Groups Geom. Dyn. 16 (2022), 1119–1164
DOI 10.4171/GGD/678

Characters of algebraic groups over number fields

Bachir Bekka and Camille Francini

Abstract. Let k be a number field, $\mathbf{G}$ an algebraic group defined over k, and $\mathbf{G}(k)$ the group of k-rational points in $\mathbf{G}$. We determine the set of functions on $\mathbf{G}(k)$ which are of positive type and conjugation invariant, under the assumption that $\mathbf{G}(k)$ is generated by its unipotent elements. An essential step in the proof is the classification of the $\mathbf{G}(k)$-invariant ergodic probability measures on an adelic solenoid naturally associated to $\mathbf{G}(k)$. This last result is deduced from Ratner's measure rigidity theorem for homogeneous spaces of S-adic Lie groups; this appears to be the first application of Ratner's theorems in the context of operator algebras.

1. Introduction

Let k be a field and $\mathbf{G}$ an algebraic group defined over k. When k is a local field (that is, a non-discrete locally compact field), the group $G = \mathbf{G}(k)$ of k-rational points in $\mathbf{G}$ is a locally compact group for the topology induced by k. In this case (and when, in addition, k is of characteristic zero), much is known [15, 21] about the unitary dual $\widehat{G}$ of G, the set of equivalence classes of irreducible unitary representations of G in Hilbert spaces. By way of contrast, if k is a global field (that is, either a number field or a function field in one variable over a finite field), then G is a countable infinite group and, unless G is abelian, the classification of $\widehat{G}$ is a hopeless task, as follows from work of Glimm and Thoma [20, 41]. In this case, a sensible substitute for $\widehat{G}$ is the set of characters of G we are going to define.

Let G be a group. Recall that a function $\varphi: G \to \mathbb{C}$ is of positive type if the complex-valued matrix $(\varphi(g_j^{-1} g_i))_{1 \leq i,j \leq n}$ is positive semi-definite for any $g_1, \ldots, g_n$ in G.

A function of positive type φ on G which is central (that is, constant on conjugacy classes) and normalized (that is, $\varphi(e) = 1$) will be called a *trace* on G. The set $\mathrm{Tr}(G)$ of traces on G is a convex subset of the unit ball of $\ell^\infty(G)$ which is compact in the topology of pointwise convergence. The extreme points of $\mathrm{Tr}(G)$ are called the *characters* of G and the set they constitute will be denoted by $\mathrm{Char}(G)$.

Besides providing an alternative dual space of a group G, characters and traces appear in various situations. Traces of G are tightly connected to representations of G in the

2020 Mathematics Subject Classification. Primary 22D10; Secondary 22D25, 22D40, 20G05.
Keywords. Algebraic groups, characters of discrete groups, von Neumann algebras, invariant random subgroups, Ratner's theory.

unitary group of tracial von Neumann algebras (see below and Section 2.2). The space $\mathrm{Tr}(G)$ of traces on G encompasses the lattice of all normal subgroups of G, since the characteristic function of every normal subgroup is a trace on G. More generally, every measure preserving action of G on a probability space gives rise to an invariant random subgroup (IRS) on G and therefore to a trace on G (see [19, §9]).

The study of characters on infinite discrete groups was initiated by Thoma [40, 41] and the space $\mathrm{Char}(G)$ was determined for various groups G (see [1, 9, 10, 14, 24, 30–32, 36, 39]).

Observe that our traces are often called characters in the literature (see, for instance, [14, 32]).

Let k be a number field (that is, a finite extension of $\mathbb{Q}$) and $\mathbf{G}$ a connected linear algebraic group defined over k. In this paper, we will give a complete description of $\mathrm{Char}(G)$ for $G = \mathbf{G}(k)$ under the assumption that G is generated by its unipotent one-parameter subgroups. A *unipotent one-parameter subgroup* of G is a subgroup of the form $\{u(t) \mid t \in k\}$, where $u\colon \mathbf{G}_a \to \mathbf{G}$ is a non-trivial k-rational homomorphism from the additive group $\mathbf{G}_a$ of dimension 1 to $\mathbf{G}$.

The case where $\mathbf{G}$ is quasi-simple over k was treated in [2] and the result is that

$$\mathrm{Char}(G) = \{\widetilde{\chi} \mid \chi \in \widehat{Z}\} \cup \{1_G\},$$

where Z is the (finite) center of G and $\widetilde{\chi}\colon G \to \mathbb{C}$ is defined by $\widetilde{\chi} = \chi$ on Z and $\widetilde{\chi} = 0$ on $G \setminus Z$. When $\mathbf{G}$ is semi-simple, the computation $\mathrm{Char}(G)$ can easily be reduced to the quasi-simple case (see [2, Proposition 5.1]; see also Corollary 2.13 below).

We now turn to a general connected linear algebraic group $\mathbf{G}$ over k. The unipotent radical $\mathbf{U}$ of $\mathbf{G}$ is defined over k, and there exists a connected reductive k-subgroup $\mathbf{L}$, called a Levi subgroup, such that $\mathbf{G} = \mathbf{L}\mathbf{U}$ (see [28]). Set $U := \mathbf{U} \cap G$ and $L := \mathbf{L} \cap G$. Then, we have a corresponding semi-direct decomposition $G = LU$, called the *Levi decomposition* of G (see [25, Lemma 2.2]).

Recall that $\mathbf{L} = \mathbf{T}\mathbf{L}'$ is an almost direct product (see Section 6.3 for this notion) of a central k-torus $\mathbf{T}$ and the derived subgroup $\mathbf{L}'$, which is a semi-simple k-group. Assume that G is generated by its unipotent one-parameter subgroups. Then the same holds for L. Since every unipotent one-parameter subgroup of $\mathbf{L}$ is contained in $\mathbf{L}'$, it follows that $G = \mathbf{L}'(k)U$, that is, the Levi subgroup $\mathbf{L}$ is semi-simple.

We will describe $\mathrm{Char}(G)$ in terms of data attached to L and the action of L on the Lie algebra $\mathrm{Lie}(\mathbf{U})$ of U.

The set $\mathfrak{u}$ of k-points of $\mathrm{Lie}(\mathbf{U})$ is a Lie algebra over k and the exponential map $\exp\colon \mathfrak{u} \to U$ is a bijective map. For every g in G, the automorphism of U given by conjugation by g induces an automorphism $\mathrm{Ad}(g)$ of the Lie algebra $\mathfrak{u}$ (see Section 3.2).

Let $\widehat{\mathfrak{u}}$ be the Pontrjagin dual of $\mathfrak{u}$, that is, the group of unitary characters of the additive group of $\mathfrak{u}$. We associate to every $\lambda \in \widehat{\mathfrak{u}}$ the following subsets $\mathfrak{k}_\lambda$, $\mathfrak{p}_\lambda$ of $\mathfrak{u}$ and L_λ of L:

- $\mathfrak{k}_\lambda$ is the set of elements $X \in \mathfrak{u}$ such that

$$\lambda(\mathrm{Ad}(g)(tX)) = 1 \quad \text{for all } g \in G,\ t \in k;$$

- $\mathfrak{p}_\lambda$ is the set of elements $X \in \mathfrak{u}$ such that

$$\lambda(\mathrm{Ad}(g)(tX)) = \lambda(tX) \quad \text{for all } g \in G,\ t \in k;$$

- L_λ is the set of $g \in L$ such that $\mathrm{Ad}(g)(X) \in X + \mathfrak{k}_\lambda$ for every $X \in \mathfrak{u}$.

Then $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ are L-invariant ideals of $\mathfrak{u}$ and L_λ is the kernel of the quotient adjoint representation of L on $\mathfrak{u}/\mathfrak{k}_\lambda$.

Observe that $\mathfrak{k}_\lambda$ is the largest L-invariant ideal of $\mathfrak{u}$ contained in $\mathrm{Ker}(\lambda)$.

The set $K_\lambda := \exp(\mathfrak{k}_\lambda)$ is a Zariski-connected normal subgroup of G. Moreover, $P_\lambda := \exp(\mathfrak{p}_\lambda)$ is the inverse image in U of the elements in U/K_λ contained in the center of G/K_λ (see Proposition 3.5). The map

$$\chi_\lambda \colon P_\lambda \to \mathbf{S}^1, \quad \exp(X) \mapsto \lambda(X)$$

is a G-invariant unitary character of P_λ, which is trivial on K_λ.

Let Ad^* denote the coadjoint action (that is, the dual action) of G on $\widehat{\mathfrak{u}}$. We say that $\lambda_1, \lambda_2 \in \widehat{\mathfrak{u}}$ have the same *quasi-orbit* under G if the closures of $\mathrm{Ad}^*(G)\lambda_1$ and $\mathrm{Ad}^*(G)\lambda_2$ in the compact group $\widehat{\mathfrak{u}}$ coincide.

We can now state our main result.

Theorem A. *Let $G = \mathbf{G}(k)$ be the group of k-rational points of a connected linear algebraic group $\mathbf{G}$ over a number field k. Assume that G is generated by its unipotent one-parameter subgroups and let $G = LU$ be a Levi decomposition of G. For $\lambda \in \widehat{\mathfrak{u}}$ and $\varphi \in \mathrm{Char}(L_\lambda)$, define $\Phi_{(\lambda,\varphi)} \colon G \to \mathbb{C}$ by*

$$\Phi_{(\lambda,\varphi)}(g) = \begin{cases} \varphi(g_1)\chi_\lambda(u) & \text{if } g = g_1 u \text{ for } g_1 \in L_\lambda,\ u \in P_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

(i) *We have*

$$\mathrm{Char}(G) = \{\Phi_{(\lambda,\varphi)} \mid \lambda \in \widehat{\mathfrak{u}},\ \varphi \in \mathrm{Char}(L_\lambda)\}.$$

(ii) *Let $\lambda_1, \lambda_2 \in \widehat{\mathfrak{u}}$ and $\varphi_1 \in \mathrm{Char}(L_{\lambda_1})$, $\varphi_2 \in \mathrm{Char}(L_{\lambda_2})$. Then $\Phi_{(\lambda_1,\varphi_1)} = \Phi_{(\lambda_2,\varphi_2)}$ if and only if λ_1 and λ_2 have the same quasi-orbit under the coadjoint action* Ad^* *and $\varphi_1 = \varphi_2$.*

A few words about the proof of Theorem A are in order. The essential step consists in the analysis of the restriction $\varphi|_U$ to U of a given character $\varphi \in \mathrm{Char}(G)$. A first crucial fact is that $\psi = \varphi|_U \circ \exp$ is a G-invariant function of positive type on $\mathfrak{u}$ (for the underlying abelian group structure) and is extremal under such functions (see Proposition 3.2 and Theorem 2.11); the Fourier transform of ψ is a G-invariant ergodic probability measure on $\widehat{\mathfrak{u}}$, which can be identified with an adelic solenoid $X = \mathbb{A}^d/\mathbb{Q}^d$, where $\mathbb{A}$ is the ring of adèles (see Section 4.2).

Using Ratner's measure rigidity results for homogeneous spaces of S-adic Lie groups (see [26, 35]), we classify all G-invariant probability measures on X; a corresponding description, based on Ratner's topological rigidity results, is given for the G-orbit closures in X. The results, which are of independent interest, are summarized as follows; for more precise statements, see Theorems 5.8 and 5.9 below.

Theorem B. *Let* $\mathbf{G}$ *be a connected algebraic subgroup of* GL_d *defined over* $\mathbb{Q}$*. Assume that* $G = \mathbf{G}(\mathbb{Q})$ *is generated by unipotent one-parameter subgroups and consider the natural action of* G *on the adelic solenoid* $X = \mathbb{A}^d/\mathbb{Q}^d$*.*

(i) *Every ergodic* G*-invariant probability measure on* X *is of the form* μ_{x+Y} *for a point* x *in* X *and a* G*-invariant subsolenoid (that is, a closed and connected subgroup)* Y *of* X*, where* μ_{x+Y} *is the normalized Haar measure on* $x + Y$*.*

(ii) *For every* $x \in X$*, the closure of the* G*-orbit of* x *in* X *coincides with* $x + Y$ *for a* G*-invariant subsolenoid* Y *of* X*.*

As far as we know, our work constitutes the first application of Ratner's rigidity results in the context of operator algebras.

Remark 1.1. (i) For every $\lambda \in \widehat{\mathfrak{u}}$, the group L_λ as defined above is the set of k-points of a normal subgroup $\mathbf{L}_\lambda$ of $\mathbf{L}$ defined over k; indeed, $\mathbf{L}_\lambda$ is the kernel of the k-rational representation of $\mathbf{L}$ on the k-vector space $\mathfrak{u}/\mathfrak{k}_\lambda$. (Observe that $\mathbf{L}_\lambda$ may be non-connected.) The set $\mathrm{Char}(L_\lambda)$ can easily be described by the results in [2] mentioned above (see Proposition 6.3 below).

(ii) Theorem A allows a full classification of $\mathrm{Char}(G)$ for any group G as above through the following procedure:

- determine the L-invariant ideals of $\mathfrak{u}$;
- fix an L-invariant ideal $\mathfrak{k}$ of $\mathfrak{u}$; determine the space $\overline{\mathfrak{p}}$ of L-fixed elements in the center of $\mathfrak{u}/\mathfrak{k}$ and let $\mathfrak{p}$ be its the inverse image in $\mathfrak{u}$;
- determine the subgroup $L(\mathfrak{k}, \mathfrak{p})$ of L of all elements which act trivially on $\mathfrak{p}/\mathfrak{k}$; determine $\mathrm{Char}(L(\mathfrak{k}, \mathfrak{p}))$;
- let $\lambda \in \widehat{\mathfrak{u}}$ with $\mathfrak{k}_\lambda = \mathfrak{k}$; then $\mathfrak{p}_\lambda = \mathfrak{p}$ and for $\varphi \in \mathrm{Char}(L(\mathfrak{k}, \mathfrak{p}))$, write $\Phi_{(\lambda,\varphi)} \in \mathrm{Char}(G)$.

See Section 7 for some examples.

(iii) The assumption that G is generated by its unipotent one-parameter subgroups is equivalent to the assumption that the Levi component L of G is semi-simple and that $L^+ = L$, where L^+ is the subgroup of L defined as in [8, §6]. A necessary condition for the equality $L^+ = L$ to hold is that every non-trivial simple algebraic normal subgroup of $\mathbf{L}$ is k-isotropic (that is, $k - \mathrm{rank}(\mathbf{L}) \geq 1$). It is known that $L^+ = L$ when $\mathbf{L}$ is simply-connected and split or quasi-split over k (see [38, Lemma 64]).

(iv) A general result about $\mathrm{Char}(G)$ cannot be expected when the condition $L = L^+$ is dropped; indeed, not even the normal subgroup structure of L is known in general when $\mathbf{L}$ is k-anisotopic (see [34, Chapter 9]).

(v) We do not know whether an appropriate version of Theorem A is valid when k is of positive characteristic (say, when $k = F(X)$ for a finite field F). The first obstacle to overcome is that a Levi subgroup of $\mathbf{G}$ does not necessarily exist; the second one is the less tight relationship between unipotent groups and their Lie algebras; finally, Ratner's measure rigidity theorem is not known in full generality (see [16] for a partial result).

(vi) In the case where $\mathbf{G}$ is unipotent, that is, $G = U$, we obtain a "Kirillov type" description of $\mathrm{Char}(U)$: the map $\Phi: \widehat{\mathfrak{u}} \to \mathrm{Char}(U)$, defined by $\Phi(\lambda)(u) = \chi_\lambda(u)$ for $u \in P_\lambda$ and $\Phi(\lambda)(u) = 0$ otherwise, factorizes to a bijection between the space of quasi-orbits in $\widehat{\mathfrak{u}}$ under Ad^* and $\mathrm{Char}(U)$. The set $\mathrm{Char}(U)$ was determined in [10, Theorem 4.2] and [37] and also implicitly in [33, Proposition 2.7].

We now rephrase Theorem A in terms of factor representations of G. Recall that a *factor representation* of a group G is a unitary representation π of G on a Hilbert space $\mathcal{H}$ such that the von Neumann subalgebra $\pi(G)''$ of $\mathcal{L}(\mathcal{H})$ is a factor (see also Section 2.2). Two such representations π_1 and π_2 are said to be *quasi-equivalent* if there exists an isomorphism $\Phi: \pi_1(G)'' \to \pi_2(G)''$ such that $\Phi(\pi_1(g)) = \pi_2(g)$ for every $g \in G$. A factor representation π of G is said to be of *finite type* if $\pi(G)''$ is a finite factor, that is, if $\pi(G)''$ admits a trace τ; in this case, $\tau \circ \pi$ belongs to $\mathrm{Char}(G)$ and the map $\pi \mapsto \tau \circ \pi$ factorizes to a bijection between the quasi-equivalence classes of factor representations of finite type of G and $\mathrm{Char}(G)$; for all this, see [12, Chapters 6 and 17].

The next result follows immediately from Theorem A, in combination with Proposition 2.4 below and [12, Corollary 6.8.10].

Let $\Gamma = L \ltimes N$ be a semi-direct product of a subgroup L and an abelian normal subgroup N. Let σ be a unitary representation of L on a Hilbert space $\mathcal{H}$ and let $\chi \in \widehat{N}$ be such that $\chi^g = \chi$ for every $g \in L$. It is straightforward to check that $\chi\sigma$ defined by $\chi\sigma(g,n) = \chi(n)\sigma(g)$ for $(g,n) \in \Gamma$ is a unitary representation of Γ on $\mathcal{H}$.

Theorem C. *Let $G = LU$ be as in Theorem A.*

(i) *For every $\lambda \in \widehat{\mathfrak{u}}$ and every factor representation σ of finite type of L_λ, the representation $\pi_{(\lambda,\sigma)} := \mathrm{Ind}_{L_\lambda P_\lambda}^G \chi_\lambda\sigma$ induced by $\chi_\lambda\sigma$ is a factor representation of finite type of G; moreover, every factor representation of finite type of G is quasi-equivalent to a representation of the form $\pi_{(\lambda,\sigma)}$ as above.*

(ii) *Let $\lambda_1, \lambda_2 \in \widehat{\mathfrak{u}}$ and let σ_1, σ_2 be factor representations of finite type of L_{λ_1}, L_{λ_2}, respectively. Then $\pi_{(\lambda_1,\sigma_1)}$ and $\pi_{(\lambda_2,\sigma_2)}$ are quasi-equivalent if and only if λ_1 and λ_2 have the same quasi-orbit under the coadjoint action* Ad^* *and σ_1 and σ_2 are quasi-equivalent.*

This paper is organized as follows. In Section 2, we establish with some detail general facts about functions of positive type on a group Γ which are invariant under a group of automorphisms of Γ; in particular, we give for two basic results (Theorems 2.11 and 2.12) new short proofs of an operator algebraic flavor. Section 3 deals with the crucial relationship (Proposition 3.2) between traces on unipotent algebraic groups and invariant traces on the associated Lie algebra. In Section 4, we show how the study of characters on an algebraic group over $\mathbb{Q}$ leads to the study of invariant probability measures on adelic solenoids. Such measures as well as orbits closures are classified in Section 5, providing the proof of Theorem B. The proof of Theorem A is completed in Section 6. In Section 7, we compute $\mathrm{Char}(G)$ for a few specific examples of algebraic groups G.

2. Invariant traces and von Neumann algebras

We consider functions of positive type on a group Γ which are invariant under a group G of automorphisms of Γ, which may be larger than the group of inner automorphisms of Γ. A systematic treatment of such functions is missing in the literature, although they have already been considered in [41] and [40]. In view of their importance in this article and for the convenience of the reader as well, we establish with more detail than necessary some general facts about them; in particular, we give new and more transparent proofs for two crucial and non-obvious properties of these functions (Theorems 2.11 and 2.12), based on the consideration of associated von Neumann algebras.

2.1. Some general facts on invariant traces

Let Γ, G be discrete groups and assume that G acts by automorphisms on Γ.

Definition 2.1. (i) A function $\varphi\colon \Gamma \to \mathbb{C}$ is called a *G-invariant trace* on Γ if

- φ is of positive type, that is, for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and all $\gamma_1, \dots, \gamma_n \in \Gamma$, we have
$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \varphi(\gamma_j^{-1}\gamma_i) \geq 0,$$
- $\varphi(g(\gamma)) = \varphi(\gamma)$ for all $\gamma \in \Gamma$ and $g \in G$, and
- φ is normalized, that is, $\varphi(e) = 1$.

We denote by $\mathrm{Tr}(\Gamma, G)$ the set of G-invariant traces on Γ. In the case where $G = \Gamma$ and Γ acts on itself by conjugation, we write $\mathrm{Tr}(\Gamma)$ instead of $\mathrm{Tr}(\Gamma, \Gamma)$.

(ii) The set $\mathrm{Tr}(\Gamma, G)$ is a compact convex set in the unit ball of $\ell^\infty(\Gamma)$ endowed with the weak*-topology. Let $\mathrm{Char}(\Gamma, G)$ be the set of extremal points in $\mathrm{Tr}(\Gamma, G)$. In case $G = \Gamma$, we write as above, $\mathrm{Char}(\Gamma)$ instead of $\mathrm{Char}(\Gamma, \Gamma)$.

(iii) Functions $\varphi \in \mathrm{Char}(\Gamma, G)$ will be called *G-invariant characters* on Γ and are characterized by the following property: if ψ is a G-invariant function of positive type on Γ which is dominated by φ (that is, $\varphi - \psi$ is a function of positive type), then $\psi = \lambda\varphi$ for some $\lambda > 0$.

Remark 2.2. Assume that Γ is countable. Then $\mathrm{Tr}(\Gamma, G)$ is metrizable and $\mathrm{Char}(\Gamma, G)$ is a Borel subset of $\mathrm{Tr}(\Gamma, G)$. By Choquet's theory, every $\varphi \in \mathrm{Tr}(\Gamma, G)$ can be written as integral

$$\varphi = \int_{\mathrm{Tr}(\Gamma,G)} \psi \, \mathrm{d}\mu_\varphi(\psi)$$

for a probability measure μ_φ on $\mathrm{Tr}(\Gamma, G)$ with $\mu_\varphi(\mathrm{Char}(\Gamma, G)) = 1$. When G contains the group of inner automorphisms of Γ, the measure μ_φ is unique, as $\mathrm{Tr}(\Gamma, G)$ is a Choquet simplex in this case [41].

The proof of the following proposition is straightforward. Observe that, if N is a G-invariant normal subgroup of Γ, then G acts by automorphisms on the quotient group Γ/N.

