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Algorithmic problems in groups with quadratic Dehn function

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Abstract. We construct and study finitely presented groups with quadratic Dehn function (QD-groups) and present the following applications of the method developed in our recent papers. (1) The isomorphism problem is undecidable in the class of QD-groups. (2) For every recursive function f, there is a QD-group G containing a finitely presented subgroup H whose Dehn function grows faster than f. (3) There exists a group with undecidable conjugacy problem but decidable power conjugacy problem; this group is QD.

Contents

1. Introduction

Every group given by a presentation $G = \langle X \mid R \rangle$ is a factor group F/N of the free group $F = F(X)$ with the set of free generators X over the normal closure $N = \langle\langle R \rangle\rangle^F$ of the set of relators R. Therefore every word w over the alphabet $X^{\pm 1}$ vanishing in G represents an element of N, and so in F, w is a product $v_1 \dots v_m$ of factors $v_i = u_i r_i^{\pm 1} u_i^{-1}$ which are conjugates of the relators $r_i \in R$ or their inverses.

The minimal number of factors $m = m(w)$ is called the *area of the word* w with respect to the presentation $G = \langle X \mid R \rangle$. M. Gromov [12, 13] introduced this concept in geometric group theory, because m is equal to the minimal number of 2-cells (counting

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with multiplicities) used in a 0-homotopy of the path $\mathbf{p}$ labeled by w in the Cayley complex of the presentation of G (or the 0-homotopy of a singular disk with boundary $\mathbf{p}$).

In other words, given equality $w = 1$ in G, one can construct a van Kampen diagram, that is, a finite, connected graph on the Euclidean plane with m bounded regions, where every edge has label from $X^{\pm 1}$, the boundary path of every region (= 2-cell) is therefore labeled. The label of it belongs in $R^{\pm 1}$, and the boundary of the whole map is labeled by w. (See more details for this visual definition of area and van Kampen diagram in Section 3.2.)

The Dehn function of a finitely presented group $G = \langle X \mid R \rangle$ is the smallest function $f(n)$ such that for every word w of length at most n in the alphabet $X^{\pm 1}$, which is equal to 1 in G, the area of w is at most $f(n)$.

It is well known (see [18]) that the Dehn functions of different finite presentations of the same finitely presented group are equivalent, where we call two functions f, g equivalent if $f \preceq g$ and $g \preceq f$. Here $f \preceq g$ means that there is a constant $c > 0$ such that $f(n) \le cg(cn) + cn$ for every $n = 1, 2, \dots$

The Dehn function is an important invariant of a group for the following reasons.

- It easily follows from the definition that if G is the fundamental group of a compact Riemannian manifold M, then the Dehn function of G is equivalent to the smallest isoperimetric function of the universal cover $\widetilde{M}$.
- The Dehn function is closely related to the solvability of the word problem in the group [9]. From the computer science point of view, the Dehn function of a group G is equivalent to the time function of a non-deterministic Turing machine 'solving' the word problem in G (see [30, Introduction] for details). Moreover, as was shown in [5]:

  A (not necessarily finitely presented) finitely generated group has word problem in **NP** *if and only if it is a subgroup of a finitely presented group with at most polynomial Dehn function (a similar result holds for other computational complexity classes* [5]*).*

- From the geometric point of view the Dehn function measures the 'curvature' of the group: linear Dehn functions correspond to negative curvature, quadratic Dehn function correspond to non-positive curvature, etc.

More precisely, a finitely presented group is hyperbolic if and only if it has a subquadratic (hence linear) Dehn function [7, 12, 21]. In particular, the conjugacy problem in such groups is decidable [12]. In contrast, we recently constructed a group with quadratic Dehn function and undecidable conjugacy problem [27]. This result answers Rips' question of 1994. The present paper is based on the constructions of groups with small Denn functions from [23, 27] as well as on the application of S-machines introduced in [30].

The affirmative solution of the isomorphism problem was obtained in [31] for the class of torsion free hyperbolic groups and in [11] for the class of all hyperbolic groups. This means that there exists an algorithm recognizing whether two hyperbolic groups G_1 and G_2 are isomorphic or not, provided G_1 and G_2 are given by their finite presentations.

We show that the linearity is the only possible restriction of Dehn functions providing a positive solution of the isomorphism problem.

