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Invariable generation does not pass to finite index subgroups

Gil Goffer and Nir Lazarovich

Abstract. Using small cancellation methods, we show that *invariable generation* does not pass to finite index subgroups, answering questions of Wiegold (1977) and Kantor–Lubotzky–Shalev (2015). We further show that a finitely generated group that is invariably generated is not necessarily finitely invariably generated, answering a question of Cox (2021). The same results were also obtained independently by Minasyan (2021).

1. Introduction

Definition 1.1 (Dixon [5]). Let G be a group. A subset $S \subseteq G$ *invariably generates* G if for every function $S \to G$, $s \mapsto g_s$, the set of conjugates $\{s^{g_s} \mid s \in S\}$ generates G. (We denote $g^h := h^{-1}gh$.)

A group G is *invariably generated* (or *IG*) if it has an invariably generating set, or equivalently, if G invariably generates itself. A group G is *finitely invariably generated* (or *FIG*) if it has a finite invariably generating set.

Dixon's original definition referred to finite groups. However, an equivalent definition was previously studied by Wiegold in the context of general (finite or infinite) groups [25]. Kantor, Lubotzky and Shalev [15] were the first to consider Dixon's definition for infinite groups, and to notice that it coincides with Wiegold's definition.

It is shown in [15, 25] that the classes of IG groups and FIG groups are closed under extensions and include all finite groups. It follows that a group with a finite index normal IG (resp. FIG) subgroup is IG (resp. FIG). The following slight generalization is probably known to experts, yet we include a proof of this theorem in Section 2.

Theorem A. *A group containing a finite index IG (resp. FIG) subgroup is IG (resp. FIG).*

In contrast, we prove the following theorem, answering questions of Wiegold [26] and Kantor–Lubotzky–Shalev [15].

Theorem B. *There exists a FIG group with an index* 2 *non-IG subgroup.*

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In the context of topological groups, it was shown in [15] that a topologically finitely generated group that is topologically invariably generated is not necessarily topologically finitely invariably generated. We therefore find it relevant to state the following theorem, answering a question of Cox [3].

Theorem C. *There exists a finitely generated group that is invariably generated, but not finitely invariably generated.*

The proofs of Theorem B and Theorem C rely on an iterative small cancellation construction. The same results were obtained independently by Minasyan [21] using similar methods.

Invariable generation has been studied for various groups and classes of groups. IG groups include virtually solvable groups [25], the Lamplighter group [15] and Thompson's group F [8]. There are uncountably many non-IG groups [12] which include convergence groups (and in particular hyperbolic groups) [7], Thompson's groups T and V [8], Osin's infinite group with only two conjugacy classes [23], and certain arithmetic groups with the congruence subgroup property [9]. For linear groups FIG is equivalent to solvability [15]. Invariable generation of wreath products was studied in [3]. As finite groups are always FIG, questions addressed in this context regard the minimal size of a (perhaps random) invariably generating set. In particular, it was thoroughly studied for S_n [4–6,24] and other finite groups [14, 18].

Organization of the paper. In Section 2 we include the proof of Theorem A. In Section 3 we give a brief statement of the tools used in the proofs of Theorems B and C. In Sections 4 and 5 we prove Theorems B and C, respectively. In Section 6 we review the necessary preliminaries on the geometry of hyperbolic groups and prove relevant lemmas on quasiconvex subgroups. In Section 7 we review the small cancellation theory of hyperbolic groups developed by Ol'shanskii [22]. We also show that one can find small cancellation words with specific properties, and prove the main lemmas of Section 3. Section 8 is devoted to the hexagon property which is an ingredient of the proof of Theorem B.

2. Proof of Theorem A

Definition 2.1. Let G be a group, and $S \subseteq G$ a subset. A subgroup $H \leq G$ is *S-conjugacy complete* if it intersects the conjugacy classes of all elements of S.

When $S = G$ we say that H is *conjugacy complete*.

The following equivalent definitions of invariable generation were observed in [15,25].

Lemma 2.2. *Let G be a group, and $S \subseteq G$ a subset. The following are equivalent:*

(1) *S invariably generates G.*

(2) *G does not contain a proper S-conjugacy complete subgroup.*

(3) *Every non-trivial transitive action $G \curvearrowright X$ has an element $s \in S$ without fixed points.*

Wiegold [25] proved that the class of IG groups is closed under extensions, in fact the following slightly stronger result holds.

Proposition 2.3. *Let G be a group, $N \leq H \leq G$ be subgroups and $N \lhd G$. Let $S \subseteq H$ and $S' \subseteq G$. If H is invariably generated by S, and G/N is invariably generated by (the image of) S', then G is invariably generated by $S \cup S'$.*

Proof. Let $G \curvearrowright X$ be a transitive action on a set with $|X| \geq 2$. We want to find an element of $S \cup S'$ which acts without fixed points on X.

Since N is normal, we know that $G/N \curvearrowright X/N$. If $|X/N| \geq 2$, then since G/N is invariably generated by S', there exists an element $s' \in S'$ that acts without fixed points on X/N and hence also on X. If $|X/N| = 1$, then N, and hence H, act transitively on X, and since H is invariably generated by S, there is an element $s \in S$ which acts without fixed points on X. ∎

In particular, we can deduce Theorem A.

Proof of Theorem A. If H is a finite index IG (resp. FIG) subgroup of G, then $N = \mathrm{Core}_G(H) = \bigcap_{g \in G} g^{-1}Hg$ is of finite index in G. Since every finite group is FIG, we get that $N \leq H \leq G$ satisfy the assumptions of Proposition 2.3 which implies that G is IG (resp. FIG). ∎

3. Toolbox

In this section we describe the toolbox for the main constructions. Since the main constructions are based on small cancellation quotients and HNN extensions, we summarize in this section the main relevant lemmas regarding these two constructions. We believe that a reader who is familiar with small cancellation theory would feel fairly comfortable with these lemmas, whose proofs follow standard techniques. We therefore postpone their proofs to later sections.

Throughout the rest of the paper we assume familiarity with notions in hyperbolic group theory (cf. for example [2, 10, 13, 22]).

3.1. Small cancellation quotients

We use the small cancellation theory developed by Ol'shanskii [22] for hyperbolic groups, which we outline more precisely in Section 7.

Roughly speaking, we say that a set of quasigeodesic words $\mathcal{R}$ in a hyperbolic group satisfies small cancellation if whenever two distinct words in $\mathcal{R}$ fellow-travel, they do so for only a small proportion of their lengths. Similarly, we say that a set of quasigeodesic words $\mathcal{R}$ has small overlap with another set of quasigeodesic words $\mathcal{K}$, if whenever a word in $\mathcal{R}$ fellow-travels with a word in $\mathcal{K}$, it does so for only a small proportion of its length. The exact form of small cancellation conditions that we use is defined in Definitions 7.3 and 7.5, and excludes relations which are powers.

Lemma 3.1. *Let G be a torsion-free hyperbolic group, and let $H, K_1, \dots, K_n$ be quasiconvex subgroups of G. If H is non-elementary and non-commensurable*[1] *into $K_1, \dots, K_n$, then for every m there exists a subset of m words $\mathcal{R} = \{w_1, \dots, w_m\} \subseteq H$ with arbitrarily small cancellation and arbitrarily small overlap with $K_1, \dots, K_n$, and every $w_i \in \mathcal{R}$ satisfies $E(w_i) = \langle w_i \rangle$ where $E(w_i) := \{x \in G \mid \exists n \neq 0\colon x w_i^n x^{-1} = w_i^{\pm n}\}$.*

If moreover G has an involution ϕ which exchanges two non-commensurable[2] *elements $a, b \in H$, and $\phi(\{K_1, \dots, K_n\}) = \{K_1, \dots, K_n\}$, then $\mathcal{R}$ can be chosen so that $\phi(\mathcal{R}) = \mathcal{R}$.*

A sketched proof for Lemma 3.1 is found in Section 7.

Remark 3.2. Let G and $H, K_1, \dots, K_n$ be as in Lemma 3.1, and let $u_1, \dots, u_m$ be quasigeodesic words in G. The words $w_1, \dots, w_m \in H$ can be chosen such that $w_1 u_1, \dots, w_m u_m$ have arbitrarily small cancellation and arbitrarily small overlap with $K_1, \dots, K_n$.

A version of the following lemma was proved by Minasyan [19, Theorem 1]. For a detailed independent proof see [11, Section 7.4] (the arXiv version of the present paper).

Lemma 3.3. *Let G be a torsion-free hyperbolic group, and let $K_1, \dots, K_n$ be quasiconvex subgroups of G. Then, for every finite set of words $\mathcal{R} = \{w_1, \dots, w_m\}$ with small enough cancellation and small enough overlap with $K_1, \dots, K_n$ the following holds:*

1. *The quotient $G/\langle\langle \mathcal{R} \rangle\rangle$ is torsion-free and hyperbolic.*
2. *For every $1 \leq i \leq n$, the subgroup K_i embeds in $G/\langle\langle \mathcal{R} \rangle\rangle$ as a quasiconvex subgroup.*
3. *For every $1 \leq i, j \leq n$, if K_i is non-commensurable into K_j in G, then the same holds in $G/\langle\langle \mathcal{R} \rangle\rangle$.*

3.2. HNN extensions

The HNN extensions which we use have cyclic edge stabilizers. In this case, one has the following theorem.

