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Euclidean Artin–Tits groups are acylindrically hyperbolic

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Abstract. In this paper, we prove that all Euclidean Artin–Tits groups are acylindrically hyperbolic. To any Garside group of finite type, Wiest and the author associated a hyperbolic graph called the *additional length graph* and they used it to show that central quotients of Artin–Tits groups of spherical type are acylindrically hyperbolic. In general, a Euclidean Artin–Tits group is not *a priori* a Garside group but McCammond and Sulway have shown that it embeds into an *infinite-type* Garside group which they call a *crystallographic Garside group*. We associate a *hyperbolic* additional length graph to this crystallographic Garside group and we exhibit elements of the Euclidean Artin–Tits group which act loxodromically and weakly properly discontinuously on this hyperbolic graph.

1. Introduction

An *Artin–Tits group* is a group defined by a presentation involving a finite set of generators S (the *standard generators*) and where all the relations are as follows: every pair (a, b) of standard generators satisfies at most one balanced relation of the form

$$\Pi(a, b; m_{a,b}) = \Pi(b, a; m_{a,b}),$$

with $m_{a,b} = m_{b,a} \geqslant 2$ and where for $j \geqslant 2$,

$$\Pi(a, b; j) = \begin{cases} (ab)^{\frac{j}{2}} & \text{if } j \text{ is even,} \\ (ab)^{\frac{j-1}{2}} a & \text{if } j \text{ is odd.} \end{cases}$$

We also write $m_{a,b} = m_{b,a} = \infty$ when a and b satisfy no relation. This presentation can be encoded by a *Coxeter graph* Γ. The vertices of Γ are in bijection with the set S. Two distinct vertices a, b of Γ are connected by an edge labeled $m_{a,b}$ if and only if either they satisfy no relation or $m_{a,b} > 2$. The Artin–Tits group defined by the Coxeter graph Γ will be denoted by A_Γ. The *rank* of A_Γ is the cardinality of S. The quotient of A_Γ by the normal subgroup generated by the squares of the elements in S is a *Coxeter group* denoted by W_Γ. The group A_Γ (W_Γ respectively) is said to be *irreducible* if Γ is connected.

The geometry of Artin–Tits groups has recently attracted a lot of attention and non-positive curvature features have been exhibited for many classes of Artin–Tits groups. In

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this paper, we focus on the acylindrical hyperbolicity, which was introduced and extensively discussed in [24]. The class of acylindrically hyperbolic groups is both sufficiently large to include a significative number of interesting groups and restrictive enough to deduce many interesting consequences. An isometric action of a group G on a metric space (X, d_X) is *acylindrical* if for every $\varepsilon > 0$, there exist $R, N > 0$ such that whenever two points $x, y \in X$ are at a distance at least R apart, then

$$\#\{g \in G,\ d_X(x, g \cdot x) \leqslant \varepsilon, d_X(y, g \cdot y) \leqslant \varepsilon\} \leqslant N.$$

A group is *acylindrically hyperbolic* if it is not virtually cyclic and admits an acylindrical isometric action on a hyperbolic metric space with unbounded orbits.

It is conjectured that the central quotient of every irreducible Artin–Tits group of rank at least 2 is acylindrically hyperbolic [14, Conjecture B]. Artin–Tits groups of rank 2 are called *dihedral* Artin–Tits groups and (when irreducible) their central quotients are virtually free hence acylindrically hyperbolic [14]. Here is a brief overview of some classes of Artin–Tits groups for which the conjecture has been proved.

- Artin's braid group, seen as the mapping class group of the punctured disk [4].
- Artin–Tits groups of spherical type (the Coxeter group is finite) [7].
- Right-angled Artin–Tits groups ($m_{a,b} \in \{2, \infty\}$ for any standard generators $a \neq b$) [17].
- Artin–Tits groups of extra extra large type ($m_{a,b} \geqslant 5$ for any standard generators $a \neq b$) [14].
- 2-dimensional Artin–Tits groups of hyperbolic type (for each triple of distinct standard generators a, b, c, we have $\frac{1}{m_{a,b}} + \frac{1}{m_{b,c}} + \frac{1}{m_{a,c}} \leqslant 1$, and the associated Coxeter group is hyperbolic) [18].
- Artin–Tits groups for which there is no partition $S = S_1 \sqcup S_2$ of the set of standard generators satisfying $m_{a,b} < \infty$ for all $a \in S_1, b \in S_2$ [9].
- 2-dimensional Artin–Tits groups [28].

In this paper, we focus on the class of *Euclidean Artin–Tits groups*. The theory of arbitrary Coxeter groups and Artin–Tits groups stems from the study of discrete groups generated by reflections which act geometrically on spheres (finite Coxeter groups) and Euclidean spaces (Euclidean Coxeter groups). Finite and Euclidean Coxeter groups are central in Lie theory and they were studied much longer before Tits gave the general definition [27] of arbitrary Coxeter and Artin–Tits groups. An Artin–Tits group has *spherical type*, or *Euclidean type*, respectively, if the associated Coxeter group is finite, or Euclidean, respectively.

There is a well-known classification of connected Coxeter graphs ((extended) Dynkin diagrams) defining irreducible finite and Euclidean Coxeter groups. Artin–Tits groups of spherical type have long been well-understood thanks to their Garside structure [6, 11]. By contrast, Artin–Tits groups of Euclidean type remained mostly mysterious (with some exceptions, see [12, 13, 26]) for decades, until their structure was elucidated by McCammond and Sulway [23] around 2015. In this paper we prove the above conjecture for irreducible Artin–Tits groups of Euclidean type (note that these groups are centerless, by [23, Proposition 11.9]):

Theorem A. *Let A be an irreducible Artin–Tits group of Euclidean type. Then A is acylindrically hyperbolic.*

To establish that a given group is acylindrically hyperbolic, it is often simpler to use an equivalent characterization due to Osin [24, Theorem 1.2]: a non-virtually cyclic group G is acylindrically hyperbolic if and only if it acts by isometries on a hyperbolic metric space and *some* element acts in a loxodromic weakly properly discontinuous (WPD) fashion, see [2]. An element $g \in G$ acts *loxodromically* on the metric space (X, d_X) if for some (any) $x \in X$ there is some $\kappa > 0$ so that for all $k \in \mathbb{Z}$,

$$d_X(x, g^k \cdot x) \geqslant |k|\kappa,$$

and it acts *weakly properly discontinuously* (WPD) if for all $x \in X$ and for all $\varepsilon > 0$, there exists $N > 0$ such that the set

$$\{h \in G,\ d_X(x, h \cdot x) \leqslant \varepsilon, d_X(g^N \cdot x, hg^N \cdot x) \leqslant \varepsilon\}$$

is finite.

We consider first the case of the *affine braid group*, or Artin–Tits group $A_{\widetilde{A}_n}$, where $\widetilde{A}_n$ is a graph with two vertices joined by an edge with label ∞ if $n = 1$ and a cyclic graph on $n+1$ vertices with all labels equal to 3 if $n \geqslant 2$.

Proposition 1.1. *Let $n \geqslant 1$. The affine braid group $A_{\widetilde{A}_n}$ is acylindrically hyperbolic.*

Proof. For $n = 1$, $A_{\widetilde{A}_1}$ is a free group of rank 2. Suppose that $n \geqslant 2$. It is known that $A_{\widetilde{A}_n}$ embeds into the central quotient of Artin's braid group on $(n+2)$ strands [16] – note that the image consists of the projections modulo the center of the so-called 1-pure braids (the braids whose first strand ends in the first position). The central quotient of the braid group on $(n+2)$ strands in turn embeds in the mapping class group of a sphere with $(n+3)$ punctures, so it has an acylindrical action on the curve graph of the punctured disk [4]. This curve graph is hyperbolic and each pseudo-Anosov braid acts on it in a loxodromic way [19]. Because the action is acylindrical, it follows that each pseudo-Anosov braid acts on the curve graph in a WPD manner. Up to taking a power, each pseudo-Anosov braid is 1-pure, so it lies in the image of the affine braid group; therefore, as the affine braid group is not virtually cyclic, the above mentioned result by Osin applies to show that the affine braid group is acylindrically hyperbolic. ∎

Our proof for a general irreducible Artin–Tits group of Euclidean type closely follows the construction of Wiest and the author to show that central quotients of Artin–Tits groups of spherical type are acylindrically hyperbolic [7]. Let us recall the strategy. They first constructed from any finite-type Garside group G a *hyperbolic* graph $\mathcal{C}_{\mathrm{AL}}(G)$ called the *additional length graph* on which the group G/ZG acts isometrically, see [8]. When G is an Artin–Tits group of spherical type, they exhibit in [7] some element x_G of G/ZG whose action on this hyperbolic graph is loxodromic and WPD. Acylindrical hyperbolicity of G/ZG then follows by the above-mentioned theorem of Osin.