Theorem 1.1. *In the class QD of finitely presented groups with quadratic Dehn function, the isomorphism problem is undecidable. Moreover, one can select a QD-group $\overline{G}$ such that there exists no algorithm deciding whether a QD-group G and $\overline{G}$ are isomorphic or not.*

It is known that the Dehn function of a finitely presented subgroup can grow faster than the Dehn function of the entire group. For example, the group $\mathrm{SL}(5, Z)$ has quadratic Dehn function [32], but it contains subgroups with exponential Dehn function. Here we prove the following result:

Theorem 1.2. *For every recursive function f, there exists a pair of finitely presented groups $H \le G$, such that $f \preceq d_H$, where d_H is the Dehn function of the subgroup H, while the Dehn function of G is quadratic.*

For a group with presentation $G = \langle X \mid R \rangle$, the power conjugacy problem is to determine, given words $u, v \in F(X)$ whether or not there exist non-zero integers k and l such that u^k is conjugate to v^l in G. The power-conjugacy problem has been the subject of extensive research, see [1–4, 6, 10, 15, 16, 28]. However to the best of our knowledge, the interconnection of this problem and the classical conjugacy problem has not been studied yet.

Theorem 1.3. (1) *There is a finitely presented group G with undecidable conjugacy problem but decidable power conjugacy problem. Moreover, G has quadratic Dehn function.*

(2) *There is a finitely presented group H with undecidable power conjugacy problem but decidable conjugacy problem.*

Notice that for the group G from Theorem 1.3 (1) and [27], there exists no algorithm recognizing the conjugacy of some *non-trivial* powers of two elements (see Remark 4.6) since elements of finite and infinite orders behave differently in G. Although G has undecidable conjugacy problem, this problem is decidable in G for elements of infinite order. The following property of G is used in the proof of Theorem 1.3 (1), and it is also interesting in itself.

Theorem 1.4. *For the group G from Theorem 1.3 (1), there is an algorithm that recognizes whether two elements g and h are conjugate in G or not, provided the orders of g and h are infinite. The order of every element of G can be also computed effectively.*

Theorems 1.1, 1.2, 1.3, and 1.4 are proved in Sections 4, 5, 7, and 6, respectively. The information needed for understanding the proofs, has been selected from earlier papers and placed in Sections 2 and 3.

2. Machine preliminaries

2.1. S-machine

Here we will use definitions which are equivalent to the definitions used in [26] and [22].

The 'hardware' of an S-machine $\mathbf{S}$ is a pair (Y, Q) of finite sets, where $Q = \bigsqcup_{i=0}^{n} Q_i$ and $Y = \bigsqcup_{i=1}^{n} Y_i$ for some $n \geq 1$. Here and below $\sqcup$ denotes the disjoint union of sets.

The elements from Q are called *state letters*, the elements from Y are *tape letters*. The sets Q_i (resp. Y_i) are called *parts* of Q (resp. Y). To unify further definitions, we may add the empty parts Y_0 and Y_{n+1} to Y.

The *language of admissible words* consists of reduced words in the free group of the form

$$q_1 u_1 q_2 \ldots u_s q_{s+1}, \tag{2.1}$$

where every q_i is a state letter from some part $Q_{j(i)}^{\pm 1}$, u_i are reduced group words in the alphabet of tape letters of the part $Y_{k(i)}$, and for every $i = 1, \ldots, s$ one of the following holds:

- If q_i is from $Q_{j(i)}$, then q_{i+1} is either from $Q_{j(i)+1}$ or is equal to q_i^{-1}; moreover $k(i) = j(i) + 1$.
- If $q_i \in Q_{j(i)}^{-1}$, then q_{i+1} is either from $Q_{j(i)-1}^{-1}$ or is equal to q_i^{-1}; moreover $k(i) = j(i)$.

Every subword $q_i u_i q_{i+1}$ of an admissible word (2.1) will be called a $Q_{j(i)}^{\pm 1} Q_{j(i+1)}^{\pm 1}$-*sector* of that word. An admissible word may contain many $Q_{j(i)}^{\pm 1} Q_{j(i+1)}^{\pm 1}$-sectors.

We denote by $\|W\|$ the length of word W. For every word W, if we delete all non-$Y^{\pm 1}$ letters from W, we get the Y-projection of the word W. The length of the Y-projection of W is called the Y-*length* and is denoted by $|W|_Y$. Usually parts of the set Q of state letters are denoted by capital letters. For example, a part P would consist of letters p with various indices.

If an admissible word W has the form (2.1), $W = q_1 u_1 q_2 u_2 \ldots q_s$, and $q_i \in Q_{j(i)}^{\pm 1}$, $i = 1, \ldots, s$, u_i are group words in tape letters, then we shall say that the *base* of W is the word $Q_{j(1)}^{\pm 1} Q_{j(2)}^{\pm 1} \ldots Q_{j(s)}^{\pm 1}$. Here Q_i are just symbols which denote the corresponding parts of the set of state letters. Note that, by the definition of admissible words, the base is not necessarily a reduced word.

