

# Topological dynamics of groupoid actions

Felipe Flores and Marius Măntoiu

**Abstract.** Some basic notions and results in topological dynamics are extended to continuous groupoid actions in topological spaces. We focus mainly on recurrence properties. Besides results that are analogous to the classical case of group actions, but which have to be put in the right setting, there are also new phenomena. Mostly for groupoids whose source map is not open (and there are many), some properties which were equivalent for group actions become distinct in this general framework; we illustrate this with various counterexamples.

## 1. Introduction

Classically, topological dynamics is understood as the study of group and semigroup actions on topological spaces. It is an important chapter of modern mathematics originating from physics and the theory of differential equations, and its theoretical and practical outreach need not be outlined here. The point of view we adopt is that of the abstract theory, as exposed in references such as [2, 6, 9, 13].

Basic to topological dynamics in the classical sense is the idea of global symmetry. However, many interesting systems only present *local (or partial) forms of symmetry*. Such systems have acquired an increasing role in modern mathematics, and their impact in applications is already largely acknowledged.

Partial symmetry is treated using concepts such as *groupoids, partial group actions or inverse semigroup actions*. Several aspects are already very well developed. In fact, our interest was raised by the applications to  $C^*$ -algebras, as in [10, 11, 16, 18, 19]. However, up to our knowledge, the basic theory in the spirit of classical topological dynamics has not yet been developed systematically. In the literature, when notions and results are needed, they are briefly introduced and used in an ad hoc way. In addition, certain basic concepts barely appear in the partial symmetry setting.

The present article is dedicated to a systematic study of the most elementary dynamical notions in the framework of *continuous groupoid actions on topological spaces*. We emphasize *recurrence phenomena*. Some connections with partial group actions are also made, but the interaction with inverse semigroup dynamical systems, as well as a deeper study of more advanced dynamical notions, are deferred to a subsequent article.

---

2020 *Mathematics Subject Classification*. Primary 22A22; Secondary 37B20, 37B05, 58H05.

*Keywords*. Groupoid action, dynamical system, orbit, minimal, transitive, almost periodic, mixing, wandering, factor.

As a general convention, all the topological spaces (including the groupoids) are Hausdorff. Local compactness is required only when needed. In certain cases we ask all the fibers of the groupoid to be non-compact, in order to avoid triviality.

In most cases (but not always), guessing the analog of the standard notions in the groupoid case is quite straightforward. Both in the statements and the proofs, one has to take into account the fibered nature of a groupoid action. When a groupoid  $\Xi$  with unit space  $X$  acts on a topological space  $\Sigma$ , the action  $\theta$  is only defined on a subset of the product  $\Xi \times \Sigma$ . The element  $\theta_\xi(\sigma)$  is defined only if the domain  $d(\xi)$  of  $\xi \in \Xi$  coincides with the image  $\rho(\sigma)$  of  $\sigma \in \Sigma$  by the *anchor map*  $\rho : \Sigma \rightarrow X$ , which is part of the definition of the action. There are also some technical difficulties that we discuss briefly:

(a) When dealing with  $\theta_\xi(\sigma)$ , sometimes one needs to approximate  $\xi$  by a net  $\{\xi_i \mid i \in I\}$ . Usually  $\theta_{\xi_i}(\sigma)$  makes no sense, and this requires more involved arguments.

(b) When a group  $\mathcal{G}$  acts over a topological space  $\Sigma$ , if  $A \subset \mathcal{G}$  and  $U \subset \Sigma$  are open, then  $AU \subset \Sigma$  is also open. Rather often, this plays an important role in the proofs. For groupoid actions, this is true only in the special case when the domain (source) map  $d : \Xi \rightarrow \Xi$  is open; we say that the groupoid is *open* if this holds. This may fail even in simple examples. In such cases some expected properties do not hold, as counterexamples will show.

(c) Even if the map  $d$  is open, in most cases the translation  $\theta_\xi(U)$  of an open set is no longer open. For group actions this phenomenon is absent, since the action is composed of (global) homeomorphisms. Therefore, extra care will be needed in some proofs.

Let us describe the content of the article.

Section 2 introduces the basic definitions. To read the article, the reader only needs to know what a (topological) groupoid is, and to be familiar with the elementary properties a classical dynamical system might have. The notations involving groupoids and continuous groupoid actions are introduced in Section 2.1. The canonical action on the unit space will be very important; it is a terminal action, by Lemma 7.2. Notions such as orbit, orbit closure, invariant set and saturation are obvious extensions of the standard ones. However, *if the groupoid  $\Xi$  is not open, the interior, the closure or the boundary of an invariant subset of  $\Sigma$  might not be invariant*. This will have far-reaching negative consequences. The central notion of *recurrence set*  $\tilde{\Xi}_M^N$  appears in Section 2.2. For  $M, N \subset \Sigma$ , this consists in the elements  $\xi \in \Xi$  under which some point of  $M$  is sent inside  $N$  (taking into account domain issues).

Section 3. In certain situations, a groupoid (or a groupoid action) is canonically constructed from other related mathematical objects. We present here a couple of such situations, stressing the way recurrence sets are determined by these objects. Our selection is probably inspired by our interest in groupoid  $C^*$ -algebras; geometrically oriented readers could prefer others. *Equivalence relations* are important in groupoid theory; for us, they will often serve as counterexamples. In fact, the most general of our counterexamples are subgroupoids of direct products between equivalence relations and groups. In Section 3.2, the usual *group actions* are encoded in two different ways to fit our setting. It is shown that the second way yields a better grasp at the level of recurrence. The *Deaconu–Renault*

groupoid of a local homeomorphism appears in Section 3.3. In Sections 3.4 and 3.5, we arrive at groupoid actions making *non-invariant restrictions* in larger, maybe global, dynamical systems; this includes partial group actions. In Section 3.6, to a groupoid action  $(\Xi, \theta, \Sigma)$  we associate a larger *transformation (crossed product) groupoid*  $\Xi \times_{\theta} \Sigma$ . It only has a partial relevance for recurrence issues. But it will also be used later, in connection with factors and extensions. A *pullback construction* appears in Section 3.7.

Section 4 mainly treats various types of *topological transitivity*. Consider the following properties of a continuous action  $\theta$  of the groupoid  $\Xi$  in the topological space  $\Sigma$ :

- (i)  $\Sigma$  is not the union of two proper invariant closed subsets,
- (ii) non-empty open invariant subsets of  $\Sigma$  are dense,
- (iii) for any  $U, V \subset \Sigma$  open and non-void,  $\theta_{\xi}(U) \cap V \neq \emptyset$  for some  $\xi \in \Xi$ ,
- (iv) invariant subsets of  $\Sigma$  are either dense, or nowhere dense.

For group actions, these properties (suitably reformulated) are equivalent and go under the common name of “topological transitivity”. This also happens for groupoid actions, if  $\Xi$  is open (i.e. the domain map is open). But if it is not, then one only has  $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ . After proving this and some connected results in Theorem 4.1, we indicate a couple of counterexamples showing that in general none of the implications can be reverted. It is also shown that pointwise transitivity (existence of a dense orbit) implies (iii) but not (iv). The failure of some implications can be tracked back to the fact that the closure of an invariant set could not be invariant if the groupoid is not open. We decided to call (iii) *recurrent transitivity* and (iv) (the strongest notion) *topological transitivity*; it is debatable whether the terminology is the best one. Properties (i) and (ii) did not receive a name. Proposition 4.5 makes the necessary specifications for the case of a Baire second-countable space  $\Sigma$ . Then we particularize to some of the examples from Section 3. In Section 4.2, we introduce and study a final transitivity notion, the *weakly pointwise transitivity*. It is similar to pointwise transitivity, but the orbit closure of a point (that might not be invariant if  $\Xi$  is not open) is replaced by the smallest closed invariant set containing the point.

Section 5 is concerned with limit points, recurrent points and wandering and non-wandering properties. *Limit sets* are mostly studied for actions of one of the groups  $\mathbb{Z}$  or  $\mathbb{R}$  on topological spaces, where one distinguishes between positive and negative limit points. A good reference is [6, Chapter II.2]. We are going to adapt the basic part of the theory to groupoids. But besides being groups,  $\mathbb{Z}$  and  $\mathbb{R}$  have two extra features: (1) they have ‘two ends’ and (2) they are not compact. The first feature leads to various ramifications, such as distinguishing between positive and negative limit sets. Trying to imitate this for groupoid actions is possible, but would require a specialized and rather intricate setting. On the other hand, too much compactness in a groupoid would make some constructions and results trivial or non-interesting. So in Section 5.1, we will restrict our attention to *strongly non-compact groupoids*, those for which all the fibers  $\Xi_x$  are non-compact. The limit points of a given point  $\sigma$  are defined by their asymptotic behavior under the action, where “asymptotic” is encoded, equivalently, by complements of compact sets, diverging

nets (both in  $\Xi$ ) or suitable recurrence sets. They are contained in the orbit closure of  $\sigma$  and form a closed subset having an attractor behavior. This subset is invariant if the groupoid is open, but a counterexample shows that in general this can fail. In Section 5.2, a point is called *recurrent* if it is a limit point of itself. In Proposition 5.8, this is related to other types of descriptions, as in classical dynamical systems, but once again full equivalences hold under the openness assumption (a counterexample is presented). This assumption is also needed to insure invariance of the family of all recurrent points. *Wandering* and *non-wandering* are similar to the group case notions, but defined in terms of groupoid recurrence sets  $\tilde{\Xi}_W^W$  and compactness, where  $W$  is a neighborhood of the point we study. The family of non-wandering points is closed; to be invariant one also requires  $\Xi$  to be open and locally compact. If  $\Sigma$  is compact, it is non-void and attracts the points of  $\Sigma$  in a suitable sense. We indicate examples involving pullbacks and action groupoids.

Section 6. We start in Section 6.1 with the set  $\Sigma_{\text{fix}} \subset \Sigma$  of *fixed points* of a groupoid action. It is invariant; it is also closed if  $\Xi$  happens to be open, but not in general, as counterexamples show. We inspect the origin of the fixed points for global and partial group actions, for non-invariant restrictions, for the Deaconu–Renault groupoid and for pullbacks. In Section 6.2, we show the inclusion

$$\Sigma_{\text{fix}} \subset \Sigma_{\text{per}} \subset \Sigma_{\text{wper}} \cap \Sigma_{\text{alper}} \subset \Sigma_{\text{wper}} \cup \Sigma_{\text{alper}} \subset \Sigma_{\text{rec}} \subset \Sigma_{\text{nw}}. \tag{1.1}$$

Besides fixed, recurrent and non-wandering points, already defined, we introduce three other types of points: *periodic*, *weakly periodic* and (most important) *almost periodic*. For the first two, terminology could be debated even in the group case. The precise meaning can be found in Definition 6.8, where an adaptation to groupoids of the standard notion of *syndeticity* is also included. We discuss issues such as invariance and closure for the new sets. In [3], Beckus, Bellissard and De Nittis work with locally compact, second countable, open groupoids with compact unit space. A unit  $x$  is called a *periodicoid* if its orbit is closed (i.e. compact). Among others they show that if, in addition, the groupoid is étale, the orbit of a periodicoid point is actually finite. Obviously, the notion makes sense and is relevant also for general groupoid actions. In Section 6.3, we connect it with our periodicity. Proposition 6.16 proves that a periodic point  $\sigma$  has a compact orbit, and in Proposition 6.18, we prove the converse implication, adding some extra conditions. *Minimality* is explored in Section 6.4. After stating the definition and providing the most elementary properties, we exhibit in Theorem 6.23 the connection between minimality and almost periodicity. For group actions, this is the standard result that can be found in every textbook [2, 6, 9, 13]. The proof is similar, but slightly more involved, because of the groupoid setting. Corollary 6.28 deals with semisimplicity and point almost periodicity. In Proposition 6.30, we provide conditions under which minimality implies the non-wandering property.

Section 7. *Homomorphisms* (equivariant maps) are an important topic in topological dynamics. In Section 7.1, we study surjective homomorphisms (epimorphisms) in the framework of groupoid actions (equivariance now also requires a compatibility of the two anchor maps). They lead to the usual concepts of *extension* and *factor*. The canonical

action of a groupoid  $\mathfrak{E}$  on the unit space is terminal, being a factor of any other  $\mathfrak{E}$ -action. We study the fate under epimorphisms of most of the dynamical properties already introduced. Two main results are Theorem 7.6 (referring to the sets (1.1)) and Proposition 7.7 (treating various types of transitivity, including recurrent transitivity). Note that, by Example 7.9, topological transitivity does not always transfer to factors when the groupoid is not open. Proposition 7.10 refers to the behavior of minimality under epimorphisms. It also contains conditions under which a minimal subsystem of the factor admits a minimal pre-image and an almost periodic point of the factor is reached from an almost periodic element of the extension. Section 7.2 connects extensions of groupoid actions as in Section 7.1 with action (crossed product) groupoids as in Section 3.6. In [8], Edeko and Kreidler use transformation groupoids to study group action extensions. They develop a very interesting theory, that however has little in common with the content of the present article. By a straightforward generalization of a construction from [8], given a groupoid action  $\Theta' = (\mathfrak{E}, \theta', \Sigma')$ , we show that *there is a one-to-one correspondence between extensions of  $\Theta'$  and actions of the crossed product groupoid  $\mathfrak{E}(\Theta') := \mathfrak{E} \rtimes_{\theta'} \Sigma'$* . We also exhibit an isomorphism between two different crossed products. In Propositions 7.13 and 7.14, we prove that the dynamical properties are the same under the mentioned bijective correspondence. Various other types of morphisms are studied in [12].

Section 8. We dedicate this short final section to convince the reader that mixing, as presented in Definition 8.1, is not an interesting concept outside the classical group case, if the anchor map  $\rho : \Sigma \rightarrow X$  is surjective and the unit space  $X$  is Hausdorff (standing assumptions in the present paper).

## 2. Groupoids and groupoid actions

### 2.1. The framework

We deal with groupoids  $\mathfrak{E}$  over a unit space  $\mathfrak{E}^{(0)} \equiv X$ , seen as small categories in which all the morphisms (arrows) are invertible. The source and range maps are denoted by  $d, r : \mathfrak{E} \rightarrow \mathfrak{E}^{(0)}$  and the family of composable pairs by  $\mathfrak{E}^{(2)} \subset \mathfrak{E} \times \mathfrak{E}$ . For  $M, N \subset X$  one uses the standard notations

$$\mathfrak{E}_M := d^{-1}(M), \quad \mathfrak{E}^N := r^{-1}(N), \quad \mathfrak{E}_M^N := \mathfrak{E}_M \cap \mathfrak{E}^N.$$

Particular cases are the d-fiber  $\mathfrak{E}_x \equiv \mathfrak{E}_{\{x\}}$ , the r-fiber  $\mathfrak{E}^x \equiv \mathfrak{E}^{\{x\}}$  and the isotropy group  $\mathfrak{E}_x^x \equiv \mathfrak{E}_{\{x\}}^{\{x\}}$  of a unit  $x \in X$ . Clearly  $\mathfrak{E}_x^x \mathfrak{E}_x^x \subset \mathfrak{E}_x^x$  and  $\mathfrak{E}_x \mathfrak{E}_x^x \subset \mathfrak{E}_x$ . The disjoint union

$$\text{Iso}(\mathfrak{E}) := \bigsqcup_{x \in X} \mathfrak{E}_x^x = \{ \xi \in \mathfrak{E} \mid d(\xi) = r(\xi) \}$$

is called *the isotropy bundle of the groupoid*.

The subset  $\Delta$  of the topological groupoid  $\mathfrak{E}$  is called a *subgroupoid* if for every  $(\xi, \eta) \in (\Delta \times \Delta) \cap \mathfrak{E}^{(2)}$  one has  $\xi\eta \in \Delta$  and  $\xi^{-1} \in \Delta$ . This subgroupoid is *wide* if  $\Delta^{(0)} = \mathfrak{E}^{(0)}$ .

A topological groupoid is a groupoid  $\Xi$  with a topology such that the inversion  $\xi \mapsto \xi^{-1}$  and multiplication  $(\xi, \eta) \mapsto \xi\eta$  are continuous. The topology in  $\Xi^{(2)}$  is the topology induced by the product topology. Whenever the map  $d : \Xi \rightarrow X$  is open, we say that *the groupoid  $\Xi$  is open*. Equivalent conditions are: (i)  $r : \Xi \rightarrow X$  is open and (ii) the multiplication is an open map. It is known that a locally compact groupoid possessing a Haar system is open. In particular, étale groupoids and Lie groupoids are open. For this reason, many texts in groupoid theory only deal with open groupoids.

An equivalence relation on  $X$  is defined by  $x \approx y$  if  $x = r(\xi)$  and  $y = d(\xi)$  for some  $\xi \in \Xi$ . This leads to the usual notions of *orbit*, *invariant (saturated) set*, *saturation*, *transitivity*, etc. *The orbit of a point  $x$*  will be denoted by  $\mathcal{O}_x = r(\Xi_x)$ . Its closure  $\overline{\mathcal{O}}_x$  is called an *orbit closure*.

**Definition 2.1.** A *groupoid action* is a 4-tuple  $(\Xi, \rho, \theta, \Sigma)$  consisting of a groupoid  $\Xi$ , a set  $\Sigma$ , a surjective map  $\rho : \Sigma \rightarrow X$  (the *anchor*) and the action map

$$\theta : \Xi \rtimes \Sigma := \{(\xi, \sigma) \mid d(\xi) = \rho(\sigma)\} \ni (\xi, \sigma) \mapsto \theta_\xi(\sigma) \equiv \xi \bullet_\theta \sigma \in \Sigma \tag{2.1}$$

satisfying the axioms:

- (1)  $\rho(\sigma) \bullet_\theta \sigma = \sigma$  for all  $\sigma \in \Sigma$ ,
- (2) if  $(\xi, \eta) \in \Xi^{(2)}$  and  $(\eta, \sigma) \in \Xi \rtimes \Sigma$ , then

$$(\xi, \eta \bullet_\theta \sigma) \in \Xi \rtimes \Sigma \quad \text{and} \quad (\xi\eta) \bullet_\theta \sigma = \xi \bullet_\theta (\eta \bullet_\theta \sigma).$$

An *action of a topological groupoid* in a topological space is just an action  $(\Xi, \rho, \theta, \Sigma)$  where  $\Xi$  is a topological groupoid,  $\Sigma$  is a Hausdorff topological space and the maps  $\rho, \theta$  are continuous.

If the action  $\theta$  is understood, we will write  $\xi \bullet \sigma$  instead of  $\xi \bullet_\theta \sigma$ . If  $\rho$  is not supposed surjective, then  $\rho(\Sigma)$  is an invariant subset of the unit space and only the reduction  $\Xi_{\rho(\Sigma)}^{\rho(\Sigma)}$  really acts on  $\Sigma$ , so asking  $\rho$  to be onto seems convenient.

**Example 2.2.** Each topological groupoid acts continuously in a canonical way on its unit space. In this case, we have  $\Sigma = X$  and  $\rho = \text{id}_X$ , and then (note the special notation)  $\xi \circ x := \xi x \xi^{-1}$  as soon as  $d(\xi) = x$ . Putting this differently,  $\xi$  sends  $d(\xi)$  into  $r(\xi)$ . One could also name this the *terminal action*; see Lemma 7.2.

**Example 2.3.** Let  $(\Xi, \rho, \theta, \Sigma)$  be a continuous groupoid action and  $\Delta$  a wide subgroupoid of  $\Xi$ . The restricted action is defined by keeping the same anchor  $\rho$  and just restricting the map  $\theta$  to

$$\Delta \rtimes \Sigma = \{(\xi, \sigma) \in \Delta \times \Sigma \mid d(\xi) = \rho(\sigma)\} = \{(\xi, \sigma) \in \Xi \rtimes \Sigma \mid \xi \in \Delta\}.$$

**Example 2.4.** The topological groupoid  $\Xi$  also acts on itself, with  $\Sigma := \Xi, \rho := r$  and  $\xi \bullet \eta := \xi\eta$ .