Proposition 2.3. *Let N be a G-invariant normal subgroup of Γ and let $p\colon \Gamma \to \Gamma/N$ be the canonical projection.*

(i) *For every $\varphi \in \mathrm{Tr}(\Gamma/N, G)$, we have $\varphi \circ p \in \mathrm{Tr}(\Gamma, G)$.*

(ii) *The image of the map*

$$\mathrm{Tr}(\Gamma/N, \Gamma) \to \mathrm{Tr}(\Gamma, G), \quad \varphi \mapsto \varphi \circ p$$

is $\{\psi \in \mathrm{Tr}(\Gamma, G) \mid \psi|_N = 1_N\}$.

(iii) *We have $\varphi \in \mathrm{Char}(\Gamma/N, G)$ if and only if $\varphi \circ p \in \mathrm{Char}(\Gamma, G)$.*

Let φ be a normalized function of positive type on Γ. Recall (see [3, Theorem C.4.10]) that there is a so-called *GNS-triple* $(\pi, \mathcal{H}, \xi)$ associated to φ, consisting of a cyclic unitary representation of Γ on a Hilbert space $\mathcal{H}$ with cyclic unit vector ξ such that

$$\varphi(\gamma) = \langle \pi(\gamma)\xi, \xi \rangle \quad \text{for all } \gamma \in \Gamma.$$

The triple $(\pi, \mathcal{H}, \xi)$ is unique in the following sense: if $(\pi', \mathcal{H}', \xi')$ is another GNS-triple associated to φ, then there is a *unique* isomorphism $U\colon \mathcal{H} \to \mathcal{H}'$ of Hilbert spaces such that

$$U\pi(\gamma)U^{-1} = \pi'(\gamma) \quad \text{for all } \gamma \in \Gamma \quad \text{and} \quad U\xi = \xi'.$$

As the next proposition shows, invariant traces on a subgroup of Γ can be induced to invariant traces on Γ.

For a function $\psi\colon Y \to \mathbb{C}$ defined on a subset Y of a set X, we denote by $\widetilde{\psi}$ the *trivial extension* of ψ to X, that is, the function $\widetilde{\psi}\colon X \to \mathbb{C}$ given by

$$\widetilde{\psi}(x) = \begin{cases} \psi(x) & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

Proposition 2.4. *Let H be a G-invariant subgroup of Γ and $\psi \in \mathrm{Tr}(H, G)$. Then $\widetilde{\psi} \in \mathrm{Tr}(\Gamma, G)$. Moreover, if σ is a GNS-representation of H associated to ψ, then the GNS-representation of Γ associated to $\widetilde{\psi}$ is equivalent to the induced representation $\mathrm{Ind}_H^\Gamma \sigma$.*

Proof. Set $\varphi := \widetilde{\psi}$. It is obvious that φ is G-invariant. The fact that φ is a function of positive type can be checked directly from the definition of such a function (see [22, §32.43]). As we need to identify the GNS-representation associated to φ, we sketch another well-known proof for this fact.

Let $(\sigma, \mathcal{K}, \eta)$ be a GNS-triple associated to ψ. Let $\pi = \mathrm{Ind}_H^\Gamma \sigma$ be realized on $\mathcal{H} = \ell^2(\Gamma/H, \mathcal{K})$, as in [17, §6.1, Remark 2]. Let $\xi \in \mathcal{H}$ be defined by $\xi(H) = \eta$ and $\xi(\gamma H) = 0$ if $\gamma \notin H$. Then $\varphi(\gamma) = \langle \pi(\gamma)\xi, \xi\rangle$ for every $\gamma \in \Gamma$ and ξ is a cyclic vector for π. So, $(\pi, \mathcal{H}, \xi)$ is a GNS-triple for φ. ∎

Attached to a given invariant trace on Γ, there are two invariant subgroups of Γ which will play an important role in the sequel.

Proposition 2.5. *Let $\varphi \in \mathrm{Tr}(\Gamma, G)$. Define*

$$K_\varphi = \{\gamma \in \Gamma \mid \varphi(\gamma) = 1\} \quad \textit{and} \quad P_\varphi = \{\gamma \in \Gamma \mid |\varphi(\gamma)| = 1\}.$$

- (i) *K_φ and P_φ are G-invariant closed subgroups of Γ with $K_\varphi \subset P_\varphi$.*
- (ii) *For $x \in P_\varphi$ and $\gamma \in \Gamma$, we have $\varphi(x\gamma) = \varphi(x)\varphi(\gamma)$; in particular, the restriction of φ to P_φ is a G-invariant unitary character of P_φ.*
- (iii) *For $x \in P_\varphi$ and $g \in G$, we have $g(x)x^{-1} \in K_\varphi$.*

Proof. Let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple associated to φ. Using the equality case of Cauchy–Schwarz inequality, it is clear that

$$K_\varphi = \{x \in \Gamma \mid \pi(x)\xi = \xi\} \quad \text{and} \quad P_\varphi = \{x \in \Gamma \mid \pi(x)\xi = \varphi(x)\xi\}.$$

Claims (i), (ii) and (iii) follow from this. ∎

We will later need the following elementary lemma.

Lemma 2.6. *Let $\varphi \in \mathrm{Tr}(\Gamma, G)$ and $\gamma \in \Gamma$. Assume that there exists a sequence $(g_n)_{n\geq 1}$ in G such that*

$$\varphi(g_n(\gamma)g_m(\gamma)^{-1}) = 0 \quad \textit{for all } n \neq m.$$

Then $\varphi(\gamma) = 0$.

Proof. Let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple for φ. We have

$$\langle \pi(g_m(\gamma)^{-1})\xi, \pi(g_n(\gamma)^{-1})\xi\rangle = \langle \pi(g_n(\gamma)g_m(\gamma)^{-1})\xi, \xi\rangle = \varphi(g_n(\gamma)g_m(\gamma)^{-1}) = 0$$

for all m, n with $m \neq n$. Therefore, $(\pi(g_n(\gamma)^{-1})\xi)_{n\geq 1}$ is an orthonormal sequence in $\mathcal{H}$ and so converges weakly to 0. The claim follows, since $\varphi(\gamma) = \overline{\varphi(\gamma^{-1})}$ and, for all n,

$$\varphi(\gamma^{-1}) = \varphi(g_n(\gamma)^{-1}) = \langle \pi(g_n(\gamma)^{-1})\xi, \xi\rangle.$$ ∎

2.2. Invariant traces and von Neumann algebras

We relate traces on groups to traces on appropriate von Neumann algebras.

Let Γ, G be discrete groups and assume that G acts by automorphisms on Γ. The uniqueness of the GNS construction for functions of positive type has the following consequence for G-invariant traces on Γ.

Proposition 2.7. *Let $\varphi \in \mathrm{Tr}(\Gamma, G)$ and let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple associated to φ. There exists a unique unitary representation $g \mapsto U_g$ of G on $\mathcal{H}$ such that*

$$U_g \pi(\gamma) U_g^{-1} = \pi(g(\gamma)) \quad \textit{for all } g \in G,\ \gamma \in \Gamma \quad \textit{and} \quad U_g \xi = \xi.$$

Proof. Let $g \in G$. Consider the unitary representation π^g of Γ on $\mathcal{H}$ given by $\pi^g(\gamma) = \pi(g(\gamma))$ for $\gamma \in \Gamma$. Since φ is invariant under g, the triple $(\pi^g, \mathcal{H}, \xi)$ is another GNS-triple associated to φ. Hence, there is a unique unitary operator $U_g : \mathcal{H} \to \mathcal{H}$ such that

$$U_g \pi(\gamma) U_g^{-1} = \pi^g(\gamma) \quad \text{for all } \gamma \in \Gamma \quad \text{and} \quad U_g \xi = \xi.$$

Using the uniqueness of U_g, one checks that $g \mapsto U_g$ is a representation of G. ∎

We now give a necessary and sufficient condition for a G-invariant trace on Γ to be a character.

Let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple associated to φ, and let $g \mapsto U_g$ be the unitary representation of G on $\mathcal{H}$ as in Proposition 2.7. Let $\mathcal{M}_\varphi$ be the von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ generated by the set of operators $\pi(\Gamma) \cup \{U_g \mid g \in G\}$, that is,

$$\mathcal{M}_\varphi := \{\pi(\gamma), U_g \mid \gamma \in \Gamma,\ g \in G\}''.$$

Proposition 2.8. *Let $\varphi \in \mathrm{Tr}(\Gamma, G)$ with associated GNS-triple $(\pi, \mathcal{H}, \xi)$ and let $\mathcal{M}_\varphi$ be the von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ as above. For every $T \in \mathcal{L}(\mathcal{H})$ with $0 \le T \le I$, let φ_T be defined by $\varphi_T(\gamma) = \langle \pi(\gamma) T\xi, T\xi \rangle$ for $\gamma \in \Gamma$. Then $T \mapsto \varphi_T$ is a bijection between $\{T \in \mathcal{M}'_\varphi \mid 0 \le T \le I\}$ and the set of G-invariant functions of positive type on Γ which are dominated by φ. In particular, we have $\varphi \in \mathrm{Char}(\Gamma, G)$ if and only if $\mathcal{M}'_\varphi = \mathbb{C} I$.*

Proof. The map $T \mapsto \varphi_T$ is known to be a bijection between the set $\{T \in \pi(\Gamma)' \mid 0 \le T \le I\}$ and the set of functions of positive type on Γ which are dominated by φ (apply [12, Proposition 2.5.1] to the $*$-algebra $\mathbb{C}[\Gamma]$, with the convolution product and the involution given by $f^*(\gamma) = f(\gamma^{-1})$ for $f \in \mathbb{C}[\Gamma]$).

Therefore, it suffices to check that, for $T \in \pi(\Gamma)'$ with $0 \le T \le I$, the function φ_T is G-invariant if and only if $T \in \{U_g \mid g \in G\}'$.

Let $T \in \mathcal{M}'_\varphi$ with $0 \le T \le I$. For every $g \in G$, we have

$$\begin{aligned} \varphi_T(g(\gamma)) &= \langle \pi(g(\gamma)) T\xi, T\xi \rangle = \langle U_g \pi(\gamma) U_{g^{-1}} T\xi, T\xi \rangle \\ &= \langle \pi(\gamma) T U_{g^{-1}} \xi, T U_{g^{-1}} \xi \rangle = \langle \pi(\gamma) T\xi, T\xi \rangle = \varphi_T(\gamma), \end{aligned}$$

for all $\gamma \in \Gamma$; so, φ_T is G-invariant.

Conversely, let $T \in \pi(\Gamma)'$ with $0 \le T \le I$ be such that φ_T is G-invariant. Let $g \in G$. For every $\gamma \in \Gamma$, we have

$$\begin{aligned}\varphi_{U_{g^{-1}}TU_g}(\gamma) &= \langle \pi(\gamma)U_{g^{-1}}TU_g\xi, U_{g^{-1}}TU_g\xi\rangle = \langle \pi(\gamma)U_{g^{-1}}T\xi, U_{g^{-1}}T\xi\rangle \\ &= \langle U_g\pi(\gamma)U_{g^{-1}}T\xi, T\xi\rangle = \langle \pi(g(\gamma))T\xi, T\xi\rangle = \varphi_T(g(\gamma)) = \varphi_T(\gamma).\end{aligned}$$

Since $0 \le U_{g^{-1}}TU_g \le I$, it follows that $U_{g^{-1}}TU_g = T$, by uniqueness of T; therefore, $T \in \mathcal{M}'_\varphi$. ■

Let $Z(\Gamma)$ be the center of Γ. We call the subgroup

$$Z(\Gamma)^G := \{z \in Z(\Gamma) \mid g(z) = z \text{ for all } g \in G\}$$

the *G-center* of Γ. We draw a first consequence on the values taken by a G-invariant character on $Z(\Gamma)^G$.

Corollary 2.9. *Let $\varphi \in$ Char(Γ, G). The G-center $Z(\Gamma)^G$ of Γ is contained in P_φ.*

Proof. Let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple associated to φ. For every $z \in Z(\Gamma)^G$, the operator $\pi(z)$ commutes with $\pi(\gamma)$ and U_g for every $\gamma \in \Gamma$ and every $g \in G$. It follows from Proposition 2.8 that $\pi(z)$ is a scalar multiple of $I_{\mathcal{H}}$ and hence that $z \in P_\varphi$. ■

We will be mostly interested in the case where G contains the group of all inner automorphisms of Γ. Upon replacing G by the semi-direct group $G \ltimes \Gamma$, we will therefore often assume that Γ is a normal subgroup of G.

Let G be a discrete group and N a normal subgroup of G. Then Tr$(N, G) \subset$ Tr(N) denotes the convex set of G-invariant traces on N and Char(N, G) the set of extreme points in Tr(N, G). We first draw a consequence of Propositions 2.7 and 2.8 in the case $N = G$.

Recall that a (finite) trace on a von Neumann algebra $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a positive linear functional τ on $\mathcal{M}$ such that

$$\tau(TS) = \tau(ST) \quad \text{for all } S, T \in \mathcal{M}.$$

Such a trace τ is faithful if $\tau(T^*T) > 0$ for every $T \neq 0$ and normal if τ is continuous on the unit ball of $\mathcal{M}$ for the weak operator topology. A von Neumann algebra $\mathcal{M}$ which has a normal faithful trace is said to be a *finite von Neumann algebra*.

Let τ be a normal faithful trace on $\mathcal{M}$ and let T be in the center $\mathcal{M} \cap \mathcal{M}'$ of $\mathcal{M}$ with $0 \le T \le I$. Then $\tau_T : \mathcal{M} \to \mathbb{C}$, defined by

$$\tau_T(S) = \tau(ST) \quad \text{for all } S \in \mathcal{M},$$

is a normal trace on $\mathcal{M}$ which is dominated by τ (that is, $\tau_T(S) \le \tau(S)$ for every $S \in \mathcal{M}$ with $S \ge 0$). The map $T \mapsto \tau_T$ is a bijection between $\{T \in \mathcal{M} \cap \mathcal{M}' \mid 0 \le T \le I\}$ and the set of normal traces on $\mathcal{M}$ which are dominated by τ (see [13, Chapter I, §6, Theorem 3]).

Recall that a von Neumann subalgebra $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ is a *factor* if its center $\mathcal{M} \cap \mathcal{M}'$ consists only of multiples of the identity operator I.

Corollary 2.10. *Let $\varphi \in \mathrm{Tr}(G)$ and let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple associated to φ.*

(i) *The linear functional $\tau\colon T \mapsto \langle T\xi, \xi\rangle$ is a normal faithful trace on $\pi(G)''$.*

(ii) *The commutant $\mathcal{M}'_\varphi$ of $\mathcal{M}_\varphi$ coincides with the center of the von Neumann algebra $\pi(G)''$ generated by $\pi(G)$. In particular, $\varphi \in \mathrm{Char}(G)$ if and only if $\pi(G)''$ is a factor.*

Proof. (i) One checks immediately that τ, as defined above, is a trace on $\pi(G)''$. It is clear that τ is normal. Let $T \in \pi(G)''$ be such that $\tau(T^*T) = 0$. Then

$$\|T\pi(g)\xi\|^2 = \tau\big(\pi(g^{-1})T^*T\pi(g)\big) = \tau(T^*T) = 0,$$

that is, $T\pi(g)\xi = 0$ for all $g \in G$; hence, $T = 0$ since ξ is a cyclic vector for π. So, τ is faithful.

(ii) Observe first that, for every $g \in G$, we have

$$U_g\pi(x)U_{g^{-1}} = \pi(gxg^{-1}) = \pi(g)\pi(x)\pi(g^{-1}) \quad \text{for all } x \in G,$$

where $g \mapsto U_g$ is the representation of G as in Proposition 2.7. It follows that $U_gTU_{g^{-1}} = \pi(g)T\pi(g^{-1})$ for every $T \in \pi(G)''$. Hence, $\pi(G)'' \cap \pi(G)'$ is contained in $\mathcal{M}'_\varphi$.

Conversely, let $T \in \mathcal{M}'_\varphi$ with $0 \le T \le I$. By Proposition 2.8, $\varphi_{T^{1/2}}$ is a G-invariant function of positive type dominated by φ. The canonical extension of $\varphi_{T^{1/2}}$ to $\pi(G)''$ is a normal trace τ' on $\pi(G)''$. Hence, by the result recalled above, $\tau' = \tau_S$ for a unique $S \in \pi(G)' \cap \pi(G)''$ with $0 \le S \le I$. This shows that $\varphi_{T^{1/2}} = \varphi_{S^{1/2}}$. Since T and S both belong to $\pi(G)'$, it follows that $T = S$. So, $T \in \pi(G)' \cap \pi(G)''$. Therefore, $\mathcal{M}'_\varphi$ is contained in $\pi(G)' \cap \pi(G)''$. ∎

The following result, which will be crucial in the sequel, appears in [41, Lemma 14]; the proof we give here for it is shorter and more transparent than the original one.

Theorem 2.11. *Let G be a discrete group, N a normal subgroup of G and $\psi \in \mathrm{Char}(G)$. Then $\psi|_N \in \mathrm{Char}(N, G)$.*

Proof. Let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple associated to ψ. Set $\varphi := \psi|_N$ and let $\mathcal{K}$ be the closed linear span of $\{\pi(x)\xi \mid x \in N\}$. Then $(\pi|_N, \mathcal{K}, \xi)$ is a GNS-triple associated to φ.

Let $g \mapsto U_g$ be the representation of G on $\mathcal{H}$ associated to ψ as in Proposition 2.7. The subspace $\mathcal{K}$ is invariant under U_g for $g \in G$, since $U_g\pi(x)U_g^{-1} = \pi(gxg^{-1})$ and $U_g\xi = \xi$. So, the representation of G on $\mathcal{K}$ associated to φ is $g \mapsto U_g|_{\mathcal{K}}$. Let $\mathcal{M}_\varphi$ be the von Neumann subalgebra of $\mathcal{L}(\mathcal{K})$ generated by

$$\{\pi(x)|_{\mathcal{K}} \mid x \in N\} \cup \{U_g|_{\mathcal{K}} \mid g \in G\}.$$

In view of Proposition 2.8, it suffices to show that $\mathcal{M}'_\varphi = \mathbb{C}I$. Let $T \in \mathcal{M}'_\varphi$ with $0 \le T \le I$. Consider the linear functional τ' on $\pi(G)''$ given by

$$\tau'(S) = \langle ST\xi, T\xi\rangle \quad \text{for all } S \in \pi(G)''.$$

We claim that τ' is a normal trace on $\pi(G)''$. Indeed, it is clear that τ is normal; moreover, for $g, h \in G$, we have

$$\begin{aligned}\tau'(ghg^{-1}) &= \langle \pi(ghg^{-1})T\xi, T\xi\rangle = \langle U_g\pi(h)U_{g^{-1}}T\xi, T\xi\rangle \\ &= \langle \pi(h)TU_{g^{-1}}\xi, U_{g^{-1}}T\xi\rangle = \langle \pi(h)T\xi, T\xi\rangle = \tau'(h).\end{aligned}$$

Let $\tau'' := \tau + \tau'$, where τ is the faithful trace on $\pi(G)''$ defined by φ, as in Corollary 2.10. Then τ'' is a normal faithful trace on $\pi(G)''$, and τ'' dominates τ and τ'. Since $\pi(G)''$ is a factor, it follows that τ and τ' are both proportional to τ''. Hence, there exists $\lambda \geq 0$ such that $\tau' = \lambda\tau$. So,

$$\langle \pi(x)T\xi, T\xi\rangle = \langle \pi(x)\sqrt{\lambda}\xi, \sqrt{\lambda}\xi\rangle \quad \text{for all } x \in N.$$

Since $T \in \{\pi(x)|_{\mathcal{K}} \mid x \in N\}'$ and $0 \leq T \leq I$, it follows that $T = \sqrt{\lambda}I_{\mathcal{K}}$. ■

As we now show, the set of characters of a product group admits a simple description; again, this is a result due to Thoma [40, Satz 4], for which we provide a short proof.

For sets $X_1, \ldots, X_r$ and functions $\varphi_i \colon X_i \to \mathbb{C}$, $i \in \{1, \ldots, r\}$, we denote by $\varphi_1 \otimes \cdots \otimes \varphi_r$ the function on $X_1 \times \cdots \times X_r$ given by

$$\varphi_1 \otimes \cdots \otimes \varphi_r(x_1, \ldots, x_r) = \varphi_1(x_1)\cdots\varphi_r(x_r),$$

for all $(x_1, \ldots, x_r) \in X_1 \times \cdots \times X_r$.

Theorem 2.12. *Let G_1, G_2 be discrete groups. Then*

$$\mathrm{Char}(G_1 \times G_2) = \big\{\varphi_1 \otimes \varphi_2 \mid \varphi_1 \in \mathrm{Char}(G_1),\ \varphi_2 \in \mathrm{Char}(G_2)\big\}.$$

Proof. Set $G := G_1 \times G_2$.

For $i = 1, 2$, let $\varphi_i \in \mathrm{Char}(G_i)$. We claim that

$$\varphi := \varphi_1 \otimes \varphi_2 \in \mathrm{Char}(G).$$

Indeed, let $(\pi_i, \mathcal{H}_i, \xi_i)$ be a GNS-triple associated to φ_i. Then $(\pi, \mathcal{H}, \xi)$ is a GNS-triple associated to φ, where π is the tensor product representation $\pi_1 \otimes \pi_2$ on $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\xi := \xi_1 \otimes \xi_2$. In view of Corollary 2.10, we have to show that $\pi(G)''$ is a factor. For this, it suffices to show that the von Neumann algebra $\mathcal{M}$ generated by $\pi(G)'' \cup \pi(G)'$ coincides with $\mathcal{L}(\mathcal{H})$.

On the one hand, $\pi(G)''$ contains $\pi_1(G_1)'' \otimes I$ and $I \otimes \pi_2(G_2)''$, and $\pi(G)'$ contains $\pi_1(G_1)' \otimes I$ and $I \otimes \pi(G_2)'$; hence, $\mathcal{M}$ contains $\mathcal{M}_1 \otimes \mathcal{M}_2$, where $\mathcal{M}_i$ is the von Neumann algebra generated by $\pi_i(G_i)'' \cup \pi_i(G_i)'$. On the other hand, since $\varphi_i \in \mathrm{Char}(G_i)$, we have $\mathcal{M}_i = \mathcal{L}(\mathcal{H}_i)$. So, $\mathcal{M}$ contains the von Neumann algebra generated by $\{T_1 \otimes T_2 \mid T_1 \in \mathcal{L}(\mathcal{H}_1), T_2 \in \mathcal{L}(\mathcal{H}_2)\}$, which is $\mathcal{L}(\mathcal{H})$.