Instead of saying that the parts of the set of state letters of $\mathbf{S}$ are $Q_0, Q_1, \ldots, Q_n$ we will write that the *the standard base* of the S-machine is $Q_0 \ldots Q_n$.

The *software* of an S-machine with the standard base $Q_0 \ldots Q_n$ is a finite set of *rules* Θ. Every $\theta \in \Theta$ is a sequence $[q_0 \to a_0 q_0' b_0, \ldots, q_n \to a_n q_n' b_n]$ and a subset $Y(\theta) = \bigsqcup Y_j(\theta)$, where $q_i, q_i' \in Q_i$, a_i is a reduced word in the alphabet $Y_i(\theta)$, b_i is a reduced word in $Y_{i+1}(\theta)$, $Y_i(\theta) \subseteq Y_i$, $i = 0, \ldots, n$. (Recall that $Y_0 = Y_n = \emptyset$, and so the words a_0 and b_n are empty.)

Each component $q_i \to a_i q_i' b_i$ is called a *part* of the rule. In most cases the sets $Y_j(\theta)$ will be equal to either Y_j or $\emptyset$. By default $Y_j(\theta) = Y_j$.

Every rule

$$\theta = [q_0 \to a_0 q_0' b_0, \dots, q_n \to a_n q_n' b_n]$$

has an inverse

$$\theta^{-1} = [q_0' \to a_0^{-1} q_0 b_0^{-1}, \dots, q_n' \to a_n^{-1} q_n b_n]$$

which is also a rule of **S**. It is always the case that $Y_i(\theta^{-1}) = Y_i(\theta)$ for every i. Thus the set of rules Θ of an S-machine is divided into two disjoint parts, Θ^+ and Θ^-, such that for every $\theta \in \Theta^+$, $\theta^{-1} \in \Theta^-$ and for every $\theta \in \Theta^-$, $\theta^{-1} \in \Theta^+$ (in particular $\Theta^{-1} = \Theta$, that is, any S-machine is symmetric).

The rules from Θ^+ (resp. Θ^-) are called *positive* (resp. *negative*).

To apply a rule $\theta = [q_0 \to a_0 q_0' b_0, \dots, q_n \to a_n q_n' b_n]$ as above to an admissible word $p_1 u_1 p_2 u_2 \dots p_s$, where each $p_i \in Q_{j(i)}^{\pm 1}$, means:

- check if u_i is a word in the alphabet $Y_{j(i)+1}(\theta)$ when $p_i \in Q_{j(i)}$ or if it is a word in $Y_{j(i)}(\theta)$ when $p_i \in Q_{j(i)}^{-1}$ $(i = 1, \dots, s-1)$; and if this property holds, then:
- replace each $p_i = q_{j(i)}^{\pm 1}$ by $(a_{j(i)} q_{j(i)}' b_{j(i)})^{\pm 1}$,
- if the resulting word is not reduced or starts (ends) with Y-letters, then reduce the word and trim the first and last Y-letters to obtain an admissible word again.

If a rule θ is applicable to an admissible word W (i.e., W belongs to the *domain* of θ), then we say that W is a *θ-admissible word* and denote the result of application of θ to W by $W \cdot \theta$. Hence each rule defines an invertible partial map from the set of admissible words to itself, and one can consider an S-machine as an inverse semigroup of partial bijections of the set of admissible words.

We call an admissible word with the standard base a *configuration* of an S-machine.

We usually assume that every part Q_i of the set of state letters contains a *start state letter* and an *end state letter*. Then a configuration is called a *start* (*end*) configuration if all state letters in it are start (end) letters. As Turing machines, some S-machines are *recognizing a language*. In that case we choose an *input* sector, usually the $Q_0 Q_1$-sector, of every configuration. The Y-projection of that sector is called the *input* of the configuration. In that case, the end configuration with empty Y-projection is called the *accept* configuration. If the S-machine (viewed as a semigroup of transformations as above) can take an input configuration with input u to the accept configuration, we say that u is *accepted* by the S-machine. We define *accepted* configurations (not necessarily start configurations) similarly.

A *computation* of *length* $t \geq 0$ is a sequence of admissible words

$$W_0 \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_t} W_t$$

such that for every $i = 0, \dots, t-1$ the S-machine passes from W_i to W_{i+1} by applying the rule θ_i from Θ. The word $H = \theta_1 \dots \theta_t$ is called the *history* of the computation, and the word W_0 is called *H-admissible*. Since W_t is determined by W_0 and the history H, we use the notation $W_t = W_0 \cdot H$.

A computation is called *reduced* if its history is a reduced word.