For  $\xi \in \Xi, A, B \subset \Xi, M \subset \Sigma$  we use the notations

$$\begin{aligned} AB &:= \{\xi\eta \mid \xi \in A, \eta \in B, d(\xi) = r(\eta)\}, \\ \xi \bullet M &:= \{\xi \bullet \sigma \mid \sigma \in M \cap \rho^{-1}[d(\xi)]\}, \\ A \bullet M &:= \{\xi \bullet \sigma \mid \xi \in A, \sigma \in M, d(\xi) = \rho(\sigma)\} = \bigcup_{\xi \in A} \xi \bullet M. \end{aligned}$$

These sets could be void in non-trivial situations.

**Remark 2.5.** In [19], it is shown that if the groupoid  $\Xi$  is open, then  $A \bullet M \subset \Sigma$  is open whenever the sets  $A \subset \Xi$  and  $M \subset \Sigma$  are open. For topological group actions the product  $AM$  is open provided that only the subset  $M$  is open. In addition, if  $A, B$  are subsets of the group, then  $AB$  is open whenever at least one of the subsets is. Examples below will show that for groupoids this is not true, and this will require some special care in some of our proofs. Note that, even for open groupoids, the translation  $\xi \bullet M$  of an open subset  $M$  of  $\Sigma$  could not be open.

**Definition 2.6.** We are going to use *orbits*  $\mathcal{D}_\sigma := \Xi_{\rho(\sigma)} \bullet \sigma$  and *orbit closures*  $\bar{\mathcal{D}}_\sigma$ . The *orbit equivalence relation* will be denoted by  $\sim$ . A subset  $M \subset \Sigma$  is called *invariant* if  $\xi \bullet M \subset M$ , for every  $\xi \in \Xi$ . If  $N \subset \Sigma$ , its *saturation*

$$\text{Sat}(N) = \Xi \bullet N = \bigcap_{\substack{N \subset M \\ M \text{ invariant}}} M$$

is the smallest invariant subset of  $\Sigma$  containing  $N$ .

**Proposition 2.7.** Assume that  $\Xi$  is an open groupoid. The saturation of an open set is also open. The interior  $M^\circ$ , the closure  $\bar{M}$  and the boundary  $\partial M$  of an invariant subset  $M$  of  $\Sigma$  are also invariant.

*Proof.* If  $N$  is an open set,  $\text{Sat}(N) = \Xi \bullet N$  is open, as  $d$  is an open map; see Remark 2.5.

One has  $(M^\circ)^c = \bar{M}^c$  and  $\partial M = \bar{M} \setminus M^\circ$ . Since the difference of two invariant sets is clearly invariant, it is enough to show that  $M^\circ$  is invariant. If  $\sigma \in M$  is an interior point, there exists some open set  $U \subset M$  containing  $\sigma$ , so we have

$$\xi \bullet \sigma \in \Xi \bullet U \subset \Xi \bullet M = M,$$

implying that  $\xi \bullet \sigma$  is also an interior point of  $M$ , since  $\Xi \bullet U$  is open. ■

Remark 3.1 and Example 4.2 will show that the openness assumption cannot be removed.

**Lemma 2.8.** For every  $\sigma \in \Sigma$  one has  $\rho(\mathcal{D}_\sigma) = \mathcal{O}_{\rho(\sigma)}$ . The map  $\rho$  sends  $\bullet$ -invariant subsets of  $\Sigma$  into  $\circ$ -invariant subsets of the unit space  $X$  (see Example 2.2).

*Proof.* If  $y \in \rho(\Omega_\sigma)$ , then for some  $\xi \in \Xi_{\rho(\sigma)}$  one has

$$y = \rho(\xi \bullet \sigma) = r(\xi) \in \mathcal{O}_{\rho(\sigma)}.$$

On the other hand, if  $y \in \mathcal{O}_{\rho(\sigma)}$ , there exists  $\xi \in \Xi_{\rho(\sigma)}$  with  $y = r(\xi) = \rho(\xi \bullet \sigma) \in \rho(\Omega_\sigma)$  and the equality is proven. From this, the last part is trivial. ■

### 2.2. Recurrence sets

**Definition 2.9.** For any  $M, N \subset \Sigma$  one defines the *recurrence set* as

$$\tilde{\Xi}_M^N = \{\xi \in \Xi \mid (\xi \bullet M) \cap N \neq \emptyset\}.$$

The set  $\tilde{\Xi}_M^N$  is increasing in  $M$  and  $N$ . In the group case, one also uses the term “dwelling set”.

The recurrence set can be also described in terms of the function

$$\Xi \bowtie \Sigma \ni (\xi, \sigma) \xrightarrow{\vartheta} (\xi \bullet \sigma, \sigma) \in \Sigma \times \Sigma.$$

If we denote by  $q$  the projection on the first variable  $\Xi \times \Sigma \rightarrow \Xi$  and by  $q$  its restriction to  $\Xi \bowtie \Sigma$ , then

$$\tilde{\Xi}_M^N := q[\vartheta^{-1}(N \times M)].$$

**Remark 2.10.** Note that  $\tilde{\Xi}_M^N = \bigcup_{\sigma \in M} \tilde{\Xi}_\sigma^N$ , where

$$\tilde{\Xi}_\sigma^N \equiv \tilde{\Xi}_{\{\sigma\}}^N = \{\xi \in \Xi_{\rho(\sigma)} \mid \xi \bullet \sigma \in N\} \subset \Xi_{\rho(\sigma)}^{\rho(N)} \subset \Xi_{\rho(\sigma)}.$$

It follows immediately that  $\tilde{\Xi}_M^N \subset \Xi_{\rho(M)}^{\rho(N)}$ . Actually, when  $\rho$  is also injective, one has  $\tilde{\Xi}_M^N = \Xi_{\rho(M)}^{\rho(N)}$ . This applies, in particular, to Example 2.2. *The stabilizer*

$$\tilde{\Xi}_\sigma^\sigma = \{\xi \in \Xi \mid \xi \bullet \sigma = \sigma\}$$

is a closed subgroup of the isotropy group  $\Xi_{\rho(\sigma)}^{\rho(\sigma)}$ . They coincide whenever  $\rho$  is injective.

**Example 2.11.** In the setting of Example 2.4 one has

$$\tilde{\Xi}_M^N = \{\xi \in \Xi \mid \Xi^{d(\xi)} \cap M \cap \xi^{-1}N \neq \emptyset\}.$$

**Example 2.12.** Given a topological space  $\Sigma$ , the *fundamental groupoid*  $\Xi$ , typically denoted by  $\Pi_1(\Sigma)$ , is just the set of homotopy classes of paths between pairs of points. The space of all paths is given the compact-open topology and this induces in  $\Xi$  the quotient topology. We observe that  $X$  is basically  $\Sigma$  and the action  $\xi \circ \sigma$  moves the starting point  $\sigma$  through the path  $\xi$ . The recurrence sets are expressed in terms of the path-connectedness of  $\Sigma$ :

$$\begin{aligned} \tilde{\Xi}_M^N &= \Xi_M^N \\ &= \{\gamma : [0, 1] \rightarrow \Sigma \mid \gamma \text{ continuous, } \gamma(0) \in M, \gamma(1) \in N\} \text{ modulo homotopy.} \end{aligned}$$

The stabilizer of  $\sigma \in X$  is the fundamental group  $\Xi_\sigma^\sigma = \pi(\Sigma, \sigma)$  of  $\Sigma$  rooted at  $\sigma$ .



The next straightforward results will be useful in the next sections.

**Lemma 2.13.** *If  $M, N \subset \Sigma$  and  $\eta_1, \eta_2 \in \Xi$ , then*

$$\tilde{\Xi}_{\eta_1 \bullet M}^{\eta_2 \bullet N} = \eta_2 \tilde{\Xi}_M^N \eta_1^{-1} \quad \text{and} \quad (\tilde{\Xi}_M^N)^{-1} = \tilde{\Xi}_N^M.$$

*Proof.* We only show the first equality:

$$\begin{aligned} \xi \in \tilde{\Xi}_{\eta_1 \bullet M}^{\eta_2 \bullet N} &\Leftrightarrow \exists \sigma \in M, \tau \in N \text{ such that } \xi \bullet (\eta_1 \bullet \sigma) = \eta_2 \bullet \tau \\ &\Leftrightarrow \exists \sigma \in M, \tau \in N \text{ such that } (\eta_2^{-1} \xi \eta_1) \bullet \sigma = \tau \\ &\Leftrightarrow \eta_2^{-1} \xi \eta_1 \in \tilde{\Xi}_M^N \\ &\Leftrightarrow \xi \in \eta_2 \tilde{\Xi}_M^N \eta_1^{-1}. \quad \blacksquare \end{aligned}$$

**Lemma 2.14.** *Let  $M, N \subset \Sigma$ . Then*

$$\text{Sat}(M) \cap N \neq \emptyset \Leftrightarrow \text{Sat}(M) \cap \text{Sat}(N) \neq \emptyset \Leftrightarrow \tilde{\Xi}_M^N \neq \emptyset.$$

*Proof.* One has  $\text{Sat}(M) \cap \text{Sat}(N) \neq \emptyset$  if and only if there exist  $\xi_1, \xi_2 \in \Xi$ ,  $\sigma \in M$  and  $\tau \in N$  such that  $\xi_1 \bullet \sigma = \xi_2 \bullet \tau$ , which is equivalent to  $(\xi_2^{-1} \xi_1) \bullet \sigma = \tau \in N$ , so we have  $\xi_2^{-1} \xi_1 \in \tilde{\Xi}_M^N$  and  $(\xi_2^{-1} \xi_1) \bullet \sigma \in \text{Sat}(M)$ . We used the fact that

$$d(\xi_2^{-1}) = r(\xi_2) = \rho(\xi_2 \bullet \tau) = \rho(\xi_1 \bullet \sigma) = r(\xi_1).$$

For the converse: If  $\tilde{\Xi}_M^N \neq \emptyset$ , there exists  $\xi \in \Xi$  such that  $\xi \bullet \sigma = \tau$ , with  $\sigma \in M$  and  $\tau \in N$ . Then  $\text{Sat}(M) \cap N \neq \emptyset$ , from which  $\text{Sat}(M) \cap \text{Sat}(N) \neq \emptyset$  follows.  $\blacksquare$

**Remark 2.15.** Let  $(\Xi, \rho, \theta, \Sigma)$  be a continuous groupoid action and  $\Delta$  a wide subgroupoid of  $\Xi$ . In terms of the restricted action from Example 2.3, if  $M, N \subset \Sigma$ , the contention  $\tilde{\Delta}_M^N \subset \tilde{\Xi}_M^N$  between the corresponding recurrence sets is obvious. It is also clear that the  $\Delta$ -orbit of any point of  $\Sigma$  is contained in the  $\Xi$ -orbit of this point and that the invariant sets under  $\Xi$  are also invariant under  $\Delta$ . From this one deduces many simple connections between dynamical properties of the two actions, that we will not write down.

### 3. Some examples

#### 3.1. Equivalence relations

If  $\Pi \subset X \times X$  is an equivalence relation on the Hausdorff topological space  $X$ , one can make  $\Pi$  into a topological groupoid by using the product topology in  $X \times X$  and the operations

$$d(x, y) = (y, y), \quad r(x, y) = (x, x), \quad (x, y)(y, z) = (x, z), \quad (x, y)^{-1} = (y, x).$$

The unit space is  $\text{Diag}(X)$  and we identify it with  $X$ , via the homeomorphism  $(x, x) \mapsto x$ .

As a particular case of Example 2.2, the groupoid  $\Xi := \Pi$  acts in a canonical way on  $X$  by

$$(x, y) \circ y = x \quad \text{for all } x, y \in X.$$

For any  $M, N \subset X$  we get

$$\tilde{\Xi}_M^N = \Xi_M^N = \Pi \cap (N \times M). \tag{3.1}$$

There are two extreme particular cases: (i)  $\Pi = \text{Diag}(X)$  (the *trivial groupoid*), for which  $\Xi_M^N$  may be identified with  $N \cap M$ , and (ii)  $\Pi = X \times X$  (the *pair groupoid*), when  $\Xi_M^N = N \times M$ . Actually, an equivalence relation on  $X$  is a wide subgroupoid of the pair groupoid.

**Remark 3.1.** We keep the same notations. The first projection  $X \times X \rightarrow X$  is always open. The restriction to a subset ( $\Pi$  in our case) may not be an open function in general. (However, when  $\Pi$  is an open subset in  $X \times X$ , it is also an open groupoid.) One special example [19, Ex. 6.2] consists of setting  $X = \mathbb{R}$ , the relation being

$$x \Pi y \Leftrightarrow x, y \in [-1, 1] \text{ or } x = y.$$

This groupoid is Hausdorff and locally compact, but it is not open: observe that  $\Xi \circ (-\frac{1}{2}, \frac{1}{2}) = [-1, 1]$ . This is relevant for Remark 2.5, Proposition 2.7 and their subsequent consequences.

Even if the equivalence relation is open, very often “translations” of open sets are not open. For the pair groupoid, if  $(x, y) \in X \times X$  and  $y \in U \subset X$  is open, then  $(x, y) \circ U = \{x\}$ .

Non-open equivalence relations will be used repeatedly as counterexamples. It is true, however, that in some situations one considers on equivalence relations topologies which are different from the one induced from the Cartesian product.

### 3.2. Group actions

We indicate two ways to encode group actions by groupoids. The second one will be convenient for our purposes. We thought it would be interesting to also mention the first one, since it seems natural.

**Example 3.2.** As a particular case of Example 2.2, the *transformation groupoid*  $\Xi \equiv \mathcal{G} \times_{\gamma} X$  associated to the continuous action  $\gamma$  of the topological group  $\mathcal{G}$  on the topological space  $X$  naturally acts on  $X$  by  $(a, x) \circ x := \gamma_a(x)$ . We recall that, as a topological space, it is just  $\mathcal{G} \times X$ . The composition is  $(b, \gamma_a(x))(a, x) := (ba, x)$  and inversion reads  $(a, x)^{-1} := (a^{-1}, \gamma_a(x))$ . If  $M, N \subset \Sigma \equiv X$ , then

$$\tilde{\Xi}_M^N = \Xi_M^N = \{(a, x) \in \mathcal{G} \times M \mid \gamma_a(x) \in N\} \subset \mathcal{G} \times X.$$

The first projection  $p : \mathcal{G} \times X \rightarrow \mathcal{G}$  restricts to a surjection

$$p : \tilde{\Xi}_M^N \rightarrow \text{Rec}_{\gamma}(M, N) := \{a \in \mathcal{G} \mid \gamma_a(M) \cap N \neq \emptyset\} \subset \mathcal{G}, \tag{3.2}$$

where  $\text{Rec}_\gamma(M, N)$  is the usual recurrence set for dynamical systems [2, 6, 9]. Injectivity fails in general: for instance,  $\tilde{\Xi}_X^X = \mathcal{G} \times X$  while  $\text{Rec}_\gamma(X, X) = \mathcal{G}$ . On the other hand, if  $M$  or  $N$  are singletons, injectivity holds. Using slightly simplified notations, one has  $\tilde{\Xi}_{x_0}^N \cong \text{Rec}_\gamma(x_0, N)$  and  $\tilde{\Xi}_M^{x_0} \cong \text{Rec}_\gamma(M, x_0)$ . Although the relation (3.2) is quite concrete, it will not be precise enough to make suitable connections between dynamical properties in the group and in the groupoid framework.

**Example 3.3.** So we implement differently the classical dynamical system  $(\mathcal{G}, \gamma, \Sigma)$  (for a better correspondence of notations, we set  $\Sigma$  for the space of the group action). The group is an open groupoid in the obvious way; so we have  $\Xi := \mathcal{G}$  and the unit space  $X = \{e\}$  is only composed of the unit of the group. The source and the domain maps are constant, the same being true for  $\rho : \Sigma \rightarrow \{e\}$ ; it follows that  $\mathcal{G} \bowtie \Sigma = \mathcal{G} \times \Sigma$ . One sets  $a \bullet \sigma := \gamma_a(\sigma)$  for every  $a \in \mathcal{G}$ ,  $\sigma \in \Sigma$  (thus  $\theta = \gamma$ ). Note that this is not covered by Examples 2.2 or 2.4. A simple inspection of the definitions shows that

$$\tilde{\Xi}_S^T \equiv \tilde{\mathcal{G}}_M^N = \text{Rec}_\gamma(M, N), \tag{3.3}$$

and this will be very convenient below.

### 3.3. The Deaconu–Renault groupoid

Let  $v : X \rightarrow X$  be a local homeomorphism of the Hausdorff topological space  $X$ . To unify many constructions in groupoid  $C^*$ -algebras, one defines the *Deaconu–Renault groupoid* [7, 18]

$$\Xi(v) := \{(x, k - l, y) \mid x, y \in X, k, l \in \mathbb{N}, v^k(x) = v^l(y)\},$$

with structure maps

$$\begin{aligned} (x, n, y)(y, m, z) &:= (x, n + m, z), & (x, n, y)^{-1} &:= (y, -n, x), \\ d(x, n, y) &:= y, & r(x, n, y) &:= x. \end{aligned}$$

With a suitable topology (not needed here), it turns into a Hausdorff étale groupoid over  $X$ . If  $v$  is a (global) homeomorphism, this results in the transformation groupoid of Example 3.2 for the group  $\mathcal{G} = \mathbb{Z}$ .

An important tool is the *canonical cocycle* (a groupoid morphism)

$$c : \Xi(v) \rightarrow \mathbb{Z}, \quad c(x, n, y) := n,$$

which is, of course, the restriction to  $\Xi(v)$  of the middle projection. For any  $M, N \subset X$  set

$$\mathbb{Z}_v(N, M) := \{k - l \mid v^k(N) \cap v^l(M) \neq \emptyset\} \subset \mathbb{Z}.$$

The orbit of  $y$  is  $\mathcal{O}_y = \{x \in X \mid \mathbb{Z}_v(x, y) \neq \emptyset\}$ , where notationally we identify singletons to points. The canonical cocycle restricts to a surjection

$$c : \widehat{\Xi(v)}_M^N \equiv \Xi(v)_M^N \rightarrow \mathbb{Z}_v(N, M). \tag{3.4}$$

In contrast with the global case, if  $M = \{y\}$ , then (3.4) could still fail to be injective. But clearly  $c$  restricts to a one-to-one map allowing to identify  $\Xi(v)_y^x$  with  $\mathbb{Z}_v(x, y)$  for any  $x, y \in X$ .

### 3.4. Non-invariant restrictions

Let  $(\Xi, \rho, \bullet, \Sigma)$  be a continuous groupoid action. Let  $\underline{\Sigma}$  be an open subset of  $\Sigma$ ; we do not suppose it to be invariant. One defines  $\underline{X} := \rho(\underline{\Sigma})$ , so  $\underline{\rho} := \rho|_{\underline{\Sigma}} : \underline{\Sigma} \rightarrow \underline{X}$  is a continuous surjection. We also have the (maybe non-invariant) groupoid restriction

$$\underline{\Xi} := \Xi|_{\underline{X}} = \{\xi \in \Xi \mid d(\xi), r(\xi) \in \underline{X}\},$$

which acts naturally by the restriction of  $\bullet$  on  $\underline{\Sigma}$ . Obviously

$$\underline{\Xi} \rtimes \underline{\Sigma} = (\Xi \rtimes \Sigma) \cap (\underline{\Xi} \times \underline{\Sigma}).$$

If  $M \subset \Sigma$  is  $\bullet$ -invariant,  $M \cap \underline{\Sigma}$  is clearly  $\bullet$ -invariant. If  $M, N \subset \Sigma$ , one easily checks that

$$\tilde{\Xi}_{M \cap \underline{\Sigma}}^{N \cap \underline{\Sigma}} \subset \underline{\Xi} \cap \tilde{\Xi}_M^N = \tilde{\Xi}_{M \cap \rho^{-1}[\rho(\underline{\Sigma})]}^{N \cap \rho^{-1}[\rho(\underline{\Sigma})]}. \tag{3.5}$$

One gets equality whenever  $\underline{\Sigma}$  is  $\rho$ -saturated, i.e. when  $\rho^{-1}[\rho(\underline{\Sigma})] = \underline{\Sigma}$ , in particular, when  $\rho$  is a bijection. On the other hand, if  $M, N \subset \underline{\Sigma}$ , one always gets equality in (3.5), and in this case

$$\tilde{\Xi}_M^N = \underline{\Xi} \cap \tilde{\Xi}_M^N = \tilde{\Xi}_M^N. \tag{3.6}$$

To get the last term, one notices that  $\tilde{\Xi}_M^N \subset \Xi_{\rho(M)}^{\rho(N)} \subset \underline{\Xi}$ , since  $\rho(M), \rho(N) \subset \underline{X}$ , or proceeds directly.