Conversely, let $\varphi \in \mathrm{Char}(G)$. Let $(\pi, \mathcal{H}, \xi)$ be a GNS-triple associated to φ. By Corollary 2.10, $\mathcal{M} := \pi(G)''$ is a factor.

For $i = 1, 2$, set $\mathcal{M}_i := \pi(G_i)''$, where we identify G_i with the subgroup $G_i \times \{e\}$ of G. We claim that $\mathcal{M}_1$ and $\mathcal{M}_2$ are factors. Indeed, since $\mathcal{M}_1 \subset \mathcal{M}_2'$, the center $\mathcal{M}_1 \cap \mathcal{M}_1'$ of $\mathcal{M}_1$ is contained in $\mathcal{M}_2' \cap \mathcal{M}_1'$. As $\mathcal{M}_1 \cup \mathcal{M}_2$ generate $\mathcal{M}$, it follows that $\mathcal{M}_1 \cap \mathcal{M}_1'$ is contained in $\mathcal{M}'$ and so in $\mathcal{M} \cap \mathcal{M}'$. Hence, $\mathcal{M}_1 \cap \mathcal{M}_1' = \mathbb{C}I$, since $\mathcal{M}$ is a factor. So, $\mathcal{M}_1$ and, similarly, $\mathcal{M}_2$ are factors.

Next, recall (Corollary 2.10) that $\mathcal{M}$ has a normal faithful trace τ given by $\tau(T) = \langle T\xi, \xi\rangle$ for $T \in \mathcal{M}$. The restriction $\tau^{(1)}$ of τ to $\mathcal{M}_1$ is a normal faithful trace on $\mathcal{M}_1$.

Let $T_2 \in \mathcal{M}_2$ with $0 \leq T_2 \leq I$ and $T_2 \neq 0$. Define a positive and normal linear functional $\tau^{(1)}_{T_2}$ on $\mathcal{M}_1$ by

$$\tau^{(1)}_{T_2}(S) = \tau(ST_2) \quad \text{for all } S \in \mathcal{M}_1.$$

For $S, T \in \mathcal{M}_1$, we have

$$\tau^{(1)}_{T_2}(ST) = \tau(STT_2) = \tau(ST_2T) = \tau((ST_2)T) = \tau(T(ST_2)) = \tau^{(1)}_{T_2}(TS).$$

So, $\tau^{(1)}_{T_2}$ is a normal trace on $\mathcal{M}_1$. Clearly, $\tau^{(1)}_{T_2}$ is dominated by $\tau^{(1)}$. Since $\mathcal{M}_1$ is a factor, it follows from the result quoted before Corollary 2.10 that there exists a scalar $\lambda(T_2) \geq 0$ such that $\tau^{(1)}_{T_2} = \lambda(T_2)\tau^{(1)}$, that is,

$$\tau(T_1T_2) = \lambda(T_2)\tau(T_1) \quad \text{for all } T_1 \in \mathcal{M}_1.$$

Taking $T_1 = I$, we see that $\lambda(T_2) = \tau(T_2)$. It follows that

$$\tau(T_1T_2) = \tau(T_1)\tau(T_2) \quad \text{for all } T_1 \in \mathcal{M}_1,\ T_2 \in \mathcal{M}_2,$$

and, in particular, $\varphi = \varphi_1 \otimes \varphi_2$, for $\varphi_i = \varphi|_{G_i}$. ∎

The following result is an immediate consequence of Proposition 2.12.

Recall (see Proposition 2.3) that, when N is a normal subgroup of a group G, we can identify $\mathrm{Char}(G/N)$ with the subset $\{\varphi \in \mathrm{Char}(G) \mid \varphi|_N = 1\}$ of $\mathrm{Char}(G)$.

Corollary 2.13. *For discrete groups $G, G_1, \ldots, G_r$, let*

$$p\colon G_1 \times \cdots \times G_r \to G$$

be a surjective homomorphism. Then

$$\mathrm{Char}(G) = \big\{\varphi = \varphi_1 \otimes \cdots \otimes \varphi_r \mid \varphi|_N = 1 \text{ and } \varphi_i \in \mathrm{Char}(G_i),\ i = 1, \ldots, r\big\},$$

where N is the kernel of p. In particular, for $\varphi \in \mathrm{Char}(G)$, we have

$$\varphi(g_1 \ldots g_n) = \varphi(g_1) \ldots \varphi(g_n)$$

for all $g_i \in p(\{e\} \times \cdots \times G_i \times \cdots \times \{e\})$.

3. Traces on unipotent groups

In this section, we will show that traces on a unipotent algebraic group U are in a one-to-one correspondence with $\mathrm{Ad}(U)$-invariant positive definite functions on the Lie algebra of U.

3.1. Invariant traces on abelian groups

Let A be a discrete abelian group and $\widehat{A}$ the Pontrjagin dual of A, which is a compact abelian group. Then $\mathrm{Tr}(A)$ is the set of normalized functions of positive type on A and $\mathrm{Char}(A) = \widehat{A}$.

Let $\mathrm{Prob}(\widehat{A})$ denote the set of regular probability measures on the Borel subsets of $\widehat{A}$. For $\mu \in \mathrm{Prob}(\widehat{A})$, the Fourier–Stieltjes transform $\mathcal{F}(\mu)\colon A \to \mathbb{C}$ of μ is given by

$$\mathcal{F}(\mu)(a) = \int_{\widehat{A}} \chi(a)\, \mathrm{d}\mu(\chi) \quad \text{for all } \chi \in \widehat{A}.$$

By Bochner's theorem (see, e.g., [22, §33]), the map $\mathcal{F}\colon \mu \mapsto \mathcal{F}(\mu)$ is a bijection between $\mathrm{Prob}(\widehat{A})$ and $\mathrm{Tr}(A)$.

Let G be a group acting by automorphisms on A. Then G acts by continuous automorphisms on $\mathrm{Tr}(A)$ and on $\widehat{A} = \mathrm{Char}(A)$, via the dual action given by

$$\varphi^g(a) = \varphi(g^{-1}(a)) \quad \text{for all } \varphi \in \mathrm{Tr}(A),\ g \in G,\ a \in A.$$

Let $(g, \mu) \mapsto g_*(\mu)$ be the induced action of G on $\mathrm{Prob}(\widehat{A})$; so, $g_*(\mu)$ is the image of $\mu \in \mathrm{Prob}(\widehat{A})$ under the map $\chi \mapsto \chi^g$.

Let $\mathrm{Prob}(\widehat{A})^G$ be the subset of $\mathrm{Prob}(\widehat{A})$ consisting of G-invariant probability measures and denote by $\mathrm{Prob}(\widehat{A})^G_{\mathrm{erg}}$ the measures in $\mathrm{Prob}(\widehat{A})^G$ which are ergodic.

Proposition 3.1. *Let A be a discrete abelian group and G a group acting by automorphisms on A. The Fourier–Stieltjes transform $\mathcal{F}$ restricts to bijections*

$$\mathcal{F}\colon \mathrm{Prob}(\widehat{A})^G \to \mathrm{Tr}(A, G) \quad \textit{and} \quad \mathcal{F}\colon \mathrm{Prob}(\widehat{A})^G_{\mathrm{erg}} \to \mathrm{Char}(A, G).$$

Proof. The claims follow from the fact that $\mathcal{F}\colon \mathrm{Prob}(\widehat{A}) \to \mathrm{Tr}(A)$ is an affine G-equivariant map and that $\mathrm{Prob}(\widehat{A})^G_{\mathrm{erg}}$ is the set of extreme points in the convex compact set $\mathrm{Prob}(\widehat{A})^G$. ■

3.2. Invariant traces on unipotent groups

Let k be a field of characteristic 0. Let U_n be the group of upper triangular unipotent $n \times n$ matrices over k, for $n \geq 1$. Then U_n is the group of k-points of an algebraic group over k and its Lie algebra is the Lie algebra $\mathfrak{u}_n$ of the strictly upper triangular matrices. The exponential map $\exp\colon \mathfrak{u}_n \to U_n$ is a bijection and, by the Campbell–Hausdorff formula, there exists a polynomial map $P\colon \mathfrak{u}_n \times \mathfrak{u}_n \to \mathfrak{u}_n$ with coefficients in k such that

$\exp(X)\exp(Y) = \exp(P(X,Y))$ for all $X, Y \in \mathfrak{u}_n$. Denote by $\log\colon U_n \to \mathfrak{u}_n$ the inverse map of exp.

Let $\mathfrak{u}$ be a nilpotent Lie algebra over k. Then, by the theorems of Ado and Engel, $\mathfrak{u}$ can be viewed as Lie subalgebra of $\mathfrak{u}_n$ for some $n \geq 1$ and $\exp(\mathfrak{u})$ is an algebraic subgroup of U_n.

Let U be the group of k-points of a unipotent algebraic group over k, that is, an algebraic subgroup of U_n for some $n \geq 1$. Then $\mathfrak{u} = \log(U)$ is a Lie subalgebra of $\mathfrak{u}_n$ and $\exp\colon \mathfrak{u} \to U$ is a bijection (for all this, see [27, Chapter 14]).

For every $u \in U$, the automorphism of U given by conjugation with u induces an automorphism $\mathrm{Ad}(u)$ of the Lie algebra $\mathfrak{u}$ determined by the property

$$\exp(\mathrm{Ad}(u)(X)) = u\exp(X)u^{-1} \quad \text{for all } X \in \mathfrak{u}.$$

Observe that a function φ on U is central (that is, constant on the U conjugacy classes) if and only if the corresponding function $\varphi \circ \exp$ on $\mathfrak{u}$ is $\mathrm{Ad}(U)$-invariant.

The following proposition will be a crucial tool in our proof of Theorem A.

Proposition 3.2. *Let U be the group of k-points of a unipotent algebraic group over a field k of characteristic zero. Let $\varphi\colon U \to \mathbb{C}$. Then $\varphi \in \mathrm{Tr}(U)$ if and only if $\varphi \circ \exp \in \mathrm{Tr}(\mathfrak{u}, \mathrm{Ad}(U))$. So, the map $\varphi \mapsto \varphi \circ \exp$ is a continuous affine bijection between $\mathrm{Tr}(U)$ and $\mathrm{Tr}(\mathfrak{u}, \mathrm{Ad}(U))$.*

Proof. Set $\varphi' := \varphi \circ \exp$. Since φ and φ' are invariant, we have to show that φ is of positive type on U if and only if φ' is of positive type on $\mathfrak{u}$.

Let $Z(U)$ be the center of U and $\mathfrak{z}$ the center of $\mathfrak{u}$. Set $\chi := \varphi|_{Z(U)}$ and $\chi' := \varphi'|_{\mathfrak{z}}$.

First step. Assume either that φ is of positive type on U or that φ' is of positive type on $\mathfrak{u}$. Then χ is of positive type on $Z(U)$ and χ' is of positive type on $\mathfrak{z}$.

Indeed, this follows from the fact that $\exp\colon \mathfrak{z} \to Z$ is a group isomorphism.

We will reduce the proof of Proposition 3.2 to the case where φ has the following multiplicativity property:

$$\varphi(gz) = \varphi(g)\chi(z) \quad \text{for all } g \in U,\ z \in Z(U). \tag{$*$}$$

Observe that property ($*$) is equivalent to

$$\varphi'(X + Z) = \varphi'(X)\chi'(Z) \quad \text{for all } X \in \mathfrak{u},\ Z \in \mathfrak{z}, \tag{$*'$}$$

since $\exp(X + Z) = \exp(X)\exp(Z)$ for $X \in \mathfrak{u}$ and $Z \in \mathfrak{z}$.

Second step. To prove Proposition 3.2, we may assume that φ has property ($*'$).

Indeed, since $\mathrm{Tr}(U)$ is the closed convex hull of $\mathrm{Char}(U)$ and $\mathrm{Tr}(\mathfrak{u}, U)$ is the closed convex hull of $\mathrm{Char}(\mathfrak{u}, U)$, it suffices to prove that if $\varphi \in \mathrm{Char}(U)$, then φ' is of positive type on $\mathfrak{u}$, and that if $\varphi' \in \mathrm{Char}(\mathfrak{u}, U)$, then φ is of positive type on U. Moreover, by Corollary 2.9 and Proposition 2.5, φ has property ($*$) if $\varphi \in \mathrm{Char}(U)$, and φ' has property ($*'$) if $\varphi' \in \mathrm{Char}(\mathfrak{u}, U)$. This proves the claim.

In view of the second step, we may and will assume in the sequel that $\varphi\colon U \to \mathbb{C}$ is a central function, normalized by $\varphi(e) = 1$, with property $(*)$. If, moreover, either φ is of positive type or φ' is of positive type, then $\chi \in \widehat{Z(U)}$ and $\chi' \in \widehat{\mathfrak{u}}$, by the first step.

Third step. Assume that either φ is of positive type or that φ' is of positive type. Assume also that Ker χ' contains no non-zero linear subspace. We claim that both φ and φ' are of positive type.

To show this, it suffices to prove that $\varphi' = \widetilde{\chi}'$ (that is, $\varphi' = 0$ on $\mathfrak{u} \setminus \mathfrak{z}$). Indeed, since this last statement is equivalent to $\varphi = \widetilde{\chi}$ and since $\chi \in \widehat{Z(U)}$ and $\chi' \in \widehat{\mathfrak{z}}$, Proposition 2.4 will imply that φ and φ' are of positive type.

Let $(\mathfrak{z}^i)_{1\leq i\leq r}$ be the ascending central series of $\mathfrak{u}$; so, $\mathfrak{z}^1 = \mathfrak{z}$, $\mathfrak{z}^{i+1}$ is the inverse image in $\mathfrak{u}$ of the center of $\mathfrak{u}/\mathfrak{z}^i$ under the canonical map $\mathfrak{u} \to \mathfrak{u}/\mathfrak{z}^i$ for every i, and $\mathfrak{z}^r = \mathfrak{u}$.

Let $(Z^i(U))_{1\leq i\leq r}$ be the corresponding ascending central series of U that is given by $Z^i(U) = \exp \mathfrak{z}^i$.

We show by induction on i that $\varphi' = 0$ on $\mathfrak{z}^i \setminus \mathfrak{z}$ for every $i \in \{2, \dots, r\}$.

Indeed, let $X \in \mathfrak{z}^2 \setminus \mathfrak{z}$. There exists $Y \in \mathfrak{u}$ with $[Y, X] \neq 0$. Since $[Y, X] \in \mathfrak{z}$ and since Ker χ' contains no non-zero linear subspace, there exists $t \in k$ such that $\chi'(t[Y, X]) = \chi'([tY, X]) \neq 1$. Upon replacing Y by tY, we can assume that $\chi'([Y, X]) \neq 1$. Since φ' is $\mathrm{Ad}(U)$-invariant, it follows from property $(*)$ that

$$\varphi'(X) = \varphi'(\mathrm{Ad}(\exp Y)(X)) = \varphi'(X + [Y, X]) = \varphi'(X)\chi'([Y, X]).$$

As $\chi'([Y, X)] \neq 1$, we have $\varphi'(X) = 0$; so, the case $i = 2$ is settled.

Assume now $\varphi' = 0$ on $\mathfrak{z}^i \setminus \mathfrak{z}$ for some $i \in \{2, \dots, r\}$. Let $X \in \mathfrak{z}^{i+1} \setminus \mathfrak{z}^i$. Then there exists $Y \in \mathfrak{u}$ such that $[Y, X] \notin \mathfrak{z}^{i-1}$. Let $(t_n)_{n\geq 1}$ be a sequence of pairwise distinct elements in k. Set $y_n = \exp(t_n Y) \in U$. Denoting by $p_{i-1}\colon \mathfrak{u} \to \mathfrak{u}/\mathfrak{z}^{i-1}$ the canonical projection, we have

$$p_{i-1}(\mathrm{Ad}(y_n)X - X) = p_{i-1}(t_n[Y, X]),$$

since $[\mathfrak{u}, [\mathfrak{u}, X]] \subset \mathfrak{z}^{i-1}$. As $(t_n - t_m)[Y, X] \notin \mathfrak{z}^{i-1}$, it follows that

$$(\mathrm{Ad}(y_n)X - X) - (\mathrm{Ad}(y_m)X - X) \notin \mathfrak{z} \quad \text{for all } n \neq m.$$

Since $\mathrm{Ad}(y_n)X - X \in \mathfrak{z}^i$ and $\varphi' = 0$ on $\mathfrak{z}^i \setminus \mathfrak{z}$ by the induction hypothesis, we have therefore

$$\varphi'(\mathrm{Ad}(y_n)X - \mathrm{Ad}(y_m)X) = 0 \quad \text{for all } n \neq m. \tag{$**'$}$$

We also have, by the Campbell–Hausdorff formula,

$$\begin{aligned} p_{i-1}(\log([y_n, \exp(X)])) &= p_{i-1}(\log(\exp(t_n Y)\exp(X)\exp(-t_n Y)\exp(-X))) \\ &= p_{i-1}([t_n Y, X]), \end{aligned}$$

where $[u, v] = uvu^{-1}v^{-1}$ is the commutator of $u, v \in U$. As $[t_n Y, X]$ commutes with $[t_m Y, X]$, it follows that

$$[y_n, \exp(X)][y_m, \exp(X)]^{-1} \notin Z(U) \quad \text{for all } n \neq m.$$

Since $[y_n, \exp(X)] \in Z^i(U)$ and $\varphi = 0$ on $Z^i(U) \setminus Z(U)$ by the induction hypothesis, we have therefore

$$\varphi\big([y_n, \exp(X)][y_m, \exp(X)]^{-1}\big) = 0 \quad \text{for all } n \neq m. \tag{$**$}$$

If φ' is of positive type, it follows from Lemma 2.6 and from $(**')$ that $\varphi'(X) = 0$. If φ is of positive type, then Lemma 2.6 and $(**)$ imply that $\varphi(\exp X) = 0$, that is, $\varphi'(X) = 0$.

As a result, $\varphi' = 0$ on $\mathfrak{z}^i \setminus \mathfrak{z}$ for every $i \in \{2, \ldots, r\}$. Since $\mathfrak{z}^r = \mathfrak{u}$, the claim is proved.

Fourth step. Assume that either φ is of positive type or φ' is of positive type. Then both φ and φ' are of positive type.

We proceed by induction on $\dim_k \mathfrak{u}$. The case $\dim_k \mathfrak{u} = 0$ being obvious, assume that the claim is true for every unipotent algebraic group with a Lie algebra of dimension strictly smaller than $\dim_k \mathfrak{u}$.

In view of the third step, we may assume that there exists a subspace $\mathfrak{k}$ of $\mathfrak{z}$ with $\dim_k \mathfrak{k} > 0$ contained in $\operatorname{Ker} \chi'$. Then φ' can be viewed as a function on the nilpotent Lie algebra $\mathfrak{u}/\mathfrak{k}$ and φ as a function of positive type on the corresponding unipotent algebraic group $U/\exp(\mathfrak{k})$. Since $\dim_k \mathfrak{u}/\mathfrak{k}$ is strictly smaller than $\dim_k \mathfrak{u}$, the claim follows from the induction hypothesis. ∎

Remark 3.3. Using induction on $\dim_k \mathfrak{u}$ as well as the arguments used in the third step of the proof of Proposition 3.2, one can easily obtain the description of $\operatorname{Char}(U)$ given in Theorem A for the special case $G = U$.

Let G be a group acting by automorphisms on U. Every $g \in G$ induces an automorphism $X \mapsto g(X)$ of $\mathfrak{u}$ determined by the property

$$\exp(g(X)) = g(\exp(X)) \quad \text{for all } X \in \mathfrak{u}.$$

Let $\widehat{\mathfrak{u}}$ be the Pontrjagin dual of the additive group $\widehat{\mathfrak{u}}$. Then G acts by automorphisms $\widehat{\mathfrak{u}}$, induced by the dual action.

Since the map $\psi \mapsto \psi \circ \log$ from the space of functions on $\mathfrak{u}$ to the space of functions on U is tautologically G-equivariant, the following result is an immediate consequence of Propositions 3.2 and 3.1.

Corollary 3.4. *Let U be as in Proposition 3.2 and let G be a group acting as automorphisms of U. Assume that the image of G in* $\operatorname{Aut}(\mathfrak{u})$ *contains* $\operatorname{Ad}(U)$.

(i) *The map*

$$\operatorname{Char}(\mathfrak{u}, G) \to \operatorname{Char}(U, G), \quad \psi \mapsto \psi \circ \log$$

is a bijection.

(ii) *The map*

$$\operatorname{Prob}(\widehat{\mathfrak{u}})^G_{\mathrm{erg}} \to \operatorname{Char}(U, G), \quad \mu \mapsto \mathcal{F}(\mu) \circ \log$$

is a bijection.

Let G be a group of automorphisms of U containing $\mathrm{Ad}(U)$. Let $\lambda \in \widehat{\mathfrak{u}}$. Recall (see Section 1) that we associated to λ the following two G-invariant ideals of $\mathfrak{u}$

$$\mathfrak{k}_\lambda = \left\{X \in \mathfrak{u} \mid \lambda(\mathrm{Ad}(g)(tX)) = 1 \text{ for all } g \in G,\ t \in k\right\}$$

and

$$\mathfrak{p}_\lambda = \left\{X \in \mathfrak{u} \mid \lambda(\mathrm{Ad}(g)(tX)) = \lambda(tX) \text{ for all } g \in G,\ t \in k\right\}.$$

Proposition 3.5. *Let $\lambda \in \widehat{\mathfrak{u}}$, $p\colon \mathfrak{u} \to \mathfrak{u}/\mathfrak{k}_\lambda$, and $P_\lambda = \exp \mathfrak{p}_\lambda$.*

(i) *We have*

$$\mathfrak{p}_\lambda = p^{-1}(Z(\mathfrak{u}/\mathfrak{k}_\lambda)^G),$$

where $Z(\mathfrak{u}/\mathfrak{k}_\lambda)^G$ is the central ideal of G-fixed elements in $\mathfrak{u}/\mathfrak{k}_\lambda$.

(ii) *The map*

$$\chi_\lambda\colon P_\lambda \to \mathbf{S}^1, \quad \exp(X) \to \lambda(X)$$

is a G-invariant unitary character of P_λ.