Note, though, that in [27] and in this paper, unlike the previous ones, we consider non-reduced computations too because these may correspond to reduced van Kampen diagrams (trapezia) under our present interpretation of S-machines in groups.

If for some rule $\theta = [q_0 \to a_0 q_0' b_0, \dots, q_n \to a_n q_n' b_n] \in \Theta$ of an S-machine $\mathbf{S}$ the set $Y_{i+1}(\theta)$ is empty (hence in every admissible word in the domain of θ every $Q_i Q_{i+1}$-sector has no Y-letters), then we say that θ locks the $Q_i Q_{i+1}$-sector. In that case we always assume that b_i, a_{i+1} are empty and we denote by

$$q_i \xrightarrow{\ell} a_i q_i'$$

the i-th part (the $(i+1)$-st part) of the rule. (We also have $q_{i+1} \to q_{i+1}' b_{i+1}$, where the $Q_{i+1} Q_{i+2}$-sector can be unlocked.)

Remark 2.1. For the sake of brevity, the substitution $[q_i \xrightarrow{\ell} a q_i', q_{i+1} \to q_{i+1}' b]$ can be written in the form $[q_i q_{i+1} \to a q_i' q_{i+1}' b]$.

The above definition of S-machines resembles the definition of multi-tape Turing machines (see [30]). The main differences are that every state letter of an S-machine is blind: it does not 'see' tape letters next to it (two state letters can see each other if they stay next to each other). Also S-machines are symmetric (every rule has an inverse), can work with words containing negative letters, and words with 'non-standard' order of state letters.

It is important that S-machines can simulate the work of Turing machines. This non-trivial fact, especially if one tries to get a polynomial time simulation, was first proved in [30]. But we do not need a restriction on time, and it would be more convenient for us to use an easier S-machine from [26].

Let $\mathbf{M}_0$ be a deterministic Turing machine accepting a non-empty language $\mathcal{L}$ of words in the one-letter alphabet $\{\alpha\}$. In different sections we will use two versions of an equivalent S-machine $\mathbf{M}_1$. For both of them there is a unique start rule, replacing the start state letters that do not occur in other rules; similarly, there is a unique end rule, the only one involving end state letters. The first version of $\mathbf{M}_1$ is borrowed from [26], where [26, Lemmas 3.25 and 3.27] provide the following additional properties of $\mathbf{M}_1$.

Lemma 2.2. *The language of accepted input words of the recognizing S-machine $\mathbf{M}_1$ is $\mathcal{L}$. In every input configuration of $\mathbf{M}_1$, there is exactly one input sector, the first sector of the word, and all other sectors are empty of Y-letters.*

If a non-empty reduced computation $C_0 \to \cdots \to C_t$ of $\mathbf{M}_1$ starts with an input configuration containing a negative letter, then C_t is neither an input nor the accept configuration. ∎

The following statement can be found in [22, Lemmas 4.15 and 4.16 (a)], although below we denote the machine by $\mathbf{M}_1$ instead of M_2 in [22].

Lemma 2.3. *The language of accepted input words of the recognizing S-machine $\mathbf{M}_1$ is $\mathcal{L}$. In every input configuration of $\mathbf{M}_1$, there is exactly one input sector, the first sector of the word, and all other sectors are empty of Y-letters.*

For every reduced computation $W_0 \to \cdots \to W_t$ of $\mathbf{M}_1$ with the standard base and a non-empty history H, we have $W_t \neq W_0$. ■

Lemma 2.4 is a modified formulation of [27, Lemma 2.8]. (In [27], it was formulated for reduced computations, but the proof did not use that the history was reduced.)

Lemma 2.4. *Suppose that a computation $W_0 \to W_1 \to \cdots \to W_t$ of an S-machine $\mathbf{S}$ has a 2-letter base and the history of the form $H \equiv H_1 H_2^k H_3$ ($k \geq 0$). Then for every $i = 0, 1, \ldots, t$, we have the inequality*

$$\|W_i\| \leq \|W_0\| + \|W_t\| + 2\|H_1\| + 3\|H_2\| + 2\|H_3\|.$$ ■

Recall that a word w is called a *periodic word* with period v if w is a subword of some power of v.

Lemma 2.5. *There is an exponential function f with the following property. Suppose a computation $\mathcal{C} : W_0 \to W_1 \to \cdots \to W_t$ of an S-machine $\mathbf{S}$ has a periodic history with period H. Assume that $\mathcal{C}$ has no subcomputations $W_i \to \cdots \to W_j$ with history H and $W_i \equiv W_j$. Then $t \leq f(\|W_0\|(\|W_0\| + \|W_t\| + \|H\|))$.*

Proof. Since the history is H-periodic, there are words $W_{i_1}, \ldots, W_{i_s}$, where $i_{k+1} - i_k = \|H\|$ ($k = 1, \ldots, s-1$), the history of every subcomputation $W_{i_k} \to W_{i_k+1} \to \cdots \to W_{i_{k+1}}$ is H, and $s \geq t\|H\|^{-1} - 1$.