### 3.5. Partial group actions

Let  $\mathcal{G}$  be a group assumed, for simplicity, to be discrete. The unit is denoted by  $e$ .

**Definition 3.4.** A partial action [11] of  $\mathcal{G}$  on the topological space  $Y$  is a family of homeomorphisms

$$\beta := \{Y_{a^{-1}} \xrightarrow{\beta_a} Y_a \mid a \in \mathcal{G}\}$$

between open subsets of  $Y$  satisfying

- (i)  $Y_e = Y$  and  $\beta_e = \text{id}_Y$ ,
- (ii)  $\beta_a \circ \beta_b$  is a restriction of  $\beta_{ab}$  for any  $a, b \in \mathcal{G}$ .

The domain of the composition above is

$$\text{dom}(\beta_a \circ \beta_b) := \{y \in Y_{b^{-1}} \mid \beta_b(y) \in Y_{a^{-1}}\} = \beta_b^{-1}(Y_{a^{-1}}).$$

It is easily shown that  $\beta_a^{-1} = \beta_{a^{-1}}$  for any element  $a$ . To such a partial action, one associates [1] an étale groupoid  $\Xi[\beta] \equiv \mathcal{G} \ltimes_{[\beta]} Y$  over  $Y$ . As a set it is

$$\mathcal{G} \ltimes_{[\beta]} Y := \{(a, y) \mid a \in \mathcal{G}, y \in Y_{a^{-1}}\},$$

on which one considers the induced product topology. The structure maps are

$$\begin{aligned} d(a, y) &:= y, & r(a, y) &:= \beta_a(y), \\ (a, \beta_b(y))(b, y) &:= (ab, y), & (a, y)^{-1} &:= (a^{-1}, \beta_a(y)). \end{aligned}$$

If the action is global, meaning that  $Y_a = Y$  for every  $a$ , then we recover the situation of Example 3.2.

The subset  $T \subset Y$  is said to be  $\beta$ -invariant if  $\beta_a(y) \in T$  whenever  $a \in \mathcal{G}$  and  $y \in T \cap Y_{a^{-1}}$ . Of course, this happens exactly when  $T$  is invariant, seen as a set of units of  $\Xi[\beta]$ .

For  $S, T \subset Y$  it seems natural to define the *recurrence set*

$$\text{Rec}_{[\beta]}(S, T) := \{a \in \mathcal{G} \mid \exists y \in Y_{a^{-1}} \cap S \text{ such that } \beta_a(y) \in T\}. \tag{3.7}$$

Very much as in Example 3.2, one gets

$$\widetilde{\Xi[\beta]}_S^T = \Xi[\beta]_S^T = \{(a, y) \in \mathcal{G} \times S \mid y \in Y_{a^{-1}}, \beta_a(y) \in T\} \subset \mathcal{G} \times_{[\beta]} Y.$$

The first projection  $(a, y) \rightarrow p(a, y) := a$  restricts to a surjection

$$p : \widetilde{\Xi[\beta]}_S^T \rightarrow \text{Rec}_{[\beta]}(S, T). \tag{3.8}$$

If  $S = \{y_0\}$  is a singleton, injectivity holds and one has  $\widetilde{\Xi[\beta]}_{y_0}^T \cong \text{Rec}_{[\beta]}(y_0, T)$ .

A very direct way to get a partial group action is to make a non-invariant restriction in a group (global) action. In a certain sense this is the most general situation, since any partial action can be extended to a global one [10].

So let  $(\mathcal{G}, \gamma, X)$  be a continuous group action and  $Y$  an open, maybe non-invariant, subset of  $X$ . For every  $a \in \mathcal{G}$ , set  $Y_a := Y \cap \gamma_a(Y) \subset Y$ . It is straightforward to check that

$$\beta := \{\beta_a := \gamma_a|_{Y_{a^{-1}}} : Y_{a^{-1}} \rightarrow Y_a \mid a \in \mathcal{G}\}$$

becomes a partial action. The groupoid  $\Xi[\beta] \equiv \mathcal{G} \times_{[\beta]} Y$  may be seen as the non-invariant restriction to  $Y$  of the transformation groupoid  $\mathcal{G} \times_\gamma X$  from Example 3.2, as described in Section 3.4. For this notice that, if  $(a, x) \in \mathcal{G} \times X$ , the conditions  $d(a, x), r(a, x) \in Y$  mean exactly that  $x \in Y_{a^{-1}}$ . The relationship between the two types of groupoid recurrence sets may be read off from (3.5), or from (3.6) in the most interesting case, when the two sets are already contained in  $Y$ . In this last case, one gets

$$(\mathcal{G} \times_{[\beta]} X)_S^T = (\mathcal{G} \times_\gamma X)_S^T, \tag{3.9}$$

which can also be checked directly. In terms of the actions themselves, using a notation from (3.2), the relation (3.7) converts into

$$\text{Rec}_{[\beta]}(S, T) = \{a \in \mathcal{G} \mid \exists y \in \gamma_{a^{-1}}(Y) \cap S \text{ such that } \gamma_a(y) \in T\} = \text{Rec}_\gamma(S, T), \tag{3.10}$$

since  $\gamma_a(y) \in T \subset Y$  already implies that  $y \in \gamma_{a^{-1}}(Y)$ . It is also easy to verify that (3.10) follows from (3.8), (3.9) and (3.2).

### 3.6. The action groupoid of a groupoid action

A continuous groupoid action  $(\Xi, \rho, \bullet, \Sigma)$  being given, one constructs *the action (or transformation, or crossed product) groupoid* which, as a set, is the closed subspace  $\Xi \bowtie \Sigma$  of  $\Xi \times \Sigma$  introduced in (2.1) and the structure maps are

$$\begin{aligned} (\eta, \xi \bullet \sigma)(\xi, \sigma) &:= (\eta\xi, \sigma), & (\xi, \sigma)^{-1} &:= (\xi^{-1}, \xi \bullet \sigma), \\ d(\xi, \sigma) &:= (\rho(\sigma), \sigma) \equiv \sigma, & r(\xi, \sigma) &:= (r(\xi), \xi \bullet \sigma) \equiv \xi \bullet \sigma. \end{aligned} \tag{3.11}$$

To stress the origin of the construction, we are going to denote by  $\Xi \ltimes_{\theta} \Sigma$  this groupoid, in analogy with the group case which is a particular example. The space of units  $(\Xi \ltimes_{\theta} \Sigma)^{(0)}$  identifies with  $\Sigma$ . The canonical action of  $\Xi \ltimes_{\theta} \Sigma$  on  $\Sigma$  (see Example 2.2)

$$(\xi, \sigma) \circ (\rho(\sigma), \sigma) = (\xi, \sigma)(\rho(\sigma), \sigma)(\xi, \sigma)^{-1} = (r(\xi), \xi \bullet \sigma) = (\rho(\xi \bullet \sigma), \xi \bullet \sigma)$$

may also be written  $(\xi, \sigma) \circ \sigma = \xi \bullet \sigma$ , and thus reproduces in some way the initial action. The invariant subsets, in particular the orbits, are the same. This has a series of obvious consequences on the way dynamical properties are preserved when passing from  $(\Xi, \rho, \bullet, \Sigma)$  to  $\Xi \ltimes_{\theta} \Sigma$ .

The connection between the relevant recurrent sets is similar with (and in fact generalizes) that of Example 3.2: if  $M, N \subset \Sigma$ , we get

$$(\Xi \ltimes_{\theta} \Sigma)_M^N = \{(\xi, \sigma) \in \Xi \ltimes_{\theta} \Sigma \mid \sigma \in M, \xi \bullet \sigma \in N\}.$$

Consequently,

$$\tilde{\Xi}_M^N = p[(\Xi \ltimes_{\theta} \Sigma)_M^N], \tag{3.12}$$

where  $p$  is the restriction of the first projection of the product  $\Xi \times \Sigma$ . This restriction is not always injective, being constant on any set of the type  $\{\xi\} \times \rho^{-1}[d(\xi)]$ . It follows that *for recurrence phenomena it is not always a good idea to replace the initial action by the action groupoid.*

### 3.7. Groupoid pullbacks

There is a powerful method to construct new more sophisticated groupoids from simpler ones, which, however, does not loose control over the orbit structure or the recurrence sets. Let  $d, r : \Xi \rightarrow X$  be the domain and the range maps of a topological groupoid,  $\Omega$  a topological space and  $h : \Omega \rightarrow X$  an open continuous surjection. Let

$$h^{\downarrow}(\Xi) := \{(\omega, \xi, \omega') \in \Omega \times \Xi \times \Omega \mid r(\xi) = h(\omega), d(\xi) = h(\omega')\}$$

be the *associated pullback groupoid* [5, 14]. We recall its structural maps:

$$\begin{aligned} (\omega_1, \xi, \omega_2)(\omega_2, \eta, \omega_3) &:= (\omega_1, \xi\eta, \omega_3), & (\omega, \xi, \omega')^{-1} &:= (\omega', \xi^{-1}, \omega), \\ d(\omega, \xi, \omega') &:= \omega', & r(\omega, \xi, \omega') &:= \omega. \end{aligned}$$

Note the relations between orbits and orbit closures in the two groupoids (closures commute with open continuous surjections):

$$\mathcal{O}_\omega^{\downarrow\downarrow} = h^{-1}(\mathcal{O}_{h(\omega)}), \quad \overline{\mathcal{O}_\omega^{\downarrow\downarrow}} = h^{-1}(\overline{\mathcal{O}_{h(\omega)}}).$$

For  $M, N \subset \Omega$ , by inspection one gets

$$h^{\downarrow\downarrow}(\Xi)_M^N = N \times \Xi_{h(M)}^{h(N)} \times M. \tag{3.13}$$

### 4. Topological transitivity

#### 4.1. The standard notions

Let us fix a continuous groupoid action  $(\Xi, \rho, \theta, \Sigma)$ . We start with the simplest notions. If there is just one orbit, the action is *transitive*. This happens for the pair groupoid, for instance. (In [8, Prop. 3.18], it is shown that a compact transitive groupoid is open.) A point having a dense orbit is called a *transitive point*. If there is a dense orbit, i.e. if a transitive point does exist, one says that the action is *pointwise transitive*. For the more refined notions, one needs first to prove the next result.

**Theorem 4.1.** *Let us consider the following conditions:*

- (i)  $\Sigma$  is not the union of two proper invariant closed subsets.
- (i') Any two open non-void invariant subsets of  $\Sigma$  have non-trivial intersection.
- (ii) Each non-empty open invariant subset of  $\Sigma$  is dense.
- (iii) For any  $U, V \subset \Sigma$  open and non-void,  $\tilde{\Xi}_U^V \neq \emptyset$  holds (recurrent transitivity).
- (iv) Each invariant subset of  $\Sigma$  is either dense, or nowhere dense (topological transitivity).

Then the following implications hold:

- (1) Transitivity  $\Rightarrow$  pointwise transitivity  $\Rightarrow$  recurrent transitivity (iii).
- (2) (iv)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i')  $\Leftrightarrow$  (i). None of the other implications holds in general. Pointwise transitivity does not imply topological transitivity.
- (3) If the groupoid  $\Xi$  is open, the conditions (i) to (iv) are all equivalent.

*Proof.* (1) The first implication is trivial (and obviously it is not an equivalence). We verify now the second one. Assume that  $\Sigma$  has a dense orbit  $\mathcal{D}_\sigma$  and let  $\emptyset \neq V_1, V_2$  be open sets. One has

$$\xi_1 \cdot \sigma \in \mathcal{D}_\sigma \cap V_1 \neq \emptyset \quad \text{and} \quad \xi_2 \cdot \sigma \in \mathcal{D}_\sigma \cap V_2 \neq \emptyset$$

for elements  $\xi_1, \xi_2 \in \Xi$  with  $r(\xi_1^{-1}) = d(\xi_1) = \rho(\sigma) = d(\xi_2)$ . Since  $(\xi_2 \xi_1^{-1})(\xi_1 \cdot \sigma) = \xi_2 \cdot \sigma$ , we infer that  $\xi_2 \xi_1^{-1} \in \tilde{\Xi}_{V_1}^{V_2} \neq \emptyset$ .

(2) (i)  $\Leftrightarrow$  (i'). If (i) fails, i.e.  $\Sigma = C \cup D$  with  $C$  and  $D$  proper closed invariant subsets, then  $C^c \cap D^c = \emptyset$ . This contradicts (i'), since  $C^c$  and  $D^c$  are open, non-void

and invariant. On the other hand, if  $A, B$  are open non-empty invariant sets such that  $A \cap B = \emptyset$ , then  $A^c \cup B^c = \Sigma$  with  $A^c, B^c$  proper closed invariant subsets, finishing the proof of the equivalence.

(ii)  $\Rightarrow$  (i'). If a non-void open invariant subset is dense, it meets every other (invariant) non-void open set.

(iii)  $\Rightarrow$  (ii). Let  $\emptyset \neq U \subset \Sigma$  be open and invariant. By assumption, for every non-void open set  $V \subset \Sigma$  there exists some  $\xi \in \Xi$  making  $(\xi \bullet U) \cap V$  non-void. But  $\xi \bullet U \subset U$ , implying  $U \cap V \neq \emptyset$ . Thus,  $U$  meets every other non-void open set and must be dense.

(iv)  $\Rightarrow$  (iii). Suppose (iv) holds. Let  $\emptyset \neq U, V \subset \Sigma$  be open sets.  $\text{Sat}(U)$  is an invariant set containing  $U$ , so it cannot be nowhere dense, meaning that it is dense. Hence  $\text{Sat}(U) \cap V \neq \emptyset$  and we conclude by Lemma 2.14.

The fact that all the other implications fail without extra assumptions will be shown in a series of counterexamples below.

(3) Provided that  $d$  is open, it is enough to prove that (i') implies (iv). So let us assume (i'), but let  $A \subset \Sigma$  be invariant, neither dense, nor nowhere dense. Then  $(\overline{A})^\circ$  and  $(A^c)^\circ = (\overline{A}^c)$  are both non-void open sets, which are invariant by Proposition 2.7. They should meet, by (i'), but this is obviously false. ■

Let us indicate now the necessary counterexamples for groupoids that are not open.

**Example 4.2.** This example shows that (ii)  $\not\Rightarrow$  (iii), (iv) (redundantly but explicitly). Let

$$\Sigma = X \equiv \mathbb{R}^x := (-\infty, 0) \cup (0, \infty).$$

Define the sets

$$X_1 = \{-1\} \cup ((0, \infty) \setminus \{1\}) \quad \text{and} \quad X_2 = ((-\infty, 0) \setminus \{-1\}) \cup \{1\}.$$

Forming a partition of  $\mathbb{R}^x$ , they induce an equivalence relation  $\Pi$ , viewed as a topological groupoid acting on  $\mathbb{R}^x$  as in Section 3.1. It is easy to check that  $\Pi$  is not open. The orbits are precisely  $X_1$  and  $X_2$  (neither open nor closed), with closures

$$\overline{X}_1 = \{-1\} \cup (0, \infty) \quad \text{and} \quad \overline{X}_2 = (-\infty, 0) \cup \{1\}.$$

These orbit closures are not invariant, showing already that for non-open groupoids the conclusion of Proposition 2.7 does not necessarily hold. (And we see that  $\Pi$  is not an open groupoid in an indirect way.)

It is easy to see that  $X = X_1 \cup X_2$  is the single (non-void) open invariant set, since any invariant set is a union of orbits, and  $X_1, X_2$  are not open. Thus condition (ii) from Theorem 4.1 holds. On the other hand condition (iv) definitely fails, since the invariant sets  $X_1, X_2$  are neither dense, nor nowhere dense. Of course recurrent transitivity also fails. To see this directly, take for instance  $U = (2, \infty)$  and  $V = (-\infty, -2)$ , for which

$$\tilde{\Xi}_U^V = \{(x, y) \in \Pi \mid y \in U, x \in V\} = \emptyset,$$

by (3.1) or by a simple computation.



**Example 4.3.** The previous example can be easily modified to show that (i')  $\not\Rightarrow$  (ii). In  $\mathbb{R}^x$ , define the equivalence relation  $\Upsilon$  associated to the partition

$$Y_1 = \{-1\} \cup (0, 1) \cup (1, 2) \cup (3, \infty), \quad X_2 = ((-\infty, 0) \setminus \{-1\}) \cup \{1\}, \quad Y_3 = [2, 3]$$

and consider the canonical action of  $\Xi = \Upsilon$  on  $\mathbb{R}^x$ . As invariant sets are union of orbits, the only non-void invariant open sets are  $\mathbb{R}^x$  and

$$Y_1 \cup X_2 = (-\infty, 0) \cup (0, 2) \cup (3, \infty),$$

which have non-empty intersection, so (i') holds. But the set  $Y_1 \cup X_2$  is not dense, hence (ii) fails.

**Example 4.4.** Define on  $\Sigma = X = \mathbb{R}$  the equivalence relation

$$x \Pi y \Leftrightarrow x, y \in \mathbb{Q} \text{ or } x = y.$$

This provides a non-open groupoid acting on  $\mathbb{R}$ , and this action is pointwise transitive ( $\mathbb{Q}$  is a dense orbit), hence recurrently transitive. The invariant set  $(0, 1) \setminus \mathbb{Q}$  is neither nowhere dense nor dense, so this action is not topologically transitive. This example shows that *pointwise transitivity*  $\not\Rightarrow$  (iv) and (consequently) (iii)  $\not\Rightarrow$  (iv).

**Proposition 4.5.** *If  $\Xi$  is open and  $\Sigma$  is a Baire second-countable space, then topological transitivity, recurrent transitivity and pointwise transitivity are equivalent (and also equivalent to the properties (i) and (ii)).*

*Proof.* Having in view Theorem 4.1, what remains is to show that (ii) implies pointwise transitivity. Since  $\Sigma$  is second-countable, its topology has a countable basis  $\{V_n \neq \emptyset\}_{n \in \mathbb{N}}$ . By defining  $U_n = \Xi \cdot V_n$  we get countably many dense open subsets of a Baire space, so  $U = \bigcap_n U_n$  is also a dense (invariant) set. Let  $W \neq \emptyset$  be an open subset of  $\Sigma$ . By the definition of a basis, there exists some  $V_n \subset W$ . Hence we have  $U \subset U_n = \Xi \cdot V_n \subset \Xi \cdot W$ . Therefore, if  $\sigma \in U$ , then  $\xi^{-1} \cdot \sigma \in W$  for some  $\xi \in \Xi$ . Hence  $\sigma$  has a dense orbit and the action is pointwise transitive. ■

We recall that Hausdorff locally compact spaces and complete metric spaces are Baire.

**Remark 4.6.** What we used in the proof of Proposition 4.5 is the fact that the intersection  $U$  is non-void. Actually, one could improve: *under the given requirements, the set of points with dense orbit is a dense  $G_\delta$ -set.*

**Example 4.7.** In the classical dynamical system case of Example 3.3 we recover known results [2, 6, 13]. Since the source map  $\mathcal{S} \rightarrow \{e\}$  is clearly open, Theorem 4.1 simplifies a lot. Even in this particular case, without second countability the full equivalence from Proposition 4.5 fails.

**Example 4.8.** The Deaconu–Renault groupoid is recurrently transitive if and only if for any open sets  $\emptyset \neq U, V \subset X$ , there exist  $x \in V, y \in U, k, l \in \mathbb{N}$  such that  $v^k(x) = v^l(y)$ .

**Example 4.9.** We say that the partial group action  $(\mathcal{G}, \beta, Y)$  from Section 3.5 is *recurrently transitive* if  $\text{Rec}_{[\beta]}(U, V) \neq \emptyset$  for any  $U, V \subset Y$  non-void and open. From (3.8) one deduces that this happens precisely when the groupoid  $\mathcal{G} \times_{[\beta]} Y$  is recurrently transitive.