[1] We use the term "commensurable" to refer to the equivalence of subgroups up to conjugation and passing to finite index. That is, two subgroups H, H' in G are *commensurable* if there exists $g \in G$ such that $H^g \cap H'$ has finite index in both H^g and H'. Similarly, H is *commensurable into* H' if there exists $g \in G$ such that $H^g \cap H'$ has finite index in H^g.

[2] Elements are *commensurable* if they generate cyclic subgroups which are commensurable. Similarly, an element is *commensurable into* a subgroup H if the cyclic subgroup it generates is commensurable into H.

Theorem 3.4 ([17, Theorem 4] or [16, Theorem 1.2]). *Let G be a hyperbolic group acting on a tree with cyclic edge stabilizers, then the vertex stabilizers of G are quasiconvex. In particular, in hyperbolic HNN extensions with cyclic edge stabilizers, quasiconvex subgroups of vertex groups are quasiconvex in the HNN extension.*

Since we will need more control over the possible conjugations of elements, we recall the definition of k-acylindrical HNN extensions.

Definition 3.5. Let $k \in \mathbb{N}$. An action $G \curvearrowright T$ of a group on a tree is k-acylindrical if for every $1 \neq g \in G$ the fixed-point set of g in T has diameter $\leq k$. Equivalently, the pointwise stabilizer in G of a geodesic path of length $k+1$ in T is trivial.

An HNN extension (and more generally a graph of groups) is k-acylindrical if the action on its associated Bass–Serre tree is k-acylindrical.

It is easy to verify the following sufficient condition for 2-acylindricity of a double HNN extension.

Lemma 3.6. *Let A be a group, and C, C', D, D' be distinct subgroups of A. Assume that for all $g \in A$, $X \in \{C', D'\}$, and $Y \in \{C, C', D, D'\}$,*

$$gXg^{-1} \cap Y \neq 1 \implies X = Y,\ g \in X.$$

Then, the (double) HNN extension $G = \langle A, s, t \mid C^s = C', D^t = D' \rangle$ is 2-acylindrical. ∎

Under the condition of 2-acylindricity it is easy to see the following.

Lemma 3.7. *Let A, C, C', D, D', G be as in Lemma 3.6. Let U, V be two non-commensurable subgroups of A. Assume that U and V are not commensurable into C' and D'. Then, U and V are non-commensurable in G.*

Proof. The lemma follows from Britton's lemma and the assumption on C, C', D, D' in Lemma 3.6. ∎

3.3. Quasiconvex subgroups

Lemma 3.8 (Ol'shanskii [22]). *Let $x, y \in G$ be non-commensurable elements. Then there exists $N > 0$ such that $\langle x^N, y^N \rangle \leq G$ is a free quasiconvex subgroup.*

Theorem 3.9 ([20, Theorem 1]). *Let G be a hyperbolic group, let $H, K_1, \dots, K_k$ be quasiconvex subgroups, and suppose H is not commensurable into any of the K_i. Then there exists $h \in H$ which is not commensurable into any of the K_i.*

Theorem 3.9 was proved by Minasyan in [20, Theorem 1]. A short independent proof is also given in the arXiv version of the present paper, see [11, Section 7.1].

Lemma 3.10. *Let $Q_1, \dots, Q_m \leq G$ be some non-elementary quasiconvex subgroups of G, and let $g_1, \dots, g_n$ be infinite order elements of G. Then, there exist $r_1, \dots, r_n \in G$ such that $K = \langle g_1^{r_1}, \dots, g_n^{r_n} \rangle$ is a quasiconvex free subgroup and $Q_1, \dots, Q_m$ are not commensurable into K.*

The proof of this lemma appears in Section 6.2.

4. Proof of Theorem B

We prove that the following proposition implies Theorem B.

Proposition 4.1. *There exist a finitely generated non-IG group G, an element $x \in G$, and an involution $\phi \in \mathrm{Aut}(G)$ such that for all $g \in G$, $\langle x^g, \phi(x^g) \rangle = G$.*

Proof of Theorem B. Let G and ϕ be as in Proposition 4.1. Consider the group $\widetilde{G} = G \rtimes \langle \phi \rangle$. By construction, G contains an index 2 non-IG subgroup. It remains to show that $\widetilde{G}$ is FIG. We claim that $\widetilde{G}$ is invariably generated by $S = \{x, \phi\}$. That is, $\langle x^{\tilde{g}}, \phi^{\tilde{g}'} \rangle = \widetilde{G}$ for all $\tilde{g}, \tilde{g}' \in \widetilde{G}$.

Let $\widetilde{H} = \langle x^{\tilde{g}}, \phi^{\tilde{g}'} \rangle$. We may assume that $\phi \in \widetilde{H}$, by conjugating $\widetilde{H}$ by $(\tilde{g}')^{-1}$ if necessary.

We can write $\tilde{g} = g\phi^{\varepsilon} \in \widetilde{G}$ where $g \in G$ and $\varepsilon \in \{0, 1\}$. Since $\phi, x^{g\phi^{\varepsilon}} \in \widetilde{H}$, it follows that both x^g and $x^{g\phi} = \phi(x^g)$ are in $\widetilde{H}$. By the assumption, $G = \langle x^g, \phi(x^g) \rangle \subseteq \widetilde{H}$, but since also $\phi \in \widetilde{H}$ we get that $\widetilde{H} = \widetilde{G}$. ∎

Proof of Proposition 4.1. We construct by induction a group G with the desired properties. Let us start with $G(0) = F(x, x', y, y')$, the free group generated by the letters x, x', y, y', and let $\phi \in \mathrm{Aut}(G(0))$ be the involution exchanging $x \leftrightarrow y, x' \leftrightarrow y'$. Enumerate the elements of $G(0) = \{g_1, g_2, g_3, \dots\}$.

Assume we have constructed a sequence $G(0) \twoheadrightarrow G(1) \twoheadrightarrow \cdots$ of quotients, $G(n) = G(0)/N_i$ where $N_1 \leq N_2 \leq \cdots$ is an increasing sequence of normal subgroups, and such that each group $G(n)$ satisfies the following:[3]

- (B1) The subgroup $\langle x, x' \rangle$ contains some conjugates of $g_1, \dots, g_n$.
- (B2) $\langle x, x' \rangle$ is a proper subgroup.
- (B3) The automorphism ϕ descends to $G(n)$.
- (B4) The conjugate x^{g_n} *ϕ-generates* $G(n)$, i.e., $G(n) = \langle x^{g_n}, \phi(x^{g_n}) \rangle$.

Consider the limit
$$G = \varinjlim G(n) = G(0)/\big(\bigcup N_n\big).$$
It is a finitely generated group by construction. The subgroup $\langle x, x' \rangle$ is conjugacy complete by (B1) and proper by (B2), implying that G is non-IG. In addition, ϕ is an involution of G by (B3), and for all $g \in G$, $\langle x^g, \phi(x^g) \rangle = G$ by (B4).

[3]We abuse notation and think of elements of $G(0)$ as their images in $G(n)$.

To complete the proof of Proposition 4.1 it remains to construct a sequence of quotients as above. To build the sequence $G(n)$ we will use small cancellation, and therefore we would like to assume more on the groups in the process.

(B5) The group $G(n)$ is a torsion-free hyperbolic group.

(B6) $\langle x, x' \rangle$ is free and quasiconvex.

(B7) $\langle x, x' \rangle$ and $\langle y, y' \rangle$ are not commensurable.

(B8) The elements x, y are non-commensurable. In particular, $\langle x, y \rangle$ is non-elementary.

(B9) (*The hexagon property*) If $\xi, \xi' \in \langle x, x' \rangle$ and $z \in G(n)$ satisfy $\xi^z = \phi((\xi')^z)$, then $\xi' = \xi^{\pm 1}$.

Remark 4.2. Property (B7) implies (B2). In fact, it follows from (B7) that $\langle x, x' \rangle$ has infinite index in $G(n)$.

Property (B9) implies that if $a, b \in G(n)$ are non-commensurable and $\phi(b) = a$, then $\langle a, b \rangle$ is not commensurable into $\langle x, x' \rangle$. Otherwise, there exist $z \in G(n)$, $\xi, \xi' \in \langle x, x' \rangle$ and $N \in \mathbb{N}$ such that $a^N = \xi^z$ and $b^N = (\xi')^z$. Applying ϕ on the second equation gives $a^N = \phi((\xi')^z)$, from which $\xi^z = \phi((\xi')^z)$ follows. Property (B9) then implies that $\xi' = \xi^{\pm 1}$, contradicting the assumption that a, b are non-commensurable.

It is easy to verify that $G(0)$ satisfies the above (B1)–(B9). Note that (B1) and (B4) are vacuous for $G(0)$.