Assume that $W_{i_k} \equiv W_{i_l}$ for some $l > k$. Then we have $V_{i_k} \equiv V_{i_l}$ for arbitrary restriction of $\mathcal{C}$ to a subbase B of length 2. Arbitrary computation with base B and history H multiplies the Y-projection v from the left by a word a and from the right by a word b, where the words a and b depend on B and H only. Therefore for the Y-projection v of the equal words V_{i_k} and V_{i_l}, we obtain the equality $v = a^m v b^m$, where $m = l - k \geq 1$. Hence we have $(v^{-1}a^{-1}v)^m = b^m$, which implies in the free group that $v^{-1}a^{-1}v = b$, i.e., $avb = v$. Hence $V_{i_k} \equiv V_{i_{k+1}}$ for every 2-letter subbase B. It follows that $W_{i_k} \equiv W_{i_{k+1}}$, contrary to the lemma assumption.

Therefore we obtain s different admissible words in the computation $\mathcal{C}$. Lemma 2.4 bounds their lengths by a linear function of $\|W_0\|(\|W_0\| + \|W_t\| + \|H\|)$ since every word W_i is covered by at most $\|W_0\|$ admissible subwords with 2-letter bases. Hence the number s and the number $t \leq (s+1)\|H\|$ are bounded from above by an exponential function. ■

2.2. Running state letters

For every alphabet Y we define a 'running state letters' S-machine $\mathbf{LR}(Y)$. We will omit Y if it is obvious or irrelevant. The standard base of $\mathbf{LR}(Y)$ is $Q^{(1)} P Q^{(2)}$ where

$$Q^{(1)} = \{q^{(1)}\}, \quad P = \{p^{(i)}, i = 1, 2\}, \quad Q^{(2)} = \{q^{(2)}\}.$$

The state letter p with indices runs from the state letter $q^{(2)}$ to the state letter $q^{(1)}$ and back. The S-machine **LR** will be used to check the 'structure' of a configuration (whether the state letters of a configuration are in the appropriate order), and to recognize a computation by its history.

The alphabet of tape letters Y of $\mathbf{LR}(Y)$ is $Y^{(1)} \sqcup Y^{(2)}$, where $Y^{(2)}$ is a (disjoint) copy of $Y^{(1)}$. The positive rules of **LR** are defined by (2.2)–(2.4):

$$\zeta^{(1)}(a) = [q^{(1)} \to q^{(1)}, p^{(1)} \to a^{-1} p^{(1)} a', q^{(2)} \to q^{(2)}], \tag{2.2}$$

where a is any positive letter from $Y = Y^{(1)}$ and a' is the corresponding letter in the copy $Y^{(2)}$ of $Y^{(1)}$. *Comment.* The state letter $p^{(1)}$ moves left, replacing letters a from $Y^{(1)}$ by their copies a' from $Y^{(2)}$.

$$\zeta^{(12)} = [q^{(1)} p^{(1)} \to q^{(1)} p^{(2)}, q^{(2)} \to q^{(2)}]. \tag{2.3}$$

Comment. When $p^{(1)}$ meets $q^{(1)}$, $p^{(1)}$ turns into $p^{(2)}$.

$$\zeta^{(2)}(a) = [q^{(1)} \to q^{(1)}, p^{(2)} \to a p^{(2)} (a')^{-1}, q^{(2)} \to q^{(2)}]. \tag{2.4}$$

Comment. The state letter $p^{(2)}$ moves right towards $q^{(2)}$, replacing letters a' from $Y^{(2)}$ by their copies a from $Y^{(1)}$.

The start (resp. end) state letters of **LR** are $\{q^{(1)}, p^{(1)}, q^{(2)}\}$ (resp. $\{q^{(1)}, p^{(2)}, q^{(2)}\}$).

Remark 2.6. For some large integer m, we will also need the S-machine $\mathbf{LR}_m$ from [27], that repeats the work of **LR** m times. That is the S-machine $\mathbf{LR}_m$ runs the state letter p back and forth between $q^{(2)}$ and $q^{(1)}$ m times. Every time p meets $q^{(1)}$ or $q^{(2)}$, the upper index of p increases by 1 after the application of the rule $\zeta^{(i,i+1)}$ ($i = 1, \dots, 2m-1$), so the highest upper index of p is $(2m)$.