**Example 4.10.** As in Section 3.4, let  $(\Xi, \rho, \theta, \Sigma)$  be a groupoid action and let  $(\underline{\Xi}, \underline{\rho}, \underline{\theta}, \underline{\Sigma})$  be the action obtained by restriction to the open, maybe non-invariant subset  $\underline{\Sigma} \subset \Sigma$ . If  $\theta$  is recurrently transitive, then  $\underline{\theta}$  is also recurrently transitive. This follows from formula (3.6) and from the fact that open subsets of  $\underline{\Sigma}$  are open in  $\Sigma$ . It is clear that in general there is no converse to the statement. In particular,  $\underline{\Sigma}$  might happen to be invariant and recurrently transitive, while the restriction to  $\Sigma \setminus \underline{\Sigma}$  is not constrained in any way. The next proposition outlines a favorable situation.

**Proposition 4.11.** *Suppose that  $\Xi$  is open and  $\text{Sat}(\underline{\Sigma})$  is dense in  $\Sigma$ . Then  $\theta$  is recurrently transitive if and only if  $\underline{\theta}$  is recurrently transitive.*

*Proof.* Suppose that  $\underline{\theta}$  is recurrently transitive and let  $U, V$  be two open non-void subsets of  $\Sigma$ . By density,  $U$  and  $V$  both intersect the saturation  $\Xi \bullet \underline{\Sigma}$ . So Lemma 2.14 tells us that the open set  $\underline{\Sigma}$  intersects the open sets  $\Xi \bullet U$  and  $\Xi \bullet V$ . Use the fact that  $\underline{\theta}$  is recurrently transitive and Lemma 2.14 (again) to conclude that  $\tilde{\Xi}_U^V \neq \emptyset$ . ■

**Corollary 4.12.** *Let  $(\mathcal{G}, \gamma, X)$  be a global group action and  $(\mathcal{G}, \beta, Y)$  the partial action obtained as in Section 3.5 by restricting to the open subset  $Y$ . We assume that  $X$  is the smallest invariant set containing  $Y$ . Then one of the actions is recurrently transitive if and only if the other is so.*

In [11, Prop. 5.5], it is shown that any partial action  $(\mathcal{G}, \beta, Y)$  may be obtained from a global one, with the extra condition that the total space  $X$  is the saturation of the initial one, in an essentially unique way. In general,  $X$  could fail to be Hausdorff, cf. [11, Prop. 5.6]. The results are attributed to F. Abadie [1].

**Example 4.13.** In the setting of Section 3.6, the groupoid action  $(\Xi, \rho, \bullet, \Sigma)$  is recurrently transitive if and only if the action groupoid  $\Xi \times_{\theta} \Sigma$  is so ( $p$  is surjective, hence  $p(B)$  is void if and only if  $B$  is void).

**Example 4.14.** Although the pullback groupoid of Section 3.7 might be much more complicated than the initial one,  $h^{\downarrow\downarrow}(\Xi)$  and  $\Xi$  are simultaneously recurrently transitive. This follows from (3.13).

### 4.2. Weakly pointwise transitivity

**Definition 4.15.** Define the *invariant closure* of  $A \subset \Sigma$  by

$$\mathfrak{C}(A) := \bigcap_{\substack{A \subset M \\ M \text{ closed, invariant}}} M.$$

This is the smallest closed and invariant set including  $A$ . One very special case (deserving its own notation) is  $\mathfrak{C}_{\sigma} = \mathfrak{C}(\{\sigma\})$ , the *invariant orbit closure* of  $\sigma$ .

**Proposition 4.16.** *The invariant closure verifies*

$$\mathfrak{C}(A) = \mathfrak{C}(\text{Sat}(A)) = \mathfrak{C}(\overline{\text{Sat}(A)}).$$

*In particular,*

$$\mathfrak{C}_\sigma = \mathfrak{C}(\mathfrak{D}_\sigma) = \mathfrak{C}(\overline{\mathfrak{D}_\sigma}) \supset \overline{\mathfrak{D}_\sigma}.$$

*If  $\Xi$  is open, then  $\mathfrak{C}(A) = \overline{\text{Sat}(A)}$  and  $\mathfrak{C}_\sigma = \overline{\mathfrak{D}_\sigma}$ .*

*Proof.* Since  $A \subset \text{Sat}(A) \subset \overline{\text{Sat}(A)}$ , one readily gets

$$\mathfrak{C}(A) \subset \mathfrak{C}(\text{Sat}(A)) \subset \mathfrak{C}(\overline{\text{Sat}(A)}).$$

The set  $\mathfrak{C}(A)$  is closed and invariant, so it contains  $\overline{\text{Sat}(A)}$ , implying that  $\mathfrak{C}(\overline{\text{Sat}(A)}) \subset \mathfrak{C}(A)$ .

If  $\Xi$  is open, then by Proposition 2.7  $\overline{\text{Sat}(A)}$  is a closed and invariant set containing  $A$ , so  $\mathfrak{C}(A) = \overline{\text{Sat}(A)}$ . Then use the fact that  $\mathfrak{D}_\sigma = \text{Sat}(\{\sigma\})$ . ■

It is not difficult to see that  $\mathfrak{C}(A)$  contains the sets

$$\overline{\text{Sat}(A)}, \overline{\overline{\text{Sat}(\text{Sat}(A))}}, \overline{\overline{\overline{\text{Sat}(\text{Sat}(\text{Sat}(A)))}}}, \dots$$

Whenever the groupoid is open, this ascending chain of sets collapses in the first step. The relevance of the set  $\mathfrak{C}(A)$  in groupoids that are not open is a new phenomenon.

**Definition 4.17.** An action  $(\Xi, \rho, \theta, \Sigma)$  is called *weakly pointwise transitive* (wpt) if a point  $\sigma \in \Sigma$  exists such that  $\mathfrak{C}_\sigma = \Sigma$ . In this case, we say that  $\sigma$  is a *weakly transitive point*.

**Remark 4.18.** Since  $\overline{\mathfrak{D}_\sigma} \subset \mathfrak{C}_\sigma$ , it is clear that *pointwise transitive*  $\Rightarrow$  *weakly pointwise transitive*. When  $\Xi$  is an open groupoid, *weakly pointwise transitive*  $\Leftrightarrow$  *pointwise transitive*.

**Example 4.19.** We compute the invariant closures in the previous examples.

- In Example 4.2, one has  $\mathfrak{C}_\sigma = \mathbb{R}^x$  for all  $\sigma \in \Sigma = \mathbb{R}^x$ . So *every point is weakly transitive*. We recall that this example is not topologically transitive or recurrently transitive.
- In Example 4.3,

$$\mathfrak{C}_\sigma = \begin{cases} [2, 3] & \text{if } \sigma \in [2, 3], \\ \mathbb{R}^x & \text{if } \sigma \in \mathbb{R}^x \setminus [2, 3]. \end{cases}$$

- In Example 4.4,

$$\mathfrak{C}_\sigma = \begin{cases} \mathbb{R} & \text{if } \sigma \in \mathbb{Q}, \\ \{\sigma\} & \text{if } \sigma \notin \mathbb{Q}. \end{cases}$$

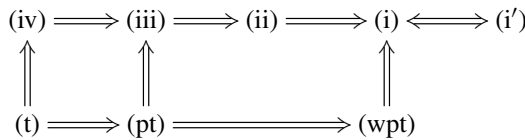
All of these are *weakly pointwise transitive* systems. The examples also show that the sets  $\{\mathfrak{C}_\sigma \mid \sigma \in \Sigma\}$  do not need to be disjoint: some could intersect or even (strictly)

contain others. Of course, this often happens for orbit closures, but for invariant closures the overlaps tend to be larger.

**Proposition 4.20.** *Weakly pointwise transitivity implies condition (i) of Theorem 4.1.*

*Proof.* Suppose that  $\Sigma = N_1 \cup N_2$ , where  $N_1$  and  $N_2$  are closed and invariant sets. Without loss of generality, there exists  $\sigma \in N_1$  such that  $\mathfrak{C}_\sigma = \Sigma$ . As  $N_1$  is closed, invariant and contains  $\sigma$ , the relation  $\Sigma = \mathfrak{C}_\sigma \subset N_1$  follows. We conclude that (i) holds. ■

**Remark 4.21.** Example 4.3 shows that *weakly pointwise transitivity does not imply condition (ii) of Theorem 4.1, in general.* We recall that pointwise transitivity does imply (iii), which is stronger than (ii), so none of these properties is implied by weakly topological transitivity. We summarize most of the implications in the following diagram:



## 5. Recurrence of points

### 5.1. Limit sets

A continuous action  $(\mathfrak{E}, \rho, \theta, \Sigma)$  will be fixed, with  $\mathfrak{E}$  strongly non-compact. We recall from the introduction that this means that none of the d-fibers of the groupoid is compact. Of course, the topological spaces  $\Sigma$  and  $X = \mathfrak{E}^{(0)}$  are allowed (but not required) to be compact. Note that closed equivalence relations on compact spaces  $X$ , with the induced topology, are excluded. Let  $\mathcal{K}(T)$  denote the family of compact subsets of the topological space  $T$ .

**Definition 5.1.** The *limit set of the point*  $\sigma \in \Sigma$  is the closed subset of  $\Sigma$

$$\mathfrak{L}_\sigma^\theta \equiv \mathfrak{L}_\sigma := \bigcap_{K \in \mathcal{K}(\mathfrak{E})} \overline{(\mathfrak{E} \setminus K) \cdot \sigma}.$$

Note that

$$(\mathfrak{E} \setminus K) \cdot \sigma = [(\mathfrak{E} \setminus K) \cap \mathfrak{E}_{\rho(\sigma)}] \cdot \sigma = [\mathfrak{E}_{\rho(\sigma)} \setminus K_{\rho(\sigma)}] \cdot \sigma,$$

where  $K_{\rho(\sigma)} := K \cap \mathfrak{E}_{\rho(\sigma)}$  is compact in  $\mathfrak{E}_{\rho(\sigma)}$ . It follows that one can also write

$$\mathfrak{L}_\sigma = \bigcap_{K \in \mathcal{K}(\mathfrak{E}_{\rho(\sigma)})} \overline{(\mathfrak{E}_{\rho(\sigma)} \setminus K) \cdot \sigma}.$$

The limit set would be void if  $\mathfrak{E}_{\rho(\sigma)}$  was allowed to be compact (which is not), but it can also be void in other situations.

For any unit  $x$  we say that the net  $(\xi_i)_{i \in I} \subset \Xi_x$  diverges if for every compact  $k \subset \Xi_x$  there exists  $i_k \in I$  such that  $\xi_i \notin k$  if  $i \geq i_k$ . The existence of divergent nets relies on our strongly non-compact assumption.

**Lemma 5.2.** *The following statements for  $\sigma, \tau \in \Sigma$  are equivalent:*

- (i)  $\tau$  belongs to the limit set  $\mathfrak{L}_\sigma$ .
- (ii) For every neighborhood  $V$  of  $\tau$  there exists a divergent net  $(\xi_i)_{i \in I}$  in  $\Xi_{\rho(\sigma)}$  such that  $\xi_i \cdot \sigma \in V$  for any  $i \in I$ .
- (ii') For every neighborhood  $V$  of  $\tau$ , the recurrence set  $\tilde{\Xi}_\sigma^V$  is not relatively compact.
- (iii) There is a divergent net  $(\eta_i)_{i \in I}$  in  $\Xi_{\rho(\sigma)}$  such that  $\eta_i \cdot \sigma \rightarrow \tau$ .

*Proof.* (i)  $\Rightarrow$  (ii). Let  $V$  be a neighborhood of  $\tau$ . If  $\tau$  belongs to the limit set  $\mathfrak{L}_\sigma$ , then  $\tau \in \overline{(\Xi_{\rho(\sigma)} \setminus k) \cdot \sigma}$  for every  $k \in \mathcal{K}(\Xi_{\rho(\sigma)})$ . Hence we can choose elements  $\xi_k \in \Xi_{\rho(\sigma)} \setminus k$  such that  $\xi_k \cdot \sigma \in V$ . The net  $(\xi_k)_{k \in \mathcal{K}(\Xi_{\rho(\sigma)})}$  is obviously divergent.

(ii)  $\Rightarrow$  (iii). Consider the set  $\mathfrak{N}_\tau$  of neighborhoods of  $\tau$ , and order it by reversing the inclusions. For each neighborhood  $V$  of  $\tau$ , select some divergent net  $(\xi_{i,V})_{i \in I}$  such that  $\xi_{i,V} \cdot \sigma \in V$  (it can be built over the same labels). Observe that, for each  $k \in \mathcal{K}(\Xi_{\rho(\sigma)})$ , there exists  $i_k^V$  such that  $\xi_{i,V} \notin k$  for every  $i \geq i_k^V$ . Define  $\eta_{k,V} = \xi_{i_k^V,V}$ , which forms a net when  $\mathfrak{N}_\tau \times \mathcal{K}(\Xi_{\rho(\sigma)})$  is given the product order. By construction, we get a divergent net and  $\tau = \lim_{k,V} \eta_{k,V} \cdot \sigma$ .

(iii)  $\Rightarrow$  (i). Let  $k \in \mathcal{K}(\Xi_{\rho(\sigma)})$ . Since  $\eta_i$  is divergent, there exists  $i_k \in I$  such that  $\eta_i \in \Xi_{\rho(\sigma)} \setminus k$  for all  $i \geq i_k$ . So we have that

$$\tau = \lim_{i \in I} \eta_i \cdot \sigma = \lim_{i \geq i_k} \eta_i \cdot \sigma \in \overline{(\Xi_{\rho(\sigma)} \setminus k) \cdot \sigma}.$$

It follows that  $\tau \in \mathfrak{L}_\sigma$ .

(ii)  $\Leftrightarrow$  (ii') follows from the definitions. ■

The next easy lemma is sometimes useful to compute limit sets.

**Lemma 5.3.** *Suppose that there exists a family of compact sets  $\{k_\lambda\}_{\lambda \in \Lambda}$  that exhausts  $\Xi_{\rho(\sigma)}$ . That is,  $\Lambda$  is a directed set,  $\Xi_{\rho(\sigma)} = \bigcup_{\lambda \in \Lambda} k_\lambda$  and  $k_{\lambda_1} \subset k_{\lambda_2}^\circ$  whenever  $\lambda_1 \leq \lambda_2$ . Then the limit set  $\mathfrak{L}_\sigma$  can be computed as*

$$\mathfrak{L}_\sigma = \bigcap_{\lambda \in \Lambda} \overline{(\Xi_{\rho(\sigma)} \setminus k_\lambda) \cdot \sigma}.$$

*Proof.* Obviously,  $\mathfrak{L}_\sigma \subset \bigcap_{\lambda \in \Lambda} \overline{(\Xi_{\rho(\sigma)} \setminus k_\lambda) \cdot \sigma}$ . For the opposite inclusion, it is enough to find for every compact subset  $k$  of  $\Xi_\rho$  an index  $\mu \in \Lambda$  such that  $k \subset k_\mu$ . Indeed,  $k$  is covered by the family of interiors of the sets  $k_\lambda$ , so, by compactness, it is also covered by a finite subfamily  $\{k_{\lambda_1}^\circ, \dots, k_{\lambda_m}^\circ\}$ . Since  $\Lambda$  is directed, that index exists. So we have  $k \subset k_\mu$  implying that

$$\overline{(\Xi_{\rho(\sigma)} \setminus k_\mu) \cdot \sigma} \subset \overline{(\Xi_{\rho(\sigma)} \setminus k) \cdot \sigma}.$$

The conclusion follows. ■

**Example 5.4.** Consider the pair groupoid  $\Xi = \mathbb{Z}^2$  over  $\mathbb{Z}$ . In this case,  $\Xi_n = \mathbb{Z} \times \{n\}$  can be exhausted by the family  $\{-\lambda, \dots, \lambda\} \times \{n\}_{\lambda \in \mathbb{N}}$ , so, by Lemma 5.3,

$$\mathfrak{L}_n = \bigcap_{\lambda \in \mathbb{N}} \{k \in \mathbb{Z} \mid |k| > \lambda\} = \emptyset.$$

**Proposition 5.5.** (i) *All the points in the orbit of  $\sigma$  have the same limit set  $\mathfrak{L}_\sigma$ . One has*

$$\bar{\mathfrak{D}}_\sigma = \mathfrak{D}_\sigma \cup \mathfrak{L}_\sigma. \tag{5.1}$$

- (ii) *If  $\Xi$  is open and locally compact, the closed set  $\mathfrak{L}_\sigma$  is invariant.*
- (iii) *If the orbit of  $\sigma$  is relatively compact,  $\mathfrak{L}_\sigma$  is non-empty and it attracts the points of the orbit  $\mathfrak{D}_\sigma$ : for every neighborhood  $W$  of  $\mathfrak{L}_\sigma$ , there is a compact subset  $k$  of  $\Xi_{\rho(\sigma)}$  such that*

$$(\Xi_{\rho(\sigma)} \setminus k) \cdot \sigma \subset W. \tag{5.2}$$

*Proof.* (i) We observe that, for  $k \subset \Xi_{\rho(\xi \cdot \sigma)}$ , with rigorous computations based on the definitions

$$\begin{aligned} (\Xi_{\rho(\xi \cdot \sigma)} \setminus k) \cdot (\xi \cdot \sigma) &= [(\Xi_{\rho(\xi \cdot \sigma)} \setminus k)\xi] \cdot \sigma \\ &= (\Xi_{\rho(\xi \cdot \sigma)}\xi \setminus k\xi) \cdot \sigma = (\Xi_{\rho(\sigma)} \setminus k\xi) \cdot \sigma, \end{aligned}$$

implying that  $\mathfrak{L}_{\xi \cdot \sigma} \subset \mathfrak{L}_\sigma$  (because  $k\xi$  is compact), from which the first statement follows. The  $\supset$  inclusion in (5.1) is obvious. For each  $K \in \mathcal{K}(\Xi)$  one can write

$$\bar{\mathfrak{D}}_\sigma = \overline{\Xi \cdot \sigma} = \overline{(K \cup K^c) \cdot \sigma} = (K \cdot \sigma) \cup \overline{K^c \cdot \sigma} \subset \mathfrak{D}_\sigma \cup \overline{K^c \cdot \sigma},$$

from which  $\subset$  follows.

(ii) Let  $(\xi, \tau) \in \Xi \bowtie \Sigma$  with  $\tau \in \mathfrak{L}_\sigma$ . Using the convergence criterion (iii) of Lemma 5.2, we see that there exists some divergent net  $(\eta_i)_{i \in I}$  in  $\Xi_{\rho(\sigma)}$  such that  $\eta_i \cdot \sigma \rightarrow \tau$ . A priori, there is no reason to think that  $(\xi, \eta_i) \in \Xi^{(2)}$ . But because of the openness of  $\Xi$  and Fell’s criterion [19, Prop. 1.1] applied to the source map, there exist a net  $(\xi_j)_{j \in J}$  and a subnet  $(\eta_{i_j})_{j \in J}$  such that  $d(\xi_j) = r(\eta_{i_j})$  and  $\xi_j \rightarrow \xi$ , so one may write

$$\xi \cdot \tau = \lim_j \xi_j \cdot (\eta_{i_j} \cdot \sigma) = \lim_j (\xi_j \eta_{i_j}) \cdot \sigma.$$

Since  $\Xi$  is locally compact,  $(\xi_j \eta_{i_j})_{j \in J}$  is also divergent.

(iii) It is enough to show that, for a fixed open neighborhood  $W$  of  $\mathfrak{L}_\sigma$ , there is a compact subset  $k$  of  $\Xi_{\rho(\sigma)}$  such that (5.2) holds: if  $\mathfrak{L}_\sigma$  were void, the empty set would be a neighborhood, which contradicts the inclusion ( $\Xi$  is strongly non-compact). For any  $k \in \mathcal{K}(\Xi_{\rho(\sigma)})$ , we set

$$\Sigma_\sigma(k) := \overline{(\Xi_{\rho(\sigma)} \setminus k) \cdot \sigma};$$

complements will refer to  $\bar{\mathfrak{D}}_\sigma$ . Since  $\mathfrak{L}_\sigma \subset W \cap \bar{\mathfrak{D}}_\sigma$ , the family  $\{\Sigma_\sigma(k)^c \mid k \in \mathcal{K}(\Xi_{\rho(\sigma)})\}$  is an open cover of the complement of  $W \cap \bar{\mathfrak{D}}_\sigma$  in the compact space  $\bar{\mathfrak{D}}_\sigma$ . We extract a

finite subcover  $\{\Sigma_\sigma(k_i)^c \mid i = 1, \dots, n\}$ . Then

$$\left(\Xi_{\rho(\sigma)} \setminus \bigcup_{i=1}^n k_i\right) \cdot \sigma \subset \bigcap_{i=1}^n \overline{(\Xi_{\rho(\sigma)} \setminus k_i) \cdot \sigma} \subset W \cap \bar{\mathfrak{D}}_\sigma \subset W$$

and the proof is finished, since a finite union of compact sets is compact. ■

**Example 5.6.** We indicate now an example that is relevant for the problem of invariance of the limit sets. Let  $X$  be a topological space,  $\mathcal{G}$  a non-compact group and  $\Pi \subset X^2$  an equivalence relation. The obvious product groupoid  $\Xi = \Pi \times \mathcal{G}$  has unit space

$$\Sigma \equiv \Xi^{(0)} = \text{Diag}(X) \times \{e\} \cong X.$$

The d-fiber of  $x$  is  $\Xi_x = \{(y, x) \in \Pi\} \times \mathcal{G}$ .