Starting with $G(n-1)$, we will build $G(n)$ in a three step process:

Step 1. Conjugating g_n into $\langle x, x' \rangle$ using HNN. Let $g = g_n$. If $g = 1$, set $G''(n) = G(n-1)$ and skip to Step 3. Otherwise, the assumptions of Lemma 3.1 with $H = \langle x, x' \rangle$, $K_1 = \langle g \rangle$, $K_2 = \langle \phi(g) \rangle$, $K_3 = \langle y, y' \rangle$ are satisfied by (B5), (B6), (B7) and (B8). Therefore, we can find a word $w \in \langle x, x' \rangle$ such that w satisfies arbitrarily small cancellation in $G(n-1)$, and has arbitrarily small overlap with $\langle g \rangle$, $\langle \phi(g) \rangle$ and $\langle y, y' \rangle$. Since $\phi(w) \in \langle y, y' \rangle$ it follows that $w, \phi(w)$ satisfy arbitrarily small cancellation and small overlap with $\langle g \rangle, \langle \phi(g) \rangle$.

Let $G'(n)$ be the (double) HNN extension

$$G'(n) = \langle G(n-1), s, t \mid g^s = w,\ \phi(g)^t = \phi(w) \rangle.$$

Extend ϕ by setting it to exchange $s \leftrightarrow t$.

Even though $G'(n)$ is not a quotient of $G(n-1)$ one can make sense of properties (B1)–(B9) for $G'(n)$. By the induction hypothesis $g_1, \dots, g_n$ are conjugate into $\langle x, x' \rangle$ in $G(n-1)$ and therefore also in $G'(n)$; the new HNN relations also conjugate $g = g_n$ to $\langle x, x' \rangle$, hence $G'(n)$ satisfies (B1). It is also immediate that $G'(n)$ satisfies (B2), (B3).

Since $w, \phi(w)$ satisfy arbitrarily small cancellation in $G(n-1)$, by Remark 3.2 we see that $g^s = w$ and $\phi(g)^t = \phi(w)$ are also small cancellation relations (in the hyperbolic group $G(n-1) * F(s,t)$). It follows that w can be chosen so that $G'(n)$ satisfies (B5) by Lemma 3.3 (1).

Moreover, the groups $C = \langle g \rangle, D = \langle \phi(g) \rangle, C' = \langle w \rangle, D' = \langle \phi(w) \rangle$ satisfy the conditions of Lemma 3.6 as we know that $E(C') = C'$ and $E(D') = D'$ by Lemma 3.1. Therefore the HNN extension $G'(n)$ will satisfy (B6) by Theorem 3.4. It will also satisfy (B7) and (B8) by Lemma 3.7 applied to $U = \langle x, x' \rangle, V = \langle y, y' \rangle$ and to $U = \langle x \rangle, V = \langle y \rangle$. The proof that the hexagon property (B9) is preserved is slightly more technical and appears in Lemma 8.2.

Note that at this point $G'(n)$ is not a quotient of $G(n-1)$, and it satisfies all properties except for (B4). In the next step, we introduce new relations to $G'(n)$, to make it a quotient of $G(n-1)$.

Step 2. Absorbing $G'(n)$ in a quotient of $G(n-1)$ using small cancellation. As explained in Remark 4.2, it follows from (B7) that $\langle x, x' \rangle$ has infinite index in $G(n-1)$, and both are quasiconvex in $G'(n)$ by (B6) and Theorem 3.4. Using Lemma 3.7, we see that the conditions of Lemma 3.1 are satisfied for $H = G(n-1), K_1 = \langle x, x' \rangle, K_2 = \langle y, y' \rangle$ in $G'(n)$. Hence, by the "moreover" part of the lemma, we can find $u \in \langle x, x', y, y' \rangle$ such that $u, \phi(u)$ have arbitrarily small cancellation in $G'(n)$, and such that $u, \phi(u)$ have arbitrarily small overlap with the subgroups $\langle x, x' \rangle$ and $\langle y, y' \rangle$. Set

$$G''(n) = G'(n)/\langle\langle s = u, t = \phi(u) \rangle\rangle.$$

By the way it is defined the composition $G(n-1) \hookrightarrow G'(n) \twoheadrightarrow G''(n)$ is onto. It also follows that $G''(n)$ satisfies (B1) and (B3). By Remark 3.2 the relations $s = u$ and $t = \phi(u)$ can be chosen to satisfy arbitrarily small cancellation and small overlap with $\langle x, x' \rangle$ and $\langle y, y' \rangle$. Properties (B5), (B6), (B7) and (B8) then follow from Lemma 3.3, and the hexagon property (B9) is postponed to Lemma 8.1. As explained in Remark 4.2, property (B2) follows.

At this point, $G''(n)$ is a quotient of $G(n-1)$ that satisfies all properties except for (B4), which will be taken care of in the last step of the construction.

Step 3. Forcing ϕ-generation using small cancellation. Recall that we denote $g = g_n$. By (B8) x, y are non-commensurable. It follows that so are x^g and $\phi(x^g) = y^{\phi(g)}$. As explained in Remark 4.2, it follows from property (B9) that $\langle x^g, \phi(x^g) \rangle$ is not commensurable into $\langle x, x' \rangle$. Using this and (B6), we see that $H = \langle x^g, \phi(x^g) \rangle$ and $K_1 = \langle x, x' \rangle$, $K_2 = \langle y, y' \rangle$ satisfy the assumptions for the "moreover" part of Lemma 3.1. Hence, there exist $v, v' \in \langle x^g, \phi(x^g) \rangle$ such that $v, v', \phi(v), \phi(v')$ satisfy arbitrarily small cancellation in $G''(n)$ and have arbitrarily small overlap with $\langle x, x' \rangle$ and $\langle y, y' \rangle$.

In order to take care of property (B4), we set

$$G(n) = G''(n)/\langle\langle x = v, x' = v', y = \phi(v), y' = \phi(v') \rangle\rangle.$$

We have $G(n-1) \twoheadrightarrow G''(n) \twoheadrightarrow G(n)$. It follows from the construction that $G(n)$ satisfies (B1), (B3) and (B4). As in Step 2, properties (B5), (B6), (B7) and (B8) follow from Lemma 3.3. The hexagon property (B9) holds by Lemma 8.1, and (B2) follows. ■

5. Proof of Theorem C

In the following section we construct a finitely generated IG group that is not FIG, proving Theorem C.

Let $F = F(a, b)$ be the free group generated by a, b, and let $F = \{g_1, g_2, \ldots\}$ be an enumeration of its elements. Assume we have found a function $h : F \times F \to F$, elements $\{r_{ij}\}_{i \geq j} \subseteq F$, and a quotient $F \twoheadrightarrow G$ that satisfy:

(P1) For all $s, t, u \in F$, $\langle a^s, b^t, h(s,t)^u \rangle = G$.[4]

(P2) For all $n \in \mathbb{N}$, $\langle g_1^{r_{n1}}, \ldots, g_n^{r_{nn}} \rangle \neq G$.

It is then easy to see that (P1) implies that G is IG, while (P2) implies that it is not FIG. We therefore wish to find such data.

We first establish some notation. Enumerate

$$F \times F = \{(s_1, t_1), (s_2, t_2), \ldots\},$$
$$(F \times F) \times F = \{((s_{j_1}, t_{j_1}), u_1), ((s_{j_2}, t_{j_2}), u_2), \ldots\}.$$

Let

$$\mathbb{N}^\natural = \{i^\natural \in \mathbb{N} \mid j_{i^\natural} \notin \{j_1, \ldots, j_{i^\natural - 1}\}\},$$

i.e., the set of indices of the enumeration of $(F \times F) \times F$ for which a pair (s, t) is introduced for the first time. When using the notation $i^\natural$, we implicitly assume that the element $i^\natural$ is in the set $\mathbb{N}^\natural$.

Set $F = G(0)$. In the n-th step of the induction, $n \geq 1$, we will construct:

- a group $G(n)$ which is a quotient $G(n-1) \twoheadrightarrow G(n)$;
- an image for the pair (s_{j_n}, t_{j_n}) under h, in case this pair has not yet appeared in a previous level; that is, in case $n \in \mathbb{N}^\natural$;
- elements $r_{nk} \in F$ for all $1 \leq k \leq n$, and a subgroup $K_n := \langle g_1^{r_{n1}}, \ldots, g_n^{r_{nn}} \rangle$;
- elements $x_{ni^\natural} \in F$ for all $1 \leq i^\natural \leq n$;

such that the following properties hold in $G(n)$:

(C0) $G(n)$ is a torsion-free hyperbolic group.

(C1) $\langle a^{s_{j_n}}, b^{t_{j_n}}, h(s_{j_n}, t_{j_n})^{u_n} \rangle = G(n)$.

(C2) a, b are non-commensurable.

(C3) For all $1 \leq i^\natural \leq n$, $h(s_{j_{i^\natural}}, t_{j_{i^\natural}})$ is not commensurable into $K_1, \ldots, K_{i^\natural - 1}$.

(C4) For all $1 \leq i \leq n$, K_i is free and quasiconvex. Since G is torsion-free but not free, it follows from Stallings' theorem that K_i have infinite index in $G(n)$, and in particular it is proper.