Remark 2.7. We will also use the right analog **RL** of **LR**. The base of **RL** is $Q_1 R Q_2$. The state letter r first moves right from $q^{(1)}$ to $q^{(2)}$ and then left. A lemma 'left-right dual' to Lemma 2.11 is true for **RL** as well.

Remark 2.8. The constant m defining the machine $\mathbf{LR}_m$ is one of the big constants used in [27]. In the present paper we will use just few of them. Here they are:

$$m, N \ll c_4 \ll L. \tag{2.5}$$

The constant N defined in Section 3 is the number of parts in the base of main machine **M**, while L is the length of the hub-relation in the presentation of the group G.

The sign $\ll$ means 'much smaller' in (2.5), and it can be explained as follows. For an arbitrary inequality from [27] involving several of these constants, let D be the highest constant appearing there. The inequality always can then be rewritten in the form

$$D \geq \text{some expression involving only lower constants.}$$

This *highest parameter principle* [20] makes the finite systems of all inequalities from [23] or [27] consistent. One can effectively select the constants starting with the smallest one, because after smaller constants are chosen, one can define D to be sufficiently large to satisfy each of the inequalities, where D is the highest parameter.

2.3. Adding history sectors

We will add new (history) sectors to an S-machine $\mathbf{M}_1$ provided by Lemma 2.2 or by Lemma 2.3. The history sectors split the base letters of $\mathbf{M}_1$. (See the definition below.) If we ignore the new sectors, in essence, we get the hardware and the software of the S-machine $\mathbf{M}_1$. The new S-machine $\mathbf{M}_2$ will start with a configuration where in every history sector a copy of the history H of a computation of $\mathbf{M}_1$ is written. Then it will execute H on the other (working) sectors simulating the work of $\mathbf{M}_1$, while in the history sector, state letters scan the history, one symbol at a time. Thus if a computation of $\mathbf{M}_2$ with the standard base starts with a configuration W and ends with configuration W', then the length of the computation does not exceed $\|W\| + \|W'\|$.

Here is a precise definition of $\mathbf{M}_2$. Let the S-machine $\mathbf{M}_1$ have hardware (Q, Y), where $Q = \bigsqcup_{i=0}^{n} Q_i$, and the set of rules Θ. The new S-machine $\mathbf{M}_2$ has hardware

$$Q_{0,r} \sqcup Q_{1,\ell} \sqcup Q_{1,r} \sqcup Q_{2,\ell} \sqcup Q_{2,r} \sqcup \cdots \sqcup Q_{n,\ell},$$
$$Y_h = Y_1 \sqcup X_1 \sqcup Y_2 \sqcup \cdots \sqcup X_{n-1} \sqcup Y_n,$$

where $Q_{i,\ell}$ and $Q_{i,r}$ are (left and right) copies of Q_i, and X_i is a disjoint union of two copies of Θ^+, namely $X_{i,\ell}$ and $X_{i,r}$. (The sets $Q_{0,\ell}$, $Q_{n,r}$ are empty.) Every letter q from Q_i has two copies $q^{(\ell)} \in Q_{i,\ell}$ and $q^{(r)} \in Q_{i,r}$. The new sectors with tape letters from X_i $(i = 1, \dots, n)$ are called *history sectors*. By definition, the start (resp. end) state letters of $\mathbf{M}_2$ are copies of the corresponding start (end) state letters of $\mathbf{M}_1$. The $Q_{0,r}Q_{1,\ell}$-sector is the *input sector* of configurations of $\mathbf{M}_2$.

The positive rules θ_h of $\mathbf{M}_2$ are in one-to-one correspondence with the positive rules θ of $\mathbf{M}_1$. If $\theta = [q_0 \to a_0 q_0' b_0, \dots, q_n \to a_n q_n' b_n]$ is a positive rule of $\mathbf{M}_1$, then each part $q_i \to a_i q_i' b_i$ is replaced in θ_h by two parts

$$q_{i,\ell} \to a_i q_{i,\ell}' h_{\theta,i}^{-1} \quad \text{and} \quad q_{i,r} \to \bar{h}_{\theta,i} q_{i,r}' b_i,$$

where $h_{\theta,i}$ (resp. $\bar{h}_{\theta,i}$) is a copy of θ in the alphabet $X_{i,\ell}$ (resp. in $X_{i,r}$).

If θ is the start (resp. end) rule of $\mathbf{M}_1$, then for any word in the domain of θ_h (resp. θ_h^{-1}) all Y-letters in history sectors are from $\bigsqcup_i X_{i,\ell}$ (resp. $\bigsqcup_i X_{i,r}$).

