms is called *expansive* if there exists δ such that for every $x \neq y \in X$, there is $g \in G$ such that $d(g \cdot x, g \cdot y) > \delta$. The real δ is then called an *expansiveness constant* of α .

Two elements $x, y \in X$ are called *homoclinic* if $\lim_{g \rightarrow \infty} d(g \cdot x, g \cdot y) \rightarrow 0$. Clearly, homoclinicity is an equivalence relation which we shall denote by \sim . It coincides with the relation \sim for the Bernoulli topological shifts. Note that since all compatible metrics on X are uniformly equivalent, being expansive and homoclinic does not depend on the choice of the metric.

Having the relation ‘ \sim ’ at our disposal, we can define the notion of strong post-surjectivity in the same way as for subshifts.

Definition 5.1. Let $\alpha : G \curvearrowright X$ have an expansiveness constant $\delta > 0$. Let $T : X \rightarrow X$ be a continuous G -equivariant map. We say that T is *strongly post-surjective* if there is a finite subset $F \subseteq G$ such that for every $x, y \in X$ such that $T(x) \sim y$, i.e., in particular,

the set $D := \{g \in G : d(g \cdot T(x), g \cdot y) > \delta\}$ is finite, there exists $z \sim x$ such that $T(z) = y$ and $\{g \in G : d(g \cdot x, g \cdot z) > \delta\} \subseteq FD$.

The finite set F is called a *post-surjectivity set* for T .

Remark 5.2. A good question asked by the referee is whether the strong post-surjectivity of T depends on the expansiveness constant δ , which is not canonical and can be always replaced by a smaller constant. We claim that by choosing another smaller expansiveness constant $0 < \gamma < \delta$, we may need to enlarge the post-surjectivity set; nevertheless, a post-surjectivity set still exists and is finite, answering the question. This follows from the following observation. There exists a finite set $D_\gamma \subseteq G$ such that for every $x, y \in X$, if $d(x, y) > \gamma$, then there is $g \in D_\gamma$ such that $d(g \cdot x, g \cdot y) > \delta$. Indeed, otherwise, for every finite set $E \subseteq G$ there are elements $x_E, y_E \in X$ satisfying $d(x_E, y_E) > \gamma$ and $d(g \cdot x_E, g \cdot y_E) \leq \delta$, for all $g \in E$. By passing to a subsequence if necessary, we may assume that $x_E \rightarrow x$ and $y_E \rightarrow y$, and we then clearly have $d(x, y) \geq \gamma$, so in particular $x \neq y$. By expansiveness, there is $g \in G$ such that $d(g \cdot x, g \cdot y) > \delta$, but then also $d(g \cdot x_E, g \cdot y_E) > \delta$ for all sufficiently large E , which is a contradiction.

It follows that by changing the expansiveness constant to γ , we may replace F by $D_\gamma^{-1} \cdot F$.

Obviously, the previous definition can only be reasonable provided the reversible maps satisfy it. We show that this is indeed the case.

First, we need an analogue of Lemma 1.9.

Lemma 5.3. *Let $\alpha: G \curvearrowright X$ have an expansiveness constant $\delta > 0$. A continuous G -equivariant map T is injective if and only if there exists a finite set $F \subseteq G$ such that for every $x \neq y \in X$ with $d(x, y) > \delta$, there is $f \in F$ such that $d(f \cdot T(x), f \cdot T(y)) > \delta$.*

Proof. Suppose that a continuous G -equivariant map T satisfies such a condition. We show that it is injective. Choose $x \neq y \in X$. By expansiveness, there is $g \in G$ such that $d(g \cdot x, g \cdot y) > \delta$, so there must be $f \in F$ such that $d(fg \cdot T(x), fg \cdot T(y)) > \delta$; in particular, $T(x) \neq T(y)$.

We now show the converse. Suppose it does not satisfy the condition. Then for every finite set $F \subseteq G$ there are $x_F, y_F \in X$ such that $d(x_F, y_F) > \delta$, yet $d(f \cdot T(x_F), f \cdot T(y_F)) \leq \delta$ for all $f \in F$. Since X is compact, we may assume that the nets $(x_F)_F$ and $(y_F)_F$ converge to elements x and y , respectively. We have $d(x, y) \geq \delta$, so $x \neq y$. If $T(x) = T(y)$, then by expansiveness there exists $f \in G$ such that $d(f \cdot T(x), f \cdot T(y)) > \delta$. Then however $d(f \cdot T(x_F), f \cdot T(y_F)) > \delta$ for all sufficiently large sets F containing f . This contradiction shows that $T(x) \neq T(y)$, so T is not injective. ■

Definition 5.4. For an injective continuous G -equivariant map T , the finite set F from Lemma 5.3 is called an *injectivity set* for T .