(C5) For all $1 \leq i^\natural \leq k \leq n$, $x_{ki^\natural} \in \langle a^{s_{j_{i^\natural}}}, b^{t_{j_{i^\natural}}} \rangle$ is not commensurable into K_k.

[4] As usual we interpret elements of F as their image under the quotient map in G.

Finally, we set $G = \varinjlim G(n)$. Notice that property (C1) for $G(n)$ implies that

$$\langle a^{s_{j_i}}, b^{t_{j_i}}, h(s_{j_i}, t_{j_i})^{u_i} \rangle = G(n) \quad \text{for all } i \le n,$$

since $G(n)$ is a quotient of $G(i)$. In particular, we get that (P1) holds for G. Furthermore, by the definition of the groups K_i, property (C4) implies (P2) for G.

It is easy to see that $G(0) = F$ satisfies the above assumptions. Notice however that most conditions are vacuous in this case, as they are defined for $i \ge 1$ only. We now describe the inductive step. Suppose we have defined the groups $G(0), \dots, G(n-1)$ with the auxiliary data described above such that they satisfy (C0)–(C5).

Step 1. Defining $h(s_{j_n}, t_{j_n})$. If $n \notin \mathbb{N}^\natural$, skip this step. Otherwise, $n \in \mathbb{N}^\natural$ and hence the image of the pair (s_{j_n}, t_{j_n}) under h was not previously defined. By Theorem 3.9, there exists an element in $G(n-1)$ that is not commensurable into $K_1, \dots, K_{n-1}$. Set $h(s_{j_n}, t_{j_n})$ to be such an element.

At this point, (C3) holds also for $i = n$, in $G(n-1)$.

Step 2. Constructing $G(n)$. By the induction hypothesis and Step 1, (C3) for $1 \le i^\natural \le n$ and (C5) for $1 \le i^\natural \le k < n$ hold in $G(n-1)$. It follows that $\langle a^{s_{j_n}}, b^{t_{j_n}}, h(s_{j_n}, t_{j_n})^{u_n} \rangle$ contains an element which is not commensurable into $K_1, \dots, K_{n-1}$.

By Lemma 3.1, there exist words $w_a, w_b \in H = \langle a^{s_{j_n}}, b^{t_{j,n}}, h(s_{j_n}, t_{j_n})^{u_n} \rangle$ with arbitrarily small cancellation in $G(n-1)$ and arbitrarily small overlap with $K_1, \dots, K_{n-1}$, $\langle a \rangle$, $\langle b \rangle$, $\{\langle h(s_{j_i}, t_{j_i}) \rangle\}_{i \le n}$, and $\{\langle x_{ki^\natural} \rangle\}_{i^\natural \le k \le n-1}$. Define

$$G(n) = G(n-1)/\langle\langle w_a = a, w_b = b \rangle\rangle.$$

By Lemma 3.3 (1), property (C0) persists under small cancellation quotients, and so it holds in $G(n)$. Moreover, it follows from the new relations that $\langle a^{s_{j_n}}, b^{t_{j_n}}, h(s_{j_n}, t_{j_n})^{u_n} \rangle = G(n)$, and so (C1) holds for $G(n)$ as well. Similarly, properties (C2) and (C3) hold in the quotient $G(n)$ by Lemma 3.3 (3) and the induction hypothesis.

Regarding the other two properties: For all $1 \le i \le n-1$, (C4) holds in $G(n)$ by Lemma 3.3 (2), since the relations have small overlap with $K_1, \dots, K_{n-1}$. Similarly, (C5) for $1 \le i^\natural \le k \le n-1$ holds in $G(n)$ by Lemma 3.3 (3). It remains to construct K_n and show (C4) for $i = n$, and (C5) for $k = n$. This is done in the next step.

Step 3. Constructing $r_{n1}, \dots, r_{nn}$ *and* $x_{ni^\natural}$. We have seen that (C2) holds in $G(n)$, i.e., a, b are non-commensurable in $G(n)$. Hence for every $1 \le i^\natural \le n$, $\langle a^{s_{j_{i^\natural}}}, b^{t_{j_{i^\natural}}} \rangle$ is non-elementary. Let $Q_{i^\natural} \le \langle a^{s_{j_{i^\natural}}}, b^{t_{j_{i^\natural}}} \rangle$ be some non-elementary quasiconvex subgroup which exists by Lemma 3.8. By Lemma 3.10, there exist $r_{n1}, \dots, r_{nn}$ such that the group $K_n := \langle g_1^{r_{n1}}, \dots, g_n^{r_{nn}} \rangle$ is quasiconvex and free, and such that for every $1 \le i^\natural \le n$, $Q_{i^\natural}$ is not commensurable into K_n. By Theorem 3.9, for every $1 \le i^\natural \le n$ there exists $x_{ni^\natural} \in Q_{i^\natural}$ that is not commensurable into K_n.

The construction of $r_{n1}, \dots, r_{nn}$ ensures (C4) for $i = n$, while the choice of $x_{ni^\natural}$ ensures (C5) for $k = n$. This completes the proof of Theorem C.

6. Geometry of hyperbolic groups and quasiconvex subgroups

Let G be generated by a finite set S. We denote by $\Gamma(G,S)$ the Cayley graph of G with respect to S. Let w be a word over S. We write $\|w\|$ to denote the length of w as a word. We use the same notation, $\|p\|$, to denote the length of a path p. We often abuse notation and identify a path in $\Gamma(G,S)$ with its label. For an element $g \in G$, we denote by $|g|$ the distance in $\Gamma(G,S)$ between g and 1_G. Given $\lambda \in (0,1]$ and $c \geq 0$, a path p is a (λ,c)-*quasigeodesic* if for every subpath $p' \subset p$, $\lambda\|p'\| - c \leq |p'|$.

Throughout this section, G is assumed to be a δ-hyperbolic group.

6.1. Basic geometry of hyperbolic groups

In this subsection we collect some standard lemmas regarding the geometry of hyperbolic groups. The proofs of Lemmas 6.1 and 6.2 can be found in Ol'shanskii's article [22].

Two paths at Hausdorff distance d from each other are said to d-*fellow-travel*, or simply *fellow-travel*, if d is independent of their lengths. The following lemma shows that two quasigeodesics whose endpoints are close must fellow-travel.

Lemma 6.1 (Fellow-traveling). *Given $\lambda \in (0,1]$ and $c \geq 0$, there exists $\delta' \geq 0$ such that for every $\varepsilon \geq 0$, there exists $\varepsilon' \geq 0$ with the following property. If $p_1q_1p_2q_2$ is a (λ,c)-quasigeodesic rectangle and $\|p_1\|, \|p_2\| \leq \varepsilon$, then there exist subpaths $q_i' \subset q_i$ of length $\|q_i'\| > \|q_i\| - \varepsilon'$ such that q_1' and q_2' are of Hausdorff distance at most δ' from each other.*

In the lemma, q_1' and q_2' δ'-fellow-travel, and thus q_1 and q_2 $(\delta' + 2\varepsilon')$-fellow-travel.

A group H is called *elementary* if it is virtually cyclic, i.e., contains a finite index cyclic subgroup. When G is hyperbolic, every infinite order element $g \in G$ is contained in a unique maximal elementary subgroup $E(g) \leq G$, which is given by

$$E(g) = \{x \in G \mid \exists n \neq 0\colon xg^nx^{-1} = g^{\pm n}\}.$$

If G is moreover torsion-free, then $E(g)$ is cyclic by Stallings' theorem.

Lemma 6.2. *Suppose that G is torsion-free, and let $g,h \in G$ be non-trivial elements. There exist constants $M > 0$ and $\theta > 0$ such that: If for some $m \geq M$, $xg^my = h^n$ and $\max\{|x|,|y|\} \leq \theta m$, then g,h are commensurable and $g \neq h^{-1}$. If moreover $g = h$, then $x,y \in E(g)$.*

It follows from the lemma that if large powers of g and h fellow-travel, then g and h must be commensurable.

Lemma 6.3 (Corner trimming). *For all $\lambda \in (0,1]$, $c \geq 0$ and $k \in \mathbb{N}$ there exist $\delta' \geq 0$, $\lambda' \in (0,1]$ and $c' \geq 0$ such that if $p_1,\dots,p_k$ are (λ,c)-quasigeodesic words, then there exist (possibly empty) words $v_1,\dots,v_{k-1}$ with $\|v_i\| \leq \delta'$ and (possibly empty) subwords $p_1',\dots,p_k'$ of $p_1,\dots,p_k$, respectively, such that*

$$p_1 \dots p_k = p_1'v_1p_2'v_2\dots v_{k-1}p_k'$$

in G, and the word on the right-hand side is a (λ',c')-quasigeodesic in G.