Proof. (i) Let $X \in \mathfrak{u}$. We have

$$\begin{aligned} p(X) \in Z(\mathfrak{u}/\mathfrak{k}_\lambda)^G &\Leftrightarrow \mathrm{Ad}(g)X - X \in \mathfrak{k}_\lambda \quad \text{for all } g \in G \\ &\Leftrightarrow \mathrm{Ad}(g)(tX) - tX \in \mathfrak{k}_\lambda \quad \text{for all } g \in G,\ t \in k \\ &\Leftrightarrow \lambda(\mathrm{Ad}(g)(tX)) = \lambda(tX) \quad \text{for all } g \in G,\ t \in k \\ &\Leftrightarrow X \in \mathfrak{p}_\lambda. \end{aligned}$$

(ii) This claim is a special case of Proposition 3.2. ∎

We will later need the following elementary lemma.

Lemma 3.6. *Let U be as in Proposition 3.2 and $g \in \mathrm{Aut}(U)$. Let N be a normal subgroup of U. For $X \in \mathfrak{u}$, the set*

$$A := \left\{t \in k \mid \exp(-tX)\exp(g(tX)) \in N\right\}$$

is a subgroup of the additive group of the field k.

Proof. Observe first that $0 \in A$. Let $t, s \in A$. Then

$$\begin{aligned} &\exp(-(t-s)X)\exp(g((t-s)X)) \\ &\quad = \exp(sX)\exp(-tX)\exp(g(tX))\exp(g(-sX)) \\ &\quad = \exp(sX)\big(\exp(-tX)\exp(g(tX))\exp(g(-sX))\exp(sX)\big)\exp(-sX) \\ &\quad = \exp(sX)\big(\exp(-tX)\exp(g(tX))(\exp(-sX)\exp(g(sX)))^{-1}\big)\exp(-sX). \end{aligned}$$

Since N is a normal subgroup of U, it follows that $t - s \in A$. ∎

4. Characters and invariant probability measures

In this section, we show how a character on an algebraic group over $\mathbb{Q}$ gives rise to an invariant ergodic probability measure on an appropriate adelic solenoid.

4.1. Reduction to the case $k = \mathbb{Q}$

Let k be a number field and $G = \mathbf{G}(k)$ be the group of k-rational points of a connected linear algebraic group $\mathbf{G}$ over k. By Weil's restriction of scalars (see [45, Proposition 6.1.3], [7, §§6.17–6.21]), there is an algebraic group $\mathbf{G}'$ over $\mathbb{Q}$ such that G is naturally isomorphic to the group $G' = \mathbf{G}'(\mathbb{Q})$ of $\mathbb{Q}$-points of $\mathbf{G}'$. If $G = LU$ is a Levi decomposition of G over k, then $G' = L'U'$ is a Levi decomposition of G' over $\mathbb{Q}$, where L' and U' are the images of L and U under the isomorphism $G \to G'$. Moreover, G' is generated by its unipotent one-parameter subgroups if G is generated by unipotent one-parameter subgroups.

The isomorphism $G \to G'$ induces an isomorphism $\mathfrak{u} \to \mathfrak{u}'$ between the additive groups of the Lie algebras of U and of U' as well as an isomorphism $\widehat{\mathfrak{u}'} \to \widehat{\mathfrak{u}}$ between their Pontrjagin duals, which are equivariant for the adjoint and co-adjoint actions of G and G'.

Assume that Theorem A holds for G'. Then every element in $\mathrm{Char}(G')$ is of the form $\Phi_{\lambda',\varphi'}$ for some $\lambda' \in \widehat{\mathfrak{u}'}$ and $\varphi' \in \mathrm{Char}(L'_{\lambda'})$. For the corresponding $\lambda \in \widehat{\mathfrak{u}}$ and $\varphi \in \mathrm{Char}(L_\lambda)$, we have $\Phi_{\lambda,\varphi} \in \mathrm{Char}(G)$. So, $\mathrm{Char}(G) = \{\Phi_{\lambda,\varphi} \mid \lambda \in \widehat{\mathfrak{u}}, \varphi \in \mathrm{Char}(L_\lambda)\}$. Since the quasi-orbits of G' in $\widehat{\mathfrak{u}'}$ correspond to the quasi-orbits of G in $\widehat{\mathfrak{u}}$, this shows that Theorem A holds for G.

4.2. Restriction to the unipotent radical

Let G be the group of $\mathbb{Q}$-rational points of a connected linear algebraic group over $\mathbb{Q}$ and let $G = LU$ be a Levi decomposition of G.

Let $\psi \in \mathrm{Char}(G)$. Set $\varphi := \psi|_U$. By Theorem 2.11, we have $\varphi \in \mathrm{Char}(U, G)$. So, by Corollary 3.4, $\varphi = \mathcal{F}(\mu) \circ \log$ for a unique $\mu \in \mathrm{Prob}(\widehat{\mathfrak{u}})^G_{\mathrm{erg}}$, where $\mathfrak{u}$ is the Lie algebra of U.

We want to determine the set $\mathrm{Prob}(\widehat{\mathfrak{u}})^G_{\mathrm{erg}}$. In the following discussion, the Lie algebra structure of $\mathfrak{u}$ will play no role, only its linear structure being relevant. So, we let E be a finite-dimensional vector space over $\mathbb{Q}$ and recall how the Pontrjagin dual $\widehat{E}$ can be described in terms of adèles.

Let $\mathcal{P}$ be the set of primes of $\mathbb{N}$. Recall that, for every $p \in \mathcal{P}$, the additive group of the field $\mathbb{Q}_p$ of p-adic numbers is a locally compact group containing the subring $\mathbb{Z}_p$ of p-adic integers as a compact open subgroup. The ring $\mathbb{A}$ of adèles of $\mathbb{Q}$ is the restricted product $\mathbb{A} = \mathbb{R} \times \prod_{p\in\mathcal{P}} (\mathbb{Q}_p, \mathbb{Z}_p)$ relative to the subgroups $\mathbb{Z}_p$; thus,

$$\mathbb{A} = \left\{(a_\infty, a_2, a_3, \dots) \in \mathbb{R} \times \prod_{p\in\mathcal{P}} \mathbb{Q}_p \mid a_p \in \mathbb{Z}_p \text{ for almost every } p \in \mathcal{P}\right\}.$$

The field $\mathbb{Q}$ can be viewed as a discrete and cocompact subring of the locally compact ring $\mathbb{A}$ via the diagonal embedding

$$\mathbb{Q} \to \mathbb{A}, \quad q \mapsto (q, q, \dots).$$

Let $b_1, \dots, b_d$ be a basis of E over $\mathbb{Q}$. Fix a non-trivial unitary character e of $\mathbb{A}$ which is trivial on $\mathbb{Q}$. For every $a = (a_1, \dots, a_d) \in \mathbb{A}^d$, let $\lambda_a \in \widehat{E}$ be defined by

$$\lambda_a(x) = e\left(\sum_{i=1}^{d} a_i q_i\right) \quad \text{for all } x = \sum_{i=1}^{d} q_i b_i \in E.$$

The map $a \mapsto \lambda_a$ factorizes to an isomorphism of topological groups

$$\mathbb{A}^d/\mathbb{Q}^d \to \widehat{E}, \quad a + \mathbb{Q}^d \mapsto \lambda_a$$

(see [43, Chapter IV, §3, Theorem 3]). So, $\widehat{E}$ can be identified with the *adelic solenoid* $\mathbb{A}^d/\mathbb{Q}^d$. We examine now how this identification behaves under the action of $\mathrm{GL}(E)$ on $\widehat{E}$.

Set $\mathbb{Q}_\infty = \mathbb{R}$. Then $\mathrm{GL}_d(\mathbb{Q}) \subset \mathrm{GL}_d(\mathbb{Q}_p)$ acts on $\mathbb{Q}_p^d$ for every $p \in \mathcal{P} \cup \{\infty\}$ in the usual way; the induced diagonal action of $\mathrm{GL}_d(\mathbb{Q})$ on $\mathbb{A}^d$ preserves the lattice $\mathbb{Q}^d$, giving rise to a (left) action of $\mathrm{GL}_d(\mathbb{Q})$ on $\mathbb{A}^d/\mathbb{Q}^d$.

Let $\theta \in \mathrm{GL}(E)$ and let $A \in \mathrm{GL}_n(\mathbb{Q})$ its matrix with respect to the basis $b_1, \dots, b_d$. One checks that

$$\lambda_a \circ \theta = \lambda_{A^t a} \quad \text{for all } a \in \mathbb{A}^d.$$

We summarize the previous discussion as follows.

Proposition 4.1. *Let E be a finite-dimensional vector space over $\mathbb{Q}$ of dimension d. The choice of a basis of E defines an isomorphism of topological groups $\mathbb{A}^d/\mathbb{Q}^d \to \widehat{E}$, which is equivariant for the action of* $\mathrm{GL}_d(\mathbb{Q})$ *given by inverse matrix transpose on $\mathbb{A}^d/\mathbb{Q}^d$ and the dual action of* $\mathrm{GL}(E)$ *on $\widehat{E}$. This isomorphism induces a bijection*

$$\mathrm{Prob}(\mathbb{A}^d/\mathbb{Q}^d)^G_{\mathrm{erg}} \to \mathrm{Prob}(\widehat{E})^G_{\mathrm{erg}},$$

for every subgroup G of $\mathrm{GL}(E) \cong \mathrm{GL}_d(\mathbb{Q})$.

5. Invariant probability measures and orbit closures on solenoids

For an algebraic $\mathbb{Q}$-subgroup of GL_d which is generated by unipotent subgroups, we will determine in this section the invariant probability measures as well as the orbits closures on the adelic solenoid $\mathbb{A}^d/\mathbb{Q}^d$. We have first to treat the case of S-adic solenoids.

5.1. Invariant probability measures and orbit closures on S-adic solenoids

Let $\mathbf{G}$ be an algebraic subgroup of GL_d defined over $\mathbb{Q}$. For every subring R of an overfield of $\mathbb{Q}$, we denote by $\mathbf{G}(R)$ the group of elements of $\mathbf{G}$ with coefficients in R and determinant invertible in R. In particular, $\mathbf{G}(\mathbb{Q}) = \mathbf{G} \cap \mathrm{GL}_d(\mathbb{Q})$.

Fix an integer $d \geq 1$ and let S be a finite subset of $\mathcal{P} \cup \{\infty\}$ with $\infty \in S$. Set

$$\mathbb{Q}_S^d := \prod_{p \in S} \mathbb{Q}_p^d$$

and let $\mathbb{Z}[1/S]$ denote the subring of $\mathbb{Q}$ generated by 1 and $(1/p)_{p \in S \cap \mathcal{P}}$. Then $\mathbb{Z}[1/S]^d$ embeds diagonally as a cocompact discrete subring of $\mathbb{Q}_S^d$.

The product group

$$\mathbf{G}(\mathbb{Q}_S) := \prod_{p \in S} \mathbf{G}(\mathbb{Q}_p)$$

is a locally compact group and acts on $\mathbb{Q}_S^d$ in the obvious way. The group $\mathbf{G}(\mathbb{Z}[1/S])$ embeds diagonally as a discrete subgroup of $\mathbf{G}(\mathbb{Q}_S)$. As $\mathbf{G}(\mathbb{Z}[1/S])$ preserves $\mathbb{Z}[1/S]^d$, this gives rise to an action of $\mathbf{G}(\mathbb{Z}[1/S])$ on the *S-adic solenoid*

$$X_S := \mathbb{Q}_S^d / \mathbb{Z}[1/S]^d,$$

which is a compact connected abelian group.

A *unipotent one-parameter subgroup* of $\mathbf{G}(\mathbb{Q}_S)$ is a subgroup of $\mathbf{G}(\mathbb{Q}_S)$ of the form $\{(u_p(t_p))_{p \in S} \mid t_p \in \mathbb{Q}_p, p \in S\}$ for $\mathbb{Q}$-rational homomorphisms $u_p \colon \mathbf{G}_a \to \mathbf{G}$ from the additive group $\mathbf{G}_a$ of dimension 1 to $\mathbf{G}$.

We aim to describe the $\mathbf{G}(\mathbb{Z}[1/S])$-invariant probability measures on X_S as well as orbit closures of points in X_S. Our results will be deduced from Ratner's measure rigidity and topological rigidity theorems in the S-adic setting (see [35] and [26]); actually, we will need the more precise version of Ratner's results in the S-arithmetic case from [42].

5.1.1. Invariant probability measures. For a closed subgroup Y of X_S and for $x \in X$, we denote by $\mu_{x+Y} \in \mathrm{Prob}(X_S)$ the image of the normalized Haar μ_Y under the map $X_S \to X_S$ given by translation by x.

Let V be a linear subspace of $\mathbb{Q}^d$. Denote by $V(\mathbb{Q}_p)$ the linear span of V in $\mathbb{Q}_p^d$ for $p \in S$. Then $V(\mathbb{Q}_S) := \prod_{p \in S} V(\mathbb{Q}_p)$ is a subring of $\mathbb{Q}_S^d$ and $V(\mathbb{Z}[1/S]) := V \cap \mathbb{Z}[1/S]^d$ is a cocompact lattice in $V(\mathbb{Q}_S)$. So, $V(\mathbb{Q}_S)/V(\mathbb{Z}[1/S])$ is a *subsolenoid* of X_S, that is, a closed and connected subgroup of X_S.

Proposition 5.1. *Assume that $\mathbf{G}(\mathbb{Q}_S)$ is generated by unipotent one-parameter subgroups. Let μ be an ergodic $\mathbf{G}(\mathbb{Z}[1/S])$-invariant probability measure on the Borel subsets of X_S. There exists a pair (a, V) consisting of a point $a \in \mathbb{Q}_S^d$ and a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace V of $\mathbb{Q}^d$ with the following properties:*

(i) *$g(a) \in a + V(\mathbb{Q}_S)$ for every $g \in \mathbf{G}(\mathbb{Q}_S)$;*

(ii) *$\mu = \mu_{x+Y}$, where x and Y are the images of a and $V(\mathbb{Q}_S)$ in X_S.*

Proof. We consider the semi-direct product

$$\widetilde{G} := \mathbf{G}(\mathbb{Q}_S) \ltimes \mathbb{Q}_S^d,$$

given by the natural action of $\mathbf{G}(\mathbb{Q}_S)$ on $\mathbb{Q}_S^d$. Then $\widetilde{G}$ is a locally compact group containing

$$\widetilde{\Gamma} := \mathbf{G}(\mathbb{Z}[1/S]) \ltimes \mathbb{Z}[1/S]^d$$

as a discrete subgroup. Since $\mathbf{G}(\mathbb{Q}_S)$ is generated by unipotent one-parameter subgroups, there is no non-trivial morphism $\mathbf{G} \to \mathrm{GL}_1$ defined over $\mathbb{Q}$. It follows (see [6, Theorem 5.6]) that $\Gamma := \mathbf{G}(\mathbb{Z}[1/S])$ has finite covolume in $\mathbf{G}(\mathbb{Q}_S)$, and so $\widetilde{\Gamma}$ is an S-arithmetic lattice in $\widetilde{G}$.

We now use the "suspension technique" from [44] to obtain an ergodic $\mathbf{G}(\mathbb{Q}_S)$-invariant probability measure $\widetilde{\mu}$ on $\widetilde{G}/\widetilde{\Gamma}$. Specifically, we embed X_S as a subset of $\widetilde{G}/\widetilde{\Gamma}$ in the obvious way. Observe that the action of $\mathbf{G}(\mathbb{Z}[1/S])$ by automorphisms on X_S becomes the action of $\mathbf{G}(\mathbb{Z}[1/S])$ by translations on $\widetilde{G}/\widetilde{\Gamma}$ under this embedding.

View μ as a $\mathbf{G}(\mathbb{Z}[1/S]))$-invariant probability measure on $\widetilde{G}/\widetilde{\Gamma}$ which is supported on the image of X_S. Let $\widetilde{\mu}$ be the probability measure on $\widetilde{G}/\widetilde{\Gamma}$ defined by

$$\widetilde{\mu} = \int_{\mathbf{G}(\mathbb{Q}_S)/\Gamma} t_g(\mu)\, \mathrm{d}\nu(g\Gamma),$$

where ν be the unique $\mathbf{G}(\mathbb{Q}_S)$-invariant probability measure on $\mathbf{G}(\mathbb{Q}_S)/\Gamma$ and $t_g(\mu)$ denotes the image of μ under the translation by g. Then $\widetilde{\mu}$ is $\mathbf{G}(\mathbb{Q}_S)$-invariant and is ergodic under this action.

By the refinement [42, Theorem 2] of Ratner's theorem, there exists a $\mathbb{Q}$-algebraic subgroup $\mathbf{L}$ of $\mathbf{G}$, an $\mathbf{L}(\mathbb{Q})$-invariant vector subspace V of $\mathbb{Q}^d$, a finite index subgroup H of $\mathbf{L}(\mathbb{Q}_S) \ltimes V(\mathbb{Q}_S)$, and an element $g \in \widetilde{G}$ with the following properties:

- $\mathbf{G}(\mathbb{Q}_S) \subset H^g := gHg^{-1}$;
- $H \cap \widetilde{\Gamma}$ is a lattice in H;
- $\widetilde{\mu}$ is the unique H^g-invariant probability measure on $\widetilde{G}/\widetilde{\Gamma}$ supported on $gH\widetilde{\Gamma}/\widetilde{\Gamma} = H^g g\widetilde{\Gamma}/\widetilde{\Gamma}$.

Since $\widetilde{G} = \mathbf{G}(\mathbb{Q}_S) \ltimes \mathbb{Q}_S^d$, there exists $g' \in \mathbf{G}(\mathbb{Q}_S)$ such that $a := g'g$ belongs to $\mathbb{Q}_S^d$. Then $\mathbf{G}(\mathbb{Q}_S) \subset H^a$ and, since $\widetilde{\mu}$ is $\mathbf{G}(\mathbb{Q}_S)$-invariant, $\widetilde{\mu}$ coincides with the H^a-invariant probability measure supported on $H^a a\widetilde{\Gamma}/\widetilde{\Gamma}$. As a result, we may assume above that $g = a \in \mathbb{Q}_S^d$.

The image $t_{a^{-1}}(\widetilde{\mu})$ of $\widetilde{\mu}$ under the translation by a^{-1} coincides with the unique H-invariant probability measure on $\widetilde{G}/\widetilde{\Gamma}$ supported on $H\widetilde{\Gamma}/\widetilde{\Gamma}$. Observe that $\mathbf{G}(\mathbb{Q}_S) \subset H^a$ implies that

$$g(a) - a \in V(\mathbb{Q}_S) \quad \text{for every } g \in \mathbf{G}(\mathbb{Q}_S),$$

where we write $g(a)$ for gag^{-1}.

Let

$$p\colon \widetilde{G}/\widetilde{\Gamma} \to \mathbf{G}(\mathbb{Q}_S)/\Gamma$$

be the natural $\mathbf{G}(\mathbb{Q}_S)$-equivariant map. We have

$$t_{a^{-1}}(\widetilde{\mu}) = \int_{\mathbf{G}(\mathbb{Q}_S)/\Gamma} t_{a^{-1}}(t_g(\mu))\, \mathrm{d}\nu(g\Gamma), \tag{$*$}$$

and $t_{a^{-1}}(t_g(\mu))(p^{-1}(g\Gamma/\Gamma)) = 1$ for every $g \in \mathbf{G}(\mathbb{Q}_S)$. So, formula ($*$) provides a decomposition of $t_{a^{-1}}(\widetilde{\mu})$ as an integral over $\mathbf{G}(\mathbb{Q}_S)/\mathbf{G}(\mathbb{Z}[1/S])$ of probability measures supported on the fibers of p.

Knowing that $t_{a^{-1}}(\widetilde{\mu})$ is the H-invariant probability measure supported on $H\widetilde{\Gamma}/\widetilde{\Gamma}$, we can perform a second such decomposition of $t_{a^{-1}}(\widetilde{\mu})$ over $\mathbf{G}(\mathbb{Q}_S)/\Gamma$. The measures supported on the fibers of p in this last decomposition are translates of the normalized Haar measure μ_Y of the image Y of $H \cap \mathbb{Q}_S^d$ in $X_S \cong \mathbb{Q}_S^d\widetilde{\Gamma}/\widetilde{\Gamma}$. By uniqueness, it follows that $t_{a^{-1}}(\mu) = \mu_Y$, that is, $\mu = \mu_{x+Y}$, where a is the image of x in X_S (for more details, see the proof of [44, Corollary 5.8]).

To finish the proof, observe that, since $V(\mathbb{Q}_S)$ is divisible, it has no proper subgroup of finite index and so $H \cap \mathbb{Q}_S^d = H \cap V(\mathbb{Q}_S) = V(\mathbb{Q}_S)$. ■

The pairs (x, Y) as in Proposition 5.1 for which μ_{x+Y} is ergodic are characterized by the following general result.

For a compact group X, we denote by $\mathrm{Aut}(X)$ the group of continuous automorphisms of X and by $\mathrm{Aff}(X) = \mathrm{Aut}(X) \ltimes X$ the group of affine transformations of X.

Proposition 5.2. *Let G be a countable group, X be a compact abelian group and $\alpha\colon G \to \mathrm{Aut}(X)$ an action of G by automorphisms of X. Let $x_0 \in X$ and let Y be a connected closed subgroup of X such $x_0 + Y$ is G-invariant. Then Y is G-invariant and $\alpha_g(x_0) - x_0 \in Y$ for every $g \in G$. Moreover, the following properties are equivalent:*

- (i) *μ_{x_0+Y} is not ergodic under the restriction of the G-action to $x_0 + Y$;*
- (ii) *there exists a proper closed connected subgroup Z of Y and a finite index subgroup H of G such that $\alpha_h(x) - x \in Z$ for every $x \in x_0 + Y$ and $h \in H$;*
- (iii) *for every $x \in x_0 + Y$, the set $\{\alpha_g(x) - x \mid g \in G\}$ is not dense in Y;*
- (iv) *there exists a subset A of $x_0 + Y$ with $\mu_{x_0+Y}(A) > 0$ such that $\{\alpha_g(x) - x \mid g \in G\}$ is not dense in Y for every $x \in A$.*

Proof. The fact that Y is G-invariant and that $\alpha_g(x) - x \in Y$ for every $g \in G$ is obvious.

The homeomorphism $t\colon x_0 + Y \to Y$ given by the translation by $-x_0$ intertwines the action α of G on $x_0 + Y$ with the action $\beta\colon G \to \mathrm{Aff}(Y)$ by affine transformations of Y, given by

$$\beta_g(y) = \alpha_g(y) + \alpha_g(x_0) - x_0 \quad \text{for all } g \in G,\ y \in Y.$$

Moreover, the image of μ_{x_0+Y} under t is the Haar measure μ_Y on Y.