On another hand, McCammond and Sulway have established that each irreducible Artin–Tits group of Euclidean type A embeds in a so-called *crystallographic group* $\mathfrak{C}$ with a Garside structure of *infinite type* [23]; such a group is sometimes also called a *quasi-Garside group*. It turns out that the construction and the proof of the hyperbolicity of the additional length graph given in [8] adapt immediately in this more general context and we obtain again a hyperbolic graph $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$ with an isometric action of $\mathfrak{C}$. Then, we construct elements of $A < \mathfrak{C}$ which act loxodromically and in a WPD fashion on this additional length graph. We conclude in the same way, using Osin's characterization of acylindrically hyperbolic groups.

The paper is organized as follows. In Section 2, we give the suitable definition of a Garside group, we recall the construction of the additional length graph and its hyperbolicity. In Section 3, we recall a number of facts on Euclidean Coxeter groups used in the sequel. In Section 4, we construct the desired loxodromic elements. In Section 5, we prove Theorem A.

2. Garside structure and the additional length graph

The reader is referred to [10] for a detailed account on Garside theory; the unpublished text [20] can also be useful.

Definition 2.1 (Garside monoid). A monoid M is a *Garside monoid* if it satisfies the following conditions:

(1) M is left and right cancellative, that is, for all $a, b, c \in M$, either of the conditions $ab = ac$ or $ba = ca$ implies $b = c$.

(2) There exists a map $\rho : M \to \mathbb{N} \cup \{0\}$ satisfying $\rho(ab) \geqslant \rho(a) + \rho(b)$ for all $a, b \in M$ and $\rho(a) = 0$ if and only if $a = 1$.

(3) Both relations $\preccurlyeq$ and $\succcurlyeq$ in M are lattice orders on M:
   - $a \preccurlyeq b$ if and only if there is $c \in M$ so that $b = ac$ (a is a *prefix* or *left divisor* of b or b is a *right multiple* of a).
   - $a \succcurlyeq b$ if and only if there is $c \in M$ so that $a = cb$ (b is a *suffix* or *right divisor* of a or a is a *left multiple* of b).

(4) There is an element $\Delta \in M$, called the *Garside element*, such that the left and right divisors of Δ are the same and generate M. These elements are called *simple elements*. A simple element is *proper* if it is distinct from 1 and Δ.

Remark 2.2. (i) In the usual definition of a Garside monoid, the set of simple elements is assumed to be finite; a Garside monoid with a finite number of simple elements is said to be of *finite type*. In absence of this condition, the monoid M is also sometimes called a *quasi*-Garside monoid, see [10, Definitions 2.1 and 2.2].

(ii) For some of the Garside monoids considered in this paper, it will be convenient to modify slightly the condition (2), so that the function ρ has values in $\mathbb{N} \cup \frac{2}{3}\mathbb{N} \cup \{0\}$ instead of $\mathbb{N} \cup \{0\}$.

(iii) An element $a \in M$ is an *atom* if a is indecomposable, that is, if the condition $a = bc$ with $b, c \in M$ implies $b = 1$ or $c = 1$. The set of atoms generates M.

Definition 2.3 (gcds, lcms, complements and weightedness). The *left/right greatest common divisor* of $a, b \in M$ is denoted by $a \wedge b / a \wedge^{\Lsh} b$; the right/left least common multiple of a, b is denoted by $a \vee b / a \vee^{\Lsh} b$. Let s be a simple element. Owing to condition (1), there exists a *unique* $\partial(s) \in M$ such that $s\partial(s) = \Delta$. By condition (4), this element is still a simple element, called the *right complement* of s. Similarly, we have the *left complement* $\partial^{-1}(s)$ of s, which is the unique simple element satisfying $\partial^{-1}(s)s = \Delta$. Conjugation by Δ induces a bijection of the set of simple elements which we denote by τ: $\tau(s) = \partial(\partial(s))$ and $s\Delta = \Delta\tau(s)$. This map extends to an automorphism of M; when M is of finite type, this automorphism has finite order but this need not be the case in our more general context. An ordered pair (s, s') of simple elements is *left-weighted* if $\partial s \wedge s' = 1$. In other words, this means that no non-trivial prefix a of s' satisfies that sa is still a simple element. Similarly, the ordered pair (s, s') of simple elements is called *right-weighted* if $\partial^{-1}(s') \wedge^{\Lsh} s = 1$.

Proposition 2.4 (Normal forms). *A Garside monoid M embeds in its group of fractions G and G is called a* Garside group*; in this context, the elements of M are called* positive*. Each element g in G admits a unique decomposition $g = a^{-1}b$, where $a, b \in M$ and $a \wedge b = 1$. This is called the* negative-positive normal form *of g. Each element $g \in G$ also admits a unique decomposition $g = \Delta^p s_1 \dots s_q$, called its* left normal form*, where $p \in \mathbb{Z}$, $q \geqslant 0$, and the s_i are* proper *simple elements such that (s_i, s_{i+1}) is left-weighted. Similarly, each $g \in G$ admits a unique* right normal form *$g = s'_q \dots s'_1 \Delta^p$ where the s'_i are* proper *simple elements such that (s'_{i+1}, s'_i) is right-weighted. The integers p, q and $q + p$ are respectively called the* infimum*, the* canonical length *and the* supremum *of g and we denote $p = \inf(g)$, $q = \ell(g)$ and $p + q = \sup(g)$.*

The following statements can all be found in [8] for *finite-type* Garside groups. However, they extend immediately to our more general context.

Definition 2.5 (Absorbable, see [8, Definition 1]). Let G be a Garside group. An element g of G is said to be *absorbable* if the two following conditions are satisfied:

- $\inf(g) = 0$ or $\sup(g) = 0$,
- there exists some $h \in G$ such that

$$\inf(hg) = \inf(h) \quad \text{and} \quad \sup(hg) = \sup(h).$$

The following lemma is a useful technical fact about absorbable elements.

Lemma 2.6 (Subwords of absorbable elements, see [8, Lemma 2]). *Let G be a Garside group. Suppose that an absorbable element $g \in G$ factors as a product of positive (possibly trivial) elements: $g = g_1g_2g_3$. Then g_2 is absorbable.*

Definition 2.7 (Additional length graph, see [8, Definition 2] and [1, Section 2.1]). Let G be a Garside group. The *additional length graph* associated to G is the graph denoted by $\mathcal{C}_{\mathrm{AL}}(G)$ defined in the following way:

- The vertices are in bijection with the left cosets $g\Delta^{\mathbb{Z}}$, $g \in G$. Each vertex V has a unique *distinguished representative* $\underline{V}$ of infimum 0. We denote by $*$ the vertex $\Delta^{\mathbb{Z}}$.
- Two vertices V and V' are connected by an edge if and only if one of the following holds:
  - There is a proper simple element s such that $\underline{V}s$ belongs to the coset V' (this is equivalent to saying that there is some proper simple element s' so that $\underline{V'}s'$ belongs to the coset V).
  - There is an absorbable element g such that $\underline{V}g \in V'$ (equivalently, there is an absorbable element g' so that $\underline{V'}g' \in V$).

The graph is endowed with the edge-metric which we denote by d_{AL}. There is an isometric action of G by left translation on the vertices: for $g \in G$ and V a vertex of $\mathcal{C}_{\mathrm{AL}}(G)$, $g \cdot V = (g\underline{V})\Delta^{\mathbb{Z}}$.

Definition 2.8 (Preferred paths, see [8, Definition 3] and [1, Section 3.1]). Let G be a Garside group. Consider a vertex V of $\mathcal{C}_{\mathrm{AL}}(G)$ and write $\underline{V} = s_1 \dots s_r$ for the left normal form of its distinguished representative. The *preferred path* $\mathcal{A}(*, V)$ is the path

$$*,\ s_1\Delta^{\mathbb{Z}},\ \dots\ ,\ (s_1 \dots s_r)\Delta^{\mathbb{Z}} = V$$

from $*$ to V. Given any two vertices V_1, V_2 of $\mathcal{C}_{\mathrm{AL}}(G)$, the preferred path $\mathcal{A}(V_1, V_2)$ is the $\underline{V_1}$ left translate of the path $\mathcal{A}(*, \underline{V_1}^{-1} \cdot V_2)$.

Here is a summary of the properties enjoyed by the preferred paths.