Proposition 5.5. *Let an action $\alpha: G \curvearrowright X$ be as above. Let $T: X \rightarrow X$ be a continuous G -equivariant map which is moreover reversible. Then T is strongly post-surjective.*

Proof. Since T is an injective continuous G -equivariant map, let F be a finite injectivity set for T provided by Lemma 5.3. We may suppose that it is symmetric. We show it is a post-surjectivity set for T . Choose $x, z \in X$ such that $T(x) \sim z$. Set $y := T^{-1}(z)$. Since T^{-1} is, by assumption, continuous and G -equivariant, we have $x \sim y$, and obviously $T(y) = z$. Set $D := \{g \in G: d(g \cdot T(x), g \cdot z) > \delta\}$, which is finite. Suppose that there is $h \in G \setminus FD$ such that $d(h \cdot x, h \cdot y) > \delta$. Then, since F is an injectivity set for T , there must be $f \in F$ such that $d(fh \cdot T(x), fh \cdot z) > \delta$. Therefore $fh \in D$, so $h \in FD$, a contradiction. ■

Theorem 5.6. *Let $\alpha: G \curvearrowright X$ be as above. Suppose that there exists a dense class in the homoclinicity relation \sim . Let $T: X \rightarrow X$ be a pre-injective and strongly post-surjective continuous G -equivariant map. Then T is injective.*

Proof. Let $\delta > 0$ be an expansiveness constant. Suppose that such T is not injective, so there are $w_1 \neq w_2 \in X$ with $T(w_1) = T(w_2)$. By expansiveness, there is $g \in G$ such that $d(g \cdot w_1, g \cdot w_2) > \delta$, so, without loss of generality, we can assume that $d(w_1, w_2) > \delta$. Let $F \subseteq G$ be a finite symmetric post-surjectivity set for T . By the assumption, there exists $v \in X$ whose equivalence class $\{v' \in X: v' \sim v\}$ is dense in X . Therefore, by the continuity of T and of the group action, we can find $v_1 \sim v$ and $v_2 \sim v$, where $d(v_1, w_1)$ and $d(v_2, w_2)$ are small enough so that $d(v_1, v_2) > \delta$ and for every $f \in F$, $d(f \cdot T(v_1), f \cdot T(v_2)) < \delta$. Since T is pre-injective, $T(v_1) \neq T(v_2)$. So by expansiveness, there exists $g \in G$ so that $d(g \cdot T(v_1), g \cdot T(v_2)) > \delta$. By assumption, $g \notin F$. Now apply the strong post-surjectivity to $x = v_1$ and $z = T(v_2)$. We have

$$g \in D := \{h \in G: d(h \cdot T(x), h \cdot z) > \delta\} \cap F = \emptyset.$$

By strong post-surjectivity, there exists $y \sim x = v_1$ with $T(y) = z = T(v_2)$ such that $D' := \{h \in G: d(h \cdot x, h \cdot y) > \delta\} \subseteq FD$. However, pre-injectivity of T implies that $y = v_2$, so since $d(v_1, v_2) > \delta$, $1_G \in D'$. Since for every $f \in F$, $d(f \cdot T(v_1), f \cdot T(v_2)) < \delta$, we have $F \cap D = \emptyset$. Since F is symmetric, it follows that $1_G \notin FD$. This contradiction finishes the proof. ■

We shall need one more simple lemma, where we require that every class in the homoclinicity relation is dense.

Lemma 5.7. *Let $\alpha: G \curvearrowright X$ be as above, and suppose that every class $[x]_\sim$ in the homoclinicity relation is dense. Then every strongly post-surjective continuous G -equivariant map $T: X \rightarrow X$ is surjective.*

Proof. Pick an arbitrary $x \in X$ and we show that there is $y \in X$ such that $T(y) = x$. Let d be a compatible metric on X . Since $\{z \in X: z \sim T(x)\}$ is dense in X , for every $n \in \mathbb{N}$, there is $z_n \sim T(x)$ with $d(z_n, x) < 1/n$. By strong post-surjectivity, there is $y_n \in X$ for each $n \in \mathbb{N}$ such that $T(y_n) = z_n$. Their cluster point y clearly satisfies $T(y) = x$. ■

We now apply the previous results to expansive algebraic actions of amenable groups, thoroughly studied in the context of surjectivity for example in [12] and [23].

The following is the most important definition.

Definition 5.8. Let $\alpha: G \curvearrowright X$ be an expansive action of a group G on a compact metrizable space X by homeomorphisms. We say that $\alpha: G \curvearrowright X$ is *dual surjective* if every continuous G -equivariant strongly post-surjective map $T: X \rightarrow X$ is reversible.

In the sequel, we work with algebraic actions. That is, actions of countable groups on compact metrizable abelian groups by continuous automorphisms. By the Pontryagin duality, all such actions of a countable group G are in one-to-one correspondence with countable modules over the group ring $\mathbb{Z}[G]$. We refer to [22, Chapter 13] for an introduction to expansive algebraic actions and the notions of entropy from the next result. We recall that a group is *polycyclic-by-finite* if it is obtained recursively in finitely many steps by the group extension operation, using at each step a finite or a cyclic group.

Theorem 5.9. Let $\alpha: G \curvearrowright X$ be an expansive algebraic action of a countable amenable group on a compact metrizable abelian group X with completely positive entropy with respect to the normalized Haar measure on X . Suppose that at least one of the following conditions is satisfied:

- (1) G is polycyclic-by-finite,
- (2) the set $\Delta(X)$ of elements of X that are homoclinic to the identity element e_X of X is dense in X .

Then α is dual surjective.

Proof. We need (2). If (1) is satisfied, i.e., G is polycyclic-by-finite, then by [12, Theorem 1.2], the assumption on completely positive entropy implies that the set $\Delta(X)$ is dense in X ; that is, (1) implies (2). Notice also that each equivalence class in \sim is a translate of $\Delta(X)$, so actually each equivalence class is dense. So fix a strongly post-surjective continuous and G -equivariant map $T: X \rightarrow X$. By Lemma 5.7, T is surjective.

It follows that we can apply [23, Theorem 1.2] to get that T is pre-injective. Finally, we have all the ingredients to apply Theorem 5.6 to obtain that T is reversible. ■

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