Assume that the action β is not ergodic. Then there exist a proper closed connected subgroup Z of Y which is invariant under the action α of G and a finite index subgroup H of G such that the image of H in $\mathrm{Aff}(Y/Z)$, for the action induced by β, is trivial (see [4, Proposition 1]). This means that $\alpha_h(x) - x \in Z$ for every $x \in x_0 + Y$ and $h \in H$. So, (i) implies (ii).

Assume that property (ii) holds and let $x \in x_0 + Y$. In this case, the image of the set $\{\alpha_g(x) - x \mid g \in G\}$ in Y/Z is finite. However, since Y is connected, Y/Z is infinite and $\{\alpha_g(x) - x \mid g \in G\}$ is therefore not dense in Y. So, (ii) implies (iii).

The fact that (iii) implies (iv) is obvious. Assume that β is ergodic. Since the support of μ_Y is Y, for μ_Y-almost every $y \in Y$, the $\beta(G)$-orbit of y is dense in Y, that is, $\{\alpha_g(x) - x \mid g \in G\}$ is dense in Y for μ_{x_0+Y}-almost every $x \in x_0 + Y$. Therefore, (iv) implies (i). ∎

We will need a description of the $\mathbf{G}(\mathbb{Z}[1/S])$-invariant (not necessarily ergodic) probability measures on X_S. For this, we adapt for our situation some ideas from [29, §2], where such description was given in the context of real Lie groups.

Let $\varphi\colon \mathbb{Q}_S^d \to X_S = \mathbb{Q}_S^d/\mathbb{Z}[1/S]^d$ denote the canonical projection. Observe that, if V is a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace of $\mathbb{Q}^d$ and if $a \in \mathbb{Q}_S^d$ is such that $g(a) \in a + V(\mathbb{Q}_S)$ for all $g \in \mathbf{G}(\mathbb{Z}[1/S])$, then $\varphi(a + V(\mathbb{Q}_S))$ is a closed and $\mathbf{G}(\mathbb{Z}[1/S])$-invariant subset of X_S.

Denote by $\mathcal{H}$ the set of $\mathbf{G}(\mathbb{Q})$-invariant linear subspaces of $\mathbb{Q}^d$. For $V \in \mathcal{H}$, define $\mathcal{N}(V,S) \subset \mathbb{Q}_S^d$ to be the set of $a \in \mathbb{Q}_S^d$ with the following properties:

- $g(a) \in a + V(\mathbb{Q}_S)$ for every $g \in \mathbf{G}(\mathbb{Z}[1/S])$ and
- $\varphi(\{g(a) - a \mid g \in \mathbf{G}(\mathbb{Z}[1/S])\})$ is dense in $\varphi(V(\mathbb{Q}_S))$.

Lemma 5.3. *For $V, W \in \mathcal{H}$, we have $\varphi(\mathcal{N}(V,S)) \cap \varphi(\mathcal{N}(W,S)) \neq \emptyset$ if and only if $V = W$.*

Proof. Assume that $\varphi(\mathcal{N}(V,S)) \cap \varphi(\mathcal{N}(W,S)) \neq \emptyset$. So, there exist $a \in \mathcal{N}(V,S)$ and $b \in \mathbb{Z}[1/S]^d$ such that $a + b \in \mathcal{N}(W,S)$. It follows that $\varphi(\{g(a) - a \mid g \in \mathbf{G}(\mathbb{Z}[1/S])\})$ is a dense subset of $\varphi(V(\mathbb{Q}_S))$ and of $\varphi(W(\mathbb{Q}_S))$. Hence, $\varphi(V(\mathbb{Q}_S)) = \varphi(W(\mathbb{Q}_S))$. This implies that the $\mathbb{R}$-vector space $V(\mathbb{R})$ is contained in $W(\mathbb{R}) + \mathbb{Z}[1/S]^d$ and so in $W(\mathbb{R})$, by connectedness. Hence, $V \subset W$. Similarly, we have $W \subset V$. ∎

We can now give a description of the finite $\mathbf{G}(\mathbb{Z}[1/S])$-invariant measures on X_S.

Proposition 5.4. *Assume that $\mathbf{G}(\mathbb{Q}_S)$ is generated by unipotent one-parameter subgroups and let $\mu \in \mathrm{Prob}(X_S)$ be a $\mathbf{G}(\mathbb{Z}[1/S])$-invariant probability measure on X_S. For $V \in \mathcal{H}$, denote by μ_V the restriction of μ to $\varphi(\mathcal{N}(V,S))$.*

(i) *We have*

$$\mu\Big(\bigcup_{V \in \mathcal{H}} \varphi(\mathcal{N}(V,S))\Big) = 1;$$

moreover, $\mu_V(\varphi(\mathcal{N}(V',S)) = 0$ for all $V, V' \in \mathcal{H}$ with $V \neq V'$; so, we have a decomposition

$$\mu = \bigoplus_{V \in \mathcal{H}} \mu_V.$$

(ii) *Let $V \in \mathcal{H}$ be such that $\mu_V \neq 0$. Then μ_V is $\mathbf{G}(\mathbb{Z}[1/S])$-invariant. Moreover, if*

$$\mu_V = \int_\Omega \nu_{V,\omega}\, \mathrm{d}\omega$$

is a decomposition of μ_V into ergodic $\mathbf{G}(\mathbb{Z}[1/S])$-invariant components $\nu_{V,\omega}$, then, for every $\omega \in \Omega$, we have $\nu_{V,\omega} = \mu_{x_\omega + Y}$, where $Y = \varphi(V(\mathbb{Q}_S))$ and $x_\omega = \varphi(a_\omega)$ for some $a_\omega \in \mathcal{N}(V, S)$.

Proof. Let

$$\mu = \int_\Omega \nu_\omega\, \mathrm{d}\omega$$

be a decomposition of μ into $\mathbf{G}(\mathbb{Z}[1/S])$-invariant ergodic probability measures ν_ω on X_S.

Fix $\omega \in \Omega$. By Proposition 5.1, there exists $a_\omega \in \mathbb{Q}_S^d$ and $V_\omega \in \mathcal{H}$ such that $g(a_\omega) \in a_\omega + V_\omega(\mathbb{Q}_S)$ for every $g \in \mathbf{G}(\mathbb{Q}_S)$ and $\nu_\omega = \mu_{x_\omega + Y_\omega}$, where $Y_\omega = \varphi(V_\omega(\mathbb{Q}_S))$ and $x_\omega = \varphi(a_\omega)$. Since $\mu_{x_\omega + Y_\omega}$ is ergodic, there exists a subset A_ω of $x_\omega + Y_\omega$ with $\mu_{x_\omega + Y_\omega}(A_\omega) = 1$ such that the $\mathbf{G}(\mathbb{Z}[1/S])$-orbit of x is dense in $x_\omega + Y_\omega$ for every $x \in A_\omega$ (see Proposition 5.2). It is clear that $x \in \varphi(\mathcal{N}(V_\omega, S))$ for every $x \in A_\omega$. It follows that $\nu_\omega(\varphi(\mathcal{N}(V_\omega, S)) = 1$ for every $\omega \in \Omega$ and hence $\mu(\bigcup_{V \in \mathcal{H}} \varphi(\mathcal{N}(V, S)) = 1$. Since the measurable subsets $\varphi(\mathcal{N}(V, S))$ of X_S are mutually disjoint (Lemma 5.3) and since $\mathcal{H}$ is countable, we have a direct sum decomposition

$$\mu = \bigoplus_{V \in \mathcal{H}} \int_{\omega : V_\omega = V} \nu_\omega\, \mathrm{d}\omega$$

and $\mu_V = \int_{\omega : V_\omega = V} \nu_\omega\, \mathrm{d}\omega$ with $\nu_\omega = \mu_{x_\omega + Y}$, where $Y = V(\mathbb{Q}_S)$ and $x_\omega = \varphi(a_\omega)$ for some $a_\omega \in \mathcal{N}(V, S)$. ∎

We will later need to know that the linear subspace of points in $\mathbb{Q}_S^d$ satisfying the first condition defining a set $\mathcal{N}(V, S)$ as above is rational.

Lemma 5.5. *Let V be a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace of $\mathbb{Q}^d$ and S a finite subset of $\mathcal{P} \cup \{\infty\}$. There exists a linear subspace W^S of $\mathbb{Q}^d$ containing V such that*

$$\{a \in \mathbb{Q}_S^d \mid g(a) \in a + V(\mathbb{Q}_S) \text{ for all } g \in \mathbf{G}(\mathbb{Z}[1/S])\} = W^S(\mathbb{Q}_S).$$

Proof. Choose a linear complement V_0 of V in $\mathbb{Q}^d$ and let $\pi_0 \colon \mathbb{Q}^d \to V_0$ be the corresponding projection. Let $p \in \mathcal{P} \cup \{\infty\}$. Then

$$\mathbb{Q}_p^d = V(\mathbb{Q}_p) \oplus V_0(\mathbb{Q}_p)$$

and the linear extension of π_0, again denoted by π_0, is the corresponding projection $\mathbb{Q}_p^d \to V_0(\mathbb{Q}_p)$.

Let $g \in \mathrm{GL}_d(\mathbb{Q})$. Denote by $W_g \subset \mathbb{Q}^d$ the kernel of $\pi_0 \circ (g - I_{\mathbb{Q}^d})$. For $a \in \mathbb{Q}_p^d$, we have

$$g(a) \in a + V(\mathbb{Q}_p) \iff a \in \mathrm{Ker}\big(\pi_0 \circ (g - I_{\mathbb{Q}_p^d})\big).$$

So, $\mathrm{Ker}(\pi_0 \circ (g - I_{\mathbb{Q}_p^d})) = W_g(\mathbb{Q}_p)$ and the linear subspace

$$W^S := \bigcap_{g \in \mathbf{G}(\mathbb{Z}[1/S])} W_g$$

of $\mathbb{Q}^d$ has the required property. ∎

5.1.2. Orbit closures. We now turn to the description of orbit closures of points in X_S. Recall that $\varphi \colon \mathbb{Q}_S^d \to X_S$ denotes the canonical projection.

Proposition 5.6. *Assume that $\mathbf{G}(\mathbb{Q}_S)$ is generated by unipotent one-parameter subgroups. Let $a \in \mathbb{Q}_S^d$ and $x = \varphi(a) \in X_S$. There exists a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace V of $\mathbb{Q}^d$ with the following properties:*

(i) *$g(a) \in a + V(\mathbb{Q}_S)$ for every $g \in \mathbf{G}(\mathbb{Q}_S)$;*

(ii) *the closure of the $\mathbf{G}(\mathbb{Z}[1/S])$-orbit of x in X_S coincides with $x + \varphi(V(\mathbb{Q}_S))$.*

Proof. As in the proof of Proposition 5.1, we consider the group $\widetilde{G} = \mathbf{G}(\mathbb{Q}_S) \ltimes \mathbb{Q}_S^d$ and embed X_S as a closed subset of $\widetilde{G}/\widetilde{\Gamma}$, where $\widetilde{\Gamma} = \mathbf{G}(\mathbb{Z}[1/S]) \ltimes \mathbb{Z}[1/S]^d$.

By the refinement [42, Theorem 1] of Ratner's theorem about orbit closures, there exist a $\mathbb{Q}$-algebraic subgroup $\mathbf{L}$ of $\mathbf{G}$, an $\mathbf{L}(\mathbb{Q})$-invariant vector subspace V of $\mathbb{Q}^d$ and a finite index subgroup H of $\mathbf{L}(\mathbb{Q}_S) \ltimes V(\mathbb{Q}_S)$ with the following properties:

- $\mathbf{G}(\mathbb{Q}_S) \subset H^a := aHa^{-1}$;
- $H \cap \widetilde{\Gamma}$ is a lattice in H;
- the closure $\overline{\mathbf{G}(\mathbb{Q}_S)x}$ of the $\mathbf{G}(\mathbb{Q}_S)$-orbit of x is $H^a x$, that is, $aH\widetilde{\Gamma}/\widetilde{\Gamma}$.

We claim that

$$\overline{\mathbf{G}(\mathbb{Z}[1/S])x} = \overline{\mathbf{G}(\mathbb{Q}_S)x} \cap X_S.$$

We only have to show that $\overline{\mathbf{G}(\mathbb{Q}_S)x} \cap X_S$ is contained in $\overline{\mathbf{G}(\mathbb{Z}[1/S])x}$, the reverse inclusion being obvious.

Set $\Gamma = \mathbf{G}(\mathbb{Z}[1/S])$ and $\Lambda = \mathbb{Z}[1/S]^d$. Choose a fundamental domain $\Omega \subset \mathbf{G}(\mathbb{Q}_S)$ for $\mathbf{G}(\mathbb{Q}_S)/\Gamma$ which is a neighbourhood of e and a compact fundamental domain $K \subset \mathbb{Q}_S^d$ for $\mathbb{Q}_S^d/\Lambda$.

Consider $y \in \overline{\mathbf{G}(\mathbb{Q}_S)x} \cap X_S$. Then there exists a sequence $g_n \in \mathbf{G}(\mathbb{Q}_S)$ such that $\lim_n (g_n, e)x = y$. Write $g_n = \omega_n \gamma_n$ for $\omega_n \in \Omega$ and $\gamma_n \in \Gamma$ and $\gamma_n(a) = k_n + \lambda_n$ for $k_n \in K$ and $\lambda_n \in \Lambda$. Then

$$\begin{aligned} y &= \lim_n (g_n, e)x = \lim_n (\omega_n, e)(\gamma_n, e)(e, a)\widetilde{\Gamma} = \lim_n (\omega_n, e)(\gamma_n, \gamma_n(a))\widetilde{\Gamma} \\ &= \lim_n (\omega_n, e)(e, k_n)\widetilde{\Gamma} = \lim_n (\omega_n, \omega_n(k_n))\widetilde{\Gamma}. \end{aligned}$$

On the one hand, it follows that $\lim_n \omega_n \delta_n = e$ for some $\delta_n \in \Gamma$. So, for large n, we have $\omega_n \delta_n \in \Omega$ and, since $\omega_n \in \Omega$, we have $\delta_n = e$, that is, $\lim_n \omega_n = e$. On the other hand, as K is compact, we can assume that $\lim_n k_n = k \in K$ exists. Therefore, we have $\lim_n(\omega_n, \omega_n(k_n)) = (e, k)$ and so $y = k + \Lambda$ and

$$y = \lim_n (k_n + \Lambda) = \lim_n (\gamma_n(a) + \Lambda),$$

that is, $y \in \overline{\mathbf{G}(\mathbb{Z}[1/S])x}$. So, the claim is proved.

We have

$$(aH\widetilde{\Gamma}/\widetilde{\Gamma}) \cap X_S = \varphi(a + H \cap V(\mathbb{Q}_S))$$

and, since (as in the proof of Proposition 5.1) $H \cap V(\mathbb{Q}_S) = V(\mathbb{Q}_S)$, this finishes the proof. ∎

5.2. Invariant probability measures and orbit closures on adelic solenoids

Let $\mathbf{G}$ be an algebraic subgroup of GL_d defined over $\mathbb{Q}$. We are now ready to deal with the description of the $\mathbf{G}(\mathbb{Q})$-invariant probability measures and the orbit closures for the adelic solenoid

$$X := \mathbb{A}^d/\mathbb{Q}^d.$$

Denote by $\mathcal{S}$ the set of finite subsets S of $\mathcal{P} \cup \{\infty\}$ with $\infty \in S$.

Let $S \in \mathcal{S}$. It is well known (see [43]) that

$$\mathbb{A}^d = \Big(\mathbb{Q}_S^d \times \prod_{p \notin S} \mathbb{Z}_p^d\Big) + \mathbb{Q}^d$$

and that

$$\Big(\mathbb{Q}_S^d \times \prod_{p \notin S} \mathbb{Z}_p^d\Big) \cap \mathbb{Q}^d = \mathbb{Z}[1/S]^d.$$

This gives rise to a well defined projection

$$\pi_S \colon X \to X_S = \mathbb{Q}_S^d/\mathbb{Z}[1/S]^d$$

given by

$$\pi_S\big((a_S, (a_p)_{p\notin S}) + \mathbb{Q}^d\big) = a_S + \mathbb{Z}[1/S]^d \quad \text{for all } a_S \in \mathbb{Q}_S^d,\ (a_p)_{p\notin S} \in \prod_{p \notin \mathcal{P}} \mathbb{Z}_p^d.$$

So, the fiber of π_S over a point $a_S + \mathbb{Z}[1/S]^d \in X_S$ is

$$\pi_S^{-1}(a_S + \mathbb{Z}[1/S]^d) = \big\{(a_S, (a_p)_{p\notin S}) + \mathbb{Q}^d \mid a_p \in \mathbb{Z}_p^d \text{ for all } p \notin S\big\}.$$

Observe that π_S is $\mathrm{GL}_d(\mathbb{Z}[1/S])$-equivariant.

Let $S' \in \mathcal{S}$ with $S \subset S'$. Then

$$\mathbb{Q}_{S'}^d = \Big(\mathbb{Q}_S^d \times \prod_{p \in S' \setminus S} \mathbb{Z}_p^d\Big) + \mathbb{Z}[1/S']^d,$$

$$\Big(\mathbb{Q}_S^d \times \prod_{p \in S' \setminus S} \mathbb{Z}_p^d\Big) \cap \mathbb{Z}[1/S']^d = \mathbb{Z}[1/S]^d,$$

and we have a similarly defined $\mathrm{GL}_d(\mathbb{Z}[1/S])$-equivariant projection $\pi_{S',S}\colon X_{S'} \to X_S$. Observe that $\pi_S = \pi_{S',S} \circ \pi_{S'}$.

Let V be a linear subspace of $\mathbb{Q}^d$. For $p \in \mathcal{P}$, we write $V(\mathbb{Z}_p)$ for the $\mathbb{Z}_p$-span of $V(\mathbb{Z})$ in $V(\mathbb{Q}_p)$; the adèle space corresponding to V is

$$V(\mathbb{A}) = \bigcup_{S \in \mathcal{S}} \left(V(\mathbb{Q}_S) \times \prod_{p \notin S} V(\mathbb{Z}_p) \right).$$

We denote by φ the canonical projection $\mathbb{A}^d \to X$. The image of $\varphi(V(\mathbb{A}))$ in X can be written as

$$\varphi(V(\mathbb{A})) = \varphi\left(V(\mathbb{R}) \times \prod_{p \in \mathcal{P}} V(\mathbb{Z}_p) \right)$$

and is a closed and connected subgroup of X. Conversely, every closed and connected subgroup of X is of the form $\varphi(V(\mathbb{A}))$ for a unique linear subspace V of $\mathbb{Q}^d$ (see Lemma 6.1 below).

The following simple fact will be useful.

Lemma 5.7. *Let V be a linear subspace of $\mathbb{Q}^d$. Set*

$$\Omega := \bigcap_{S} \varphi\left(V(\mathbb{R}) \times \prod_{p \in S} V(\mathbb{Z}_p) \times \prod_{p \notin S} \mathbb{Z}_p^d \right) \subset X,$$

where S runs over the finite subsets of $\mathcal{P}$. Then

$$\Omega = \varphi(V(\mathbb{A})).$$

Proof. It is clear that $\varphi(V(\mathbb{A}))$ is contained in Ω. Conversely, let $x \in \Omega$. Then there exists $a = (a_p)_{p \in \mathcal{P} \cup \{\infty\}} \in \mathbb{A}^d$ with $\varphi(a) = x$ such that $a_\infty \in V(\mathbb{R})$ and $a_p \in \mathbb{Z}_p^d$ for all $p \in \mathcal{P}$. We claim that $a_p \in V(\mathbb{Z}_p)$ for all $p \in \mathcal{P}$.

Indeed, let $p_0 \in \mathcal{P}$. Since $\varphi(a) \in \Omega$, there exists $q \in \mathbb{Q}^d$ such that $(a_p + q)_{p \in \mathcal{P} \cup \{\infty\}} \in \mathbb{R}^d \times \prod_{p \in \mathcal{P}} \mathbb{Z}_p^d$ with $a_\infty + q \in V(\mathbb{R})$ and $a_{p_0} + q \in V(\mathbb{Z}_{p_0})$. For every $p \in \mathcal{P}$, we have $q = (a_p + q) - a_p \in \mathbb{Z}_p^d$ and hence $q \in \mathbb{Z}^d$. Since $a_\infty \in V(\mathbb{R})$, we also have $q = (a_\infty + q) - a_\infty \in V(\mathbb{R})$. Hence, $q \in V(\mathbb{Z}) \subset V(\mathbb{Z}_{p_0})$ and therefore $a_{p_0} = (a_{p_0} + q) - q \in V(\mathbb{Z}_{p_0})$. ■

5.2.1. Invariant probability measures. We will denote by φ the canonical projection $\mathbb{A}^d \to X$ and by φ_S the projection $\mathbb{Q}_S^d \to X_S$ for a set $S \in \mathcal{S}$.

Theorem 5.8. *Let $\mathbf{G}$ be a connected algebraic subgroup of GL_d defined over $\mathbb{Q}$. Assume that $\mathbf{G}(\mathbb{Q})$ is generated by unipotent one-parameter subgroups. Let μ be an ergodic $\mathbf{G}(\mathbb{Q})$-invariant probability measure on the Borel subsets of $X = \mathbb{A}^d/\mathbb{Q}^d$. There exists a pair (a, V_0) consisting of a point $a \in \mathbb{A}^d$ and a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace V_0 of $\mathbb{Q}^d$ such that*

$$\mu = \mu_{x+Y},$$

for $x = \varphi(a)$ *and* $Y = \varphi(V_0(\mathbb{Q}))$. *Moreover, a can be chosen so that the set* $\{g(a) - a \mid g \in \mathbf{G}(\mathbb{Q})\}$ *is dense in* $V_0(\mathbb{A})$.