Proposition 2.9 (Properties of preferred paths). *Let G be a Garside group. Let V_1, V_2, V_3 be three vertices of $\mathcal{C}_{\mathrm{AL}}(G)$.*

(i) ([8, Lemma 4]) *The preferred path $\mathcal{A}(V_1, V_2)$ is the concatenation of the paths $\mathcal{A}(V_1, (\underline{V_1} \wedge \underline{V_2})\Delta^{\mathbb{Z}})$ and $\mathcal{A}((\underline{V_1} \wedge \underline{V_2})\Delta^{\mathbb{Z}}, V_2)$.*

(ii) ([8, Lemma 5]) *The preferred paths are symmetric; that is, $\mathcal{A}(V_2, V_1)$ is the reverse of $\mathcal{A}(V_1, V_2)$.*

(iii) ([7, Lemma 2]) *Let $g \in G$. We have $\mathcal{A}(g \cdot V_1, g \cdot V_2) = g \cdot \mathcal{A}(V_1, V_2)$.*

(iv) ([8, Lemma 7]) *The triangle in $\mathcal{C}_{\mathrm{AL}}(G)$ with vertices V_1, V_2, V_3 and with sides $\mathcal{A}(V_1, V_2)$, $\mathcal{A}(V_2, V_3)$ and $\mathcal{A}(V_3, V_1)$ is 2-thin: each side is at Hausdorff distance at most 2 from the union of the other two sides.*

Finally, the main result of [8] is the following.

Theorem 2.10 (Hyperbolic, see [8, Theorem 1]). *Let G be a Garside group. The graph $\mathcal{C}_{\mathrm{AL}}(G)$ is* 60*-hyperbolic and the preferred paths form a family of uniformly unparameterized quasi-geodesics: for all vertices V_1, V_2 of $\mathcal{C}_{\mathrm{AL}}(G)$, the Hausdorff distance between $\mathcal{A}(V_1, V_2)$ and any geodesic connecting V_1 and V_2 is bounded above by* 39*.*

3. Euclidean Coxeter groups

In this section we gather a number of useful facts concerning Euclidean Coxeter groups. We follow McCammond's approach developed in [5,21,23] with Brady and Sulway, see also [22] and [25]. Other useful references are [3,15].

3.1. Euclidean isometries

Definition 3.1 (Euclidean space and its isometries). We denote by $\mathbb{E} = \mathbb{R}^n$ the n-dimensional Euclidean space endowed with the usual scalar product

$$\langle \eta, \eta' \rangle = \langle (\eta_i)_{i=1}^n, (\eta'_i)_{i=1}^n \rangle = \sum_{i=1}^{n} \eta_i \eta'_i.$$

Two elements $\eta, \eta' \in \mathbb{E}$ are *orthogonal* if $\langle \eta, \eta' \rangle = 0$. Given $\eta, \eta' \in \mathbb{E}$, the *distance* between η and η' is the Euclidean norm $\|\eta' - \eta\| = \sqrt{\langle \eta' - \eta, \eta' - \eta \rangle}$. We denote by $\mathrm{ISOM}(\mathbb{E})$ the group of isometries of $\mathbb{E}$, that is, the group of distance-preserving transformations of $\mathbb{E}$. Throughout, the elements of $\mathbb{E}$ will be formally considered as vectors; however, $\mathbb{E}$ can be identified with the affine space which it underlies and sometimes it will be intuitively clearer to think of elements of $\mathbb{E}$ also as *points* rather than vectors.

Definition 3.2 (Linear subspace). A *linear subspace* of $\mathbb{E}$ is a non-empty subset closed under linear combination. We denote by $\mathrm{DIM}(U)$ the dimension of a linear subspace U of $\mathbb{E}$. Each linear subspace U of $\mathbb{E}$ has an *orthogonal complement* $U^\perp$, which is the linear subspace of $\mathbb{E}$ made of those elements in $\mathbb{E}$ which are orthogonal to all elements of U. There is a direct sum decomposition $\mathbb{E} = U \oplus U^\perp$, and $\mathrm{DIM}(U^\perp) = n - \mathrm{DIM}(U)$ is called the *codimension* of U.

Definition 3.3 (Affine subspace). A subset B of $\mathbb{E}$ is an *affine subspace* if there are a linear subspace U of $\mathbb{E}$ and an element θ of $\mathbb{E}$ such that $B = U + \theta$. Note that $U = \{\eta' - \eta,\ \eta, \eta' \in B\}$. The linear subspace U is called the *direction* of the affine subspace B and we denote it by $\mathrm{DIR}(B)$. The *dimension* of B is $\mathrm{DIM}(\mathrm{DIR}(B))$. An affine subspace is a linear subspace if and only if it contains $0_{\mathbb{E}}$ (equivalently if it is equal to its direction). Given an affine subspace B, there is a unique $\theta_0 \in B$ such that $\|\theta_0\|$ is minimal; then $B = \mathrm{DIR}(B) + \theta_0$ and this is called the *standard form* of B – note that $\theta_0 \in \mathrm{DIR}(B)^\perp$. Two affine subspaces B_1, B_2 of $\mathbb{E}$ are *parallel* if $B_1 \cap B_2 = \emptyset$ and $\mathrm{DIR}(B_1) \subset \mathrm{DIR}(B_2)$ (or vice-versa).

Definition 3.4 (Hyperplane and reflection). A *hyperplane* of $\mathbb{E}$ is an affine subspace of dimension $n-1$. Given a hyperplane H in $\mathbb{E}$, there is a unique non-trivial isometry r_H of $\mathbb{E}$ which fixes H pointwise; r_H is called the *reflection through* H. The orthogonal complement of $\textsc{Dir}(H)$ is a line in $\mathbb{E}$; a non-trivial vector in this line is called a *root* of the reflection r_H. Given a hyperplane $H \subset \mathbb{E}$ and a root α of r_H, there is a unique $c \in \mathbb{R}$ such that $H = \{\eta \in \mathbb{E},\ \langle \eta, \alpha \rangle = c\}$. With the same notation, we also write $r_{\alpha,c} = r_H$. This isometry can be described explicitly by

$$r_{\alpha,c}(\eta) = \eta - 2\frac{\langle \alpha, \eta \rangle - c}{\langle \alpha, \alpha \rangle}\alpha, \quad \forall \eta \in \mathbb{E}.$$

Definition 3.5 (Translation). Given $\lambda \in \mathbb{E}$, the map $t_\lambda : \eta \mapsto \eta + \lambda$ is called the *translation of vector* λ; this transformation belongs to $\textsc{Isom}(\mathbb{E})$. The subset of all translations in $\textsc{Isom}(\mathbb{E})$ is an abelian subgroup isomorphic to the additive group of $\mathbb{E}$.

Definition 3.6 (Basic invariants). Associated to any isometry $u \in \textsc{Isom}(\mathbb{E})$ are two basic invariants, called the *move-set* and the *min-set* of u, see [5, Definition 3.1]. Given $\eta \in \mathbb{E}$, its *displacement* under u is $\textsc{Dis}_u(\eta) = u(\eta) - \eta$ and the move-set of u is the set of displacements: $\textsc{Mov}(u) = \{\textsc{Dis}_u(\eta),\ \eta \in \mathbb{E}\}$. For every $u \in \textsc{Isom}(\mathbb{E})$, $\textsc{Mov}(u)$ is an affine subspace of $\mathbb{E}$; the min-set of u is the set $\textsc{Min}(u) = \{\eta \in \mathbb{E},\ \|\textsc{Dis}_u(\eta)\| \text{ minimal}\}$ and this is also an affine subspace of $\mathbb{E}$, see [5, Proposition 3.2]. Note that $\textsc{Mov}(u)$ is a linear subspace if and only if u has some fixed point, in which case u is called *elliptic*. Otherwise, u is called *hyperbolic*. For each isometry u, the respective directions of the move-set and the min-set of u are mutually orthogonal complementary linear subspaces of $\mathbb{E}$, see [5, Lemma 3.6]. Intuitively, the move-set of u is the set of vectors which are motions of points under the isometry u while the min-set of u is the set of points with the minimal possible motion.

Example 3.7. (i) Let H be a hyperplane in $\mathbb{E}$. The reflection r_H is an elliptic isometry, with $\textsc{Min}(r_H) = H$ and $\textsc{Mov}(r_H) = (\textsc{Dir}(H))^\perp$.

(ii) If $\lambda \in \mathbb{E}$ is non-zero, the translation of vector λ is a hyperbolic isometry with $\textsc{Min}(t_\lambda) = \mathbb{E}$ and $\textsc{Mov}(t_\lambda) = \{\lambda\}$.

(iii) Let $H \subset \mathbb{E}$ be a hyperplane; a *glide-reflection* through H is the composition of the reflection r_H and a translation t_λ by a non-zero vector $\lambda \in \textsc{Dir}(H)$; this is a hyperbolic isometry whose min-set is H and whose move-set is the affine line $(\textsc{Dir}(H))^\perp + \lambda$.