We prove now that  $\mathfrak{L}_x = \bar{\mathfrak{D}}_x$ . Let  $y \in \bar{\mathfrak{D}}_x$  and let  $(x_i)_{i \in I} \subset \mathfrak{D}_x$  be a net converging to  $y$ . As  $\mathcal{G}$  is not compact, for every  $k \in \mathcal{K}(\mathcal{G})$  there exists some  $g_k \in \mathcal{G} \setminus k$ . The net  $(x_i, x, g_k)_{(i,k) \in I \times \mathcal{K}(\mathcal{G})}$  belongs to  $\Xi_x$ , is divergent and fulfills

$$\lim_{i,k} (x_i, x, g_k) \circ x = \lim_i x_i = y.$$

Therefore  $\mathfrak{L}_x = \bar{\mathfrak{D}}_x$ , because of Lemma 5.2 and equation (5.1).

In particular, one can choose  $X = \mathbb{R}$  and  $\mathcal{G} = \mathbb{Z}$ . If  $\Pi$  is given by

$$x \Pi y \Leftrightarrow x, y \leq 0 \text{ or } x, y > 0,$$

one has  $\mathfrak{L}_1 = \bar{\mathfrak{D}}_1 = [0, \infty)$ , which is not invariant! Note that the equivalence relation is not open.

### 5.2. Recurrent points and wandering

We keep the framework of the preceding subsection.

**Definition 5.7.** When  $\sigma \in \mathfrak{L}_\sigma$  holds, we say that  $\sigma$  is a *recurrent point*. We denote by  $\Sigma_{\text{rec}}^\theta \equiv \Sigma_{\text{rec}}$  the family of all recurrent points of the groupoid action.

In (5.1) the union could be disjoint or not.

**Proposition 5.8.** For a point  $\sigma \in \Sigma$ , consider the following five conditions:

- (a)  $\mathfrak{D}_\sigma \cap \mathfrak{L}_\sigma \neq \emptyset$ ,
- (b)  $\mathfrak{L}_\sigma = \bar{\mathfrak{D}}_\sigma$ ,
- (c)  $\sigma \in \mathfrak{L}_\sigma$ ,
- (d) there is a divergent net  $(\xi_i)_{i \in I} \subset \Xi_{\rho(\sigma)}$  such that  $\xi_i \cdot \sigma \rightarrow \sigma$ ,
- (e)  $\tilde{\Xi}_\sigma^U$  is not relatively compact for any open neighborhood  $U$  of  $\sigma$ .

Then (b)  $\Rightarrow$  (c)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e)  $\Rightarrow$  (a). If  $\Xi$  is open and locally compact, the five conditions are equivalent.

*Proof.* The equivalence (c)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e) is the content of Lemma 5.2 for  $\tau = \sigma$ .

(b)  $\Rightarrow$  (c) and (c)  $\Rightarrow$  (a) are obvious.

Assume now that  $\Xi$  is open and locally compact. We will show that (a)  $\Rightarrow$  (b). We know from Proposition 5.5 (ii) that  $\mathcal{L}_\sigma$  is (closed and) invariant, so if (a) holds, it contains the closure of the orbit  $\mathcal{D}_\sigma$ . But it cannot be strictly bigger, by (5.1). ■

**Corollary 5.9.** *If  $\Xi$  is open and locally compact,  $\Sigma_{\text{rec}}$  is invariant.*

*Proof.* Since  $\Xi$  is open and locally compact, we can describe its fellowship to  $\mathcal{L}_\sigma$  by condition (b). Using Proposition 5.5 (ii) for a recurrent point  $\sigma$  and for  $\xi \in \Xi_{\rho(\sigma)}$ , we can write

$$\mathcal{L}_{\xi \bullet \sigma} = \mathcal{L}_\sigma = \bar{\mathcal{D}}_\sigma = \bar{\mathcal{D}}_{\xi \bullet \sigma},$$

so  $\xi \bullet \sigma$  is also recurrent. ■

**Definition 5.10.** The point  $\sigma \in \Sigma$  is *wandering* with respect to the action  $(\Xi, \rho, \theta, \Sigma)$  with strongly non-compact groupoid if  $\sigma$  has a neighborhood  $W$  such that  $\tilde{\Xi}_W^W$  is relatively compact. In the opposite case, we say that  $\sigma$  is *non-wandering*. We denote the family of all non-wandering points by  $\Sigma_{\text{nw}}^\theta \equiv \Sigma_{\text{nw}}$ . If  $\Sigma_{\text{nw}}^\theta = \Sigma$ , one says that *the action is non-wandering*.

**Example 5.11.** If  $(\mathcal{G}, \gamma, \Sigma)$  is a topological group dynamical system, one says that  $\sigma \in \Sigma$  is wandering if  $\text{Rec}_\gamma(W, W)$  is relatively compact for some neighborhood  $W$  of  $\sigma$  (actually, this is mainly used for  $\mathcal{G} = \mathbb{Z}, \mathbb{R}$ ). The discussion in Example 3.3 shows that by Definition 5.10 one gets the same concept.

**Remark 5.12.** Suppose that both  $\Xi$  and  $\Sigma$  are (Hausdorff and) locally compact. In this case, wandering points are involved, with a different language, in [19, Sect. 2.1], extending the group case of [15]. One says that the action is *proper* if the map

$$\Xi \rtimes \Sigma \ni (\xi, \sigma) \xrightarrow{\vartheta} (\xi \bullet \sigma, \sigma) \in \Sigma \times \Sigma$$

is proper, i.e. that inverse images of compact sets are compact. A strictly weaker concept is that of a *Cartan action*, for which each point  $\sigma \in \Sigma$  has a compact neighborhood  $W$  such that  $\vartheta^{-1}(W \times W)$  is compact. In [19, Lem. 2.23], it is shown that this happens precisely when all the points of  $\Sigma$  are wandering (without mentioning the term). Consequently, in the locally compact case, for a Cartan action one has  $\Sigma_{\text{nw}} = \emptyset$ . In addition, [19, Ex. 2.1.19] states that *for a Cartan action all orbits are closed*.

**Proposition 5.13.** *The set  $\Sigma_{\text{nw}}$  is closed. If the groupoid  $\Xi$  is open and locally compact,  $\Sigma_{\text{nw}}$  is also invariant.*

*Proof.* Let  $\tau \in \bar{\Sigma}_{\text{nw}}$ . By definition of the closure, every open neighborhood  $U$  of  $\tau$  intersects  $\Sigma_{\text{nw}}$ , so let  $\sigma \in \Sigma_{\text{nw}} \cap U$ . As  $U$  is a neighborhood of  $\sigma \in \Sigma_{\text{nw}}$ , the set  $\tilde{\Xi}_U^U$  is not relatively compact. Thus  $\tau \in \Sigma_{\text{nw}}$ .



Suppose that  $\Xi$  is open and locally compact and let  $\sigma \notin \Sigma_{\text{nw}}$ ; we are going to show that  $\eta \cdot \sigma \notin \Sigma_{\text{nw}}$ , for all  $\eta \in \Xi_{\rho(\sigma)}$ . Let  $N_\eta$  be a relatively compact open neighborhood of  $\eta$  and let  $W$  be an open neighborhood of  $\sigma$  such that  $\tilde{\Xi}_W^W$  is relatively compact. Since  $\Xi$  is open,  $N_\eta \cdot W$  is an open neighborhood of  $\eta \cdot \sigma$ . A plain application of the definitions leads to

$$\tilde{\Xi}_{N_\eta \cdot W}^{N_\eta \cdot W} \subset N_\eta \tilde{\Xi}_W^W N_\eta^{-1},$$

which is relatively compact. ■

**Example 5.14.** The invariance may fail without openness. Let  $\Xi$  be the groupoid constructed in the final part of Example 5.6, the one associated to the partition  $\mathbb{R} = (-\infty, 0] \cup (0, \infty)$  and to the group  $\mathcal{G} = \mathbb{Z}$ . Then consider the wide (locally compact) subgroupoid

$$\Delta := \{(x, y, n) \in \Xi \mid x, y > 0 \text{ or } n = 0\}.$$

If we denote by  $\theta_1$  and  $\theta_2$  the canonical actions of  $\Xi$  and  $\Delta$  in  $\Sigma = \mathbb{R}$ , respectively, we have

$$\Sigma_{\text{nw}}^{\theta_1} = \mathbb{R} \quad \text{and} \quad \Sigma_{\text{nw}}^{\theta_2} = [0, \infty). \tag{5.3}$$

To check the second equality, note that

$$\tilde{\Delta}_W^W = \begin{cases} W \times W \times \{0\} & \text{if } W \subset (-\infty, 0), \\ W \times W \times \mathbb{Z} & \text{if } W \subset (0, \infty). \end{cases}$$

The second set in (5.3) is not invariant, since 0 is orbit-equivalent with any negative number. In addition, it follows easily that

$$\mathfrak{L}_x^{\theta_2} = \begin{cases} [0, \infty) & \text{if } x > 0, \\ \emptyset & \text{if } x \leq 0. \end{cases}$$

So  $\Sigma_{\text{rec}}^{\theta_2} = (0, \infty)$ , which is *not closed*.

**Proposition 5.15.** *One has*

$$\Sigma_{\text{rec}} \subset \bigcup_{\sigma \in \Sigma} \mathfrak{L}_\sigma \subset \Sigma_{\text{nw}}. \tag{5.4}$$

*Proof.* The first inclusion follows from the definition of recurrent points. So we only need to prove that  $\mathfrak{L}_\sigma \subset \Sigma_{\text{nw}}$ . Pick  $\tau \in \mathfrak{L}_\sigma$  and let  $U$  be a neighborhood of  $\tau$ . By Lemma 5.2, we already know that  $\tilde{\Xi}_\sigma^U$  is not relatively compact. If  $\xi \in \tilde{\Xi}_\sigma^U$ , then  $\xi \cdot \sigma \in U$ ; using Lemma 2.13, we get

$$\tilde{\Xi}_\sigma^U \xi^{-1} = \tilde{\Xi}_{\xi \cdot \sigma}^U \subset \tilde{\Xi}_U^U,$$

showing that the latter set is not relatively compact. (Notice that  $d(\tilde{\Xi}_\sigma^U) = \{r(\xi^{-1})\}$ , hence  $\tilde{\Xi}_\sigma^U \xi^{-1}$  is relatively compact if and only if  $\tilde{\Xi}_\sigma^U$  is relatively compact.) ■

**Corollary 5.16.** *If at least one of the orbits is relatively compact (in particular, if  $\Sigma$  is compact),  $\Sigma_{\text{nw}}$  is non-void.*

*Proof.* This follows from (5.4) and Proposition 5.5 (iii). ■

**Proposition 5.17.** *If  $\Sigma$  is compact,  $\Sigma_{\text{nw}}$  attracts the points of  $\Sigma$ : for every  $\sigma \in \Sigma$  and for every neighborhood  $V$  of  $\Sigma_{\text{nw}}$ , one has  $\xi \cdot \sigma \in V$  for every  $\xi \in \Xi_{\rho(\sigma)}$  outside some compact set.*

*Proof.* One has to show that for every neighborhood  $V$  of  $\Sigma_{\text{nw}}$ , the complement in  $\Xi_{\rho(\sigma)}$  of the set  $\tilde{\Xi}_{\sigma}^V$  is relatively compact. If  $V$  is a neighborhood of  $\Sigma_{\text{nw}}$ , by (5.4), it is also a neighborhood of the limit set  $\mathfrak{L}_{\sigma}$ . One applies Proposition 5.5 (iii) to infer that there exists a compact subset  $k$  of  $\Xi_{\rho(\sigma)}$  such that  $(\Xi_{\rho(\sigma)} \setminus k) \cdot \sigma \subset V$ , i. e.  $\Xi_{\rho(\sigma)} \setminus k \subset \tilde{\Xi}_{\sigma}^V$ . Then the complement of  $\tilde{\Xi}_{\sigma}^V$  in  $\Xi_{\rho(\sigma)}$  is contained in  $k$  and the proof is finished. ■

**Example 5.18.** If  $\Xi \rtimes_{\theta} \Sigma$  is the action groupoid of the groupoid action  $(\Xi, \theta, \rho, \Sigma)$ , for every  $\sigma \in \Sigma$  the two limit sets that make sense are equal. The recurrent sets are also equal. These are easy consequences of the definitions. Formula (3.12) shows that if  $\sigma$  is non-wandering for the action  $\theta$ , then it is also non-wandering for the action groupoid  $\Xi \rtimes_{\theta} \Sigma$ . If  $\Sigma$  is compact, the non-wandering sets coincide.

**Example 5.19.** We refer now to the pullback construction of Section 3.7, assuming that  $\Omega$  is locally compact. We leave it to the reader to check the formula

$$\mathfrak{L}_{\omega}^{\downarrow\downarrow} = h^{-1}(\mathfrak{L}_{h(\omega)}) \quad \text{for all } \omega \in \Omega,$$

where the limit set in the left-hand side is computed in  $h^{\downarrow\downarrow}(\Xi)$ , being a subset of  $\Omega$ . It follows easily that  $\omega \in \Omega$  is  $h^{\downarrow\downarrow}(\Xi)$ -recurrent if and only if  $h(\omega) \in X$  is  $\Xi$ -recurrent. Using (3.13), one checks easily that  $\omega$  is  $h^{\downarrow\downarrow}(\Xi)$ -wandering if and only if  $h(\omega)$  is  $\Xi$ -wandering.

## 6. Minimality and almost periodicity

### 6.1. Fixed points

Let  $(\Xi, \rho, \theta, \Sigma)$  be a continuous groupoid action.

**Definition 6.1.** A *fixed point* is a point  $\sigma \in \Sigma$  such that  $\xi \cdot \sigma = \sigma$  for every  $\xi \in \Xi_{\rho(\sigma)}$ . This is equivalent to  $\tilde{\Xi}_{\sigma}^{\sigma} = \Xi_{\rho(\sigma)}$ . We write  $\sigma \in \Sigma_{\text{fix}}^{\theta} \equiv \Sigma_{\text{fix}}$ .

**Example 6.2.** For group actions, converted into groupoid actions as in Example 3.3, the notion of fixed point boils down to the usual one. The same can be said if the model is that of Example 3.2.

**Example 6.3.** If  $\underline{\Sigma} \subset \Sigma$  is an open set, then  $\underline{\Sigma}_{\text{fix}}^{\theta} \supset \underline{\Sigma} \cap \Sigma_{\text{fix}}^{\theta}$ . We used the restriction notions from Section 3.4.

**Example 6.4.** The point  $y \in Y$  is said to be a *fixed point of the partial group action* of Definition 3.4 if  $\beta_a(y) = y$  for  $a \in \mathcal{G}$  and  $y \in Y_{a^{-1}}$ . This happens if and only if  $y$  is a fixed point with respect to the canonical action of the groupoid  $\mathcal{G} \rtimes_{[\beta]} Y$ .

**Example 6.5.** Let  $\Xi(v)$  be the Deaconu–Renault groupoid attached to the local homeomorphism  $v : X \rightarrow X$ . The unit  $y$  is a fixed point if and only if for every  $x \neq y$  there are no positive integers  $k, l$  with  $v^k(x) = v^l(y)$ .

**Example 6.6.** An element  $\omega \in \Omega$  is a fixed point of the pullback groupoid  $h^{\downarrow\downarrow}(\Xi)$  introduced in Section 3.7 if and only if  $h(\omega)$  is a fixed point of the canonical action of  $\Xi$  on its unit space.

**Proposition 6.7.** *The set  $\Sigma_{\text{fix}}$  is invariant. If  $\Xi$  is open,  $\Sigma_{\text{fix}}$  is also closed.*

*Proof.* If  $\sigma \in \Sigma_{\text{fix}}$ , then  $\mathfrak{D}_\sigma = \{\sigma\}$ , which makes  $\Sigma_{\text{fix}}$  trivially invariant.

If  $\Xi$  is open and  $\sigma \in \overline{\Sigma_{\text{fix}}}$ , then there exists a net  $(\sigma_i)_{i \in I}$  of fixed points converging to  $\sigma$ . Let  $\xi \in \Xi_{\rho(\sigma)}$ . By Fell’s criterion [19, Prop. 1.1], applied to the open map  $d$  and the net  $\rho(\sigma_i) \in X$ , there exist a net  $(\xi_j)$  and a subnet  $(\sigma_{i_j})$  such that  $\xi_j \rightarrow \xi$  and  $\rho(\sigma_{i_j}) = d(\xi_j)$ . By continuity, we have

$$\xi \cdot \sigma = (\lim_j \xi_j) \cdot (\lim_j \sigma_{i_j}) = \lim_j (\xi_j \cdot \sigma_{i_j}) = \lim_j \sigma_{i_j} = \sigma.$$

So  $\sigma$  is a fixed point for the action. ■

Examples 6.9 and 7.9 will illustrate that openness of  $\Xi$  is important.

### 6.2. Periodic and almost periodic points

Although it is not always necessary, in this subsection we prefer to assume that in the continuous action  $(\Xi, \rho, \theta, \Sigma)$  the groupoid  $\Xi$  is strongly non-compact (all the  $d$ -fibers are non-compact).

**Definition 6.8.** (a) Let  $x \in X$ . The subset  $A$  of  $\Xi_x$  is called *syndetic* if  $KA = \Xi_x$  for a compact subset  $K$  of  $\Xi$ .

(b) We say that  $\sigma \in \Sigma$  is *periodic* (we write  $\sigma \in \Sigma_{\text{per}}$ ), if  $\tilde{\Xi}_\sigma^\sigma$  is syndetic in  $\Xi_{\rho(\sigma)}$ .

(c) The point  $\sigma$  is called *weakly periodic* (we write  $\sigma \in \Sigma_{\text{wper}}$ ), if the subgroup  $\tilde{\Xi}_\sigma^\sigma$  is not compact.

(d) The point  $\sigma \in \Sigma$  is said to be *almost periodic* (we write  $\sigma \in \Sigma_{\text{alper}}$ ), if  $\tilde{\Xi}_\sigma^U$  is syndetic in  $\Xi_{\rho(\sigma)}$  for every neighborhood  $U$  of  $\sigma$  in  $\Sigma$ . If  $\Sigma_{\text{alper}} = \Sigma$ , the action is *pointwise almost periodic*.

**Example 6.9.** Consider the equivalence relation  $\Pi$  introduced in Remark 3.1, and form the groupoid product  $\Xi = \Pi \times \mathbb{Z}$ . By considering the action of  $\Xi$  in  $\Sigma = \mathbb{R}$ , we see that

$$\Sigma_{\text{fix}} = (-\infty, -1) \cup (1, \infty), \quad \Sigma_{\text{per}} = \Sigma_{\text{wper}} = \Sigma_{\text{alper}} = \mathbb{R},$$

which also illustrates that, in general, the set  $\Sigma_{\text{fix}}$  is not closed.

**Example 6.10.** We fix a unit  $y \in X$  of the Deaconu–Renault groupoid of the local homeomorphism  $v : X \rightarrow X$ . We remarked that (3.4) becomes a bijection for singletons. Then

$y$  is periodic if and only if it is weakly periodic, and this happens exactly when  $\mathbb{Z}_y(x, x)$  (a subgroup of  $\mathbb{Z}$ ) does not coincide with  $\{0\}$  (then it will be both infinite and syndetic). This means that for some positive integers  $k \neq l$  one has  $v^k(x) = v^l(x)$ .