Proof. For $S \in \mathcal{S}$, let μ_S be the image of μ under the projection

$$\pi_S \colon X \to X_S.$$

Then μ_S is a $\mathbf{G}(\mathbb{Z}[1/S]$-invariant probability measure on X_S and, by Proposition 5.4, we have a decomposition

$$\mu_S = \bigoplus_{V \in \mathcal{H}} \mu_{S,V}$$

with mutually singular measures $\mu_{S,V}$ on X_S such that

$$\mu_{S,V}(X_S \setminus \varphi_S(\mathcal{N}(V,S))) = 0.$$

Fix $S_0 \in \mathcal{S}$ and $V_0 \in \mathcal{H}$ with $\mu_{S_0,V_0} \neq 0$ and such that

$$\dim V_0 = \max\{\dim V \mid \mu_{S,V} \neq 0 \text{ for some } S \in \mathcal{S}\}.$$

Write

$$\mathcal{P} \cup \{\infty\} = \bigcup_{n \geq 0} S_n$$

for an increasing sequence of subsets $S_n \in \mathcal{S}$. Denote by μ_n instead of μ_{S_n} the image of μ under the projection $\pi_{S_n} \colon X \to X_{S_n}$. Set

$$c := \mu_0(\varphi_{S_0}(\mathcal{N}(V_0, S_0))) > 0.$$

First step. We claim that

$$\mu_n(\varphi_{S_n}(\mathcal{N}(V_0, S_n))) \geq c \quad \text{for all } n \geq 1.$$

Indeed, let $V \in \mathcal{H}$ be such that $\mu_{S_n,V} \neq 0$. Recall that

$$\pi_{S_n,S_0} \colon X_{S_n} \to X_{S_0}$$

is the natural $\mathbf{G}(\mathbb{Z}[1/S_0])$-equivariant projection. Let

$$x \in \pi_{S_n,S_0}^{-1}(\varphi_{S_0}(\mathcal{N}(V_0, S_0))) \cap \varphi_{S_n}(\mathcal{N}(V, S_n)).$$

Then, on the one hand, $x_0 := \pi_{S_n,S_0}(x) \in \varphi_{S_0}(\mathcal{N}(V_0, S_0))$ and hence the set $\{g(x_0) - x_0 \mid g \in \mathbf{G}(\mathbb{Z}[1/S_0])\}$ is dense in $\varphi_{S_0}(V_0(\mathbb{Q}_{S_0}))$. On the other hand, since $x \in \varphi_{S_n}(\mathcal{N}(V, S_n))$ and since $\mathbf{G}(\mathbb{Z}[1/S_0])$ is contained in $\mathbf{G}(\mathbb{Z}[1/S_n])$, the set $\{g(x) - x \mid g \in \mathbf{G}(\mathbb{Z}[1/S_0])\}$ is contained in $\varphi_{S_n}(V(\mathbb{Q}_{S_n}))$. As

$$\pi_{S_n,S_0}(\varphi_{S_n}(V(\mathbb{Q}_{S_n}))) = \varphi_{S_0}(V(\mathbb{Q}_{S_0}))$$

and π_{S_n,S_0} is $\mathbf{G}(\mathbb{Z}[1/S_0])$-equivariant and continuous, it follows that

$$\varphi_{S_0}(V_0(\mathbb{Q}_{S_0})) \subset \varphi_{S_0}(V(\mathbb{Q}_{S_0})).$$

This implies that $V_0 \subset V$ (see the proof of Lemma 5.3). It follows that $V = V_0$, by maximality of the dimension of V_0. This shows that

$$\mu_n\big(\pi_{S_n,S_0}^{-1}(\varphi_{S_0}(\mathcal{N}(V_0,S_0))) \cap \varphi_{S_n}(\mathcal{N}(V,S_n))\big) = 0 \quad \text{for every } V \neq V_0$$

and hence that

$$\mu_n\big(\pi_{S_n,S_0}^{-1}(\varphi_{S_0}(\mathcal{N}(V_0,S_0)))\big) \leq \mu_n(\varphi_{S_n}(\mathcal{N}(V_0,S_n))).$$

Since $\mu_0 = \mu_{S_0}$ is the image of μ_n under π_{S_n,S_0}, we have

$$\mu_n\big(\pi_{S_n,S_0}^{-1}(\varphi_{S_0}(\mathcal{N}(V_0,S_0)))\big) = \mu_0(\varphi_{S_0}(\mathcal{N}(V_0,S_0))),$$

and the claim is proved.

For every $n \geq 0$, let $W^n = W^{S_n}$ be the linear subspace of $\mathbb{Q}^d$ defined by V_0 as in Lemma 5.5. It is clear that the family $(W^n)_{n\geq 0}$ of finite-dimensional linear subspaces is decreasing. So, there exists $N \geq 0$ such that $W^n = W^N$ for all $n \geq N$. Set $W := W^N$. Recall that $V_0 \subset W^n$ for every $n \geq 0$ and hence $V_0 \subset W$.

Second step. We claim that $\mu(\varphi(W(\mathbb{A})) \neq 0$.

Indeed, since $\mathcal{N}(V_0,S_n) \subset W^n(\mathbb{Q}_{S_n})$, it follows from the first step that

$$\mu_n(\varphi_{S_n}(W(\mathbb{Q}_{S_n}))) = \mu_n(\varphi_{S_n}(W^n(\mathbb{Q}_{S_n}))) \geq c$$

for every $n \geq N$. Setting

$$\Omega_n := \varphi\Big(W(\mathbb{Q}_\infty) \times \prod_{p\in S_n, p\neq\infty} W(\mathbb{Z}_p) \times \prod_{p\notin S_n} \mathbb{Z}_p^d\Big),$$

this means that

$$\mu(\Omega_n) \geq c \quad \text{for all } n \geq N,$$

since μ_n is the image of μ under π_{S_n}.

As $(\Omega_n)_{n\geq N}$ is a decreasing sequence, it follows that

$$\mu\Big(\bigcap_{n\geq N} \Omega_n\Big) \geq c > 0.$$

On the other hand, we have (see Lemma 5.7)

$$\bigcap_{n\geq N} \Omega_n = \varphi(W(\mathbb{A}))$$

and the claim is proved.

Set $Y := \varphi(V_0(\mathbb{A}))$.

Third step. We claim that there exists $x \in \varphi(W(\mathbb{A}))$ such that $\mu(x+Y)=1$.

Indeed, $\varphi(W(\mathbb{A}))$ is $\mathbf{G}(\mathbb{Q})$-invariant, since W is $\mathbf{G}(\mathbb{Q})$-invariant. By the ergodicity of μ, it follows from the second step that $\mu(\varphi(W(\mathbb{A}))=1$; so, we may view μ as probability measure on $\varphi(W(\mathbb{A}))$.

Let $Z \subset \varphi(W(\mathbb{A}))$ be the support of μ. Again by ergodicity of μ, there exists a point $a \in W(\mathbb{A})$ such that $x=\varphi(a) \in Z$ and such that the $\mathbf{G}(\mathbb{Q})$-orbit of x is dense in Z. Since $g(a) \in a + V_0(\mathbb{A})$ for all $g \in \mathbf{G}(\mathbb{Q})$, this implies that $Z \subset x+Y$ and so $\mu(x+Y)=1$.

Fourth step. We claim that μ is invariant under translations by elements from Y. Once proved, it will follow that $\mu=\mu_{x+Y}$, by the uniqueness of the Haar measure on the closed subgroup Y of X.

Indeed, the topological space $\varphi(W(\mathbb{A})) \subset X$ is the projective limit of the sequence $(\varphi(W(\mathbb{Q}_{S_n})))_{n\geq 0}$ of the topological spaces $\varphi(W(\mathbb{Q}_{S_n})) \subset X_{S_n}$, with respect to the canonical maps $\varphi(W(\mathbb{Q}_{S_n})) \to \varphi(W(\mathbb{Q}_{S_m}))$ for $n \geq m$. Consequently, the sets of the form $\varphi(B \times \prod_{p\notin S_n} W(\mathbb{Z}_p))$, where B runs over the Borel subsets of $W(\mathbb{Q}_{S_n})$, generate the Borel structure of $\varphi(W(\mathbb{A}))$.

Let $n \geq 0$. It follows from the third step and from Proposition 5.4 that the image μ_n of μ in X_{S_n} is the Haar measure on the coset $\pi_{S_n}(x+Y)$ of the subgroup $\varphi_{S_n}(V_0(\mathbb{Q}_{S_n}))$. So, μ_n is invariant under translations by elements from $\varphi_{S_n}(V_0(\mathbb{Q}_{S_n}))$. This means that, for every Borel subset B of $W(\mathbb{Q}_{S_n})$, we have

$$\mu\Big(z+\varphi\Big(B \times \prod_{p\notin S_n} W(\mathbb{Z}_p)\Big)\Big) = \mu\Big(\varphi\Big(B \times \prod_{p\notin S_n} W(\mathbb{Z}_p)\Big)\Big),$$

for every $z \in Y$. This proves the claim. ∎

5.2.2. Orbit closures. We now deduce the description of orbit closures of points in X from the corresponding description in the S-adic case.

Recall that $X_S = \mathbb{Q}_S^d/\mathbb{Z}[1/S]^d$ for $S \in \mathcal{S}$ and that $\varphi\colon \mathbb{A}^d \to X$, $\varphi_S\colon \mathbb{Q}_S^d \to X_S$, and $\pi_S\colon X \to X_S$ denote the canonical projections.

Theorem 5.9. *Let $\mathbf{G}$ be a connected algebraic subgroup of GL_d defined over $\mathbb{Q}$. Assume that $\mathbf{G}(\mathbb{Q})$ is generated by unipotent one-parameter subgroups. Let $a \in \mathbb{A}^d$ and $x=\varphi(a) \in X$. There exists a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace V_0 of $\mathbb{Q}^d$ such that the closure of the $\mathbf{G}(\mathbb{Q})$-orbit of x in X coincides with $x+\varphi(V_0(\mathbb{A}))$.*

Proof. For $S \in \mathcal{S}$, set $x_S := \pi_S(x) \in X_S$. By Proposition 5.6, there exists a unique $\mathbf{G}(\mathbb{Q})$-invariant linear subspace V_S of $\mathbb{Q}^d$ such that

$$\overline{\mathbf{G}(\mathbb{Z}[1/S])x_S} = x_S + \varphi_S(V_S(\mathbb{Q}_S)).$$

Fix $S_0 \in \mathcal{S}$ such that $V_0 := V_{S_0}$ has maximal dimension among all the subspaces V_S for $S \in \mathcal{S}$.

We claim that the closure of the $\mathbf{G}(\mathbb{Q})$-orbit of x in X coincides with $x+\varphi(V_0(\mathbb{A}))$.

Indeed, let $\mathcal{P} \cup \{\infty\} = \cup_{n\geq 0} S_n$ for an increasing sequence of subsets $S_n \in \mathcal{S}$. Let $n \geq 1$ and write V_n for V_{S_n}.

Since $\{g(x_{S_0}) - x_{S_0} \mid g \in \mathbf{G}(\mathbb{Z}[1/S_0])\}$ is dense in $\varphi_{S_0}(V_0(\mathbb{Q}_{S_0}))$ and since

$$g(x_{S_n}) - x_{S_n} \in \varphi_{S_n}(V_n(\mathbb{Q}_{S_n})) \quad \text{for all } g \in \mathbf{G}(\mathbb{Z}[1/S_0]),$$

it follows that $V_0 \subset V_n$ (see the first step in the proof of Theorem 5.8) and hence $V_n = V_0$, by maximality of the dimension of V. Therefore, we have

$$\overline{\mathbf{G}(\mathbb{Z}[1/S_n])x_{S_n}} = x_{S_n} + \varphi_{S_n}(V_0(\mathbb{Q}_{S_n})) \quad \text{for all } n \geq 0.$$

Let $n \geq 0$ and $g \in \mathbf{G}(\mathbb{Z}[1/S_n])$. For every $m \geq n$, we have

$$g(x_{S_m}) - x_{S_m} \in \varphi_{S_m}(V_0(\mathbb{Q}_{S_m}))$$

and hence

$$g(x) - x \in \varphi\left(V_0(\mathbb{R}) \times \prod_{p\in S_m,\, p\neq\infty} V_0(\mathbb{Z}_p) \times \prod_{p\notin S_m} \mathbb{Z}_p^d \right).$$

It follows (see Lemma 5.7) that $g(x) - x \in \varphi(V_0(\mathbb{A}))$ for every $g \in \mathbf{G}(\mathbb{Z}[1/S_n])$. Hence, $x + \varphi(V_0(\mathbb{A}))$ is $\mathbf{G}(\mathbb{Q})$-invariant. Since $x + \varphi(V_0(\mathbb{A}))$ is closed in X, this implies that

$$\overline{\mathbf{G}(\mathbb{Q})x} \subset x + \varphi(V_0(\mathbb{A})).$$

Conversely, let $y \in \varphi(V_0(\mathbb{A}))$. Then $y = \varphi(v)$ for $v = (v_p)_{p\in\mathcal{P}\cup\{\infty\}}$ in $V_0(\mathbb{A})$ with $v_p \in V_0(\mathbb{Z}_p)$ for all $p \in \mathcal{P}$. Let U be a neighbourhood of y in X. Then U contains a set of the form $\varphi(O_n \times \prod_{p\notin S_n} \mathbb{Z}_p^d)$ for some $n \geq 0$, where O_n is a neighbourhood of $(v_p)_{p\in S_n}$ in $\mathbb{R}^d \times \prod_{p\in S_n\setminus\{\infty\}} \mathbb{Z}_p^d$.

Since $\mathbf{G}(\mathbb{Z}[1/S_n])x_{S_n} - x_{S_n}$ is dense in $\varphi_{S_n}(V_0(\mathbb{Q}_{S_n}))$, there exists $g \in \mathbf{G}(\mathbb{Z}[1/S_n])$ such that $g(x_{S_n}) - x_{S_n} \in \varphi_{S_n}(O_n)$. As $g(\mathbb{Z}_p^d) \subset \mathbb{Z}_p^d$ for every $p \notin S_n$, it follows that

$$g(x) - x \in \varphi\left(O_n \times \prod_{p\notin S_n} \mathbb{Z}_p^d \right) \subset U.$$

This shows that $x + y \in \overline{\mathbf{G}(\mathbb{Q})x}$. ■

6. Proof of Theorem A

In this section, we will give the proof of Theorem A.

6.1. Invariant characters on $\mathbb{Q}^d$

Let $\mathbf{G}$ be a connected algebraic subgroup of GL_d defined over $\mathbb{Q}$. Using Fourier transform, we establish the dual versions of Theorems 5.8 and 5.9 in terms of $\mathbf{G}(\mathbb{Q})$-invariant characters on $\mathbb{Q}^d$.

Recall (see Section 4.2) that, after the choice of non-trivial unitary character e of $\mathbb{A}$ which is trivial on $\mathbb{Q}$, we can identify $\widehat{\mathbb{Q}^d}$ with $X = \mathbb{A}^d/\mathbb{Q}^d$ by means of the $\mathrm{GL}_d(\mathbb{Q})$-equivariant map

$$X \to \widehat{\mathbb{Q}^d}, \quad a + \mathbb{Q}^d \mapsto \lambda_a,$$

where

$$\lambda_a(q) = e(\langle a, q\rangle) \quad \text{for all } q = (q_1, \dots, q_d) \in \mathbb{Q}^d,$$

and $\langle a, q\rangle = \sum_{i=1}^d a_i q_i$ for $a = (a_1, \dots, a_d) \in \mathbb{A}^d$.

By Pontrjagin duality, the map

$$q \mapsto (a + \mathbb{Q}^d \mapsto \lambda_a(q))$$

is a $\mathrm{GL}_d(\mathbb{Q})$-equivariant isomorphism between $\mathbb{Q}^d$ and the dual group $\widehat{X}$ of X. The annihilator of a subset Y of X in $\mathbb{Q}^d \cong \widehat{X}$ is

$$Y^\perp = \{q \in \mathbb{Q}^d \mid \lambda_a(q) = 1 \text{ for all } a + \mathbb{Q}^d \in Y\}.$$

If Y is a closed subgroup of X, the map

$$q + Y^\perp \mapsto (a + \mathbb{Q}^d \mapsto \lambda_a(q))$$

is an isomorphism between $\mathbb{Q}^d/Y^\perp$ and $\widehat{Y}$.

We will need the following characterization of subsolenoids of X. Recall that φ is the canonical projection $\mathbb{A}^d \to X$.

Lemma 6.1. *Let $\mathcal{Y}(X)$ be the set of connected and closed subgroups of X and $\mathbf{Gr}(\mathbb{Q}^d)$ the set of linear subspaces of $\mathbb{Q}^d$.*

(i) *The map $Y \to Y^\perp$ is a $\mathrm{GL}_d(\mathbb{Q})$-equivariant bijection between the sets $\mathcal{Y}(X)$ and $\mathbf{Gr}(\mathbb{Q}^d)$.*

(ii) *For $V \in \mathbf{Gr}(\mathbb{Q}^d)$, we have $\varphi(V(\mathbb{A})) \in \mathcal{Y}(X)$; moreover, we have*

$$\varphi(V(\mathbb{A})) \cap \varphi(W(\mathbb{A})) = \varphi((V \cap W)(\mathbb{A}))$$

for every $V, W \in \mathbf{Gr}(\mathbb{Q}^d)$.

(iii) *The map $V \to \varphi(V(\mathbb{A}))$ is a $\mathrm{GL}_d(\mathbb{Q})$-equivariant bijection between $\mathbf{Gr}(\mathbb{Q}^d)$ and $\mathcal{Y}(X)$.*

In particular, for every $P \in \mathbf{Gr}(\mathbb{Q}^d)$, there exists a unique $V \in \mathbf{Gr}(\mathbb{Q}^d)$ such that $P = \varphi(V(\mathbb{A}))^\perp$.

Proof. (i) Let Y be a closed subgroup of X. Then Y is connected if and only if $\widehat{Y}$ is torsion-free (see [23, Corollary 24.19]), that is, if and only if $\mathbb{Q}^d/Y^\perp$ is torsion-free. It follows that Y is connected if and only if $Y^\perp$ is a linear subspace of $\mathbb{Q}^d$.

(ii) Let $V \in \mathbf{Gr}(\mathbb{Q}^d)$. Since the canonical embedding of $\mathbb{R}$ is dense in $\mathbb{A}/\mathbb{Q}$, the subgroup $\varphi(V(\mathbb{R}))$ is dense in $\varphi(V(\mathbb{A}))$ and therefore $\varphi(V(\mathbb{A}))$ is connected. Moreover, $\varphi(V(\mathbb{A}))$ is closed as it is the continuous image of the compact solenoid $V(\mathbb{A})/V(\mathbb{Q})$.

Let $V, W \in \mathbf{Gr}(\mathbb{Q}^d)$. It is clear that $\varphi((V \cap W)(\mathbb{A}))$ is contained in $\varphi(V(\mathbb{A})) \cap \varphi(W(\mathbb{A}))$. Let $a \in V(\mathbb{A})$ be such that $\varphi(a) \in \varphi(W(\mathbb{A}))$. Writing $a = (a_p)_{p \in \mathcal{P} \cup \{\infty\}}$, we may assume that $a_p \in V(\mathbb{Z}_p)$ for every $p \in \mathcal{P}$. So, there exists $q \in \mathbb{Q}^d$ such that $a_\infty + q \in W(\mathbb{R})$ and $a_p + q \in W(\mathbb{Z}_p)$ for every $p \in \mathcal{P}$. It follows that $q \in \mathbb{Z}^d$. Hence, we have $a_\infty \in W(\mathbb{R}) + \mathbb{Z}^d$. Observe that $ta \in V(\mathbb{A})$ for every $t \in \mathbb{Q}$; it follows by the same reasoning that $ta_\infty \in W(\mathbb{R}) + \mathbb{Z}^d$ for every $t \in \mathbb{Q}$ and hence for every $t \in \mathbb{R}$, since $W(\mathbb{R}) + \mathbb{Z}^d$ is closed in $\mathbb{R}^d$. By connectedness, this implies that $a_\infty \in W(\mathbb{R})$. Since $a_\infty + q \in W(\mathbb{R})$, we have $q \in W(\mathbb{Z})$ and so $a_p \in W(\mathbb{Z}_p)$ for every $p \in \mathcal{P}$. Since $V(\mathbb{A}) \cap W(\mathbb{A}) = (V \cap W)(\mathbb{A})$, this shows that $a \in (V \cap W)(\mathbb{A})$.

(iii) As already mentioned (see the proof of Lemma 5.3), we have $\varphi(V(\mathbb{A})) \neq \varphi(W(\mathbb{A}))$ for every $V, W \in \mathbf{Gr}(\mathbb{Q}^d)$ with $V \neq W$.

Let $Y \in \mathcal{Y}(X)$. Then $Y^\perp \in \mathbf{Gr}(\mathbb{Q}^d)$, by (i). Consider the linear subspace

$$V := \left\{a \in \mathbb{Q}^d \mid \langle a, q\rangle = 0 \text{ for all } q \in Y^\perp\right\}$$

of $\mathbb{Q}^d$. We claim that $Y = \varphi(V(\mathbb{A}))$.

Indeed, let $q_1, \dots, q_s$ be a basis of $Y^\perp$. For every $i \in \{1, \dots, s\}$ and $t \in \mathbb{Q}$, let $V_i = \{a \in \mathbb{Q}^d \mid \langle a, q_i\rangle = 0\}$ and choose $a_{i,t} \in \mathbb{Q}^d$ such that $\langle a_{i,t}, q_i\rangle = t$. Then

$$\left\{a \in \mathbb{A}^d \mid \langle a, q_i\rangle = t\right\} = V_i(\mathbb{A}) + a_{i,t}.$$

We have

$$\begin{aligned} Y = (Y^\perp)^\perp &= \left\{a + \mathbb{Q}^d \in X \mid \lambda_a(q) = 1 \text{ for all } q \in Y^\perp\right\} \\ &= \left\{a + \mathbb{Q}^d \in X \mid \lambda_a(tq) = 1 \text{ for all } q \in Y^\perp,\ t \in \mathbb{Q}\right\} \\ &= \left\{a + \mathbb{Q}^d \in X \mid e(t\langle a, q\rangle) = 1 \text{ for all } q \in Y^\perp,\ t \in \mathbb{Q}\right\} \\ &= \left\{a + \mathbb{Q}^d \in X \mid \langle a, q_i\rangle \in \mathbb{Q} \text{ for every } i = 1, \dots, s\right\} \\ &= \bigcap_{i=1}^{s} \bigcup_{t \in \mathbb{Q}} \varphi(V_i(\mathbb{A}) + a_{i,t}) = \bigcap_{i=1}^{s} \varphi(V_i(\mathbb{A})). \end{aligned}$$

Using (ii), it follows that $Y = \varphi((\bigcap_{i=1}^s V_i)(\mathbb{A})) = \varphi(V(\mathbb{A}))$. ■

Let $\mathbf{G}$ be a connected algebraic subgroup of GL_d defined over $\mathbb{Q}$. Fix $x = \varphi(a) \in X$ for some $a \in \mathbb{A}^d$. Let P_x be the $\mathbb{Q}$-linear span of $\{\lambda_{g(a)}\lambda_{-a} \mid g \in \mathbf{G}(\mathbb{Q})\}^\perp$ in $\mathbb{Q}^d$; so,

$$\begin{aligned} P_x &= \bigcap_{g \in \mathbf{G}(\mathbb{Q}),\, t \in \mathbb{Q}} \left\{q \in \mathbb{Q}^d \mid \lambda_{g(a)}(tq) = \lambda_a(tq)\right\} \\ &= \bigcap_{g \in \mathbf{G}(\mathbb{Q})} \left\{q \in \mathbb{Q}^d \mid \langle g(a) - a, q\rangle \in \mathbb{Q}\right\} \end{aligned}$$

and P_x is a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace of $\mathbb{Q}^d$. Define $\chi_x\colon \mathbb{Q}^d \to \mathbb{C}$ by

$$\chi_x(q) = \begin{cases} \lambda_a(q) & \text{if } q \in P_x, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that χ_x is $\mathbf{G}(\mathbb{Q})$-invariant and is of positive type (see Proposition 2.4), that is, $\chi_x \in \mathrm{Tr}(\mathbb{Q}^d, \mathbf{G}(\mathbb{Q}))$.