Definition 3.8 (Reflection length and order). By the Cartan–Dieudonné theorem, the reflections generate $\textsc{Isom}(\mathbb{E})$. Given $u \in \textsc{Isom}(\mathbb{E})$, its *reflection length* $|u|_{\textsc{Isom}}$ is the minimal number of reflections needed to write u. Note that conjugates of reflections are again reflections, so that the reflection length is invariant under conjugacy. We have a partial order on $\textsc{Isom}(\mathbb{E})$ given by $u \preccurlyeq u'$ if and only if $|u|_{\textsc{Isom}} + |u^{-1}u'|_{\textsc{Isom}} = |u'|_{\textsc{Isom}}$. Note that this is equivalent to the condition $|u'|_{\textsc{Isom}} = |u|_{\textsc{Isom}} + |u'u^{-1}|_{\textsc{Isom}}$ that u right divides u'. Given $v \in \textsc{Isom}(\mathbb{E})$, we denote by $[1, v] = [1, v]^{\textsc{Isom}(\mathbb{E})}$ the *interval* formed by those isometries u satisfying $u \preccurlyeq v$.

The main result in [5] gives a close relation between this partial order and the basic invariants. Here, we record only what will be used in the sequel:

Proposition 3.9. (i) *A hyperbolic isometry can never be smaller than an elliptic one.*

(ii) ([5, Theorem 8.7]) *Let* $v \in \mathrm{ISOM}(\mathbb{E})$. *If* $u_1, u_2 \in [1, v]$ *are elliptic isometries, then* $u_1 \preccurlyeq u_2$ *if and only if* $\mathrm{MIN}(u_2) \subset \mathrm{MIN}(u_1)$.

3.2. Euclidean Coxeter groups

Definition 3.10 (Euclidean Coxeter groups). Irreducible Euclidean Coxeter groups (hence also irreducible Artin–Tits groups of Euclidean type) are classified into four infinite families and five exceptional groups. The corresponding Coxeter graphs $\widetilde{A}_n$ ($n \geqslant 1$), $\widetilde{B}_n$ ($n \geqslant 2$), $\widetilde{C}_n$ ($n \geqslant 2$), $\widetilde{D}_n$ ($n \geqslant 4$), $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$ and $\widetilde{G}_2$ are displayed in Figure 1.

For $Z \in \{A, B, C, D, E, F, G\}$ and $n \in \mathbb{N}$, let Z_n be the full subgraph of $\widetilde{Z}_n$ consisting of the black vertices, see Figure 1. Then the Coxeter group W_{Z_n} is finite. The graphs Z_n and $\widetilde{Z}_n$ are known as *Dynkin diagrams* and *extended Dynkin diagrams* respectively (up to replacing the edges labeled 4 by a double edge and the edge labeled 6 by a triple edge). The meaning of the inequality signs will be explained later. From now on, we choose an arbitrary fixed extended Dynkin diagram $\widetilde{Z}_n$ and we denote $W = W_{\widetilde{Z}_n}$ and $W_0 = W_{Z_n}$.

Definition 3.11 (Root system). The *root system* of type Z_n is described in [3, Planches I to IX]: this is a finite subset of $\mathbb{E} = \mathbb{R}^n$ and its elements are called *roots*. Let us denote by Ξ this root system; it contains a *simple* system Ξ' (a linear basis of $\mathbb{E}$ such that each $\alpha \in \Xi$ is a linear combination of Ξ' with coefficients all of the same sign). The set of (linear) reflections $r_{\alpha,0}$, $\alpha \in \Xi'$ is the set of standard generators of the finite Coxeter group W_0.

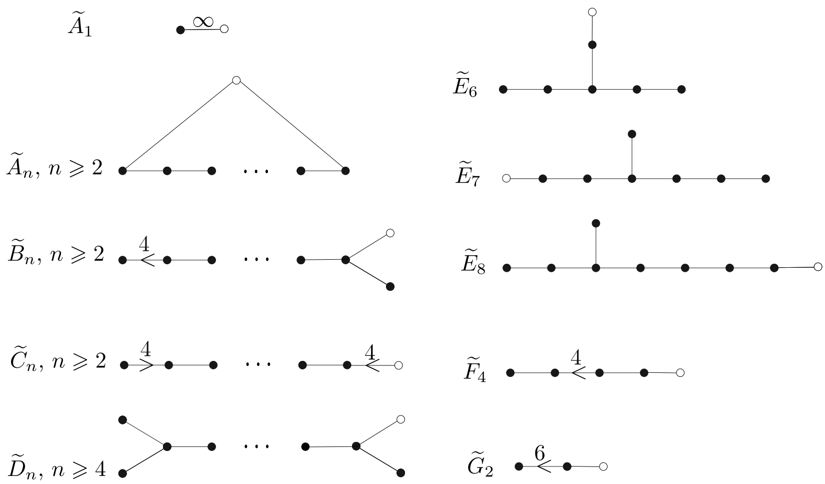

Figure 1. Coxeter graphs for irreducible Euclidean Coxeter groups. For $Z \in \{A, B, C, D, E, F, G\}$ and $n \in \mathbb{N}$, the graph named $\widetilde{Z}_n$ has $n + 1$ vertices. As usual, edge labels 3 are dropped.

Definition 3.12 (Highest root). Given $\alpha \in \Xi$, the sum of the coefficients in the – unique – expression of α as a linear combination of Ξ' is called the *height* of α. There is a unique highest root; let us denote it by μ. We have a unique linear combination

$$\mu = \sum_{\alpha \in \Xi'} m_\alpha \alpha, \tag{1}$$

where the m_α are positive integers.

Definition 3.13 (Standard generators). Let $r_{\mu,1}$ be the reflection in $\mathbb{E}$ through the hyperplane $\{\eta \in \mathbb{E},\ \langle \eta, \mu \rangle = 1\}$, let

$$S = \{r_{\alpha,0},\ \alpha \in \Xi'\} \cup \{r_{\mu,1}\};$$

then S is the set of standard generators for the Euclidean Coxeter group W and the reflection $r_{\mu,1}$ corresponds to the white vertex in the extended Dynkin diagram of Figure 1.

Definition 3.14 (Length of roots). If $\widetilde{Z}_n$ has no label on its edges, all roots in Ξ have the same length while in the other cases, the roots have two different lengths and they are called *long* or *short*, accordingly. In presence of a label 4 (6, respectively), the ratio of the two different root lengths is $\sqrt{2}$ ($\sqrt{3}$, respectively). In the extended Dynkin diagram, the inequality sign(s) indicate(s) which roots are longer.

Definition 3.15 (Elements of W, coroots and the Coxeter complex). Every reflection in W has the form $r_{\alpha,c}$ for $\alpha \in \Xi$ and $c \in \mathbb{Z}$; the corresponding hyperplanes $H_{\alpha,c}$ provide a simplicial tiling of $\mathbb{E}$ called the *Coxeter complex*. The spacing between two consecutive parallel hyperplanes $H_{\alpha,i}$ and $H_{\alpha,i+1}$ is given by the vector $\frac{\alpha}{\langle \alpha,\alpha \rangle}$, so that hyperplanes corresponding to long roots are more closely spaced. The set of translations in W is generated by the translations of the form $t_{\alpha^\vee} = r_{\alpha,i+1} r_{\alpha,i}$ where $\alpha^\vee = \frac{2\alpha}{\langle \alpha,\alpha \rangle}$ for $\alpha \in \Xi$. The vector $\alpha^\vee$ is called the *coroot* associated to α.

3.3. Coxeter elements

Definition 3.16 (Coxeter element). A *Coxeter element* for W is a product of the elements in S in any order. Every Coxeter element is a *hyperbolic* isometry whose move-set is a non-linear affine hyperplane of $\mathbb{E}$ (see [21, Proposition 7.2]).

From this point on, we make the additional assumption that W is not $W_{\widetilde{A}_n}$ – the case of $\widetilde{A}_n$ ($n \geqslant 1$) is somewhat different, as the extended Dynkin diagram is not a tree (except for $W_{\widetilde{A}_1}$ which is dihedral). The proof of the acylindrical hyperbolicity of the corresponding Artin–Tits group is given in the introduction, see Proposition 1.1.

Definition 3.17 (Bipartite Coxeter element). As the extended Dynkin diagram $\widetilde{Z}_n$ is a tree, there is a unique way of 2-coloring its vertices (say blue and green) in such a way that no two adjacent vertices have the same color. This yields a partition $S = S_b \sqcup S_g$ of S in which the reflections in each part commute pairwise. Without loss, we may assume that $r_{\mu,1} \in S_b$. Let us denote $\Phi = \Xi' \sqcup \{\mu\}$; this set of roots is partitioned accordingly: $\Phi = (\{\mu\} \sqcup \Xi'_b) \sqcup \Xi'_g$, where the vectors in each part are pairwise orthogonal. The two elements $\iota_b = \prod_{r \in S_b} r$ and $\iota_g = \prod_{r \in S_g} r$ are involutions and we obtain two special Coxeter elements, inverses of each other (namely, $\iota_b \iota_g$ and $\iota_g \iota_b$) called the *bipartite Coxeter elements*. From now on, we fix $w = \iota_b \iota_g$; this element will be referred to as *the* Coxeter element. Also, we denote by w_0 the Coxeter element of W_0 defined by $w_0 = r_{\mu,1} w$.