Proof. The case $k = 2$ follows from slimness of quasigeodesic triangles in hyperbolic groups, and for $k > 2$ it follows by inductively applying the case $k = 2$. ■

Lemma 6.4 (Local-to-global; see the remark following [2, Corollary 1.14]). *Let $\lambda \in (0,1]$, $c \geq 0$. Then there exist $\lambda' \in (0,1]$, $c' \geq 0$, $L > 0$ such that if p is a path for which every subpath of length at most L is (λ, c)-quasigeodesic, then p is (λ', c')-quasigeodesic.*

6.2. Constructing non-commensurated elements and quasiconvex subgroups

Proof of Lemma 3.10. Let $Q_1, \dots, Q_m \leq G$ be some non-elementary quasiconvex subgroups of G, and let $g_1, \dots, g_n$ be infinite order elements of G. We wish to find $r_1, \dots, r_n \in G$ such that $K = \langle g_1^{r_1}, \dots, g_n^{r_n} \rangle$ is a quasiconvex free subgroup and $Q_1, \dots, Q_m$ are not commensurable into K.

Let $\mu = \min\{\delta(Q_1), \dots, \delta(Q_n)\}$ where $\delta(Q_i)$ is the critical exponent of Q_i, i.e., the infimal α for which $\sum_{w \in Q_i} e^{-\alpha |w|}$ converges. The critical exponent is a commensurability invariant.

By Lemma 6.5 below we can choose $r_1, \dots, r_n \in G$ such that $\delta(K) < \mu$ where $K = \langle g_1^{r_1}, \dots, g_n^{r_n} \rangle$ is a quasiconvex free subgroup. It follows that $Q_1, \dots, Q_n$ are not commensurable into K, as otherwise by the monotonicity of the critical exponent $\delta(Q_i) \leq \delta(K)$, which contradicts the assumption $\delta(K) < \mu$. ■

Lemma 6.5. *Given $g_1, \dots, g_n \in G$ and $\mu > 0$, there exist $r_1, \dots, r_n \in G$ such that $K = \langle g_1^{r_1}, \dots, g_n^{r_n} \rangle$ is a quasiconvex free group with critical exponent $\delta(K) < \mu$.*

We remark that it follows from Arzhantseva [1] that given a quasiconvex free subgroup $\langle g_1^{r_1}, \dots, g_{n-1}^{r_{n-1}} \rangle$, there exists $g \in G$ such that $\langle g_1^{r_1}, \dots, g_{n-1}^{r_{n-1}}, g \rangle$ is again quasiconvex and free. However, we wish to choose g to be a conjugate of the pre-given g_n.

Proof. Let $r_1', \dots, r_n'$ be pairwise non-commensurable elements, such that r_j' is non-commensurable to g_j. We claim that for M large enough the elements $r_j = (r_j')^M$ satisfy the requirements.

Quasiconvexity of K. We first show that for large enough M, every word $w \in K = \langle g_1^{r_1}, \dots, g_n^{r_n} \rangle$ of the form

$$w = (r_{i_1}^{-1} g_{i_1}^{\pm k_1} r_{i_1})(r_{i_2}^{-1} g_{i_2}^{\pm k_2} r_{i_2}) \dots (r_{i_a}^{-1} g_{i_a}^{\pm k_a} r_{i_a}), \tag{6.1}$$

$k_1, \dots, k_a > 0$, is a quasigeodesic with some quasigeodesicity constants (independent of a, $k_1, \dots, k_a$ and M). It suffices to prove it locally, using the local-to-global principle (Lemma 6.4). More precisely, we want to show that each subpath $r_{i_j}^{-1} g_{i_j}^{\pm k_j} r_{i_j}$ and each subpath $r_{i_j} r_{i_{j+1}}^{-1}$ is a long enough quasigeodesic for some quasigeodesicity constants (independent of a, $k_1, \dots, k_a$ and M).

Let λ_1, c_1 be such that all powers of $g_1, \dots, g_n$ and of $r_1', \dots, r_n'$ are (λ_1, c_1)-quasigeodesic. Let λ_2, c_2, δ' be as in the corner trimming lemma (Lemma 6.3) applied for λ_1, c_1 and $k = 3$. The corner trimming process gives the following. For each path of the form

$p = p_1 p_2 p_3 = r_{i_j}^{-1} g_{i_j}^{\pm k_j} r_{i_j}$ there exist (possibly empty) words v_1, v_2 with $\|v_i\| \le \delta'$ and (possibly empty) subwords p_1', p_2', p_3' of p_1, p_2, p_3, respectively, such that $p_1 p_2 p_3 = p_1' v_1 p_2' v_2 p_3'$ in G, and the word on the right-hand side is a (λ_2, c_2)-quasigeodesic. Denote further by $p_1'', p_2'', p_2''', p_3''$ the subpaths of p_1, p_2, p_2, p_3 such that

$$p_1 = p_1' p_1'', \quad p_2 = p_2'' p_2' p_2''', \quad p_3 = p_3'' p_3'.$$

(In case p_2' is empty, take $p_2'' p_2''' = p_2$ to be any partition). We wish to prove that p_1'', p_2'', p_2''', p_3'' are all bounded, and therefore the original path $p = r_{i_j}^{-1} g_{i_j}^{\pm k_j} r_{i_j}$ is close to the quasigeodesic $p_1' v_1 p_2' v_2 p_3'$.

Suppose toward contradiction that at least one of $p_1'', p_2'', p_2''', p_3''$ is long. In case p_2' is not the empty word, the relation $p_1'' p_2'' = v_1$, in which $\|v_1\| \le \delta'$, implies that p_1'' fellow travels with p_2''. The relation $p_2''' p_3'' = v_2$, $\|v_2\| \le \delta'$, implies the same for p_2''' and p_3''. In case p_2' is empty, we get the relation $p_1'' p_2 p_3'' = v_1 v_2$, $\|v_1 v_2\| \le 2\delta'$, which implies that either p_2 fellow travels with one of p_1'' or p_3'', or, by Lemma 6.2, in case p_2 is itself short, that p_2 belongs to $E(r_{i_j}')$. All the cases of fellow traveling described above are impossible by Lemma 6.2 when M is large enough, since r_{i_j}' is non-commensurable with g_{i_j}. It follows that the lengths $\|p_1''\|, \|p_2''\|, \|p_2'''\|, \|p_3''\|$ are indeed bounded, and the bound only depends on $g_1, \dots, g_n$ and $r_1', \dots, r_n'$. Let B denote their common upper bound, then

$$\|p_1 p_2 p_3\| \le \|p_1' v_1 p_2' v_2 p_3'\| + 2B.$$

In particular, $p_1 p_2 p_3 = r_{i_j}^{-1} g_{i_j}^{\pm k_j} r_{i_j}$ is a (λ, c)-quasigeodesic where $\lambda = \lambda_2$ and $c = c_2 + 2B$.

Similarly, one performs corner trimming on $p = p_1 p_2 = r_{i_j} r_{i_{j+1}}^{-1}$ to obtain that $p_1 p_2 = p_1' v_1 p_2'$ in G, and the right-hand side is (λ_2, c_2)-quasigeodesic. Since $r_{i_j}, r_{i_{j+1}}$ are non-commensurable, we can use Lemma 6.2 to obtain a bound, say B again, on the length of the shortcut, namely on $\|p_1 p_2\| - \|p_1' v_1 p_2'\|$. We get that $p_1 p_2 = r_{i_j} r_{i_{j+1}}^{-1}$ is a (λ, c)-quasigeodesic as well.

By the local-to-global principle (Lemma 6.4) applied to λ, c, there exist $\lambda' \in (0, 1]$, $c' \ge 0$ and $L > 0$. For large enough M the subpaths $r_{i_j}^{-1} g_{i_j}^{\pm k_j} r_{i_j}$ and $r_{i_j} r_{i_{j+1}}^{-1}$ have length $> L$ and are (λ, c)-quasigeodesics. Thus, by Lemma 6.4, w is a (λ', c')-quasigeodesic, and λ', c' are independent of $a, k_1, \dots, k_a$ and M.

It follows that K is quasiconvex and free.

Small critical exponent. To get $\delta(K) < \mu$ we recall that the critical exponent is the infimal α such that the series $\sum_{w \in K} e^{-\alpha|w|}$ converges. In order to show convergence for $\alpha < \mu$, it is enough to find $\mu' < \mu$ such that for every large enough R,

$$\#\{w \in K \mid |w| < R\} < e^{\mu' R}.$$

Let $N > 0$. Let M be large enough so that any word w of the form (6.1) is (λ', c')-quasigeodesic. By perhaps enlarging M even more, we can assume that for some $C > 0$, every w of the form (6.1) satisfies

$$|w| > Na + Ck$$

in G, where $k = \sum_{j=1}^{a} k_j$. As the number of words of the form (6.1) is at most $(2n)^a \binom{k}{a}$, we obtain

$$\begin{aligned}
\#\{w \in K \mid |w| < R\} &\leq \#\{w \text{ is of the form (6.1)} \mid Na + Ck \leq R\} \\
&\leq \sum_{Na+Ck\leq R} (2n)^a \binom{k}{a} \\
&\leq \sum_{Na+Ck\leq R} (2n)^a \Big(\frac{ke}{a}\Big)^a \\
&\leq \sum_{a=1}^{\frac{R}{N}} \Big(\frac{2ne}{a}\Big)^a \sum_{k=1}^{\frac{R-Na}{C}} k^a.
\end{aligned} \tag{6.2}$$