Recall that two points $x, y \in X$ belong to the same $\mathbf{G}(\mathbb{Q})$-quasi-orbit if their $\mathbf{G}(\mathbb{Q})$-orbits have the same closure in X.

Theorem 6.2. *Let $\mathbf{G}$ be a connected algebraic subgroup of GL_d defined over $\mathbb{Q}$. Assume that $\mathbf{G}(\mathbb{Q})$ is generated by unipotent one-parameter subgroups. The map*

$$X \to \mathrm{Tr}(\mathbb{Q}^d, \mathbf{G}(\mathbb{Q})), \quad x \mapsto \chi_x$$

has $\mathrm{Char}(\mathbb{Q}^d, \mathbf{G}(\mathbb{Q}))$ as image and factorizes to a bijection

$$X/{\sim} \to \mathrm{Char}(\mathbb{Q}^d, \mathbf{G}(\mathbb{Q})),$$

where $X/{\sim}$ is the space of $\mathbf{G}(\mathbb{Q})$-quasi-orbits in $X = \mathbb{A}^d/\mathbb{Q}^d$.

Proof. Identifying $\mathbb{Q}^d$ with $\widehat{X}$, the Fourier transform on X is the map

$$\mathcal{F}\colon \mathrm{Prob}(X) \to \mathrm{Tr}(\mathbb{Q}^d)$$

given by

$$\mathcal{F}(\mu)(q) = \int_X \lambda_a(q)\, \mathrm{d}\mu(a + \mathbb{Q}^d) \quad \text{for all } \mu \in \mathrm{Prob}(X),\ q \in \mathbb{Q}^d.$$

Recall (see Proposition 3.1) that $\mathcal{F}$ restricts to a bijection

$$\mathcal{F}\colon \mathrm{Prob}(X)^{\mathbf{G}(\mathbb{Q})}_{\mathrm{erg}} \to \mathrm{Char}(\mathbb{Q}^d, \mathbf{G}(\mathbb{Q})).$$

Let $x = \varphi(a) \in X$ for $a \in \mathbb{A}^d$. By Lemma 6.1, there exists a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace V of $\mathbb{Q}^d$ such that $P_x = Y^\perp$ for $Y = \varphi(V(\mathbb{A}))$.

First step. We claim that $\mathcal{F}(\mu_{x+Y}) = \chi_x$.

Indeed, for every $q \in \mathbb{Q}^d$, we have

$$\mathcal{F}(\mu_{x+Y})(q) = \int_Y \lambda_{a+b}(q)\, \mathrm{d}\mu_Y(b + \mathbb{Q}^d) = \lambda_a(q) \int_Y \lambda_b(q)\, \mathrm{d}\mu_Y(b + \mathbb{Q}^d).$$

Now, $\int_Y \lambda_b(q)\, \mathrm{d}\mu_Y(b + \mathbb{Q}^d) = 0$ whenever $b + \mathbb{Q}^d \mapsto \lambda_b(q)$ is a non-trivial character of Y, by the orthogonality relations. This proves the claim.

Second step. We claim that $\chi_x \in \mathrm{Char}(\mathbb{Q}^d, \mathbf{G}(\mathbb{Q}))$. In view of the first step, it suffices to show that μ_{x+Y} is ergodic.

Assume, by contradiction, that μ_{x+Y} is not ergodic. Observe that $\mathbf{G}(\mathbb{Q})$ is connected (in the Zariski topology of $\mathrm{GL}_d(\mathbb{Q})$) and has hence no proper finite index subgroup. Therefore, by Proposition 5.2, there exists a proper closed connected subgroup Z of Y such that $g(x+y) \in x+y+Z$ for every $g \in \mathbf{G}(\mathbb{Q})$ and $y \in Y$. By Lemma 6.1, we can write $Z = \varphi(W(\mathbb{A}))$ for a $\mathbf{G}(\mathbb{Q})$-invariant proper $\mathbb{Q}$-linear subspace W of V. So, we have $g(x) - x \in \varphi(W(\mathbb{A}))$ for every $g \in \mathbf{G}(\mathbb{Q})$. In view of the definition of P_x, this implies that

$$\varphi(W(\mathbb{A}))^\perp = P_x = Y^\perp = \varphi(V(\mathbb{A}))^\perp,$$

which is a contradiction since $W \neq V$.

Third step. We claim that the closure of the $\mathbf{G}(\mathbb{Q})$-orbit of x coincides with $x + Y$.

Indeed, by Theorem 5.9, there exists a $\mathbf{G}(\mathbb{Q})$-invariant linear subspace W of $\mathbb{Q}^d$ such that the closure of the $\mathbf{G}(\mathbb{Q})$-orbit of x in X coincides with $x + \varphi(W(\mathbb{A}))$, that is, $g(x) - x \in \varphi(W(\mathbb{A}))$ for every $g \in \mathbf{G}(\mathbb{Q})$. As in the second step, this implies that $W = V$.

Fourth step. We claim that $\mathrm{Char}(\mathbb{Q}^d, \mathbf{G}(\mathbb{Q})) = \{\chi_x \mid x \in X\}$. Indeed, this follows from Theorem 5.8 and the first two steps.

Fifth step. For $i = 1, 2$, let $a_i \in \mathbb{A}^d$ and $x_i = \varphi(a_i)$. We claim that $\chi_{x_1} = \chi_{x_2}$ if and only if x_1 and x_2 belong to the same $\mathbf{G}(\mathbb{Q})$-quasi-orbit.

Indeed, assume that x_1 and x_2 belong to the same quasi-orbit. Then, by the third step, we have $x_1 + Y_1 = x_2 + Y_2$, where $Y_i = P_{x_i}^\perp$. Hence, $\chi_{x_1} = \chi_{x_2}$, by the first step.

Conversely, assume that $\chi_{x_1} = \chi_{x_2}$. Then $P_{x_1} = P_{x_2}$ and $\lambda_{a_1} = \lambda_{a_2}$ on P_{x_1}. So, $Y_1 = Y_2$ and $x_1 - x_2 \in Y_1$. Hence, $x_1 + Y_1 = x_2 + Y_2$ and so, by the third step, x_1 and x_2 belong to the same quasi-orbit. ■

6.2. Conclusion of the proof of Theorem A

Let $G = \mathbf{G}(\mathbb{Q})$ be as in the statement of Theorem A, $G = LU$ a Levi decomposition of G, and $\mathfrak{u}$ the Lie algebra of U.

Let $\psi \in \mathrm{Char}(G)$.

First step. Set $\varphi := \psi|_U \circ \exp$. There exists $\lambda \in \widehat{u}$ such that φ coincides with the trivial extension to $\mathfrak{u}$ of the restriction of λ to $\mathfrak{p}_\lambda$, where $\mathfrak{p}_\lambda$ is the G-invariant linear subspace of $\mathfrak{u}$ given by

$$\mathfrak{p}_\lambda = \big\{X \in \mathfrak{u} \mid \lambda(\mathrm{Ad}(g)(tX)) = \lambda(tX) \text{ for all } g \in G,\ t \in \mathbb{Q}\big\}.$$

Indeed, as discussed in Section 4.2, $\varphi \in \mathrm{Char}(\mathfrak{u}, G)$. Identifying $\mathfrak{u}$ with $\mathbb{Q}^d$, we can view φ as an element in $\mathrm{Char}(\mathbb{Q}^d, G)$. By Theorem 6.2, there exists $x \in X$ such that φ is the trivial extension to $\mathfrak{u}$ of the restriction to P_x of the unitary character λ of $\mathfrak{u}$ defined by $x \in X$. As $P_x = \mathfrak{p}_\lambda$, the claim follows.

Second step. We claim that

$$\psi(\exp(X)g) = \psi(\exp(X))\psi(g) \quad \text{for all } g \in G,\ X \in \mathfrak{p}_\lambda.$$

Indeed, since

$$|\psi(\exp(X))| = |\varphi(X)| = |\lambda(X)| = 1$$

for every $X \in \mathfrak{p}_\lambda$, the claim follows from Proposition 2.5 (ii).

Third step. For every $g \in G$ and $X \in \mathfrak{u}$, we have

$$\psi(g) = \psi(\exp(-X)\exp(\mathrm{Ad}(g)(X))g).$$

This is indeed the case, since

$$\psi(g) = \psi(\exp(-X)g\exp(X)) = \psi(\exp(-X)\exp(\mathrm{Ad}(g)(X))g).$$

Recall that

$$\mathfrak{k}_\lambda = \big\{X \in \mathfrak{u} \mid \lambda(\mathrm{Ad}(g)(tX)) = 1 \text{ for all } g \in G,\ t \in \mathbb{Q}\big\};$$

observe that $K_\lambda := \exp(\mathfrak{k}_\lambda)$ is in general strictly contained in $K_\psi \cap U$, where

$$K_\psi := \{g \in G \mid \psi(g) = 1\}.$$

Let

$$G_\lambda = \big\{g \in G \mid \mathrm{Ad}(g)(X) \in X + \mathfrak{k}_\lambda \text{ for all } X \in \mathfrak{u}\big\}.$$

Assume that $G_\lambda \neq G$. Observe that this implies that $\lambda \neq 1_{\mathfrak{u}}$. Let $g \in G \setminus G_\lambda$ and fix $X \in \mathfrak{u}$ such that $\mathrm{Ad}(g)(X) - X \notin \mathfrak{k}_\lambda$. Let

$$A_{g,X} := \big\{t \in \mathbb{Q} \mid \exp(-tX)\exp(\mathrm{Ad}(g)(tX)) \in K_\psi\big\}.$$

By Lemma 3.6, $A_{g,X}$ is a subgroup of $\mathbb{Q}$.

Fourth step. We claim that $A_{g,X} \neq \mathbb{Q}$.

Indeed, assume, by contradiction, that $A_{g,X} = \mathbb{Q}$, that is,

$$\exp(-tX)\exp(\mathrm{Ad}(g)(tX)) \in K_\psi \quad \text{for all } t \in \mathbb{Q}.$$

By the Campbell–Hausdorff formula, there exists $Y_1, Y_2, \dots, Y_r \in \mathfrak{u}$ such that

$$\exp(-tX)\exp(\mathrm{Ad}(g)(tX)) = \exp\left(\sum_{k=1}^{r} t^k Y_k\right) \quad \text{for all } t \in \mathbb{Q},$$

where $Y_1 = \mathrm{Ad}(g)(X) - X$. We have then

$$\lambda\left(\sum_{k=1}^{r} t^k Y_k\right) = 1 \quad \text{for all } t \in \mathbb{Q}.$$

Identifying $\mathfrak{u}$ with $\mathbb{Q}^d$ via a basis $\{X_1, \dots, X_d\}$, the character λ of $\mathfrak{u}$ is given by some $a = (a_1, \dots, a_d) \in \mathbb{A}^d$ via the formula

$$\lambda\Big(\sum_{i=1}^{d} q_i X_i\Big) = e\Big(\sum_{i=1}^{d} a_i q_i\Big) \quad \text{for all } (q_1, \dots, q_d) \in \mathbb{Q}^d$$

for a non-trivial unitary character e of $\mathbb{A}$ which is trivial on $\mathbb{Q}$ (see Section 4.2). It follows that

$$e\Big(\sum_{k=1}^{r} t^k \Big(\sum_{i=1}^{d} a_i q_{k,i}\Big)\Big) = 1 \quad \text{for all } t \in \mathbb{Q},$$

where $(q_{k,i})_{i=1}^d$ are the coordinates of Y_k in $\{X_1, \dots, X_d\}$. This implies that

$$\sum_{i=1}^{d} a_i q_{k,i} \in \mathbb{Q} \quad \text{for every } k = 1, \dots, r.$$

Indeed, otherwise the image of the set

$$\Big\{\sum_{k=1}^{r} t^k \Big(\sum_{i=1}^{d} a_i q_{k,i}\Big) \Big| t \in \mathbb{Q}\Big\}$$

would be dense in $\mathbb{A}/\mathbb{Q}$ (see [11, Theorem 2] or [5, Theorem 5.2]) and this would contradict the non-triviality of e.

In particular, we have $\sum_{i=1}^{d} a_i q_{1,i} \in \mathbb{Q}$ and, since $Y_1 = \mathrm{Ad}(g)X - X$, we obtain $\lambda(\mathrm{Ad}(g)(tX) - tX) = 1$ for all $t \in \mathbb{Q}$. So, $\mathrm{Ad}(g)X - X$ belongs to $\mathfrak{k}_\lambda$ and this is a contradiction to the choice of X.

Fifth step. Let $g \in G \setminus G_\lambda$. We claim that $\psi(g) = 0$.

Indeed, let $X \in \mathfrak{u}$ with $\mathrm{Ad}(g)(X) - X \notin \mathfrak{k}_\lambda$ and let $A_{g,X} \subset \mathbb{Q}$ be as in the fourth step. Set

$$B_{g,X} := \big\{t \in \mathbb{Q} \mid \exp(-tX)\exp(\mathrm{Ad}(g)(tX)) \in P_\psi\big\}.$$

Then $A_{g,X} \subset B_{g,X}$ and, by Lemma 3.6 again, $B_{g,X}$ is a subgroup of $\mathbb{Q}$. Two cases may occur.

– First case: $A_{g,X} \neq B_{g,X}$. So, there exists $t \in \mathbb{Q}$ such that

$$\exp(-tX)\exp(\mathrm{Ad}(g)(tX)) \in P_\psi \setminus K_\psi.$$

Then, using the third and second steps, we have

$$\psi(g) = \psi(\exp(-X)\exp(\mathrm{Ad}(g)(X)))\psi(g)$$

and hence $\psi(g) = 0$, since

$$\psi(\exp(-X)\exp(\mathrm{Ad}(g)(X))) \neq 1.$$

– Second case: $A_{g,X} = B_{g,X}$. Then $B_{g,X}$ is a proper subgroup of $\mathbb{Q}$ by the fourth step. So, $B_{g,X}$ has infinite index in $\mathbb{Q}$ and we can find an infinite sequence $(t_n)_{n\geq 1}$ in $\mathbb{Q}$ such that $t_n - t_m \notin B_{g,X}$ for all $n \neq m$.

Set

$$u_n := \exp(-t_n X)\exp(\mathrm{Ad}(g)(t_n X)) \quad \text{for all } n \geq 1.$$

For $n \neq m$, we have

$$\begin{aligned} u_n u_m^{-1} &= \exp(-t_n X)\exp(\mathrm{Ad}(g)(t_n - t_m)X)\exp(t_m X) \\ &= \exp(-t_m X)(\exp(-(t_n - t_m)X)\exp(\mathrm{Ad}(g)(t_n - t_m)X))\exp(t_m X); \end{aligned}$$

since $\exp(-(t_n - t_m)X)\exp(\mathrm{Ad}(g)(t_n - t_m)X) \notin P_\psi$ and P_ψ is a normal subgroup of U, we have $u_n u_m^{-1} \notin P_\psi$ and hence

$$\psi(u_n u_m^{-1}) = 0 \quad \text{for all } n \neq m,$$

by the first step.

As u_n coincides with the commutator $[\exp(-t_n X), g]$ in G, it follows from Lemma 2.6 that $\psi(g) = 0$.

It remains to determine the restriction of ψ to G_λ.

Since $\psi|_{K_\lambda} = 1_{K_\lambda}$, we may view ψ as a character of G/K_λ, which is the group of $\mathbb{Q}$-points of an algebraic group, and we can therefore assume that $K_\lambda = \{e\}$. Then G_λ is the centralizer of U in G and is the group of $\mathbb{Q}$-points of an algebraic normal subgroup of $\mathbf{G}$. Let $G_\lambda = L_1 U_1$ be a Levi decomposition of G_λ. Since U_1 is a unipotent characteristic subgroup of G_λ, we have $U_1 \subset U$. Moreover, L_1 is the group of $\mathbb{Q}$-points of an algebraic subgroup $\mathbf{L}_1$ of $\mathbf{G}$ and so $\mathbf{L}_1$ is contained in a Levi subgroup of $\mathbf{G}$. As two Levi subgroups of $\mathbf{G}$ are conjugate by an element of U (see [28]), we can assume that $L_1 \subset L$, that is, $L_1 = L_\lambda$ and so, $G_\lambda = L_\lambda Z(U)$.

Sixth step. We claim that there exists $\varphi_1 \in \mathrm{Char}(L_\lambda)$ such that

$$\psi(gu) = \varphi_1(g)\psi(u) \quad \text{for all } g \in L_\lambda,\ u \in U.$$

Indeed, we can find a normal subgroup H of L which centralizes L_λ such that $L_\lambda \cap H$ is finite and such that $L = L_\lambda H$ (see Proposition 6.3 below). Then $G = L_\lambda H U$ and HU centralizes L_λ. So, the claim follows from Proposition 2.12 (see also Corollary 2.13).

Seventh step. Let $\lambda \in \hat{\mathfrak{u}}$, $\varphi_1 \in \mathrm{Char}(L_\lambda)$, and let $\Phi_{(\lambda,\varphi)}\colon G \to \mathbb{C}$ be defined as in Theorem A. We claim that $\psi := \Phi_{(\lambda,\varphi)} \in \mathrm{Char}(G)$.

Indeed, it is clear (see Proposition 2.4) that $\psi \in \mathrm{Tr}(G)$. Write

$$\psi = \int_\Omega \psi_\omega \,\mathrm{d}\nu(\omega)$$

as an integral over a probability space (Ω, ν) with $\psi_\omega \in \mathrm{Char}(G)$ for every ω (see Remark 2.2).

Then $\varphi := \psi|_U \circ \exp$ coincides with the trivial extension to $\mathfrak{u}$ of the restriction of λ to $\mathfrak{p}_\lambda$, where $\mathfrak{p}_\lambda$ is defined as above. It follows from Theorem 6.2 that $\varphi \in \mathrm{Char}(\mathfrak{u}, G)$. This implies that the restriction of ψ_ω to U coincides with $\psi|_U$ for (ν-almost) every ω.

Let $\omega \in \Omega$. The fifth step, applied to $\psi_\omega \in \mathrm{Char}(G)$, shows that $\psi_\omega = 0$ on $G \setminus G_\lambda$, where G_λ is defined as above. By the sixth step, also applied to ψ_ω, there exists $\varphi_1^\omega \in \mathrm{Char}(L_\lambda)$ such that

$$\psi(gu) = \varphi_1^\omega(g)\psi(u) \quad \text{for all } g \in L_\lambda,\ u \in U.$$

As a result, we have

$$\varphi_1 = \int_\Omega \varphi_1^\omega \, d\nu(\omega).$$

Since $\varphi_1 \in \mathrm{Char}(L_\lambda)$, it follows that $\varphi_1^\omega = \varphi_1$ and hence that $\varphi = \varphi_\omega$ for (ν-almost) every ω. This shows that $\varphi \in \mathrm{Char}(G)$.

Eighth step. Let $\lambda_1, \lambda_2 \in \widehat{\mathfrak{u}}$, $\varphi_1 \in \mathrm{Char}(L_{\lambda_1})$ and $\varphi_2 \in \mathrm{Char}(L_{\lambda_2})$. Let us show that $\Phi_{(\lambda_1,\varphi_1)} = \Phi_{(\lambda_2,\varphi_2)}$ if and only if λ_1 and λ_2 have the same G-orbit closure and $\varphi_1 = \varphi_2$.

Indeed, set $\psi_i = \Phi(\lambda_i, \varphi_i)$ for $i = 1, 2$. It follows from Theorem 6.2 that $\psi_1|_U = \psi_2|_U$ if and only if the closures of the G-orbits of λ_1 and λ_2 coincide. If this is the case, then $\mathfrak{k}_{\lambda_1} = \mathfrak{k}_{\lambda_2}$ and hence $G_{\lambda_1} = G_{\lambda_2}$. The claim follows from these facts.

6.3. Characters of semi-simple algebraic groups

The following proposition, in combination with [2] and Corollary 2.13, shows how the characters of the groups L_λ appearing in Theorem A can be described.

A group L is the *almost direct product* of subgroups $H_1, \ldots, H_n$ of L if the product map $H_1 \times \cdots \times H_n \to L$ is a surjective homomorphism with finite kernel.

Proposition 6.3. *Let $\mathbf{G}$ be a connected semi-simple algebraic group defined over a field k. Assume that $\mathbf{G}(k)$ is generated by its unipotent one-parameter subgroups. Let $\mathbf{L}$ be a (not necessarily connected) algebraic normal k-subgroup of $\mathbf{G}$. Then there exist connected almost k-simple normal k-subgroups $\mathbf{G}_1, \ldots, \mathbf{G}_r$ of $\mathbf{G}$, a subgroup F of $\mathbf{L}(k)$ contained in the (finite) center of $\mathbf{G}$, and a connected normal k-subgroup $\mathbf{H}$ of $\mathbf{G}$ with the following properties:*

(i) *$\mathbf{G}(k)$ is the almost direct product of $\mathbf{L}(k)$ and $\mathbf{H}(k)$;*

(ii) *$L(\mathbf{k})$ is the almost direct product of $F, \mathbf{G}_1(k), \ldots, \mathbf{G}_r(k)$;*

(iii) *every $\mathbf{G}_i(k)$ is generated by its unipotent one-parameter subgroups.*

Proof. Let $\mathbf{L}_0$ be the connected component of $\mathbf{L}$. Let $\mathbf{G}_1, \ldots, \mathbf{G}_r$ be the connected almost k-simple normal k-subgroups of $\mathbf{G}$ contained in $\mathbf{L}_0$. Then $\mathbf{L}_0$ is the almost direct product of the $\mathbf{G}_i$'s and there exists a connected normal k-subgroup $\mathbf{H}$ of $\mathbf{G}$ such that $\mathbf{G}$ is the almost direct product of $\mathbf{L}_0$ and $\mathbf{H}$ (see [7, §2.15]). It follows that $\mathbf{L}$ is the almost direct product of $\mathbf{L}_0$ and $\mathbf{L} \cap \mathbf{H}$ and hence that $\mathbf{L} \cap \mathbf{H}$ is finite, since $\mathbf{L}_0$ has finite index in $\mathbf{L}$.