**Example 6.11.** Let  $\Xi[\beta]$  be the groupoid associated to the partial action  $\beta$  of the discrete group  $\mathcal{G}$  on the topological space  $Y$ . After (3.8), we established an identification between  $\Xi[\beta]_y^N$  and  $\text{Rec}_{[\beta]}(y, N)$  for every point  $y \in Y$  and subset  $N \subset Y$ . This allows rephrasing the periodicity properties in this case. For example,  $y$  is periodic in the groupoid  $\Xi[\beta]$  if and only if the subgroup of  $\mathcal{G}$  composed of all the elements “defined at  $y$  and leaving it invariant” is syndetic (which is equivalent to having finite index in  $\mathcal{G}$ ).

**Proposition 6.12.** *Let  $\Xi$  be a strongly non-compact groupoid. Then one has*

$$\Sigma_{\text{fix}} \stackrel{(1)}{\subset} \Sigma_{\text{per}} \stackrel{(2)}{\subset} \Sigma_{\text{wper}} \cap \Sigma_{\text{alper}} \stackrel{(3)}{\subset} \Sigma_{\text{wper}} \cup \Sigma_{\text{alper}} \stackrel{(4)}{\subset} \Sigma_{\text{rec}} \stackrel{(5)}{\subset} \bigcup_{\sigma \in \Sigma} \mathcal{L}_\sigma \stackrel{(6)}{\subset} \Sigma_{\text{nw}}. \tag{6.1}$$

*Proof.* The inclusions (1) and (3) are obvious. We proved (5) and (6) previously, in Proposition 5.15.

To deduce (2) from the definitions, note that  $\tilde{\Xi}_\sigma^\sigma \subset \tilde{\Xi}_\sigma^U$  if  $\sigma \subset U$  and that a syndetic set is not compact (since the  $d$ -fibers are not compact).

The inclusion (4) also follows easily from the definitions, by the same type of arguments: use Proposition 5.8 (e) to describe recurrent points. ■

**Example 6.13.** The situation is particularly simple for equivalence relations, outlined in Section 3.1, especially because of equation (3.1). In particular, for  $y \in X \equiv \Sigma$  one gets

$$\tilde{\Xi}_y \cong \{x \in X \mid x \Pi y\} \quad \text{and} \quad \tilde{\Xi}_y^y = \{(y, y)\} \cong \{y\}.$$

To insure that the associated groupoid is strongly non-compact, we require that for every  $y \in X$  the set  $\{x \in X \mid x \Pi y\}$  is non-compact. Then  $\Sigma_{\text{wper}} = \emptyset$  (so there are no fixed points or periodic points). With some abuse of notation and interpretation,  $y$  will be almost periodic if and only if  $\{x \in U \mid x \Pi y\}$  is syndetic in  $\{x \in X \mid x \Pi y\}$  for every neighborhood  $U$  of  $y$ . If  $X$  is locally compact, one could choose a relatively compact neighborhood, and syndeticity contradicts the fact that the fiber in  $y$  is non-compact. Therefore, in the locally compact strongly non-compact case, equivalence relations do not exhibit almost periodic points.

If the equivalence relation is not forced to lead to a strongly non-compact groupoid, the situation might be very different. In particular, some of the inclusions in (6.1) no longer hold. We leave this to the reader.

**Example 6.14.** Denoting by  $\beta$  any of the properties “periodic”, “weakly periodic” and “almost periodic”, it is easy to check that  $\omega \in \Omega$  has  $\beta$  in the pullback  $h^{\downarrow\downarrow}(\Xi)$  if and only if  $h(\omega) \in X$  has  $\beta$  in  $\Xi$ . This happens mostly because of equality (3.13).

**Proposition 6.15.** *The sets  $\Sigma_{\text{per}}$  and  $\Sigma_{\text{wper}}$  are invariant.*

*Proof.* By Lemma 2.13, one has

$$\tilde{\Xi}_{\xi \bullet \sigma}^{\xi \bullet \sigma} = \xi \tilde{\Xi}_{\sigma}^{\sigma} \xi^{-1},$$

so  $\Sigma_{\text{wper}}$  is invariant. We focus now on  $\Sigma_{\text{per}}$ ; let  $\sigma \in \Sigma_{\text{per}}$  and  $\xi \in \Xi_{\rho(\sigma)}$ . For some compact set  $K$ , we have

$$K\xi^{-1} \tilde{\Xi}_{\xi \bullet \sigma}^{\xi \bullet \sigma} \xi = K \tilde{\Xi}_{\sigma}^{\sigma} = \Xi_{\rho(\sigma)} = \Xi_{\rho(\xi \bullet \sigma)} \xi.$$

Hence

$$(K\xi^{-1}) \tilde{\Xi}_{\xi \bullet \sigma}^{\xi \bullet \sigma} = \Xi_{\rho(\xi \bullet \sigma)},$$

meaning that  $\tilde{\Xi}_{\xi \bullet \sigma}^{\xi \bullet \sigma}$  is syndetic in  $\Xi_{\rho(\xi \bullet \sigma)}$ , and thus  $\xi \bullet \sigma \in \Sigma_{\text{per}}$  holds. ■

The sets of periodic, weakly periodic or almost periodic points might fail to be closed, even for group actions.

### 6.3. Compact orbits

We connect now periodicity with the notion of a periodicoid point, introduced in [3, Definition 12] in a more restricted context.

**Proposition 6.16.** *Every periodic point  $\sigma$  has a compact orbit.*

*First proof.* If  $\sigma \in \Sigma_{\text{per}}$ , then  $K\tilde{\Xi}_{\sigma}^{\sigma} = \Xi_{\rho(\sigma)}$  for some compact set  $K \subset \Xi$ . Any net  $(\sigma_i) \subset \mathcal{D}_{\sigma}$  can be written as  $\sigma_i = (\xi'_i \xi_i) \bullet \sigma$  for some nets  $(\xi'_i) \subset K$ ,  $(\xi_i) \subset \tilde{\Xi}_{\sigma}^{\sigma}$ . By compactness, extract a subnet  $(\xi'_j) \subset (\xi'_i)$  such that  $\xi'_j \rightarrow \xi' \in K$  and get

$$\sigma_j := (\xi'_j \xi_j) \bullet \sigma = \xi'_j \bullet (\xi_j \bullet \sigma) = \xi'_j \bullet \sigma \rightarrow \xi \bullet \sigma \in \mathcal{D}_{\sigma}.$$

We conclude that  $\mathcal{D}_{\sigma}$  is compact. ■

*Second proof.* Let us give a second proof, using a construction that will also be useful for Proposition 6.18. For  $\sigma \in \Sigma$ , let us define the continuous surjective function

$$\alpha^{\sigma} : \Xi_{\rho(\sigma)} \rightarrow \mathcal{D}_{\sigma} \subset \Sigma, \quad \alpha^{\sigma}(\xi) := \theta_{\xi}(\sigma) \equiv \xi \bullet \sigma.$$

If  $\sigma \in \Sigma_{\text{per}}$ , then  $K\tilde{\Xi}_{\sigma}^{\sigma} = \Xi_{\rho(\sigma)}$  for some compact set  $K \subset \Xi$ . By using the definition of  $\tilde{\Xi}_{\sigma}^{\sigma}$ , one gets

$$\mathcal{D}_{\sigma} = \alpha^{\sigma}(\Xi_{\rho(\sigma)}) = \alpha^{\sigma}(K),$$

which is compact, as a direct continuous image of a compact set. ■

To derive a converse proposition, we will use a well-known lemma (with proof, for the convenience of the reader):

**Lemma 6.17.** *Let  $\Xi = \mathcal{G}$  be a locally compact, second countable group acting on a topological space  $\Sigma$ . If  $\sigma \in \Sigma$  has a compact orbit, then  $\tilde{\mathcal{G}}_{\sigma}^{\sigma} \equiv \text{Rec}(\sigma, \sigma)$  (see Section 3.2) is syndetic in  $\mathcal{G}$ .*

*Proof.* Let  $N_g \subset \mathcal{G}$  be a relatively compact, open neighborhood of  $g$ , for every  $g \in \mathcal{G}$ . Observe that

$$\mathcal{G} = \mathcal{G} \tilde{\mathcal{G}}_\sigma^\sigma = \left( \bigcup_{g \in \mathcal{G}} N_g \right) \tilde{\mathcal{G}}_\sigma^\sigma.$$

So

$$\mathfrak{D}_\sigma = \mathcal{G} \cdot \sigma = \left( \bigcup_{g \in \mathcal{G}} N_g \right) \tilde{\mathcal{G}}_\sigma^\sigma \cdot \sigma = \bigcup_{g \in \mathcal{G}} N_g \cdot \sigma.$$

Let us, for the sake of the argument, use (and prove it later) that the set  $N_g \cdot \sigma$  is a neighborhood of  $\sigma$ . By compactness we can extract a finite index set  $F = \{g_1, \dots, g_n\}$  such that

$$\mathfrak{D}_\sigma = \bigcup_{i=1}^n N_{g_i} \cdot \sigma \subset \left( \bigcup_{i=1}^n \bar{N}_{g_i} \right) \cdot \sigma.$$

Define  $K = \bigcup_{i=1}^n \bar{N}_{g_i}$  and notice that for every  $g \cdot \sigma \in \mathfrak{D}_\sigma$ , there exists  $h \in K$  such that

$$g \cdot \sigma = h \cdot \sigma \Rightarrow (h^{-1}g) \cdot \sigma = \sigma,$$

meaning that  $h^{-1}g \in \tilde{\mathcal{G}}_\sigma^\sigma$  and  $g \in K \tilde{\mathcal{G}}_\sigma^\sigma$ . As  $K$  is compact, we conclude that  $\tilde{\mathcal{G}}_\sigma^\sigma$  is syndetic in  $\mathcal{G}$ .

Now, we fill the remaining gap: Pick  $W \subset \mathcal{G}$ , another relatively compact, open neighborhood of  $g$  but satisfying  $\bar{W} \subset N_g$ . As  $\mathcal{G}$  is second countable, it has the Lindelöf property, so there exists a countable subset  $C \subset \mathcal{G}$  making  $\mathcal{G} = CW$  true. If  $N_g \cdot \sigma$  had void interior, so would  $c \cdot [W \cdot \sigma] = [cW] \cdot \sigma$ . But then we could decompose  $\mathfrak{D}_\sigma$  as a countable union of closed nowhere dense sets:

$$\mathfrak{D}_\sigma = \mathcal{G} \cdot \sigma = CW \cdot \sigma = \bigcup_{c \in C} [c\bar{W}] \cdot \sigma,$$

contradicting Baire’s category theorem. ■

**Proposition 6.18.** *Assume that  $\mathfrak{E}$  is locally compact, second countable and open. If  $\sigma \in \Sigma$  has a compact orbit, then it is a periodic point.*

*Proof.* Let us first treat the case of the canonical action of  $\mathfrak{E}$  in its unit space  $X$  (and notice that  $X$ , being closed in  $\mathfrak{E}$ , is a locally compact, second countable space by its own). We will use a small amount of information from [19, Sect. 22], treating the Mackey–Glimm–Ramsay dichotomy for groupoids; see also [17]. For  $x \in X$ , let us define the continuous surjective function

$$\alpha^x : \mathfrak{E}_x \rightarrow \mathcal{O}_x \subset X, \quad \alpha^x(\xi) := \xi \circ x.$$

One has  $\alpha^x(\xi) = \alpha^x(\eta)$  if and only if  $\xi^{-1}\eta \in \mathfrak{E}_x^x$ . This leads to a continuous bijection

$$\tilde{\alpha}^x : \mathfrak{E}_x / \mathfrak{E}_x^x \rightarrow \mathcal{O}_x. \tag{6.2}$$

The quotient map  $p : \mathfrak{E}_x \rightarrow \mathfrak{E}_x / \mathfrak{E}_x^x$  is (surjective, continuous and) open, cf. [19, Ex. 2.2.1]. Then  $\tilde{\alpha}^x$  is a homeomorphism if and only if  $\alpha^x$  is an open function. By [19] (see

the non-trivial implication (2)  $\Rightarrow$  (e) on pages 41–42), this happens if the orbit  $\mathcal{O}_x$  is Baire. In our case this is insured, since it is (Hausdorff and) compact. So we conclude that (6.2) is a homeomorphism and thus  $\Xi_x/\tilde{\Xi}_x^x$  is compact.

To finish this part of the proof, we show now that the compactness of the quotient implies that  $\Xi_x^x$  is syndetic in  $\Xi_x$ . Let  $\{V_i \mid i \in I\}$  be a covering of the locally compact space  $\Xi_x$  by open subsets with compact closures. Since  $p$  is open and surjective,  $\{W_i := p(V_i) \mid i \in I\}$  will be an open covering of the compact space  $\Xi_x/\Xi_x^x$ , from which we extract a finite subcovering  $\{W_i := p(V_i) \mid i \in I_0\}$ . Then  $V := \bigcup_{i \in I_0} V_i$  has a compact closure  $K$  such that  $p(K) = \Xi_x/\Xi_x^x$ . To check that  $K\Xi_x^x = \Xi_x$ , pick  $\xi \in \Xi_x$ . Then  $p(\xi) = p(\eta)$  for some  $\eta \in K$ . By the definition of  $p$ , this means  $\xi \in \eta\Xi_x^x$  and we are done.

Now, we will derive the full result: Suppose that  $\Xi$  acts on a very general space  $\Sigma$  and that  $\mathfrak{D}_\sigma \subset \Sigma$  is compact, for some  $\sigma \in \Sigma$ . As the anchor map is continuous,  $\rho(\mathfrak{D}_\sigma) = \mathcal{O}_{\rho(\sigma)} \subset X$  (Lemma 2.8) is compact and by the previous discussion, the decomposition  $\Xi_{\rho(\sigma)} = K_1\Xi_{\rho(\sigma)}^{\rho(\sigma)}$  holds for some compact set  $K_1 \subset \Xi$ . Remark that  $\mathcal{G} = \Xi_{\rho(\sigma)}^{\rho(\sigma)}$  is a (locally compact) group acting continuously on the compact space

$$\mathfrak{D}_\sigma \cap \{\tau \in \Sigma \mid \rho(\tau) = \rho(\sigma)\}.$$

By applying Lemma 6.17 (with a change of notations), we obtain another compact set  $K_2 \subset \Xi$  such that

$$\Xi_{\rho(\sigma)} = K_1\Xi_{\rho(\sigma)}^{\rho(\sigma)} = K_1(K_2\tilde{\Xi}_\sigma^\sigma) = (K_1K_2)\tilde{\Xi}_\sigma^\sigma,$$

finishing the proof. ■

There is a shorter proof, in only one step and avoiding the use of Lemma 6.16, but it requires  $\Sigma$  to be locally compact and second countable, which we succeeded to avoid.

### 6.4. Minimal sets

Minimality is a very important property in classical topological dynamics; it extends straightforwardly to groupoid actions, denoted below by  $(\Xi, \rho, \theta, \Sigma)$ . Throughout this subsection, both  $\Xi$  and  $\Sigma$  are assumed to be locally compact.

**Definition 6.19.** A closed invariant subset  $M \subset \Sigma$  is called *minimal* if it does not contain proper non-void closed invariant subsets. Equivalently,  $M$  is *minimal* if all orbits contained in  $M$  are dense in  $M$ . *The action is minimal* if  $\Sigma$  itself is minimal.

The minimal sets are the closed invariant non-empty subsets of  $\Sigma$  which are minimal under such requirements. Two minimal sets either coincide or are disjoint. Clearly,

$$\text{transitivity} \Rightarrow \text{minimality} \Rightarrow \text{pointwise transitivity,}$$

and the implications are strict in general (even for group actions).

**Remark 6.20.** For open groupoids, minimal sets are either clopen or nowhere dense. This follows from Proposition 2.7: the boundary of an invariant set is invariant. So if  $\Xi$  is open and  $M$  is minimal, the boundary  $\partial M \subset M$  is closed and invariant, therefore it should be void (i.e.  $M$  is open) or coincide with  $M$  (meaning that  $M$  is nowhere dense). But in general this is not the case: in Remark 3.1,  $[-1, 1] \subset \mathbb{R}$  is a minimal set.

**Remark 6.21.** A function  $\varphi : \Sigma \rightarrow \mathbb{R}$  is called *invariant* with respect to the groupoid action if  $\varphi(\sigma) = \varphi(\tau)$  whenever  $\sigma \overset{\theta}{\sim} \tau$ . By an obvious proof, one shows that if the action is minimal and  $\varphi$  is continuous at least at one point, then it has to be constant.

We proceed now to characterize minimality.

**Proposition 6.22.** *Let  $\emptyset \neq M \subset \Sigma$  be closed and invariant. Then  $M$  is minimal if and only if for every  $U \subset \Sigma$  open, with  $U \cap M \neq \emptyset$ , one has  $M = \text{Sat}(U \cap M)$ .*

*Proof. If:* Assume that  $\emptyset \neq N \subset M$  is closed and invariant; then  $U = N^c = \Sigma \setminus N$  is open. If we show that  $N^c \cap M = \emptyset$ , one gets  $M = N$ , i.e. the minimality of  $M$ . But  $N^c \cap M = \emptyset$  follows if we check that  $M \not\subset \text{Sat}(N^c \cap M)$ , by assumption. This would follow from  $N \cap \text{Sat}(N^c \cap M) = \emptyset$ . But  $\sigma \in N \cap \text{Sat}(N^c \cap M)$  means that  $\sigma \in N$  and it is in the orbit of some element of  $N^c$ . This is impossible since  $N$  is invariant and  $N^c$  is its complement.

*Only if:* Assuming now that  $M$  is minimal, for every  $\sigma \in M$  one has  $M = \mathfrak{D}_\sigma = \bar{\mathfrak{D}}_\sigma$ . If  $U \cap M \neq \emptyset$ ,  $U$  being open, then  $U \cap \mathfrak{D}_\sigma \neq \emptyset$ . But this means that  $\sigma$  is in the orbit of some point that belongs to  $U \cap \mathfrak{D}_\sigma \subset U \cap M$ , so  $x \in \text{Sat}(U \cap M)$ . ■

The following theorem is the main result of this section.

**Theorem 6.23.** *If the point  $\sigma \in \Sigma$  is almost periodic, its orbit closure  $\bar{\mathfrak{D}}_\sigma$  is minimal and compact. If, in addition, the groupoid  $\Xi$  is open, then  $\sigma$  is almost periodic if and only if  $\bar{\mathfrak{D}}_\sigma$  is minimal and compact.*

*Proof.* Suppose that  $\sigma$  is almost periodic; we show first that its orbit closure  $\bar{\mathfrak{D}}_\sigma$  is compact. Let  $U_0$  be a compact neighborhood of  $\sigma$ . Using the assumptions, for some compact set  $K$  one has

$$\mathfrak{D}_\sigma = \Xi_{\rho(\sigma)} \cdot \sigma = (K\tilde{\Xi}_\sigma^{U_0}) \cdot \sigma = K \cdot (\tilde{\Xi}_\sigma^{U_0} \cdot \sigma) \subset K \cdot U_0 = \text{compact},$$

so  $\bar{\mathfrak{D}}_\sigma$  is compact.

If  $\bar{\mathfrak{D}}_\sigma$  is not minimal, then it strictly contains a minimal (and compact) set  $M$ . The point  $\sigma$  does not belong to  $M$ , so there are disjoint open sets  $U, V \subset \Sigma$  such that  $\sigma \in U$  and  $M \subset V$ . For an arbitrary compact set  $K \subset \Xi$  we will now show that  $K\tilde{\Xi}_\sigma^U \neq \Xi_{\rho(\sigma)}$ , implying that in fact  $\sigma$  is not almost periodic.

The set  $M$  being invariant,  $K^{-1} \cdot M \subset M$  holds. Let  $W$  be a neighborhood of  $M$  with  $K^{-1} \cdot W \subset V$ . Since  $M \subset \bar{\mathfrak{D}}_\sigma = \overline{\Xi_{\rho(\sigma)} \cdot \sigma}$ , there exists  $\eta \in \Xi_{\rho(\sigma)}$  such that  $\eta \cdot \sigma \in W$ . Then

$$(K^{-1}\eta) \cdot \sigma = K^{-1} \cdot (\eta \cdot \sigma) \subset K^{-1} \cdot W \subset V,$$



and thus  $(K^{-1}\eta) \cdot \sigma$  is disjoint from  $U$ , meaning that  $K^{-1}\eta$  is disjoint from  $\tilde{\Xi}_\sigma^U$ . This shows that

$$\Xi_{\rho(\sigma)} \ni \eta \notin K\tilde{\Xi}_\sigma^U \neq \Xi_{\rho(\sigma)},$$

finishing the proof.