Thus for every rule θ of $\mathbf{M}_1$, the rule θ_h of $\mathbf{M}_2$ acts in the $Q_{i,r}Q_{i+1,\ell}$-sector in the same way as θ acts in the Q_iQ_{i+1}-sector. In particular, Y-letters which can appear in the $Q_{i,r}Q_{i+1,\ell}$-sector of an admissible word in the domain of θ_h are the same as the Y-letters that can appear in the Q_iQ_{i+1}-sector of an admissible word in the domain of θ. Hence if θ locks Q_iQ_{i+1}-sectors, then θ_h locks $Q_{i,r}Q_{i+1,\ell}$-sectors.

Remark 2.9. Every computation of the S-machine $\mathbf{M}_2$ with history H and the standard base coincides with the a computation of $\mathbf{M}_1$ whose history is a copy of H if one observes it only in sectors $Q_{i,r}Q_{i+1,l}$.

Let $I_1(\alpha^k)$ be a start configuration of $\mathbf{M}_1$ (an input configuration in the domain of the start rule of $\mathbf{M}_1$) with α^k written in the input sector (all other sectors do not contain Y-letters), and H be a word in the alphabet of rules of $\mathbf{M}_1$. Then the corresponding start configuration $I_2(\alpha^k, H)$ of $\mathbf{M}_2$ is obtained by first replacing each state letter q by the product of two corresponding letters $q^{(\ell)}q^{(r)}$, and then inserting a copy of H in the *left alphabet* $X_{i,\ell}$ in every history $Q_{i,\ell}Q_{i,r}$-sector. End configurations $A_2(H)$ of $\mathbf{M}_2$ are defined similarly, only the Y-letters in the history sectors must be from the *right alphabet* $X_{i,r}$.

2.4. Adding running state letters

Our next S-machine will be a composition of $\mathbf{M}_2$ with **LR** and **RL**. The running state letters will control the work of $\mathbf{M}_3$.

First we replace every part Q_i of the state letters in the standard base of $\mathbf{M}_2$ by three parts $P_iQ_iR_i$ where P_i, R_i contain the running state letters. Thus if $Q_0 \dots Q_s$ is the standard base of $\mathbf{M}_2$, then the standard base of $\overline{\mathbf{M}}_2$ is

$$P_0Q_0R_0P_1Q_1R_1 \dots P_sQ_sR_s, \tag{2.6}$$

where P_i contains copies of running P-letters of **LR**, and R_i contains copies of running R-letters of **RL**, $i = 0, \dots, s$.

For every rule θ of $\mathbf{M}_2$, its i-th part $[q_i \to a_iq_i'b_i]$ is replaced in $\overline{\mathbf{M}}_2$ with

$$[p(i)q_ir(i) \to a_ip(i)q_i'r(i)b_i], \quad i = 0, \dots, s, \tag{2.7}$$

where $p(i) \in P_i, r(i) \in R_i$ do not depend on θ, and $q_i, q_i' \in Q_i$.

Comment. Thus, the sectors P_iQ_i and Q_iR_i are always locked. Of course, such a modification is useless for solo work of $\mathbf{M}_2$. But it will be helpful when one constructs a composition of $\overline{\mathbf{M}}_2$ with **LR** and **RL** which will be turned on after certain rules of $\overline{\mathbf{M}}_2$ are applied.

If $Q_{i-1}Q_i$ is an input sector of configurations of the machine $\mathbf{M}_2$, then $R_{i-1}P_i$ is an input sector of the configurations of $\overline{\mathbf{M}}_2$.

2.5. The machine M_3

The next S-machine $\mathbf{M}_3$ is the composition of the S-machine $\overline{\mathbf{M}}_2$ with **LR** and **RL**. The S-machine $\mathbf{M}_3$ has the same base as $\overline{\mathbf{M}}_2$, although the parts of this base have more state letters than the corresponding parts of $\overline{\mathbf{M}}_2$. It works as follows. Suppose that $\mathbf{M}_3$ starts with a start configuration of $\overline{\mathbf{M}}_2$, a word α^k in the input R_0P_1-sector, copies of a history word H in the alphabets $X_{i,\ell}$ in the history sectors, all other sectors empty of Y-letters.