This implies that $\mathbf{L} \cap \mathbf{H}$ is contained in the center of $\mathbf{G}$, as $\mathbf{L} \cap \mathbf{H}$ is a normal subgroup of $\mathbf{G}$.

Since $\mathbf{G}(k)$ is generated by its unipotent one-parameter subgroups, the same is true for $\mathbf{G}_i(k)$ for every i, for $\mathbf{L}_0(k)$, and for $\mathbf{H}(k)$. Hence, $\mathbf{L}_0(k)$ is the almost direct product of $\mathbf{G}_1(k), \dots, \mathbf{G}_r(k)$, and $\mathbf{G}(k)$ is the almost direct product of $\mathbf{L}_0(k)$ and $\mathbf{H}(k)$ (see [8, Proposition 6.2]). It follows that $\mathbf{L}(k)$ is the almost direct product of $\mathbf{L}_0(k)$ and $F := \mathbf{L}(k) \cap \mathbf{H}(k)$. From what we have seen above, $F \subset \mathbf{L} \cap \mathbf{H}$ is a subgroup of the center of $\mathbf{G}$. ∎

7. A few examples

7.1. Abelian unipotent radical

Let $\mathbf{L}$ be a quasi-simple algebraic group defined over $\mathbb{Q}$. Assume that $L = \mathbf{L}(\mathbb{Q})$ is generated by its unipotent one-parameter subgroups. Let $\mathbf{L} \to \mathrm{GL}(V)$ be a finite-dimensional representation defined over $\mathbb{Q}$ of dimension at least 2; assume that $L \to \mathrm{GL}(V(\mathbb{Q}))$ is irreducible. Set $G = L \ltimes V(\mathbb{Q})$. Then G is the group of $\mathbb{Q}$-rational points of the algebraic group $\mathbf{L} \ltimes V$. Denote by F the kernel of $L \to \mathrm{GL}(V(\mathbb{Q}))$ and observe that F is a subgroup of the finite center of $\mathbf{L}$. We claim that

$$\mathrm{Char}(G) = \{\varphi \circ p \mid \varphi \in \mathrm{Char}(L)\} \cup \{\tilde{\chi} \mid \chi \in \mathrm{Char}(F)\},$$

where $p: G \to L$ is the canonical epimorphism. Indeed, let $\lambda \in \widehat{V(\mathbb{Q})}$. The sets K_λ and P_λ as in Theorem A are L-invariant linear subspaces of $V(\mathbb{Q})$ and so are equal either to $V(\mathbb{Q})$ or to $\{0\}$, by irreducibility of the representation of L on $V(\mathbb{Q})$.

- Assume that $K_\lambda = V(\mathbb{Q})$; then $\lambda = \mathbf{1}_{V(\mathbb{Q})}$ and $L_\lambda = L$. So, the characters of G associated to λ are the characters of L lifted to G.
- Assume that $K_\lambda = \{0\}$. Then $P_\lambda = \{0\}$ (see Proposition 3.5). So, $L_\lambda = F$ and every element of $\mathrm{Char}(G)$ associated to λ is of the form $\tilde{\chi}$ for some $\chi \in \mathrm{Char}(F)$.

For instance, for every faithful $\mathbb{Q}$-irreducible rational representation $\mathrm{SL}_2 \to \mathrm{GL}(V)$, we have

$$\mathrm{Char}(\mathrm{SL}_2(\mathbb{Q}) \ltimes V(\mathbb{Q})) = \{\mathbf{1}_G, \varepsilon, \delta_e\},$$

where ε is defined by $\varepsilon(I, v) = 1$, $\varepsilon(-I, v) = -1$, and $\varepsilon(g, v) = 0$ otherwise.

7.2. The Heisenberg group as unipotent radical

For an integer $n \geq 1$, consider the symplectic form β on $\mathbb{C}^{2n}$ given by

$$\beta((x, y), (x', y')) = (x, y)^t J(x', y') \quad \text{for all } (x, y), (x', y') \in \mathbb{C}^{2n},$$

where J is the $(2n \times 2n)$-matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The symplectic group

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid {}^t gJg = J\}$$

is an algebraic group which is a quasi-simple and defined over $\mathbb{Q}$.

The $(2n+1)$-dimensional Heisenberg group is the unipotent algebraic group H_{2n+1} defined over $\mathbb{Q}$, with underlying set $\mathbb{C}^{2n} \times \mathbb{C}$ and product

$$((x,y),s)((x',y'),t) = \Big((x+x', y+y'), s+t+\frac{1}{2}\beta((x,y),(x',y'))\Big),$$

for $(x,y),(x',y') \in \mathbb{C}^{2n}$, $s,t \in \mathbb{C}$.

The group Sp_{2n} acts by rational automorphisms of H_{2n+1}, given by

$$g((x,y),t) = (g(x,y),t) \quad \text{for all } g \in \mathrm{Sp}_{2n},\ (x,y) \in \mathbb{C}^{2n},\ t \in \mathbb{C}.$$

Let

$$G = \mathrm{Sp}_{2n}(\mathbb{Q}) \ltimes H_{2n+1}(\mathbb{Q})$$

be the group of $\mathbb{Q}$-points of the algebraic group $\mathrm{Sp}_{2n} \ltimes H_{2n+1}$ defined over $\mathbb{Q}$. Since Sp_{2n} is $\mathbb{Q}$-split, G is generated by its unipotent one-parameter subgroups. We claim that

$$\mathrm{Char}(G) = \{\mathbf{1}_G, \mathbf{1}_{H_{2n+1}(\mathbb{Q})}, \varepsilon\} \cup \{\widetilde{\chi} \mid \chi \in \widehat{Z}\},$$

where $Z = \{((0,0),s) : s \in \mathbb{Q}\}$ is the center of $H_{2n+1}(\mathbb{Q})$ and ε is the character of G defined by

$$\varepsilon(I,h) = 1, \quad \varepsilon(-I,h) = -1, \quad \text{and} \quad \varepsilon(g,h) = 0$$

for $g \in \mathrm{Sp}_{2n}(\mathbb{Q}) \setminus \{\pm I\}$ and $h \in H_{2n+1}(\mathbb{Q})$.

Indeed, the Lie algebra of $H_{2n+1}(\mathbb{Q})$ is the $(2n+1)$-dimensional nilpotent Lie algebra $\mathfrak{h}$ over $\mathbb{Q}$ with underlying set $\mathbb{Q}^{2n} \times \mathbb{Q}$ and Lie bracket

$$[(x,y),s),((x',y'),t] = (0, \beta((x,y),(x',y'))),$$

for $(x,y),(x',y') \in \mathbb{Q}^{2n}$, $s,t \in \mathbb{Q}$. The action of $\mathrm{Sp}_{2n}(\mathbb{Q})$ on $\mathfrak{h}$ is given by the same formula as for the action on $H_{2n+1}(\mathbb{Q})$.

The $\mathrm{Sp}_{2n}(\mathbb{Q})$-invariant ideals $\mathfrak{k}$ of $\mathfrak{h}$ are $\{0\}$, $\mathfrak{h}$, and the center $\mathfrak{z}$ of $\mathfrak{h}$. The corresponding ideals $\mathfrak{p}$, which are inverse images in $\mathfrak{h}$ of the G-fixed elements in $\mathfrak{h}/\mathfrak{k}$, are respectively $\mathfrak{z}$, $\mathfrak{h}$ and $\mathfrak{z}$.

Let $\lambda \in \widehat{\mathfrak{h}}$.

- Assume that $\mathfrak{k}_\lambda = \{0\}$. Then $\mathfrak{p}_\lambda = \mathfrak{z}$. Since no element in $\mathrm{Sp}_{2n}(\mathbb{Q}) \setminus \{e\}$ acts trivially on $\mathfrak{h}/\mathfrak{z} \cong \mathbb{Q}^{2n}$, we have $L_\lambda = \{e\}$. So, the only character of G associated to λ is $\widetilde{\chi_\lambda}$. (Observe that $\widetilde{\chi_\lambda} \neq \mathbf{1}_Z$, since $\mathfrak{k}_\lambda = \{0\}$.)

- Assume that $\mathfrak{k}_\lambda = \mathfrak{h}$; then $\lambda = \mathbf{1}_{\mathfrak{h}}$ and $L_\lambda = \mathrm{Sp}_{2n}(\mathbb{Q})$. So, the characters of G associated to λ are the characters of $\mathrm{Sp}_{2n}(\mathbb{Q})$ lifted to G, that is, $\mathbf{1}_G$, $\mathbf{1}_{H_{2n+1}(\mathbb{Q})}$, and ε.
- Assume that $\mathfrak{k}_\lambda = \mathfrak{z}$. Then $\mathfrak{p}_\lambda = \mathfrak{z}$ and $L_\lambda = \{e\}$. So, $\mathbf{1}_Z$ is the only character of G associated to λ.

7.3. Free nilpotent groups as unipotent radical

Let $\mathfrak{u} = \mathfrak{u}_{n,2}$ be the free 2-step nilpotent Lie algebra on $n \geq 2$ generators over $\mathbb{Q}$; as is well known (see [18]), $\mathfrak{u}$ can be realized as follows. Let V be an n-dimensional vector space over $\mathbb{Q}$ and set $\mathfrak{u} := V \otimes \wedge^2 V$, where $\wedge^2 V$ is the second exterior power of V. The Lie bracket on $\mathfrak{u} = V \otimes \wedge^2 V$ is defined by

$$[(v_1, w_1), (v_2, w_2)] = (0, v_1 \wedge v_2) \quad \text{for all } v_1, v_2 \in V,\ w_1, w_2 \in \wedge^2 V.$$

The center of $\mathfrak{u}$ is $\wedge^2 V$ and the associated unipotent group U is $V \oplus \wedge^2 V$ with the product

$$(v_1, w_1)(v_2, w_2) = \left(v_1 + v_2, w_1 + w_2 + \frac{1}{2} v_1 \wedge v_2\right)$$

so that the exponential mapping $\exp\colon \mathfrak{u} \to U$ is the identity. The group GL_n acts naturally on V as well as on $\wedge^2 V$, and these actions induce an action of GL_n by automorphisms on U given by

$$g(v, w) = (gv, gw) \quad \text{for all } g \in \mathrm{GL}_n(\mathbb{Q}),\ v \in V,\ w \in \wedge^2 V.$$

Since U coincides with the Heisenberg group $H_3(\mathbb{Q})$ when $n = 2$, we may assume that $n \geq 3$. Let L be the group of $\mathbb{Q}$-points of an algebraic subgroup of GL_n defined and quasi-simple over $\mathbb{Q}$. Assume that the representations of L on V and on $\wedge^2 V$ are faithful and irreducible over $\mathbb{Q}$ and that, moreover, L is generated by its unipotent one-parameter subgroups (an example of such a group is $L = \mathrm{SL}_n(\mathbb{Q})$). The group $G = L \ltimes U$ is the group of $\mathbb{Q}$-points of an algebraic group defined over $\mathbb{Q}$ and G is generated by its unipotent one-parameter subgroups.

We claim that

$$\mathrm{Char}(G) = \{\mathbf{1}_G\} \cup \left\{\widetilde{\chi} \circ p \mid \chi \in \widehat{Z(L)}\right\} \cup \{\delta_e\} \cup \{\mathbf{1}_{\wedge^2 V}\},$$

where $p\colon G \to L$ is the canonical epimorphism and $Z(L)$ the center of L.

Indeed, the L-invariant ideals $\mathfrak{k}$ of $\mathfrak{u}$ are $\{0\}$, $\mathfrak{u}$, and the center $\mathfrak{z} = \wedge^2 V$. By irreducibility of the L-action on V and on $\wedge^2 V$, the corresponding ideals $\mathfrak{p}$, which are inverse images in $\mathfrak{u}$ of the G-fixed elements in $\mathfrak{u}/\mathfrak{k}$, are respectively $\{0\}$, $\mathfrak{u}$ and $\mathfrak{z}$.

Let $\lambda \in \widehat{\mathfrak{u}}$.

- Assume that $\mathfrak{k}_\lambda = \{0\}$. Then $\mathfrak{p}_\lambda = \{0\}$ and $L_\lambda = \{e\}$. So, the only character of G associated to λ is δ_e.
- Assume that $\mathfrak{k}_\lambda = \mathfrak{u}$. Then the characters of G associated to λ are the characters of L lifted to G, that is, $\{\mathbf{1}_G\}$ and $\widetilde{\chi} \circ p$ for $\chi \in \widehat{Z(L)}$.

- Assume that $\mathfrak{k}_\lambda = \mathfrak{z}$. Then $\mathfrak{p}_\lambda = \mathfrak{z}$ and $L_\lambda = \{e\}$. So, $\mathbf{1}_{\wedge^2 V}$ is the only character of G associated to λ.

Funding. The authors acknowledge the support by the ANR (French Agence Nationale de la Recherche) through the projects Labex Lebesgue (ANR-11-LABX-0020-01) and GAMME (ANR-14-CE25-0004).

References

[1] B. Bekka, Operator-algebraic superridigity for $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$. *Invent. Math.* **169** (2007), no. 2, 401–425 Zbl 1135.22009 MR 2318561

[2] B. Bekka, Character rigidity of simple algebraic groups. *Math. Ann.* **378** (2020), no. 3-4, 1223–1243 Zbl 07262920 MR 4163525

[3] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*. New Math. Monogr. 11, Cambridge University Press, Cambridge, 2008 Zbl 1146.22009 MR 2415834

[4] B. Bekka and C. Francini, Spectral gap property and strong ergodicity for groups of affine transformations of solenoids. *Ergodic Theory Dynam. Systems* **40** (2020), no. 5, 1180–1193 Zbl 1448.37036 MR 4082259

[5] V. Bergelson and J. Moreira, Van der Corput's difference theorem: some modern developments. *Indag. Math. (N.S.)* **27** (2016), no. 2, 437–479 Zbl 1353.37011 MR 3479166

[6] A. Borel, Some finiteness properties of adele groups over number fields. *Inst. Hautes Études Sci. Publ. Math.* **16** (1963), 5–30 Zbl 0135.08902 MR 202718

[7] A. Borel and J. Tits, Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.* **27** (1965), 55–150 Zbl 0145.17402 MR 207712

[8] A. Borel and J. Tits, Homomorphismes "abstraits" de groupes algébriques simples. *Ann. of Math. (2)* **97** (1973), 499–571 Zbl 0272.14013 MR 316587

[9] R. Boutonnet and C. Houdayer, Stationary characters on lattices of semisimple Lie groups. *Publ. Math. Inst. Hautes Études Sci.* **133** (2021), 1–46 Zbl 07395792 MR 4292738

[10] A. L. Carey and W. Moran, Characters of nilpotent groups. *Math. Proc. Cambridge Philos. Soc.* **96** (1984), no. 1, 123–137 Zbl 0549.43004 MR 743708

[11] L. Corwin and C. Pfeffer, On the density of sets in $(\mathbf{A}/\mathbf{Q})^n$ defined by polynomials. *Colloq. Math.* **68** (1995), no. 1, 1–5 Zbl 0826.11036 MR 1311756

[12] J. Dixmier, *C^*-algebras*. North-Holland Math. Libr. 15, North-Holland Publishing Co., Amsterdam, 1977 Zbl 0372.46058 MR 0458185

[13] J. Dixmier, *von Neumann algebras (with a preface by E.C. Lance)*. North-Holland Math. Libr. 27, North-Holland Publishing Co., Amsterdam, 1981 Zbl 0473.46040 MR 641217

[14] A. Dudko and K. Medynets, Finite factor representations of Higman–Thompson groups. *Groups Geom. Dyn.* **8** (2014), no. 2, 375–389 Zbl 1328.20012 MR 3231220

[15] M. Duflo, Théorie de Mackey pour les groupes de Lie algébriques. *Acta Math.* **149** (1982), no. 3–4, 153–213 Zbl 0529.22011 MR 688348

[16] M. Einsiedler and A. Ghosh, Rigidity of measures invariant under semisimple groups in positive characteristic. *Proc. Lond. Math. Soc. (3)* **100** (2010), no. 1, 249–268 Zbl 1189.37035 MR 2578474

[17] G. B. Folland, *A course in abstract harmonic analysis*. Stud. Adv. Math., CRC Press, Boca Raton, FL, 1995 Zbl 0857.43001 MR 1397028

[18] M. A. Gauger, On the classification of metabelian Lie algebras. *Trans. Amer. Math. Soc.* **179** (1973), 293–329 Zbl 0267.17015 MR 325719

[19] T. Gelander, A view on invariant random subgroups and lattices. In *Proceedings of the International Congress of Mathematicians – Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 1321–1344, World Sci. Publ., Hackensack, NJ, 2018 Zbl 1441.22007 MR 3966811

[20] J. Glimm, Type I C^*-algebras. *Ann. of Math. (2)* **73** (1961), 572–612 Zbl 0152.33002 MR 124756

[21] Harish-Chandra, *Collected papers. Vols. I, II, III, IV*. Springer, New York, 1984 Zbl 0546.01013

[22] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. II. Structure and analysis for compact groups. Analysis on locally compact Abelian groups*. Grundlehren Math. Wiss. 152, Springer, New York, 1970 Zbl 0213.40103 MR 0262773

[23] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I. Structure of topological groups, integration theory, group representations*. 2nd edn., Grundlehren Math. Wiss. 115, Springer, Berlin, 1979 Zbl 0416.43001 MR 551496

[24] A. A. Kirillov, Positive-definite functions on a group of matrices with elements from a discrete field. *Soviet. Math. Dokl.* **6** (1965), 707–709 Zbl 0133.37505 MR 0193183

[25] R. L. Lipsman, The CCR property for algebraic groups. *Amer. J. Math.* **97** (1975), no. 3, 741–752 Zbl 0319.22009 MR 390123

[26] G. A. Margulis and G. M. Tomanov, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces. *Invent. Math.* **116** (1994), no. 1–3, 347–392 Zbl 0816.22004 MR 1253197

[27] J. S. Milne, *Algebraic groups. The theory of group schemes of finite type over a field*. Cambridge Stud. Adv. Math. 170, Cambridge University Press, Cambridge, 2017 Zbl 1390.14004 MR 3729270

[28] G. D. Mostow, Fully reducible subgroups of algebraic groups. *Amer. J. Math.* **78** (1956), 200–221 Zbl 0073.01603 MR 92928

[29] S. Mozes and N. Shah, On the space of ergodic invariant measures of unipotent flows. *Ergodic Theory Dynam. Systems* **15** (1995), no. 1, 149–159 Zbl 0818.58028 MR 1314973

[30] S. V. Ovchinnikov, Positive definite functions on Chevalley groups. *Funct. Anal. Appl.* **5** (1971), no. 1, 79–80 Zbl 0247.43015 MR 0291371

[31] J. Peterson, Character rigidity for lattices in higher-rank groups. Preprint, 2014, avalaible at https://math.vanderbilt.edu/peters10/rigidity.pdf

[32] J. Peterson and A. Thom, Character rigidity for special linear groups. *J. Reine Angew. Math.* **716** (2016), 207–228 Zbl 1347.20051 MR 3518376

[33] C. Pfeffer Johnston, Primitive ideal spaces of discrete rational nilpotent groups. *Amer. J. Math.* **117** (1995), no. 2, 323–335 Zbl 0857.22005 MR 1323678

[34] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*. Pure Appl. Math. 139, Academic Press, Inc., Boston, MA, 1994 Zbl 0841.20046 MR 1278263

[35] M. Ratner, Raghunathan's conjectures for Cartesian products of real and p-adic Lie groups. *Duke Math. J.* **77** (1995), no. 2, 275–382 Zbl 0914.22016 MR 1321062

[36] H.-L. Skudlarek, Die unzerlegbaren Charaktere einiger diskreter Gruppen. *Math. Ann.* **223** (1976), no. 3, 213–231 Zbl 0313.43010 MR 463356

[37] S. V. Smirnov, Positive definite functions on algebraic nilpotent groups over a discrete field. *Soviet Math. Dokl.* **7** (1966), 1240–1241 Zbl 0189.02804 MR 0201573

[38] R. Steinberg, *Lectures on Chevalley groups*. Univ. Lecture Ser. 66, Amer. Math. Soc., Providence, RI, 2016 Zbl 1361.20003 MR 3616493

[39] E. Thoma, Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. *Math. Z.* **85** (1964), 40–61 Zbl 0192.12402 MR 173169
[40] E. Thoma, Über positiv-definite Klassenfunktionen abzählbarer Gruppen. *Math. Z.* **84** (1964), 389–402 Zbl 0136.11701 MR 170217
[41] E. Thoma, Über unitäre Darstellungen abzählbarer, diskreter Gruppen. *Math. Ann.* **153** (1964), 111–138 Zbl 0136.11603 MR 160118
[42] G. Tomanov, Orbits on homogeneous spaces of arithmetic origin and approximations. In *Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997)*, pp. 265–297, Adv. Stud. Pure Math. 26, Math. Soc. Japan, Tokyo, 2000 Zbl 0960.22006 MR 1770724
[43] A. Weil, *Basic number theory*. 3rd edn., Grundlehren Math. Wiss. 144, Springer, New York, 1974 Zbl 0326.12001 MR 0427267
[44] D. Witte, Measurable quotients of unipotent translations on homogeneous spaces. *Trans. Amer. Math. Soc.* **345** (1994), no. 2, 577–594 Zbl 0831.28010 MR 1181187
[45] R. J. Zimmer, *Ergodic theory and semisimple groups*. Monogr. Math. 81, Birkhäuser, Basel, 1984 Zbl 0571.58015 MR 776417

Received 15 October 2020.

Bachir Bekka
CNRS, Université de Rennes 1, IRMAR–UMR 6625, Campus Beaulieu, 35042 Rennes Cedex, France; bachir.bekka@univ-rennes1.fr

Camille Francini
CNRS, Université de Rennes 1, IRMAR–UMR 6625, Campus Beaulieu, 35042 Rennes Cedex, France; camille.francini@ens-rennes.fr