For the converse, suppose now that  $d$  is open and  $\bar{\Delta}_\sigma = \overline{\Xi_{\rho(\sigma)} \cdot \sigma}$  is minimal and compact. Let  $U$  be an open neighborhood of  $\sigma$ . For each  $\xi \in \Xi$ , choose an open neighborhood  $N_\xi$  of  $\xi$  with compact closure. The sets  $\Xi \cdot U$  and  $N_\xi \cdot U$  are open in  $\Sigma$ , cf. [19, Ex. 2.1.11]. By minimality one has

$$\bar{\Delta}_\sigma \subset \Xi \cdot U = \bigcup_{\xi \in \Xi} \xi \cdot U \subset \bigcup_{\xi \in \Xi} N_\xi \cdot U.$$

By compactness of  $\bar{\Delta}_\sigma$  applied to the open cover above, for a finite set  $F = \{\xi_1, \dots, \xi_k\} \subset \Xi$  we get

$$\bar{\Delta}_\sigma \subset \bigcup_{i=1}^k N_{\xi_i} \cdot U.$$

If  $\eta \in \Xi_{\rho(\sigma)}$ , then  $\eta \cdot \sigma \in N_{\xi_j} \cdot U$  for some  $j$ , and then one has  $\eta \cdot \sigma \in \eta_j \cdot U$  for some  $\eta_j \in N_{\xi_j}$ . It follows immediately that  $r(\eta) = r(\eta_j) = d(\eta_j^{-1})$  and

$$\eta_j^{-1} \cdot (\eta \cdot \sigma) = (\eta_j^{-1}\eta) \cdot \sigma \in U.$$

This means that  $\eta_j^{-1}\eta \in \tilde{\Xi}_\sigma^U$  or, equivalently, that

$$\eta \in \eta_j \tilde{\Xi}_\sigma^U \subset N_{\xi_j} \tilde{\Xi}_\sigma^U \subset \bigcup_{i=1}^k (\overline{N_{\xi_i}} \tilde{\Xi}_\sigma^U) = \left( \bigcup_{i=1}^k \overline{N_{\xi_i}} \right) \tilde{\Xi}_\sigma^U.$$

Since  $\eta$  is arbitrary, one gets  $\Xi_{\rho(\sigma)} \subset K\tilde{\Xi}_\sigma^U$ , where  $K := \bigcup_{i=1}^k \overline{N_{\xi_i}}$  is compact. We checked that  $\tilde{\Xi}_\sigma^U$  is syndetic in  $\Xi_{\rho(\sigma)}$ , so  $\sigma$  is almost periodic. ■

**Example 6.24.** In the case of the transformation groupoid associated to a topological dynamical system  $(\mathcal{G}, \gamma, X)$ , one recovers the classical result ([2, p. 11] and [9, pp. 28, 38, 39]). For this, we use Example 3.3. First of all, it is clear that minimality of the group action coincides with minimality in the sense of groupoids, since the orbits are the same. The relevant recurrence sets also coincide: set  $S = \{\sigma\}$  and  $T = U$  in (3.3). Finally, syndeticity has the same meaning in the two cases.

**Example 6.25.** Consider the case of the pair groupoid  $\Xi := X \times X$  acting on its unit space  $X$ , taken to be compact. The action is transitive (only one orbit), so every  $x \in X$  should be almost periodic. And it is, since  $\Xi_x = X \times \{x\}$  and  $\tilde{\Xi}_x^U = U \times \{x\}$ . The set  $K := X \times \{x\}$  itself is compact, and

$$K\tilde{\Xi}_x^U = (X \times \{x\})(U \times \{x\}) = (X \times \{x\})\{(x, x)\} = X \times \{x\} = \Xi_x.$$

This example also shows another difference between the group and the groupoid case. For groups, the compact set  $K$  in the definition of syndeticity can always be taken finite (see for example [6, p. 271], where this property is called ‘discrete syndeticity’). In this groupoid, no finite set  $K \subset X \times \{x\}$  makes the equality  $K\tilde{\Xi}_x^U = \Xi_x$  true if  $X$  itself is infinite.

If  $\Sigma$  is a compact space, Zorn’s lemma implies that it has a minimal subset  $M \subset \Sigma$ . If in addition  $\Xi$  is open, every  $x \in M$  is almost periodic, by Theorem 6.23. Thus, in this setting, almost periodic points always exist.

**Corollary 6.26.** *If  $\Xi$  is open (and locally compact), the set  $\Sigma_{\text{alper}}$  is invariant.*

*Proof.* By the second part of Theorem 6.23, if  $\sigma \in \Sigma_{\text{alper}}$ , then  $\mathcal{D}_\sigma \subset \Sigma_{\text{alper}}$ . ■

**Corollary 6.27.** *Suppose that  $\Sigma$  is compact,  $\Xi$  is open and the action is minimal. Then  $\tilde{\Xi}_\sigma^V$  is syndetic for every  $\sigma \in \Sigma$  and every open non-void subset  $V$  of  $\Sigma$  (and not just for neighborhoods of  $\sigma$ ).*

*Proof.* By minimality, there exists  $\zeta \in \Xi_{\rho(\sigma)}$  such that  $V$  is an open neighborhood of  $\zeta \bullet \sigma$ . Hence  $\tilde{\Xi}_{\zeta \bullet \sigma}^V$  is syndetic by Theorem 6.23; it can be written as  $K\tilde{\Xi}_{\zeta \bullet \sigma}^V = \Xi_{\rho(\zeta \bullet \sigma)}$  for some compact subset  $K$  of  $\Xi$ . Then

$$\Xi_{\rho(\sigma)} = \Xi_{\text{d}(\zeta)} = \Xi_{\text{r}(\zeta)}\zeta = \Xi_{\rho(\zeta \bullet \sigma)}\zeta = K\tilde{\Xi}_{\zeta \bullet \sigma}^V\zeta = K\tilde{\Xi}_\sigma^V,$$

meaning that  $\tilde{\Xi}_\sigma^V$  is also syndetic. ■

We say that the action is *semisimple* if all orbit closures are minimal (equivalently: the orbit closures form a partition of  $\Sigma$ ).

**Corollary 6.28.** *If all orbits are closed, the action is semisimple. A pointwise almost periodic action is semisimple. If  $\Xi$  is open and all orbits are compact, the action is pointwise almost periodic.*

*Proof.* The statements are obvious or they follow easily from Theorem 6.23. ■

**Remark 6.29.** In Example 2.4 the orbits are precisely the  $\text{d}$ -fibers, automatically closed:  $\mathcal{D}_\eta = \bar{\mathcal{D}}_\eta = \Xi_{\text{d}(\eta)}$ . In particular, this action of  $\Xi$  on itself to the left is semisimple. Pointwise almost periodicity depends on compactness of the fibers.

**Proposition 6.30.** *If  $\Sigma$  is compact and minimal and  $\Xi$  is open and strongly non-compact, the action is non-wandering.*

*Proof.* We know from Proposition 5.13 and Corollary 5.16 that  $\Sigma_{\text{nw}}$  is a non-void closed invariant set. By minimality, it coincides with  $\Sigma$ . ■

## 7. Factors

### 7.1. Epimorphisms of groupoid actions

In this subsection, we indicate the fate of some of the above properties under epimorphisms. Some of the results do not use surjectivity; we leave this to the reader.

**Definition 7.1.** A homomorphism of the groupoid actions  $(\Xi, \rho, \theta, \Sigma)$ ,  $(\Xi, \rho', \theta', \Sigma')$  is a continuous function  $f : \Sigma \rightarrow \Sigma'$  such that for all  $\sigma \in \Sigma$ ,  $\xi \in \Xi_{\rho(\sigma)}$  one has

$$\rho'(f(\sigma)) = \rho(\sigma) \quad \text{and} \quad f(\theta(\xi, \sigma)) = \theta'(\xi, f(\sigma)). \tag{7.1}$$

An epimorphism is a surjective homomorphism; in such a case we say that  $(\Xi, \rho', \theta', \Sigma')$  is a factor of  $(\Xi, \rho, \theta, \Sigma)$  and that  $(\Xi, \rho, \theta, \Sigma)$  is an extension of  $(\Xi, \rho', \theta', \Sigma')$ .

Writing  $\bullet$  instead of  $\theta$  and  $\bullet'$  instead of  $\theta'$ , the second requirement in (7.1) is

$$f(\xi \bullet \sigma) = \xi \bullet' f(\sigma) \quad \text{for all } (\xi, \sigma) \in \Xi \rtimes \Sigma. \tag{7.2}$$

**Lemma 7.2.** The canonical action  $(\Xi, \text{id}_X, \circ, X)$  from Example 2.2 is a factor of any other continuous action  $(\Xi, \rho, \bullet, \Sigma)$ .

*Proof.* The continuous surjection  $f := \rho : \Sigma \rightarrow X$  is an epimorphism, since it satisfies

$$\rho(\xi \bullet \sigma) = r(\xi) = \xi \circ d(\xi) = \xi \circ \rho(\sigma) \quad \text{for all } (\xi, \sigma) \in \Xi \rtimes \Sigma. \quad \blacksquare$$

**Lemma 7.3.** If  $f$  is an epimorphism between the groupoid dynamical systems  $(\Xi, \rho, \theta, \Sigma)$  and  $(\Xi, \rho', \theta', \Sigma')$  and  $\sigma \in \Sigma$ , then  $f(\mathfrak{D}_\sigma) = \mathfrak{D}'_{f(\sigma)}$  and  $f(\bar{\mathfrak{D}}_\sigma) \subset \bar{\mathfrak{D}}'_{f(\sigma)}$ . Direct images of invariant sets are also invariant.

*Proof.* The elementary proof relies on (7.2) and on the properties of continuous functions. We recall that for continuous surjections the direct image may not commute with the closure. ■

**Proposition 7.4.** For every homomorphism  $f$  between actions  $(\Xi, \rho, \bullet, \Sigma)$ ,  $(\Xi, \rho', \bullet', \Sigma')$  and every point  $\sigma \in \Sigma$ , we have  $f(\mathfrak{L}_\sigma^\theta) = \mathfrak{L}'_{f(\sigma)}{}^{\theta'}$ . In consequence,  $f(\sigma)$  is  $\theta'$ -recurrent if  $\sigma$  is  $\theta$ -recurrent.

*Proof.* By Proposition 5.2,  $\tau' \in f(\mathfrak{L}_\sigma^\theta)$  if and only if there exists a divergent net  $(\eta_i)_{i \in I}$  in  $\Xi_{\rho(\sigma)} = \Xi_{\rho'[f(\sigma)]}$  such that

$$\tau' = f(\lim_i \eta_i \bullet \sigma) = \lim_i f(\eta_i \bullet \sigma) = \lim_i \eta_i \bullet' f(\sigma),$$

which is equivalent to  $\tau' \in \mathfrak{L}'_{f(\sigma)}{}^{\theta'}$ . Then the statement about recurrence follows from the definitions. ■

Let us see what happens with recurrence sets under epimorphisms.

**Lemma 7.5.** *If  $M, N \subset \Sigma$  then  $\tilde{\Xi}_M^N \subset \tilde{\Xi}_{f(M)}^{f(N)}$ , where the latter set is computed with respect to the factor  $(\Xi, \rho', \theta', \Sigma')$  of the groupoid dynamical system  $(\Xi, \rho, \theta, \Sigma)$ .*

*Proof.* One verifies easily that the next sequence of equivalences and implications is rigorous:

$$\begin{aligned} \xi \in \tilde{\Xi}_M^N &\Leftrightarrow (\xi \bullet M) \cap N \neq \emptyset \\ &\Leftrightarrow f[(\xi \bullet M) \cap N] \neq \emptyset \\ &\Rightarrow f(\xi \bullet M) \cap f(N) \neq \emptyset \\ &\Leftrightarrow (\xi \bullet' f(M)) \cap f(N) \neq \emptyset \\ &\Leftrightarrow \xi \in \tilde{\Xi}_{f(M)}^{f(N)}. \end{aligned}$$

The second equivalence is true because  $f$  is onto. In general, one has  $f(A \cap B) \subset f(A) \cap f(B)$  and the inclusion could be strict; this shows why (and when) there is no equality in the statement. ■

To see the usefulness of this lemma, we hurry to apply it. On many occasions we are going to use the equality  $f[f^{-1}(B')] = B'$  for  $B' \subset \Sigma'$ , valid by surjectivity.

**Theorem 7.6.** *Let  $f$  be an epimorphism between the groupoid actions  $(\Xi, \rho, \bullet, \Sigma)$  and  $(\Xi, \rho', \bullet', \Sigma')$ . For every index  $\alpha \in \{\text{fix, per, wper, alper, rec, nw}\}$  one has  $f(\Sigma_\alpha) \subset \Sigma'_\alpha$ . In particular,  $\rho(\Sigma_\alpha) \subset X_\alpha$ , where  $X_\alpha$  indicates the set of units of  $X$  having the property  $\alpha$  with respect to the canonical action.*

*Proof.* Using Lemma 7.2, the last statement follows from the first, which we now prove.

For  $\alpha = \text{rec}$  this is already known from Proposition 7.4. The statement for  $\alpha = \text{fix}$  follows immediately from (7.1).

We prove now the case  $\alpha = \text{nw}$ . Assume that  $\sigma \in \Sigma$  is non-wandering, but  $f(\sigma) \in \Sigma'$  is wandering. Then there is an open neighborhood  $W'$  of  $f(\sigma)$  such that  $\tilde{\Xi}_{W'}^{W'}$  is relatively compact. The set  $f^{-1}(W')$  is a neighborhood of  $\sigma$ , so  $\tilde{\Xi}_{f^{-1}(W')}^{f^{-1}(W')}$  is not relatively compact. By Lemma 7.5, this set is contained in  $\tilde{\Xi}_{W'}^{W'}$ , and this is a contradiction.

The other proofs are similar. For instance, the statement about almost periodicity follows easily from the definitions and from Lemma 7.5; a set containing a syndetic subset is obviously syndetic. ■

**Proposition 7.7.** *Let  $f$  be an epimorphism between the groupoid actions  $(\Xi, \rho, \bullet, \Sigma)$  and  $(\Xi, \rho', \bullet', \Sigma')$ . Suppose that the action  $(\Xi, \rho, \bullet, \Sigma)$  has one of the transitivity properties*

$$\mathcal{P} \in \{\text{transitivity, pointwise transitivity, weakly pointwise transitivity, (i), (ii), (iii)}\}$$

*(see Theorem 4.1). Then the action  $(\Xi, \rho', \bullet', \Sigma')$  also has  $\mathcal{P}$ .*

*Proof.*  $\mathcal{P} = \text{transitivity}$ . By Lemma 7.3, the epimorphism  $f$  transforms the single orbit  $\mathcal{O}_\sigma = \Sigma$  into an orbit  $\mathcal{O}'_{f(\sigma)} = f(\mathcal{O}_\sigma) = \Sigma'$ .

$\mathcal{P}$  = pointwise transitivity. This also follows immediately from Lemma 7.3, the part referring to orbit closures.

$\mathcal{P}$  = weakly pointwise transitivity. If  $C' \subset \Xi'$  is a closed and invariant set containing  $f(\sigma)$ , then  $f^{-1}(C') \subset \Xi$  is a closed and invariant set containing  $\sigma$ , hence

$$\Sigma = \mathfrak{C}_\sigma \subset f^{-1}(C') \quad \text{and} \quad \Sigma' = C'.$$

$\mathcal{P}$  = (i)  $\Leftrightarrow$  (i'). Let  $\emptyset \neq U', V' \subset \Sigma'$  be invariant open sets and let  $f^{-1}(U'), f^{-1}(V')$  be their open non-void invariant inverse images. Since  $(\Xi, \rho, \bullet, \Sigma)$  satisfies (i'), one has

$$f^{-1}(U') \cap f^{-1}(V') \neq \emptyset.$$

Consequently,

$$U' \cap V' = f[f^{-1}(U')] \cap f[f^{-1}(V')] \supset f[f^{-1}(U') \cap f^{-1}(V')] \neq \emptyset$$

and  $(\Xi, \rho', \bullet', \Sigma')$  also satisfies (i'), which is equivalent to (i).

$\mathcal{P}$  = (ii). Let  $U' \subset \Sigma'$  be open and invariant. Then  $f^{-1}(U') \subset \Sigma$  is open and invariant, so it is dense. Since  $f$  is surjective, it follows that  $U' = f(f^{-1}(U'))$ , and this one is dense in  $\Sigma'$ .

$\mathcal{P}$  = (iii). Let  $\emptyset \neq U', V' \subset \Sigma'$  be open subsets. By recurrent transitivity of the initial action, one has

$$\tilde{\Xi}_{f^{-1}(U')}^{f^{-1}(V')} \neq \emptyset.$$

Then, by Lemma 7.5, we get

$$\emptyset \neq \tilde{\Xi}_{f^{-1}(U')}^{f^{-1}(V')} \subset \tilde{\Xi}'_{f[f^{-1}(U')]}^{f[f^{-1}(V')]} = \tilde{\Xi}'_{U'}^{V'},$$

and the proof is finished. ■

**Corollary 7.8.** *We say that the groupoid  $\Xi$  has the property  $\mathcal{P}$  if its canonical action on its unit space has this property. Suppose that the topological groupoid  $\Xi$  admits a continuous action  $(\rho, \theta, \Sigma)$  having one of the properties  $\mathcal{P}$  mentioned in Proposition 7.7. Then  $\Xi$  itself has this property.*

*Proof.* This follows from Proposition 7.7 and Lemma 7.2. ■

If we require  $\Xi$  to be open, there is a direct proof that property (iv) of Theorem 4.1 (called topological transitivity) also transfer to factors; it uses Lemma 2.7. This also follows joining Theorem 4.1 and Proposition 7.7. But see the next example for a non-topologically transitive groupoid, which however possesses a topologically transitive action. It follows that topological transitivity is not preserved by factors (which is rather surprising).

**Example 7.9.** Form the product groupoid  $\Xi = \Pi \times \mathbb{R}$ , with the relation

$$x \Pi y \Leftrightarrow x, y \in \mathbb{Q} \text{ or } x = y$$

over  $X = \mathbb{R}$ , and consider the wide subgroupoid

$$\Delta = \{(x, y, g) \in \Xi \mid x, y \notin \mathbb{Q} \Rightarrow g = 0\} = (\mathbb{Q} \times \mathbb{Q} \times \mathbb{R}) \cup (\text{Diag}(\mathbb{R} \times \mathbb{R}) \times \{0\}).$$

By analogy with Example 4.4, it is easy to see that the canonical action of  $\Delta$  on  $X = \mathbb{R}$  is not topologically transitive. Actually, the orbits of rational points all coincide with  $\mathbb{Q}$ , but each irrational point is a fixed point, so  $(s, t) \setminus \mathbb{Q}$  is invariant for every  $s < t$ , without being dense or nowhere dense. Now we will exhibit a topologically transitive action of  $\Xi$ :  
Let

$$\Sigma = \{(y, h) \in \mathbb{R}^2 \mid h = 0 \text{ or } y \in \mathbb{Q}\} = (\mathbb{R} \times \{0\}) \cup (\mathbb{Q} \times \mathbb{R})$$

with the topology inherited from  $\mathbb{R}^2$  and define the continuous action

$$\rho(y, h) = y \quad \text{and} \quad (x, y, g) \bullet (y, h) = (x, g + h).$$

Notice that the orbits of  $\bullet$  are  $\mathbb{Q} \times \mathbb{R}$  and (the fixed points)  $\{(y, 0) \mid y \in \mathbb{R} \setminus \mathbb{Q}\}$ , implying that all of the invariant sets are either dense or nowhere dense, since  $\Sigma \setminus (\mathbb{Q} \times \mathbb{R}) = (\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$  is already nowhere dense.

We finish this subsection with a result on the behavior of minimality under epimorphisms, in both directions.