Definition 3.18 (Coxeter axis). The min-set of the Coxeter element w is a line called the *Coxeter axis*; let us denote it by L. According to [21, Remark 8.4], the direction of the Coxeter axis is given by

$$\gamma = \mu - \sum_{\alpha \in \Xi'_b} m_\alpha \alpha = \sum_{\alpha \in \Xi'_g} m_\alpha \alpha, \tag{2}$$

where the m_α with $\alpha \in \Xi'$ are the positive integers involved in formula (1). For all $\eta \in L$, the displacement $\textsc{Dis}_w(\eta)$ under w is the same and we denote it by γ_0, see [5, Proposition 3.7]; the standard form of $\textsc{Mov}(w)$ is $\textsc{Mov}(w) = \textsc{Dir}(L)^\perp + \gamma_0$.

Definition 3.19 (Horizontal and vertical). A vector which is orthogonal to the direction of the Coxeter axis is called *horizontal*; a vector which is not horizontal is called *vertical*. Similarly, a reflection is called *horizontal* (or *vertical*, respectively) if its root is horizontal (vertical, respectively).

Lemma 3.20. *The standard generators of W are vertical reflections.*

Proof. Let $\alpha \in \Phi$ be the root of an element in S. We need to check that α is not orthogonal to the direction of the Coxeter axis. As in Definition 3.17, write the bipartite decomposition of Φ as $\Phi = (\{\mu\} \sqcup \Xi'_b) \sqcup \Xi'_g$. According to formula (2), we see that

$$\langle \alpha, \gamma \rangle = \begin{cases} \langle \mu, \mu \rangle & \text{if } \alpha = \mu, \\ -m_\alpha \langle \alpha, \alpha \rangle & \text{if } \alpha \in \Xi'_b, \\ m_\alpha \langle \alpha, \alpha \rangle & \text{if } \alpha \in \Xi'_g, \end{cases}$$

because vectors in $\{\mu\} \sqcup \Xi'_b$ (in Ξ'_g, respectively) are pairwise orthogonal. The coefficients m_α in formula (1) cannot be zero. It follows that α is not orthogonal to the direction of the Coxeter axis. ∎

Definition 3.21 (Horizontal root system and horizontal factorizations, see [23, Definition 6.1]). Let Ξ_h be the intersection of Ξ with the orthogonal complement of $\textsc{Dir}(L)$. It turns out that Ξ_h is a root system in $\textsc{Dir}(L)^\perp$, called the *horizontal root system*. The corresponding Coxeter group W_h is a subgroup of W_0 called the *horizontal Coxeter group*.

The Coxeter element w factorizes as

$$w = r_{\mu,1} r_{\mu,0} w_h = t_{\mu^\vee} w_h,$$

where w_h is a Coxeter element of W_h. Any factorization of w of the form $w = t_\lambda r_1 \dots r_{n-1}$

where t_λ is a translation and the r_i are horizontal reflections is called a *horizontal factorization*. In any horizontal factorization, the move-set of the horizontal part is precisely the horizontal hyperplane $\mathrm{DIR}(L)^\perp$ and we have $\mathrm{MOV}(w) = \mathrm{DIR}(L)^\perp + \lambda$. It follows that if t_λ is the translation in a horizontal factorization, the projection of λ on the vertical axis $\mathrm{DIR}(L)$ is γ_0 (defined in Definition 3.18), independently of λ.

Definition 3.22 (Translation in the direction of the Coxeter axis). The element w_h has finite order (denoted by e_0) and w^{e_0} acts as a translation on $\mathbb{E}$ in the vertical direction (of vector $e_0\gamma_0$). For $u \in W$, we write $T_w(u) = w^{e_0}uw^{-e_0}$. If r is a vertical reflection, then $T_w(r)$ is a vertical reflection through a distinct parallel hyperplane. If r is a horizontal reflection, then $T_w(r) = r$.

Proposition 3.23 (Translation part of w, see [23, Proposition 6.3]). *For every $i \in \mathbb{Z}$, we have $T_w(r_{\mu,i}) = r_{\mu,i+1}$.*

3.4. Reflection length and Garside structure

Definition 3.24 (Reflection length in W). The reflection length $|u|_W$ of $u \in W$ is the minimal number of reflections in W needed to express u. For $u, v \in W$, the relation $u \preccurlyeq_W v$ if and only if $|u|_W + |u^{-1}v|_W = |v|_W$ defines a partial order on W. The *interval* $[1, v]^W$ is the set $\{u \in W,\ u \preccurlyeq_W v\}$. For any Coxeter element w, it turns out that $|w|_W = |w|_{\mathrm{ISOM}}$ and $|u|_W = |u|_{\mathrm{ISOM}}$ for every $u \in [1, w]^W$.

Proposition 3.25 (Some elements in $[1, w]^W$).

(i) ([21, Theorem 9.6] and [23, Definition 5.5]) *All vertical reflections lie in $[1, w]^W$, and $[1, w]^W$ contains exactly two horizontal reflections for each antipodal pair of horizontal roots in the root system Ξ_h.*

(ii) ([23, Proposition 6.3]) *The translations in $[1, w]^W$ are exactly those which appear in a horizontal factorization of w.*

Definition 3.26 (Dual monoid and group). The *monoid associated* to $[1, w]^W$ is the monoid M_w^W generated by $[1, w]^W$ subject to the relations $uu' = v$ whenever $u, u', v \in [1, w]^W$ satisfy $|u|_W + |u'|_W = |v|_W$ and $uu' = v$ in W. The *dual Artin–Tits group* is the group with the same presentation; it is isomorphic to the Artin–Tits group A associated to W (see [23, Theorem C]).

Theorem 3.27 (Lattice).

(i) ([23, Proposition 2.11]) *If the interval $[1, w]^W$ equipped with the restriction of the partial order $\preccurlyeq_W$ is a lattice, then M_w^W is a Garside monoid and the Artin–Tits group A associated to W is a Garside group. There is a monoid homomorphism (or a* weight function*) $\rho : M_w^W \to \mathbb{N} \cup \{0\}$ assigning 1 to each reflection; w is the Garside element whose set of left and right divisors (the set of simple elements) is the interval $[1, w]^W$.*

(ii) ([21, Theorem 10.3]) *The interval $[1, w]^W$ is a lattice if and only if the horizontal root system Ξ_h is irreducible.*

(iii) ([21, Section 11]) *When Ξ_h is not irreducible, it has k_0 irreducible components, with $k_0 = 2$ or $k_0 = 3$. The system Ξ_h is irreducible if and only if $\widetilde{Z}_n \in \{\widetilde{C}_n, \widetilde{G}_2\}$.*

In order to deal with the cases where $[1, w]^W$ is not a lattice, McCammond and Sulway define a supergroup C of W. They need first to introduce new isometries:

Definition 3.28 (Factored translation, see [23, Definition 6.7]). Suppose that the horizontal root system Ξ_h is reducible; let $\mathrm{DIR}(L)^{\perp} = U_1 \oplus \cdots \oplus U_{k_0}$ be the corresponding direct sum decomposition of the horizontal hyperplane – recall that $k_0 \in \{2, 3\}$ by Theorem 3.27 (iii). Let t_λ be a translation in $[1, w]^W$. The projection of λ onto the vertical line $\mathrm{DIR}(L)$ is γ_0 (see Definition 3.21); for $i = 1, \ldots, k_0$, let λ_i be the projection of λ onto the subspace U_i. The *factored translations* corresponding to λ are the k_0 translations $t_{\lambda_i + \frac{1}{k_0}\gamma_0}$.

Definition 3.29 (Other groups). The *crystallographic group C* is the subgroup of $\mathrm{ISOM}(\mathbb{E})$ generated by W together with the factored translations. The *diagonal group D* is the subgroup of $\mathrm{ISOM}(\mathbb{E})$ generated by the translations in $[1, w]^W$ together with horizontal reflections in $[1, w]^W$ and the *factored group F* is the subgroup of $\mathrm{ISOM}(\mathbb{E})$ generated by factored translations and horizontal reflections in $[1, w]^W$. A length is given so that a reflection has length 1, a factored translation has length $\frac{2}{k_0}$ and a translation has length 2. As for W, this yields a length and a partial order on the respective groups. The respective intervals $[1, w]^D$, $[1, w]^F$ and $[1, w]^C$ are naturally defined in the same way as $[1, w]^W$ (see for instance [22, Section 4]) and one can also associate corresponding monoids and groups as in Definition 3.26.