Replacing all elements in the inner sum by $(\frac{R-Na}{C})^a$, we get

$$\begin{aligned}
(6.2) &\leq \sum_{a=1}^{\frac{R}{N}} \Big(\frac{2ne}{a}\Big)^a \Big(\frac{R-Na}{C}\Big)^{a+1} \\
&\leq \sum_{a=1}^{\frac{R}{N}} \Big(\frac{R-Na}{C}\Big)\Big(\frac{2ne}{C}\Big)^a \Big(\frac{R-Na}{a}\Big)^a.
\end{aligned} \tag{6.3}$$

The function $(\frac{1-x}{x})^x$ obtains a maximum D in the interval $(0,1]$. Substituting $a\frac{N}{R}$ for x, one has

$$\Big(\frac{R-Na}{a}\Big)^a \leq N^a D^{\frac{R}{N}} \quad \text{for } a \in [1, \tfrac{R}{N}].$$

Thus,

$$\begin{aligned}
(6.3) &\leq \sum_{a=1}^{\frac{R}{N}} \Big(\frac{R-Na}{C}\Big)\Big(\frac{2ne}{C}\Big)^a N^a D^{\frac{R}{N}} \\
&\leq \frac{R}{N}\Big(\frac{R}{C}\Big)\Big(\frac{2ne}{C}\Big)^{\frac{R}{N}} N^{\frac{R}{N}} D^{\frac{R}{N}} \\
&\leq e^{\ln \frac{R^2}{NC} + \frac{R}{N} \ln \frac{2neND}{C}} \leq e^{\alpha(N)R},
\end{aligned}$$

where $\alpha(N) \to 0$ as $N \to \infty$. In particular, for large enough N we have that $\mu' := \alpha(N) < \mu$, and so the expression above is bounded by $e^{\mu' R}$, as required. ■

7. Small cancellations with small overlaps

In this section we review the main definitions for small cancellation theory following Ol'shanskii [22], we then prove that one can find small cancellation words with specific properties, and prove the main lemmas of Section 3.

7.1. Small cancellation conditions

A set of words $\mathcal{R}$ is *symmetrized* if it is closed under taking cyclic permutations and inverses.

Definition 7.1 (Pieces). Let $\mathcal{R}$ and $\mathcal{K}$ be symmetrized sets of words in S, and $\varepsilon > 0$. Let U be a subword of a word $R \in \mathcal{R}$. Then U is called a *$(\mathcal{K}, \varepsilon)$-piece* if there exists a word $R' \in \mathcal{K}$ such that

(1) $R = UV$, $R' = U'V'$ as words, for some words U', V, V';

(2) $U' = CUD$ in G for some words C, D in S such that $\max\{\|C\|, \|D\|\} \le \varepsilon$;

(3) $CRC^{-1} \neq R'$ in G.

U is called an *ε'-piece*[5] if

(1′) $R = UVU'V'$ for some U', V, V';

(2′) $U' = CU^{\pm 1}D$ in G for some words C, D in S such that $\max\{\|C\|, \|D\|\} \le \varepsilon$.

Remark 7.2. In case $\mathcal{K} = \mathcal{R}$, a $(\mathcal{K}, \varepsilon)$-piece is simply called an *ε-piece*, and this definition coincides with the usual definition found for example in [22, 23].

Definition 7.3 (Small cancellation conditions). Let $\mathcal{R}$ and $\mathcal{K}$ be symmetrized sets of words in G. We say that $\mathcal{R}$ satisfies the $C_1(\varepsilon, \mu, \lambda, c, \rho, \mathcal{K})$-condition for some $\varepsilon \ge 0$, $\mu > 0$, $\lambda \in (0, 1]$, $c \ge 0$, $\rho > 0$, if

(1) $\|R\| \ge \rho$ for any $R \in \mathcal{R}$;

(2) any word $R \in \mathcal{R}$ is (λ, c)-quasigeodesic;

(3) for any $(\mathcal{R}, \varepsilon)$-piece U of any word $R \in \mathcal{R}$, $\max\{\|U\|, \|U'\|\} < \mu\|R\|$;

(4) for any $(\mathcal{K}, \varepsilon)$-piece U of any word $R \in \mathcal{R}$, $\max\{\|U\|, \|U'\|\} < \mu\|R\|$;

(5) for any ε'-piece U of any word $R \in \mathcal{R}$, $\max\{\|U\|, \|U'\|\} < \mu\|R\|$.

Remark 7.4. (1) An arbitrary set of words is said to satisfy $C_1(\varepsilon, \mu, \lambda, c, \rho, \mathcal{K})$ if its symmetrized closure does.

(2) When $\mathcal{K} = \emptyset$, condition (4) trivially holds, and the $C_1(\varepsilon, \mu, \lambda, c, \rho, \mathcal{K})$-conditions coincide with the usual $C_1(\varepsilon, \mu, \lambda, c, \rho)$-conditions found for example in [22, 23].

Instead of keeping track of quantifiers, it would be convenient to use the following.

Definition 7.5. Let $G, \mathcal{K}$ be as in the definitions above. Let $\mathcal{P}$ be some property. We say that *there exists a set of words $\mathcal{R}$ satisfying $\mathcal{P}$ in G with arbitrarily small cancellation and arbitrarily small overlap with $\mathcal{K}$* if there exist λ, c such that for all ε, μ, ρ there exists a set $\mathcal{R}$ satisfying $\mathcal{P}$ and the $C_1(\varepsilon, \mu, \lambda, c, \rho, \mathcal{K})$-condition.

[5] For consistency with [19, 22, 23] we follow the standard yet confusing notation "ε'-piece" by which we mean that U is an ε'-piece if it satisfies (1′) and (2′) with respect to the parameter ε (and not (1)–(3) with respect to the parameter ε').

Similarly, we say that $\mathcal{P}$ *holds for sets of words* $\mathcal{R}$ *of* G *with small enough cancellation and small enough overlap with* $\mathcal{K}$ if for each λ, c there exist ε, μ, ρ such that $\mathcal{P}$ holds for all $\mathcal{R}$ satisfying the $C_1(\varepsilon, \mu, \lambda, c, \rho, \mathcal{K})$-condition.

Remark 7.6. Suppose G is hyperbolic and $K_1, \dots, K_n$ are quasiconvex in G. Fix some generating sets $S_1, \dots, S_n, S$ for $K_1, \dots, K_n, G$, respectively. We assume S contains $S_1, \dots, S_n$. By "small overlap with $K_1, \dots, K_n$" we mean "small overlap with $\mathcal{K}$" where $\mathcal{K} = \mathcal{K}_1 \cup \dots \cup \mathcal{K}_n$ and $\mathcal{K}_i$ is the set of all words in S_i which are geodesic in K_i.

7.2. Existence of words with arbitrarily small cancellation

The goal of this section is to prove Lemma 3.1 which states that there exist words with arbitrarily small cancellation and arbitrarily small overlap with a finite union of quasiconvex subgroups.

Given a word $R(X, Y)$, we denote by $\|R(X,Y)\|_F = \|R(X,Y)\|_F$ the norm of R in the free group $F(X,Y)$ with respect to the generating set X, X^{-1}, Y, Y^{-1}. For words g, h in S we denote by $R(g,h)$ the word obtained by substituting g, h for X, Y, and by $\|R(g,h)\|$ the length of a path labeled by $R(g,h)$ in G, with respect to the generating set S.

Given a set of words $\mathcal{R} \subseteq F(X,Y)$ and words g, h in S, we denote by $\mathcal{R}(g,h)$ the symmetrized closure of the set $\{R(g,h) \mid R \in \mathcal{R}\}$.

Lemma 7.7. *Let G be a torsion-free hyperbolic group. Let $a, b \in G$ be non-trivial elements in G that are non-commensurable. Let $\lambda_0 \in (0,1]$, $c_0 \geq 0$, and let $\mathcal{K}$ be a symmetrized set of (λ_0, c_0)-quasigeodesic words that is closed under taking subwords. Suppose that a is non-commensurable into $\mathcal{K}$. Then there exist $\lambda \in (0,1]$ and $c \geq 0$ such that for any $\varepsilon \geq 0$, $\mu > 0$, $\rho > 0$, there are μ', ρ', N with the following property: If a set of words $\mathcal{R} \subset F(X,Y)$ satisfies $C_1(0, \mu', 1, 0, \rho')$ in $F(X,Y)$, then $\mathcal{R}(a^N, b^N)$ satisfies the $C_1(\varepsilon, \mu, \lambda, c, \rho, \mathcal{K})$-condition in G;*

Moreover, for every $R \in \mathcal{R}(a^N, b^N)$ with small enough cancellation, we have that the elementary group $E(R)$ equals $\langle R \rangle$.

Lemma 7.7 will not surprise the experts in small-cancellation theory. We omit the proof of the lemma as it follows similar lines to proofs in the literature, e.g., [22, Lemma 4.2] and [19, Lemma 6.1]. However, a detailed proof of the lemma is found in the arXiv version of this paper, see [11, Lemma 7.9].

As a corollary we now prove Lemma 3.1.