**Proposition 7.10.** *Let  $f : \Sigma \rightarrow \Sigma'$  be an epimorphism between the actions  $(\Xi, \rho, \theta, \Sigma)$  and  $(\Xi, \rho', \theta', \Sigma')$ .*

- (i) *If  $M \subset \Sigma$  is minimal and  $f(M)$  is closed in  $\Sigma'$ , then  $f(M)$  is minimal.*
- (ii) *Suppose that  $\Sigma$  is compact (hence  $\Sigma'$  is also compact). If  $M' \subset \Sigma'$  is minimal, there exists  $M \subset \Sigma$  minimal such that  $f(M) = M'$ . If  $\Xi$  is open and locally compact and  $\sigma' \in \Sigma'$  is almost periodic, then  $\sigma' = f(\sigma)$  for some almost periodic point  $\sigma$  of  $\Sigma$ .*

*Proof.* (i) This is dealt with easily by Lemma 7.3: if  $\sigma \in M$ , then

$$f(M) = f(\overline{\mathcal{D}}_\sigma) \subset \overline{\mathcal{D}'_{f(\sigma)}} \subset \overline{f(M)} = f(M),$$

so the orbit of  $f(\sigma)$  is dense in the closed set  $f(M)$ .

(ii) The inverse image  $f^{-1}(M')$  is non-void closed and  $\bullet$ -invariant. By Zorn’s lemma, it contains a minimal (and compact) subsystem  $M$ . The direct image  $f(M) \subset M'$  is non-void closed and  $\bullet'$ -invariant, so it must coincide with  $M'$ . For the second part: As  $\sigma'$  is almost periodic,  $M' := \overline{\mathcal{D}'_{\sigma'}}$  is minimal, so we can find some (compact) minimal subset  $M \subset \Sigma$ , such that  $f(M) = M'$  and  $\sigma' = f(\sigma)$  for some  $\sigma \in M$ . By Theorem 6.23, any  $\sigma \in M$  is almost periodic. ■

### 7.2. The action associated to an extension

The next result is a straightforward generalization of a recent construction from [8], in which Edeko and Kreidler showed how to encode extensions of classical group actions

by groupoids. Then we are going to study the resulting correspondence with respect to recurrence and other dynamical properties.

**Proposition 7.11.** *Let  $(\Xi, \rho', \theta', \Sigma')$  be a groupoid action. There is a one-to-one correspondence between extensions  $f : \Sigma \rightarrow \Sigma'$  and actions of the groupoid  $\Xi \rtimes_{\theta'} \Sigma'$ .*

*Proof.* Let

$$(\Xi, \rho, \theta, \Sigma) \xrightarrow{f} (\Xi, \rho', \theta', \Sigma')$$

be an epimorphism. We denote by  $\Xi(\theta') = \Xi \rtimes_{\theta'} \Sigma'$  the action groupoid of the second groupoid action, described in Section 3.6. Its unit space is  $\Sigma'$ . This gives rise to a new groupoid action  $(\Xi(\theta'), f, \Theta, \Sigma)$ , where (see the notation (2.1) and use  $\rho' \circ f = \rho$ )

$$\begin{aligned} \Xi(\theta') \rtimes \Sigma &:= \{((\xi, \sigma'), \sigma) \mid d(\xi) = \rho'(\sigma'), f(\sigma) = \sigma'\} \\ &= \{(\xi, f(\sigma), \sigma) \mid d(\xi) = \rho'[f(\sigma)] = \rho(\sigma)\} \subset \Xi \times \Sigma' \times \Sigma \end{aligned}$$

and

$$\Theta_{(\xi, f(\sigma))}(\sigma) := \theta_{\xi}(\sigma), \quad \text{if } d(\xi) = \rho(\sigma). \tag{7.3}$$

It is straightforward to check that everything is well-defined and that one obtains a groupoid action.

Reciprocally, suppose that  $(\Xi, \rho', \theta', \Sigma')$  is a groupoid action and  $(\Xi(\theta'), f, \Theta, \Sigma)$  is an action of the transformation groupoid  $\Xi(\theta') = \Xi \rtimes_{\theta'} \Sigma'$  on another topological space  $\Sigma$ . In particular,  $f : \Sigma \rightarrow \Sigma' = \Xi(\theta')^{(0)}$  is a continuous surjection. The new action  $(\Xi, \rho, \theta, \Sigma)$  is constructed in the following way: The anchor is

$$\rho := \rho' \circ f : \Sigma \rightarrow X := \Xi^{(0)}$$

and the action is given by (7.3), with a modified interpretation: the right-hand side is now defined by the left-hand side. Then  $f : \Sigma \rightarrow \Sigma'$  is the epimorphism. The details are easy. To show that  $f \circ \theta_{\xi} = \theta'_{\xi} \circ f$ , for instance, pick  $\sigma \in \Sigma$  with  $d(\xi) = \rho(\sigma) = \rho'[f(\sigma)]$ . Then

$$f[\theta_{\xi}(\sigma)] \stackrel{(7.3)}{=} f[\Theta_{(\xi, f(\sigma))}(\sigma)] = r(\xi, f(\sigma)) \stackrel{(3.11)}{=} \theta'_{\xi}[f(\sigma)].$$

Here  $r$  is the range map of  $\Xi \rtimes_{\theta'} \Sigma'$ . The two procedures are inverse to each other. ■

**Remark 7.12.** It is easy to see that the projection

$$\Xi \times \Sigma' \times \Sigma \ni (\xi, \sigma', \sigma) \rightarrow (\xi, \sigma) \in \Xi \times \Sigma$$

restricts to a groupoid isomorphism  $(\Xi \rtimes_{\theta'} \Sigma') \rtimes_{\Theta} \Sigma \cong \Xi \rtimes_{\theta} \Sigma$ . The basic remark is that the elements of  $(\Xi \rtimes_{\theta'} \Sigma') \rtimes_{\Theta} \Sigma$  have the form  $((\xi, f(\sigma)), \sigma) \equiv (\xi, f(\sigma), \sigma)$ , with  $d(\xi) = \rho(\sigma)$ . The algebraic verifications are left to the reader.

**Proposition 7.13.** *The two actions  $(\Xi, \rho, \theta, \Sigma)$  and  $(\Xi(\theta'), f, \Theta, \Sigma)$  have the same invariant sets. Consequently, they have in the same time the following properties: transitivity, point transitivity, topological transitivity, the properties (i) and (ii) of Theorem 4.1 and minimality. The fixed points are the same.*

*Proof.* Let us explore the orbit structure; for  $\sigma, \tau \in \Sigma$  one has

$$\begin{aligned} \sigma \overset{\Theta}{\sim} \tau &\Leftrightarrow \exists (\xi, \sigma') \in \Xi(\theta') \text{ such that } \sigma' = f(\sigma), \Theta_{(\xi, \sigma')}(\sigma) = \tau \\ &\Leftrightarrow \exists \xi \in \Xi \text{ such that } d(\xi) = \rho(\sigma), \theta_\xi(\sigma) = \tau \\ &\Leftrightarrow \sigma \overset{\theta}{\sim} \tau. \end{aligned}$$

All the statements of the proposition follow from this. ■

Now, we discuss recurrence for the connected actions  $(\Xi(\theta'), f, \Theta, \Sigma)$  and  $(\Xi, \rho, \theta, \Sigma)$ . For  $M, N \subset \Sigma$  we have two recurrent sets:

$$\begin{aligned} \widetilde{\Xi(\theta')}_M^N &\text{ with respect to the action } \Theta, \\ \widetilde{\Xi}_M^N &\text{ with respect to the action } \theta. \end{aligned}$$

Using the definitions, one gets

$$\begin{aligned} (\xi, \sigma') \in \widetilde{\Xi(\theta')}_M^N &\Leftrightarrow d(\xi) = \rho'(\sigma') \text{ and } \exists \sigma \in M \text{ such that } f(\sigma) = \sigma', \Theta_{(\xi, \sigma')}(\sigma) \in N \\ &\Leftrightarrow \exists \sigma \in M \text{ such that } f(\sigma) = \sigma', d(\xi) = \rho(\sigma), \theta_\xi(\sigma) \in N, \end{aligned}$$

while

$$\xi \in \widetilde{\Xi}_M^N \Leftrightarrow \exists \sigma \in M \text{ such that } d(\xi) = \rho(\sigma), \theta_\xi(\sigma) \in N.$$

Then it is clear that

$$\xi \in \widetilde{\Xi}_M^N \Leftrightarrow \exists \sigma' \in \Sigma' \text{ such that } (\xi, \sigma') \in \widetilde{\Xi(\theta')}_M^N,$$

which can be written in terms of the first projection  $p : \Xi \times \Sigma' \rightarrow \Xi$  as

$$p[\widetilde{\Xi(\theta')}_M^N] = \widetilde{\Xi}_M^N. \tag{7.4}$$

Setting  $M = \{\sigma\}$ , we get the more precise version

$$\widetilde{\Xi(\theta')}_\sigma^N = \widetilde{\Xi}_\sigma^N \times \{f(\sigma)\}. \tag{7.5}$$

Two particular cases are

$$\Xi(\theta')_{\rho(\sigma)} = \Xi_{\rho(\sigma)} \times \{f(\sigma)\} \tag{7.6}$$

and

$$\widetilde{\Xi(\theta')}_\sigma^\sigma = \widetilde{\Xi}_\sigma^\sigma \times \{f(\sigma)\}, \tag{7.7}$$

obtained by setting in (7.5)  $N = \Sigma$  or  $N = \{\sigma\}$ . With this, we can state the following proposition.

**Proposition 7.14.** *Consider the actions  $(\Xi(\theta'), f, \Theta, \Sigma)$  and  $(\Xi, \rho, \theta, \Sigma)$ . The action  $(\Xi(\theta'), f, \Theta, \Sigma)$  is recurrently transitive if and only if  $(\Xi, \rho, \theta, \Sigma)$  is recurrently transitive. One has the equalities*

$$\Sigma_{\text{per}}^\Theta = \Sigma_{\text{per}}^\theta, \quad \Sigma_{\text{wper}}^\Theta = \Sigma_{\text{wper}}^\theta, \quad \Sigma_{\text{alper}}^\Theta = \Sigma_{\text{alper}}^\theta, \quad \Sigma_{\text{rec}}^\Theta = \Sigma_{\text{rec}}^\theta. \tag{7.8}$$



*Proof.* The statement about recurrent transitivity follows from (7.4), taking non-void open sets  $M = U, N = V$ , since  $p(R) \neq \emptyset$  if and only if  $R \neq \emptyset$ .

The equality  $\Sigma_{\text{wper}}^\Theta = \Sigma_{\text{wper}}^\theta$  follows from (7.7):  $\widetilde{\Xi}(\theta')_\sigma^\sigma$  is compact if and only if  $\widetilde{\Xi}_\sigma^\sigma$  is compact.

To prove  $\Sigma_{\text{per}}^\Theta = \Sigma_{\text{per}}^\theta$ , let  $K \subset \Xi$  be a compact set with  $K\widetilde{\Xi}_\sigma^\sigma = \Xi_{\rho(\sigma)}$ . Then  $K' = K \times \{f(\sigma)\} \subset \widetilde{\Xi}(\theta')$  is compact and

$$\begin{aligned} K' \widetilde{\Xi}(\theta')_\sigma^\sigma &\stackrel{(7.7)}{=} K'(\widetilde{\Xi}_\sigma^\sigma \times \{f(\sigma)\}) \\ &= (K\widetilde{\Xi}_\sigma^\sigma) \times \{f(\sigma)\} \\ &= \Xi_{\rho(\sigma)} \times \{f(\sigma)\} \\ &\stackrel{(7.6)}{=} \Xi(\theta')_{\rho(\sigma)}. \end{aligned}$$

The second equality is a consequence of a computation of the form

$$(\xi, f(\sigma))(\eta, f(\sigma)) = (\xi\eta, f(\sigma)), \quad \text{for } \eta \in \widetilde{\Xi}_\sigma^\sigma,$$

implying

$$\eta \cdot' f(\sigma) = f(\sigma).$$

So syndeticity of  $\widetilde{\Xi}_\sigma^\sigma$  implies the syndeticity of  $\widetilde{\Xi}(\theta')_\sigma^\sigma$ . On the other hand, if  $\widetilde{\Xi}(\theta')_\sigma^\sigma$  is syndetic with compact set  $K'$ , to obtain  $K$  one takes the projection onto the first coordinate:  $K = p(K')$ . The same type of argument works to show that  $\Sigma_{\text{alper}}^\Theta = \Sigma_{\text{alper}}^\theta$ .

The relation  $\Sigma_{\text{rec}}^\Theta = \Sigma_{\text{rec}}^\theta$  holds because of (7.5), by using Proposition 5.8. ■

**Remark 7.15.** Proving (7.8), we took advantage of the fact that  $\Sigma_{\text{per}}, \Sigma_{\text{wper}}, \Sigma_{\text{alper}}, \Sigma_{\text{rec}}$  are defined by recurrent sets involving a couple of subsets of  $\Sigma$ , one of them being a singleton; this allowed the application of (7.5). One also gets  $\Sigma_{\text{nw}}^\Theta = \Sigma_{\text{nw}}^\theta$  if  $\Sigma'$  is supposed compact, using (7.4). Without compactness, only one obvious inclusion is true.

### 8. A disappointing notion: mixing

The next definition seems a legitimate generalization of the classical one:

**Definition 8.1.** The action  $(\Xi, \rho, \theta, \Sigma)$  is called *weakly mixing* whenever for any non-empty open sets  $U, U', V, V' \subset \Sigma$ , one has  $\widetilde{\Xi}_U^V \cap \widetilde{\Xi}_{U'}^{V'} \neq \emptyset$ . It is called *strongly mixing* if the complement of  $\widetilde{\Xi}_U^V$  is relatively compact for any open sets  $U, V \neq \emptyset$ .

If  $\Xi$  itself is not compact, strongly mixing implies weakly mixing. This follows from the equality

$$(\widetilde{\Xi}_U^V \cap \widetilde{\Xi}_{U'}^{V'})^c = (\widetilde{\Xi}_U^V)^c \cup (\widetilde{\Xi}_{U'}^{V'})^c$$

and the fact that the union of two relatively compact sets is relatively compact. On the other hand, *weakly mixing always implies recurrent transitivity*: just take  $U = U'$  and  $V = V'$ .

**Example 8.2.** For classical group actions one recovers the usual concepts; see Example 3.3.

The next result shows that there is no point in exploring the notion of mixing besides the group case, which is already extensively treated in all standard textbooks in topological dynamics; see also [4]. Recall that we assumed  $\rho$  surjective and  $X$  Hausdorff.

**Proposition 8.3.** *Suppose that  $(\Xi, \rho, \theta, \Sigma)$  is weakly mixing. Then  $\rho(\Sigma) = X$  consists of a single point, so the groupoid is a group.*

*Proof.* Suppose there exist two distinct points  $x, y \in \rho(\Sigma)$ , and let  $U_0, V_0 \subset X$  be disjoint open sets separating them. Form the non-void open sets  $U = \rho^{-1}(U_0)$  and  $V = \rho^{-1}(V_0)$ . By weakly mixing, there exists some  $\xi \in \tilde{\Xi}_U^U \cap \tilde{\Xi}_V^V$ . Thus, there are points  $\sigma_1, \sigma_2, \sigma_3 \in U$ ,  $\tau \in V$  such that

$$\xi \cdot \sigma_1 = \sigma_2 \quad \text{and} \quad \xi \cdot \sigma_3 = \tau,$$

implying that

$$\rho(\sigma_2) = \rho(\xi \cdot \sigma_1) = r(\xi) = \rho(\xi \cdot \sigma_3) = \rho(\tau).$$

But  $\rho(\sigma_2) \in U_0$  and  $\rho(\tau) \in V_0$ . This contradiction shows that  $\rho(\Sigma)$  consist of a single point. ■

**Funding.** M.M. has been supported by the Fondecyt Project 1200884.

## References

- [1] F. Abadie, On partial actions and groupoids. *Proc. Amer. Math. Soc.* **132** (2004), no. 4, 1037–1047 Zbl [1040.46035](#) MR [2045419](#)
- [2] J. Auslander, *Minimal flows and their extensions*. North-Holland Math. Stud. 153, North-Holland Publishing Co., Amsterdam, 1988 Zbl [0654.54027](#) MR [956049](#)
- [3] S. Beckus, J. Bellissard, and G. De Nittis, Spectral continuity for aperiodic quantum systems I. General theory. *J. Funct. Anal.* **275** (2018), no. 11, 2917–2977 Zbl [1406.81023](#) MR [3861728](#)
- [4] G. Cairns, A. Kolganova, and A. Nielsen, Topological transitivity and mixing notions for group actions. *Rocky Mountain J. Math.* **37** (2007), no. 2, 371–397 Zbl [1208.37005](#) MR [2333376](#)
- [5] C. Carvalho, V. Nistor, and Y. Qiao, Fredholm conditions on non-compact manifolds: theory and examples. In *Operator theory, operator algebras, and matrix theory*, pp. 79–122, Oper. Theory Adv. Appl. 267, Birkhäuser/Springer, Cham, 2018 Zbl [1446.58011](#) MR [3837632](#)
- [6] J. de Vries, *Elements of topological dynamics*. Math. Appl. 257, Kluwer Academic Publishers Group, Dordrecht, 1993 Zbl [0783.54035](#) MR [1249063](#)
- [7] V. Deaconu, Groupoids associated with endomorphisms. *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1779–1786 Zbl [0826.46058](#) MR [1233967](#)
- [8] N. Edeko and H. Kreidler, Uniform enveloping semigroupoids for groupoid actions. *J. Anal. Math.*, to appear

- [9] D. B. Ellis and R. Ellis, *Automorphisms and equivalence relations in topological dynamics*. London Math. Soc. Lecture Note Ser. 412, Cambridge University Press, Cambridge, 2014  
Zbl [1322.37001](#) MR [3242163](#)
- [10] R. Exel, Inverse semigroups and combinatorial  $C^*$ -algebras. *Bull. Braz. Math. Soc. (N.S.)* **39** (2008), no. 2, 191–313 Zbl [1173.46035](#) MR [2419901](#)
- [11] R. Exel, *Partial dynamical systems, Fell bundles and applications*. Math. Surveys Monogr. 224, American Mathematical Society, Providence, RI, 2017 Zbl [1405.46003](#) MR [3699795](#)
- [12] F. Flores and M. Măntoiu, Morphisms of groupoid actions and recurrence. *Topology Appl.* **314** (2022), Paper No. 108122 Zbl [07530406](#) MR [4415579](#)
- [13] E. Glasner, *Ergodic theory via joinings*. Math. Surveys Monogr. 101, American Mathematical Society, Providence, RI, 2003 Zbl [1038.37002](#) MR [1958753](#)
- [14] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*. London Math. Soc. Lecture Note Ser. 124, Cambridge University Press, Cambridge, 1987 Zbl [0683.53029](#) MR [896907](#)
- [15] R. S. Palais, On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2)* **73** (1961), 295–323 Zbl [0103.01802](#) MR [126506](#)
- [16] A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*. Progr. Math. 170, Birkhäuser, Boston, MA, 1999 Zbl [0913.22001](#) MR [1724106](#)
- [17] A. Ramsay, The Mackey-Glimm dichotomy for foliations and other Polish groupoids. *J. Funct. Anal.* **94** (1990), no. 2, 358–374 Zbl [0717.57016](#) MR [1081649](#)
- [18] J. Renault, *A groupoid approach to  $C^*$ -algebras*. Lecture Notes in Math. 793, Springer, Berlin, 1980 Zbl [0433.46049](#) MR [584266](#)
- [19] D. P. Williams, *A tool kit for groupoid  $C^*$ -algebras*. Math. Surveys Monogr. 241, American Mathematical Society, Providence, RI, 2019 Zbl [1456.46002](#) MR [3969970](#)

Received 23 November 2020.

### Felipe Flores

Departamento de Ingeniería Matemática, Universidad de Chile, Beauchef 851, Torre Norte, Oficina 436, Santiago, Chile; [fflores@dim.uchile.cl](mailto:fflores@dim.uchile.cl), [hmy3f@virginia.edu](mailto:hmy3f@virginia.edu)

### Marius Măntoiu

Departamento de Matemáticas, Universidad de Chile, Las Palmeras 3425, Casilla 653, Santiago, Chile; [mantoiu@uchile.cl](mailto:mantoiu@uchile.cl)