Theorem 3.30 (Garside, see [23, Proposition 7.4] and [23, Theorems A and B]). *The interval $[1, w]^C$ is a balanced lattice (which contains $[1, w]^W$). The monoid M_w^C and the group $\mathfrak{C}$ associated to $[1, w]^C$ are Garside and the Artin–Tits group A associated to W is a subgroup of $\mathfrak{C}$. The group $\mathfrak{C}$ is called the* crystallographic Garside group.

Remark 3.31. M_w^C is equipped with a monoid homomorphism ρ extending the lengths given in Definition 3.29; if $k_0 = 3$, then ρ takes values in $\mathbb{N} \cup \frac{2}{3}\mathbb{N} \cup \{0\}$: it sends each reflection to 1, and each factored translation to $\frac{2}{3}$, see Remark 2.2 (ii). In any case, reflections and factored translations are the atoms and w is the Garside element.

4. The loxodromic elements

In this section, an irreducible Euclidean Coxeter group W distinct of $W_{\widetilde{A}_n}$ is fixed; we keep all notations from the previous section.

The set $\mathcal{S}$ of simple elements of the crystallographic Garside group $\mathfrak{C}$ is in bijection with $[1, w]^C$ (and contains a copy of $[1, w]^W$). We shall use the same notation for a simple element in $\mathcal{S}$ and for the corresponding isometry of $\mathbb{E}$. For any simple element $s \in \mathcal{S}$, we denote by $\mathcal{A}(s)$ (and $\mathcal{A}'(s)$, respectively) the set of atoms which left divide s (right divide s, respectively).

When dealing with elliptic isometries, the factored translations do not play an important role:

Lemma 4.1. *Let $u \in [1,w]^W$ be an elliptic isometry. Then*

$$\mathcal{A}(u) = \mathcal{A}'(u) = \{r,\ r \text{ is a reflection in } W \text{ whose fixed hyperplane contains } \mathrm{MIN}(u)\}.$$

Proof. First, we shall see that $\mathcal{A}(u)$ and $\mathcal{A}'(u)$ do not contain any factored translation. Suppose on the contrary that $\mathcal{A}(u)$ or $\mathcal{A}'(u)$ contains some factored translation t_F. By the proof of [23, Lemma 7.2], we know that there does not exist a minimal length factorization of w in C which includes both a factored translation and a vertical reflection. It follows that $u \in [1,w]^D$. Because all the factored translations have vectors with the same vertical projection $\frac{\gamma_0}{k_0}$, there is some $\eta \in \mathbb{E}$ such that the displacement $\mathrm{DIS}_u(\eta)$ has a vertical component, so u cannot be a product of only horizontal reflections. Therefore, there is a translation $t \in [1,w]^W$ so that $u = tu'$ (or $u = u't$), with $|u'|_W = |u|_W - 2$, which is impossible as u was supposed to be elliptic, see Proposition 3.9 (i). Now, the first equality follows from the fact that conjugates of reflections are again reflections. The fact that $\mathrm{MIN}(u)$ is contained in the fixed hyperplane of every reflection in $\mathcal{A}(u) = \mathcal{A}'(u)$ follows from Proposition 3.9 (ii). Conversely, if $\mathrm{MIN}(r) \supset \mathrm{MIN}(u)$, we obtain $r \preccurlyeq_W u$ by [25, Lemma 2.16]. ■

In the sequel, we shall consider the set $\mathcal{A}(u)$ for different elliptic elements $u \in [1,w]^W$ and we will use Lemma 4.1 without explicit reference. Also, according to our convention using the same symbol for an isometry in $[1,w]^C$ and the corresponding simple element in $\mathcal{S}$, we shall write, for $u \in [1,w]^C$, $\partial(u) = u^{-1}w$ and $\partial^{-1}(u) = wu^{-1}$. This is consistent with the notation in Definition 2.3. In this context, the left-weightedness of a pair of simple elements (s,s') is equivalent to $\mathcal{A}(\partial(s)) \cap \mathcal{A}(s') = \emptyset$. Similarly, (s,s') is right-weighted if and only if $\mathcal{A}'(\partial^{-1}(s')) \cap \mathcal{A}'(s) = \emptyset$.

Recall the "translation" T_w defined by $T_w(u) = w^{e_0}uw^{-e_0}$ for all $u \in W$ (Definition 3.22). Note that for $u \in [1,w]^W$, $\mathcal{A}(T_w(u)) = T_w(\mathcal{A}(u))$. Recall also the elliptic elements ι_b and ι_g – blue and green – from Definition 3.17, which satisfy $\iota_b\iota_g = w$ (that is, $\partial(\iota_b) = \iota_g$ and $\partial^{-1}(\iota_g) = \iota_b$). In what follows we will denote $\iota'_b = T_w(\iota_b)$ and $\iota'_g = T_w(\iota_g)$, so that $w = \iota'_b\iota'_g$. Observe also that $\mathcal{A}(\iota_b) = \{r_{\mu,1}\} \sqcup \{r_{\alpha,0},\ \alpha \in \Xi'_b\}$ and $\mathcal{A}(\iota_g) = \{r_{\alpha,0},\ \alpha \in \Xi'_g\}$. Finally, recall that the elliptic element w_0 is defined by $w_0 = r_{\mu,1}w$.

Lemma 4.2. *The pair (ι'_b, w_0) is left- and right-weighted.*

Proof. As $\mathcal{A}(\iota_b) = \{r_{\mu,1}\} \sqcup \{r_{\alpha,0},\ \alpha \in \Xi'_b\}$, we have

$$\mathcal{A}(\iota'_b) = \{T_w(r_{\mu,1})\} \sqcup \widehat{\mathcal{A}} = \{r_{\mu,2}\} \sqcup \widehat{\mathcal{A}},$$

(the equality $T_w(r_{\mu,1}) = r_{\mu,2}$ comes from Proposition 3.23), where $\widehat{\mathcal{A}}$ is a set of reflections through hyperplanes orthogonal to roots which are distinct from μ. Recall that $\partial^{-1}(w_0) = r_{\mu,1}$; the previous discussion shows that $r_{\mu,1} \notin \mathcal{A}(\iota'_b)$, whence $\mathcal{A}(\iota'_b) \cap \mathcal{A}(\partial^{-1}(w_0)) = \emptyset$ and the right-weightedness follows. Also, $\partial(\iota'_b) = \iota'_g$. Fixed hyperplanes of reflections in $\mathcal{A}(\iota'_g)$ do not contain 0 while fixed hyperplanes of reflections in $\mathcal{A}(w_0)$ do contain 0. Therefore $\mathcal{A}(\iota'_g) \cap \mathcal{A}(w_0) = \emptyset$, whence left-weightedness. ∎

Lemma 4.3. *The pair (w_0, ι'_g) is left- and right-weighted.*

Proof. Let us describe $\partial(w_0)$. We have $w = r_{\mu,1} w_0 = w_0(w_0^{-1} r_{\mu,1} w_0)$, whence $\partial(w_0) = w_0^{-1} r_{\mu,1} w_0$. But recall that

$$w_0 = \prod_{\alpha \in \Xi'_b} r_{\alpha,0} \prod_{\alpha \in \Xi'_g} r_{\alpha,0}.$$

In the first product, all reflections commute with $r_{\mu,1}$ and in the second (which is a product in which all reflections commute pairwise), all reflections commute with $r_{\mu,1}$, except one (as $r_{\mu,1}$ corresponds to a leaf in the extended Dynkin diagram). Therefore, for some $\alpha_0 \in \Xi'_g$, we have $w_0^{-1} r_{\mu,1} w_0 = r_{\alpha_0,0} r_{\mu,1} r_{\alpha_0,0}$. The root of this reflection is

$$r_{\alpha_0,0}(\mu) = \sum_{\alpha \in \Xi'} m_\alpha r_{\alpha_0,0}(\alpha) = -m_{\alpha_0}\alpha_0 + \sum_{\alpha \in \Xi'_g \setminus \{\alpha_0\}} m_\alpha \alpha + \sum_{\alpha \in \Xi'_b} m_\alpha r_{\alpha_0,0}(\alpha),$$

which has a positive component along each root in Ξ'_b. As all reflections in $\mathcal{A}(\iota'_g)$ have their roots in Ξ'_g, we obtain $\mathcal{A}(\partial(w_0)) \cap \mathcal{A}(\iota'_g) = \emptyset$, which shows left-weightedness.