Proof of Lemma 3.1. Say we are given $H, K_1, \dots, K_n$ as in the statement of the lemma. Suppose without loss of generality that the generators of each of $K_1, \dots, K_n$ belong to S. Since $K_1, \dots, K_n$ are quasiconvex, the set $\mathcal{K}$ of all elements in $K_1 \cup \dots \cup K_n$ is closed under taking subwords, and all words in $\mathcal{K}$ are (λ_0, c_0)-quasigeodesic with respect to some uniform $\lambda_0 \in (0,1]$, $c_0 \geq 0$.

Since H is not commensurable into $K_1, \dots, K_n$, Lemma 3.9 ensures there exists an element $a' \in H$ non-commensurable into $\mathcal{K}$. Since H is non-elementary, there exists $b' \in H$ such that a' and b' are non-commensurable.

The first part of Lemma 3.1 then follows immediately by applying Lemma 7.7 on a' and b'. Indeed, given parameters $(\varepsilon, \mu, \lambda, c, \rho)$, it is enough to construct an arbitrarily large set of words satisfying $C_1(0, \mu', 1, 0, \rho')$ in the free group $F(X, Y)$. Such a set is easy to construct. For example, take $N > \max\{\rho', \frac{3}{\mu'}\}$, and for $1 \le i \le m$ set

$$w_i = X^{iN} Y X^{iN+1} Y X^{iN+2} Y \dots X^{iN+N} Y,$$
$$w_i' = Y^{iN} X Y^{iN+1} X Y^{iN+2} X \dots Y^{iN+N} X.$$

For the “moreover” part, suppose ϕ is an involution of G exchanging two non-commensurable elements $a, b \in H$, and suppose further that $\mathcal{K} = \phi(\mathcal{K})$. It is enough to find elements $a', b' \in H$ non-commensurable in G, such that ϕ exchanges $a' \leftrightarrow b'$ and such that a' is non-commensurable into $\mathcal{K}$. Indeed, given such elements, one can then apply Lemma 7.7 with a', b', and take the words $w_1, \dots, w_m, w_1', \dots, w_m'$ as suggested above.

We will now find such elements. By Theorem 3.9, there exists an element $h \in H$ that is not commensurable into $\mathcal{K}' = \mathcal{K} \cup \langle a \rangle \cup \langle b \rangle$. For large enough integers s, t, the elements $a' = (a^s h^s)^t$ and $b' = (b^s \phi(h)^s)^t$ satisfy the requirements. Indeed, suppose that for some integer l and $g \in G$ we had that $g^{-1} a'^l g = U$ is either a power of b' or a word in $\mathcal{K}$. We may assume that a'^l is much longer than g, by replacing l by a large multiple. By Lemma 6.1 there exists a major part of a'^l that is contained in a small neighborhood of U. In particular, by largeness of t, this major part must contain a subpath labeled by $a^s h^s$. However, for s large enough, this is impossible by Lemma 6.2, as a is non-commensurable with b and $\phi(h)$, and h is non-commensurable into $\mathcal{K}$. ∎

7.3. Greendlinger’s lemma

In this subsection we review Ol’shanskii’s version of Greendlinger’s lemma and the relevant definitions, to be used in Section 8.

Let $G = \langle S \mid \mathcal{O} \rangle$ be a presentation of G, $\mathcal{R}$ a set of words and $G' = \langle S \mid \mathcal{O} \cup \mathcal{R} \rangle$. Let Δ be a van Kampen diagram over $G' = \langle S \mid \mathcal{O} \cup \mathcal{R} \rangle$ and q a subpath of $\partial\Delta$. Let Π be an $\mathcal{R}$-cell of Δ, i.e., a cell whose boundary is labeled by a word in $\mathcal{R}$. Suppose Γ is a subdiagram of Δ, containing no $\mathcal{R}$-cells, and such that $\partial\Gamma = s_1 q_1 s_2 q_2$ where q_1 is a subpath of $\partial\Pi$, q_2 a subpath of q and $\max\{|s_1|, |s_2|\} \le \varepsilon$ for some $\varepsilon > 0$. Then Γ is called an *ε-contiguity subdiagram of* Π *to* q, and the ratio $\|q_1\|/\|\partial\Pi\|$ is called *the contiguity degree* of Π to q, denoted by (Π, Γ, q).

Let Σ, Σ' be subdiagrams of Δ containing no $\mathcal{R}$-cells and such that $\partial\Sigma$ and $\partial\Sigma'$ have the same label. In this case, replacing Σ by Σ' will not affect the label of $\partial\Delta$ and the number of $\mathcal{R}$-cells in Δ. Diagrams over $\langle S \mid \mathcal{O} \cup \mathcal{R} \rangle$ that can be obtained from each other by a sequence of such replacements are called $\mathcal{O}$-equivalent.

Lemma 7.8 (Greendlinger's lemma, Ol'shanskii [22, Lemma 6.6]). *Let $G = \langle S \mid \mathcal{O} \rangle$ be hyperbolic and torsion-free. Then for any $\lambda \in (0, 1]$ and $c \geq 0$ there exist $\mu > 0$, $\varepsilon \geq 0$ and $\rho > 0$ with the following property. Let $\mathcal{R}$ be a symmetrized set of words satisfying $C_1(\varepsilon, \mu, \lambda, c, \rho)$ and let Δ be a reduced van-Kampen diagram over $\langle S \mid \mathcal{O} \cup \mathcal{R} \rangle$ whose boundary is (λ, c)-quasigeodesic. Assume that Δ has at least one $\mathcal{R}$-cell. Then there exist a diagram Δ' which is $\mathcal{O}$-equivalent to Δ, an $\mathcal{R}$-cell Π in Δ' and an ε-contiguity subdiagram Γ of Π to $\partial\Delta'$ such that*

$$(\Pi, \Gamma, \partial\Delta') > 1 - 13\mu.$$

8. The hexagon property

Let G be a group with an involution ϕ, let $X \leq G$ be a subgroup. Recall that G has the *hexagon property* with respect to X, ϕ if for all $\xi, \xi' \in X$ and $z \in G$: $\xi^z = \phi((\xi')^z)$ implies $\xi' = \xi^{\pm 1}$.

8.1. Hexagon property for small cancellation quotients

Lemma 8.1. *Let G be a torsion-free hyperbolic group with an involution ϕ, let $X \leq G$ be a quasiconvex subgroup. For all $\mathcal{R}$ such that $\phi(\mathcal{R}) = \mathcal{R}$ with small enough cancellation and small enough overlap with X, if G has the hexagon property with respect to X, ϕ, then so does $G/\langle\langle \mathcal{R} \rangle\rangle$.*

Proof. Assume for contradiction that there exist $\xi, \xi' \in X$, $z \in G$ such that

$$\xi^z = \phi((\xi')^z) \in G/\langle\langle \mathcal{R} \rangle\rangle$$

but $\xi' \neq \xi^{\pm 1}$. Let us assume that ξ, ξ' are (λ, c)-quasigeodesics in G, and that z is a geodesic in $G/\langle\langle \mathcal{R} \rangle\rangle$. The word $q := z^{-1}\xi z \phi(z)^{-1}\phi(\xi')^{-1}\phi(z)$ is trivial in $G/\langle\langle \mathcal{R} \rangle\rangle$ but is not trivial in G since G is assumed to satisfy the hexagon property. We would like to apply Greendlinger's lemma to the path q. However, even though the path q is a concatenation of 6 quasigeodesic paths in G, it might not be a quasigeodesic because of "backtracking". One can fix this by trimming the backtracking corners as described in Lemma 6.3. There exist (possibly empty) subwords z_1, z_2, z_3, z_4 of z and subwords η, η' of ξ, ξ', respectively, and words $v_1, \ldots, v_6$ of length $\leq \delta'$ such that the path

$$p := z_1^{-1} v_1 \eta v_2 z_2 v_3 \phi(z_3)^{-1} v_4 \phi(\eta')^{-1} v_5 \phi(z_4) v_6$$

is a conjugate of q in G, and the path p is a (λ', c')-quasigeodesic, where δ', λ', c' depend only on λ, c and G. See Figure 1. Moreover, by symmetry of $z\phi(z)^{-1}$ we may assume that z_2 and z_3 end at the same place in z (i.e., $z = z' z_2 u = z'' z_3 u$ as words, for some z', z'', u). A similar statement holds for z_4, z_1. By replacing ξ, ξ' with large enough powers, we may assume that η and η' are arbitrarily long, and in particular non-empty.

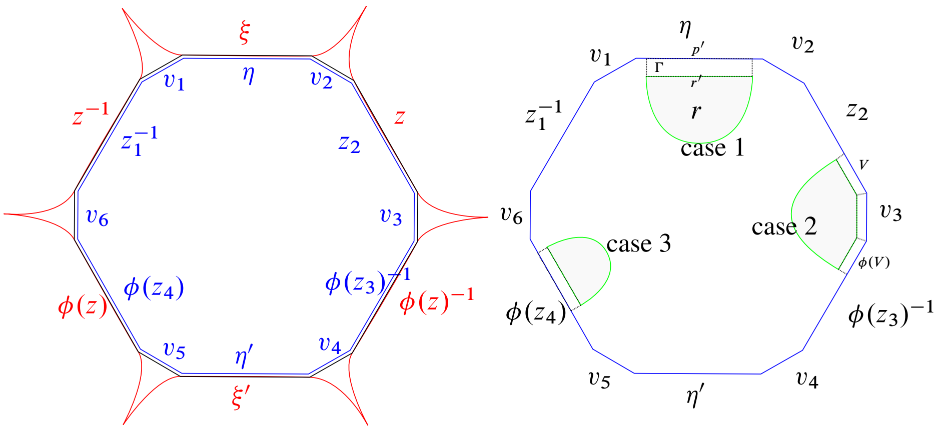

Figure 1. The trimmed hexagon, and the 3 cases of the contiguous cell in the proof of Lemma 8.1.