For right-weightedness, note that $\partial^{-1}(\iota'_g) = \iota'_b$. On the one hand, $\mathcal{A}(w_0)$ consists of reflections whose fixed hyperplane contains 0, on the other hand, $\mathcal{A}(\iota'_b)$ contains no reflection whose fixed hyperplane contains 0. Therefore $\mathcal{A}(\iota'_b) \cap \mathcal{A}(w_0) = \emptyset$ and we are done. ∎

Lemma 4.4. *Let r_v be a vertical reflection and let $\alpha \in \Xi$ be a vertical root. Then there is at most one $k \in \mathbb{Z}$ such that $r_{\alpha,k} r_v$ is a simple element. Also, there is at most one $l \in \mathbb{Z}$ such that $r_v r_{\alpha,l}$ is a simple element.*

Proof. First, observe that for a pair of *atoms* r, r', rr' is a simple element if and only if $r \in \mathcal{A}'(\partial^{-1}(r'))$ if and only if $r' \in \mathcal{A}(\partial(r))$. By [21, Lemma 9.3], $\partial^{-1}(r_v) = w r_v^{-1}$ is an elliptic isometry whose min-set is just a point. There is at most one $k \in \mathbb{Z}$ such that $H_{\alpha,k}$ contains this point, that is, there is at most one $k \in \mathbb{Z}$ such that $r_{\alpha,k} \in \mathcal{A}(\partial^{-1}(r_v))$. Similarly, $\partial(r_v)$ is an elliptic isometry whose min-set is just a point. There is at most one $l \in \mathbb{Z}$ such that $H_{\alpha,l}$ contains this point, that is, there is at most one $l \in \mathbb{Z}$ such that $r_{\alpha,l} \in \mathcal{A}(\partial(r_v))$. ∎

Lemma 4.5. *There is a vertical reflection r_0 such that both (r_0, ι'_b) and (ι'_g, r_0) are left- and right-weighted.*

Proof. Fix a vertical root α. Let $r_1, \dots, r_p$ be an enumeration of $\mathcal{A}(\iota'_b)$ and let $s_1, \dots, s_q$ be an enumeration of $\mathcal{A}(\iota'_g)$. By Lemma 3.20 (and Definition 3.22), all these reflections are vertical. By Lemma 4.4, for each $i = 1, \dots, p$ and each $j = 1, \dots, q$, there is at most one k_i such that $r_{\alpha,k_i} r_i$ is simple and at most one l_j such that $s_j r_{\alpha,l_j}$ is a simple element. If we choose $m_0 \notin \{k_1, \dots, k_p, l_1, \dots, l_q\}$, and $r_0 = r_{\alpha,m_0}$, then both (r_0, ι'_b) and (ι'_g, r_0) are left- and right-weighted. ∎

5. Proof of Theorem A

In this section, an Artin–Tits group A of Euclidean type distinct from the affine braid group is fixed (except in the proof of Theorem A). We keep notations from the previous sections with the following exception. As it is a standard notation for the Garside element in Garside groups, we will use the letter Δ for the Garside element of $\mathfrak{C}$ – this is the same that was denoted above by w.

Definition 5.1. For the remainder of the paper, we define the following element of $\mathfrak{C}$, which is also an element of A. Let r_0 be as in Lemma 4.5. Define

$$x = r_0 \cdot \iota'_b \cdot w_0 \cdot \iota'_g \cdot r_0.$$

First, we gather some facts about the element x. For any $g \in \mathfrak{C}$ with $\inf(g) = 0$, we denote $\partial(g) = g^{-1}\Delta^{\sup(g)}$ – this matches the notation for the right complement of a simple element s (in which case $\sup(s) = 1$).

Proposition 5.2. (i) *The left and right normal form of x are the same and we just call it the "normal form"; this normal form is given by the formula in Definition 5.1.*

(ii) *The first and last factor of the normal form of x coincide; thus x is* rigid*: for every $m \in \mathbb{N}$, the left – and right – normal form of x^m is the concatenation of m copies of the normal form of x.*

(iii) *Both normal forms of x and $\partial(x)$ contain a factor which is the right complement of a reflection; the elements x and $\partial(x)$ are not absorbable.*

(iv) *For each $m \geqslant 0$, $\partial(x^m) = \prod_{i=0}^{m-1} \tau^{5i}(\partial(x))$ and this is in left and right normal form as written.*

(v) *No non-trivial power of Δ commutes with x.*

Proof. (i) follows from Lemmas 4.2, 4.3 and 4.5.

(ii) is immediate.

(iii) To see that x is not absorbable, it suffices to notice that w_0 is not absorbable and to use Lemma 2.6. Recall that ρ is the weight function of the monoid M_w^C (Remark 3.31). If w_0 was absorbable, we would have some $s \in \mathcal{S}$ such that $w_0 s$ is a *proper* simple element. We then would have $\rho(w_0 s) = \rho(w_0) + \rho(s) = n + \rho(s) < n + 1$. Then $\rho(s) < 1$ and the only possibility is that $k_0 = 3$ and $\rho(s) = \frac{2}{3}$. But then $\rho(\partial(w_0 s)) = \frac{1}{3}$, which is impossible since there is no simple element with weight $\frac{1}{3}$. For the same reason, $\partial(r_0)$ is not absorbable and $\partial(x)$ is not absorbable.

(iv) In any Garside group, if (x_1, x_2) is both a right- and left-weighted pair of simple elements, then $(\partial(x_2), \tau(\partial(x_1)))$ is right- and left-weighted as well. To see this, it is enough to compute $\partial(\partial(x_2)) \wedge \tau(\partial(x_1)) = \tau(x_2) \wedge \tau(\partial(x_1)) = \tau(x_2 \wedge \partial(x_1)) = 1$, whence left-weightedness; right-weightedness is obtained analogously. The claim then follows from (i).

(v) Otherwise, there would be some power $l \neq 0$ of Δ commuting with r_0 (see [23, Proposition 2.14]) and hence r_0 would also commute with Δ^{le_0} (e_0 is given in Definition 3.22), which is impossible as r_0 is vertical and all isometries $T_w^k(r_0), k \in \mathbb{Z}$ are distinct (Definition 3.22). ∎

5.1. Loxodromic

Theorem 5.3. *The element x acts in a loxodromic way on the additional length graph $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$. More precisely, $d_{\mathrm{AL}}(*, X^k) \geqslant \frac{|k|}{2}$ for all k in $\mathbb{Z}$. As a consequence, $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$ has infinite diameter.*

Here, X^k, $k \in \mathbb{Z}$ stands for the vertex $x^k \Delta^{\mathbb{Z}} = x^k \cdot *$ of $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$. Throughout, we shall use the symbol $\preccurlyeq$ for the order on $\mathfrak{C}$ which extends the order on $[1, w]^C$ to M_w^C and then to $\mathfrak{C}$ as follows: for $g, h \in \mathfrak{C}$, $g \preccurlyeq h$ if and only if $g^{-1}h \in M_w^C$. The main tool for the proof is a projection map from $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$ to $\{X^k,\ k \in \mathbb{Z}\}$:

Lemma 5.4 ([7, Definition 3]). *There is a well-defined map Λ from the set of vertices of $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$ to $\mathbb{Z}$ given by the formula*

$$\Lambda(V) = -\max\bigl\{k \in \mathbb{Z},\ x \not\preccurlyeq \underline{x^k \cdot V}\bigr\}.$$

This yields a projection from the set of vertices of $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$ onto the set of vertices $\{X^k,\ k \in \mathbb{Z}\}$, given by $V \mapsto X^{\Lambda(V)}$.

This projection has the following key-property:

Proposition 5.5 ([7, Proposition 4]). *Let V_1, V_2 be two vertices of $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$; let $\Lambda_1 = \Lambda(V_1)$ and $\Lambda_2 = \Lambda(V_2)$. Suppose that $\Lambda_2 - \Lambda_1 \geqslant 3$. Then the preferred path $\mathcal{A}(V_1, V_2)$ contains the subpath $\mathcal{A}(X^{\Lambda_1+1}, X^{\Lambda_2-1})$.*

The technical ground for proving Proposition 5.5 is achieved in [7, Lemmas 5, 6, 7]. The proof of these lemmas is unchanged in our context, with the exception that $\Delta^{\sup(x)}$ is not central, so $\partial(x^k) \neq \partial(x)^k$ for $k \in \mathbb{N}$. As a consequence, the expression "k copies of the normal form of ∂x" in [7] must be replaced by "the normal form of $\partial(x^k)$".

Proof of Theorem 5.3. See the proof of [7, Proposition 5]. ∎

5.2. WPD

Theorem 5.6. *The action of $x \in A < \mathfrak{C}$ on $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$ is WPD, that is, for each vertex V of $\mathcal{C}_{\mathrm{AL}}(\mathfrak{C})$ and for each $\kappa > 0$, there exists an integer N such that the set*

$$\bigl\{g \in \mathfrak{C},\ d_{\mathrm{AL}}(V, g \cdot V) \leqslant \kappa,\ d_{\mathrm{AL}}(x^N \cdot V, gx^N \cdot V) \leqslant \kappa\bigr\}$$

is finite. Equivalently, for each $\kappa > 0$, there exists an integer N such that the set

$$\bigl\{g \in \mathfrak{C},\ d_{\mathrm{AL}}(*, g \cdot *) \leqslant \kappa,\ d_{\mathrm{AL}}(X^N, g \cdot X^N) \leqslant \kappa\bigr\}$$

is finite.

The proof follows very closely the proof of [7, Proposition 6]. The first step is to see that the projection Λ defined in Lemma 5.4 is coarsely Lipschitz. In what follows, K is the maximal Hausdorff distance between a geodesic and a preferred path between a pair of vertices of $\mathcal{C}_{AL}(\mathfrak{C})$; one can take $K = 39$, see Theorem 2.10.

Lemma 5.7 ([7, Proposition 7]). *Suppose that V_1, V_2 are vertices of $\mathcal{C}_{AL}(\mathfrak{C})$. Then*

$$|\Lambda(V_2) - \Lambda(V_1)| \leqslant 2(d_{AL}(V_1, V_2) + 2K + 1).$$

Proof of Theorem 5.6. Fix any $\kappa > 0$; let $\xi = \kappa + 2K + 1$ and fix $N \geqslant 4\xi + 3$. Suppose that $g \in \mathfrak{C}$ satisfies

$$d_{AL}(*, g \cdot *) \leqslant \kappa \quad \text{and} \quad d_{AL}(X^N, g \cdot X^N) \leqslant \kappa. \tag{3}$$

Claim 5.8. *We have*

$$\Lambda(g \cdot X^N) - \Lambda(g \cdot *) \geqslant N - 4\xi \geqslant 3.$$

The preferred path between $g \cdot *$ and $g \cdot X^N$ contains the subpath

$$\mathcal{A}(X^{\Lambda(g\cdot *)+1}, X^{\Lambda(g \cdot X^N)-1}).$$

Proof. Under our hypothesis, we have according to Lemma 5.7 (observe that $\Lambda(X^N) = N$ and $\Lambda(*) = 0$),

$$|N - \Lambda(g \cdot X^N)| \leqslant 2(\kappa + 2K + 1) = 2\xi,$$
$$|\Lambda(g \cdot *)| \leqslant 2(\kappa + 2K + 1) = 2\xi;$$

the first part of the claim follows. Due to our choice of N, the second part of the claim follows immediately from Proposition 5.5. ∎

Write for short $\Lambda_1 = \Lambda(g \cdot *)$ and $\Lambda_2 = \Lambda(g \cdot X^N)$. By Proposition 2.9 (iii), the preferred path $\mathcal{A}(g \cdot *, g \cdot X^N)$ is the g left translate of the preferred path $\mathcal{A}(*, X^N)$, so by Claim 5.8, there are vertices P and Q along $\mathcal{A}(*, X^N)$ so that $g \cdot P = X^{\Lambda_1+1}$ and $g \cdot Q = X^{\Lambda_2-1}$.

Claim 5.9. *P is represented by some power of x, say $P = X^a$, thus $Q = X^{a+\Lambda_2-\Lambda_1-2}$.*

Proof. Write $x = x_1 \dots x_5$ for the normal form of x (see Definition 5.1) and recall that x is rigid (Proposition 5.2 (ii)). Notice that the distinguished representatives of the $(5N + 1)$ vertices along the path $\mathcal{A}(*, X^N)$ are

$$*, x_1, x_1x_2, x_1x_2x_3, x_1x_2x_3x_4, x, xx_1, \dots, x^2, \dots, X^N.$$

We know that the path $\mathcal{A}(P, Q)$ has the same length as the path $\mathcal{A}(X^{\Lambda_1+1}, X^{\Lambda_2-1})$, that is $5(\Lambda_2 - \Lambda_1 - 2)$, so if $P = X^a$, then $Q = X^{a+\Lambda_2-\Lambda_1-2}$.

Suppose, in contradiction with the claim, that $P = (x^a x_1 \dots x_j)\Delta^{\mathbb{Z}}$ for some a and $1 \leqslant j \leqslant 4$. Then as the length of $\mathcal{A}(P, Q)$ is $5(\Lambda_2 - \Lambda_1 - 2)$,

$$Q = (x^{a+\Lambda_2-\Lambda_1-2} x_1 \dots x_j)\Delta^{\mathbb{Z}}.$$

Therefore we have (notice that $\underline{P}$ is a prefix of $\underline{Q}$)

$$\underline{P}^{-1}\underline{Q} = x_{j+1} \dots x_5 x^{\Lambda_2-\Lambda_1-3} x_1 \dots x_j, \tag{4}$$

and this is the left normal form, by construction of x.

On the other hand,

$$x^{\Lambda_1+1} = \underline{g \cdot P} = g\underline{P}\Delta^{-\inf(g\underline{P})}$$

and

$$x^{\Lambda_2-1} = \underline{g \cdot Q} = g\underline{Q}\Delta^{-\inf(g\underline{Q})},$$

from which we deduce (recall that τ is the conjugation by Δ)

$$x^{\Lambda_2-\Lambda_1-2} = \Delta^{\inf(g\underline{P})}\underline{P}^{-1}g^{-1}g\underline{Q}\Delta^{-\inf(g\underline{Q})} = \tau^{-\inf(g\underline{P})}(\underline{P}^{-1}\underline{Q})\Delta^{\inf(g\underline{P})-\inf(g\underline{Q})}.$$

Considerations on the infimum show that $\inf(g\underline{P}) = \inf(g\underline{Q})$. We see also that $x^{\Lambda_2-\Lambda_1-2}$ and $\underline{P}^{-1}\underline{Q}$ are conjugate by $\Delta^{-\inf(g\underline{P})}$. By [23, Proposition 2.14] (conjugation of normal forms by Δ), we see in particular that the first factor (last factor, respectively) of the normal form of $x^{\Lambda_2-\Lambda_1-2}$ and the first factor (last factor, respectively) of the normal form of $\underline{P}^{-1}\underline{Q}$ are conjugate by $\Delta^{-\inf(g\underline{P})}$.

By construction of x, the first and last factor of the normal form of $x^{\Lambda_2-\Lambda_1-2}$ are x_1 and x_5 respectively. In view of equation (4) the first and last factor of the normal form of $\underline{P}^{-1}\underline{Q}$ are x_{j+1} and x_j respectively. As τ commutes with the weight function ρ, we obtain $\rho(x_{j+1}) = \rho(x_1) = 1$ and $\rho(x_j) = \rho(x_5) = 1$, by construction of x, which contradicts the choice of j. ∎

Claim 5.10. *Any element $g \in \mathfrak{C}$ which satisfies the conditions* (3) *is a power of x.*

Proof. As $\Lambda_2 - \Lambda_1 \geqslant 3$, we have along the path $\mathcal{A}(P, Q) = \mathcal{A}(X^a, X^{a+\Lambda_2-\Lambda_1-2})$ *at least two* consecutive vertices X^a and X^{a+1} such that

$$g \cdot X^a = X^b \quad \text{and} \quad g \cdot X^{a+1} = X^{b+1}$$

(with $b = \Lambda_1 + 1$). By the first equality, we obtain $gx^a = x^b\Delta^l$ for some $l \in \mathbb{Z}$. By the second equality, we obtain $gx^{a+1} = x^{b+1}\Delta^{l'}$ for some $l' \in \mathbb{Z}$. Combining both assertions yields $x^b\Delta^l x = x^{b+1}\Delta^{l'}$, which is equivalent to $\Delta^l x = x\Delta^{l'}$. This forces $l = l'$ (by considering the infimum) and also $l = 0$ since no non-trivial power of Δ commutes with x (Proposition 5.2 (v)). We deduce that $g = x^{b-a}$ is a power of x as desired. ∎

By Lemma 5.7, if $d_{\mathrm{AL}}(*, x^l \cdot *) \leqslant \kappa$, we have $|l| \leqslant 2\xi$; hence Claim 5.10 shows that the elements of $\mathfrak{C}$ which satisfy the conditions (3) are contained in the *finite* set $\{x^l,\ |l| \leqslant 2\xi\}$. This achieves the proof of Theorem 5.6. ∎

Proof of Theorem A. This follows from Proposition 1.1, Theorem 5.6 and a result by Osin [24, Theorem 1.2]. ∎

With the same line of arguments, we have also shown:

Corollary B. *The crystallographic Garside group $\mathfrak{C}$ is acylindrically hyperbolic.*

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