Since p and q are conjugates, we have that $p = 1$ in $G/\langle\langle \mathcal{R} \rangle\rangle$ while $p \neq 1$ in G. By Greendlinger's lemma there exists a cell labeled $r \in \mathcal{R}$ with contiguity degree $> (1 - 13\mu)$ assuming $\mathcal{R}$ satisfies small enough cancellation. Let us denote by Γ the contiguity subdiagram, and by r', p' the subwords of r, p, respectively, which label the opposite sides of Γ.

Applying Lemma 6.1 to the quasigeodesic rectangle $\partial\Gamma$, we see that there exist ε' and subpaths r'', p'' of r', p', respectively, of length $\|r''\| > \|r'\| - \varepsilon'$, $\|p''\| > \|p'\| - \varepsilon'$. Let $r'' = r_{z1} r_\eta \dots r_{z4}$, where $r_{z1}, r_\eta \dots, r_{z4}$ are the (possibly empty) subwords of r'' which correspond to the paths that δ''-fellow-travel with $z_1^{-1}, \eta, \dots, \phi(z_4)$, respectively. Since $\|r''\| > \|r'\| - \varepsilon'$, we deduce

$$\|r_{z1}\| + \|r_\eta\| + \dots + \|r_{z4}\| > (1 - 13\mu)\|r\| - \varepsilon' =: \omega.$$

In particular, it follows that at least one of the summands on the left-hand side must be relatively large. We therefore consider the three following cases.

Case 1. $\|r_\eta\| > \mu\|r\| =: \omega_1$ *or* $\|r_{\eta'}\| > \mu\|r\|$. This is impossible when $\mathcal{R}$ has small enough overlap with X since $\eta, \eta' \in X$.

Case 2. $\min\{\|r_{z2}\|, \|r_{z3}\|\} > \lambda^{-1}(\mu\|r\| + 2\delta'' + c) + 2\delta'' =: \omega_2$ *and the path* p'' *contains* v_3. In this case, let

$$p''_{z2} = p'' \cap z_2, \quad p''_{v3} = p'' \cap v_3, \quad p''_{z3} = p'' \cap \phi(z_3)^{-1}.$$

Since $\|r\|$ is a (λ, c)-quasigeodesic and z_2 and $\phi(z_3)^{-1}$ are geodesics, we get that

$$\min\{\|p''_{z2}\|, \|p''_{z3}\|\} > \mu\|r\| + 2\delta''.$$

Recall that z_2 and z_3 end at the same place in z, thus there is a subword V of z of length $\|V\| > \mu\|r\| + 2\delta''$ such that V is in p''_{z2} and $\phi(V)^{-1}$ is in p''_{z3}. Let U and U' be the subwords of r that δ''-fellow-travel with V and $\phi(V)$. We have $\|U\| \geq \|V\| - 2\delta'' > \mu\|r\|$ and similarly $\|U'\| > \mu\|r\|$. Since $\phi(r) \in \mathcal{R}$, we get that r has a $(2\delta'')$-piece (and hence an ε-piece) with $\phi(r)$ of length $> \mu\|r\|$ which is impossible if $\mathcal{R}$ has $C_1(\varepsilon, \mu, \lambda, c, \rho, X)$.

Similarly one proves the case

$$\min\{\|r_{z1}\|, \|r_{z4}\|\} > \lambda^{-1}(\mu\|r\| + 2\delta'' + c)$$

and the path p'' contains v_6.

Case 3. $\|r_{z4}\| > \omega - \omega_1 - \omega_2 =: \omega_3$ *(and similarly for* r_{z1}, r_{z2} *and* r_{z3}*).* For small enough μ and large enough ρ we can assume that $\omega_3/\|r\|$ is arbitrarily close to 1, and thus we can assume

$$\|r\| - \omega_3 + 2\delta'' < \lambda'\omega_3 - c' - 2\delta''. \tag{8.1}$$

We now show that $\phi(z)$ can be shortened, contradicting the assumption that z is a geodesic. Let z''_4 be the subpath of $\phi(z_4)$ that δ''-fellow-travels with r_{z4}, and let t_1, t_2 be paths of length at most δ'' such that $z''_4 = t_1 r_{z4} t_2$. Let r_c be the subpath of r which is complementary to r_{z4}, that is, such that r is a cyclic conjugate of $r_{z4}^{-1} r_c$. Then $z''_4 = t_1 r_c t_2$ in G. But

$$\begin{aligned}
\|t_1 r_c t_2\| &\leq \|t_1\| + \|r_c\| + \|t_2\| \\
&\leq \|r\| - \omega_3 + 2\delta'' \\
&< \lambda'\omega_3 - c' - 2\delta'' \\
&< \lambda'\|r_{z4}\| - c' - 2\delta'' \\
&\leq \|r_{z4}\| - \|t_1\| - \|t_2\| \leq \|z''_4\|,
\end{aligned}$$

where the third inequality is by (8.1) and the fifth inequality is by (λ', c')-quasiconvexity of r_{z4}. This contradicts the assumption that z is a geodesic in $G/\langle\langle\mathcal{R}\rangle\rangle$, as $t_1 r_c t_2$ is a shortcut of a subpath of $\phi(z)$. ■

8.2. Hexagon condition for HNN extensions

Lemma 8.2. *Let A be a group with an involution ϕ, and $X \leq A$ a subgroup. Let $C \leq X$ and $C' \leq A$ be such that $C, C', D = \phi(C), D' = \phi(C')$ satisfy the conditions of Lemma 3.6. Set $G = \langle A, s, t \mid C^s = C', D^t = D\rangle$. Extend ϕ to an involution of G by setting $\phi(s) = t$. If A satisfies the hexagon property with respect to X, ϕ, then so does G.*

Proof. Assume $\xi^z = \phi(\xi'^z)$, for some $\xi, \xi' \in X$ and $z \in G$.

Write z in normal form as $z = a_0 x_1 a_1 \ldots x_n a_n \in G$, where $a_i \in A$, $x_i \in \{s, s^{-1}, t, t^{-1}\}$. Without loss of generality, assume that z has the minimal n among all elements in G that satisfy $\xi^z = \phi(\xi'^z)$.

By the assumption on A, if $z = a_0 \in A$, then $\xi' = \xi^{\pm}$ and we are done. Hence, we may assume that $n \geq 1$. The word $z\phi(z)^{-1}$ is reduced in the HNN extension. By Lemma 3.6, the extension G is 2-acylindrical. It follows that $n \leq 1$.

Write $z = axb$ where $a, b \in A$, $x \in \{s, s^{-1}, t, t^{-1}\}$. The relation $\xi^z = \phi(\xi'^z)$ becomes

$$b^{-1}x^{-1}a^{-1}\xi axb\phi(b^{-1}x^{-1}a^{-1}\xi'^{-1}axb) = 1.$$

By symmetry, there are two cases to consider.

Case 1. $x = s^{-1}$. Here the relation becomes

$$b^{-1} \overbrace{s \underbrace{a^{-1}\xi a}_{\in A} s^{-1}}^{\heartsuit} \underbrace{b\phi(b)^{-1}}_{\in A} \overbrace{t \underbrace{\phi(a^{-1}\xi'^{-1}a)}_{\in A} t^{-1}}^{\heartsuit} \phi(b) = 1.$$

By Britton's lemma, the word must be non-reduced at both expressions marked with $\heartsuit$. After reducing and rearranging we get $c^b = d^{\phi(b)}$ where $c = sa^{-1}\xi as^{-1} \in C$ and $d = t\phi(a^{-1}\xi' a)t^{-1} \in D = \phi(C)$. Since $c \in C \leq X$ and $d = \phi(c')$ for some $c' \in C \leq X$, we can apply the hexagon condition of A to deduce that $c' = c^{\pm 1}$. Tracing back the definition of c, c', it follows that $\xi' = \xi^{\pm 1}$, as desired.

Case 2. $x = s$. Applying the same argument, we get $(c')^b = (d')^{\phi(b)}$ for some $c' \in C'$ and $d' \in D'$. However, this contradicts the assumption that $gC'g^{-1} \cap D' = \{1\}$ for all $g \in A$. ∎

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Gil Goffer
Weizmann Institute of Science, Herzl St 234, Rehovot 7610001, Israel; gil.goffer@weizmann.ac.il

Nir Lazarovich
Technion – Israel Institute of Technology, Haifa 3200003, Israel; nirlazarovich@gmail.com