Then $\mathbf{M}_3$ first executes **RL** in all history sectors (moves the running state letter from R_i in the history sectors right and left), then it executes the history H of $\overline{\mathbf{M}}_2$. After that the Y-letters in the history sectors are in $X_{i,r}$ and $\mathbf{M}_3$ executes copies of **LR** in the history sectors (moves the running state letters left then right). After that $\mathbf{M}_3$ executes a copy of H backwards, getting to a copy of the same start configuration of $\overline{\mathbf{M}}_2$, runs **RL**, executes a copy of the history H of $\overline{\mathbf{M}}_2$, runs a copy of **LR**, etc. It stops after m times running **RL**, $\overline{\mathbf{M}}_2$, **LR**, $\overline{\mathbf{M}}_2^{-1}$ and running **RL** one more time.

Thus the S-machine $\mathbf{M}_3$ is a concatenation of $4m + 1$ S-machines $\mathbf{M}_{3,1}$ to $\mathbf{M}_{3,4m+1}$. After one of these S-machines terminates, a transition rule changes its end state letters to the start state letters of the next S-machine. All these S-machines have the same standard bases as $\overline{\mathbf{M}}_2$.

The configuration $I_3(\alpha^k, H)$ of $\mathbf{M}_3$ is obtained from $I_2(\alpha^k, H)$ by adding the control state letters $r_i^{(1)}$ and $p_i^{(1)}$ according to (2.7) in Section 2.4.

Set (of the rules of machine) $\mathbf{M}_{3,1}$. It is a copy of the set of rules of the S-machine **RL**, with *parallel work* in all history sectors, i.e., every subword $Q_{i-1}R_{i-1}P_i$ of the standard base, where $Q_{i-1}Q_i$ is a history sector of $\mathbf{M}_2$, is treated as the base of a copy of **RL**, that is, R_{i-1} contains the running state letters which run between state letters from Q_{i-1} and P_i. Each rule of set $\mathbf{M}_{3,1}$ executes the corresponding rule of **RL** simultaneously in each history sector of $\mathbf{M}_2$. The partition of the set of state letters of these copies of **RL** in each history sector is $X_{i,\ell} \sqcup X_{i,r}$ for some i (that is, state letters from R_{i-1} first run right, replacing letters from $X_{i,\ell}$ by the corresponding letters of $X_{i,r}$ and then run left, replacing letters from $X_{i,r}$ by the corresponding letters of $X_{i,\ell}$).

The transition rule $\chi(1,2)$ changes the state letters to the state letters of start configurations of $\overline{\mathbf{M}}_2$. The admissible words in the domain of $\chi(1,2)^{\pm 1}$ have all Y-letters from the left alphabets $X_{i,\ell}$. The rule $\chi(1,2)$ locks all sectors except the history sectors $R_{i-1}P_i$ and the input sector. It does not apply to admissible words containing Y-letters from right alphabets.

Set $\mathbf{M}_{3,2}$. It is a copy of the set of rules of the S-machine $\overline{\mathbf{M}}_2$.

The transition rule $\chi(2,3)$ changes the state letters of the stop configuration of $\overline{\mathbf{M}}_2$ to their copies in a different alphabet. The admissible words in the domain of $\chi(2,3)^{\pm 1}$ have no Y-letters from the left alphabets $X_{i,\ell}$. The rule $\chi(2,3)$ locks all sectors except for the history sectors $R_{i-1}P_i$. It does not apply to admissible words containing Y-letters from right alphabets.

Set (of the rules of machine) $\mathbf{M}_{3,3}$. It is a copy of the set of rules of the S-machine **LR**, with *parallel work* in all history sectors, i.e., every subword $R_{i-1}P_iQ_i$ of the standard base, where $Q_{i-1}Q_i$ is a history sector of $\mathbf{M}_2$, is treated as the base of a copy of **LR**, that is, P_i contains the running state letters which run between state letters from R_{i-1} and Q_i. Each rule of set $\mathbf{M}_{3,3}$ executes the corresponding rule of **LR** simultaneously in each history sector of $\mathbf{M}_2$.

The transition rule $\chi(3,4)$ changes the state letters of the stop configuration of $\overline{\mathbf{M}}_2$ to their copies in a different alphabet. The admissible words in the domain of $\chi(3,4)^{\pm 1}$ have no Y-letters from the left alphabets $X_{i,l}$. The rule $\chi(3,4)$ locks all non-history sectors.

Set $\mathbf{M}_{3,4}$. The positive rules of set $\mathbf{M}_{3,4}$ are the copies of the negative rules of the S-machine $\overline{\mathbf{M}}_2$.

The transition rule $\chi(4,5)$ changes the state letters of the start configuration of $\overline{\mathbf{M}}_2$ to their copies in a different alphabet. The admissible words in the domain of $\chi(4,5)^{\pm 1}$ have no Y-letters from the right alphabets $X_{i,r}$. The rule $\chi(4,5)$ locks all non-history and non-input sectors.

Sets $\mathbf{M}_{3,5}, \dots, \mathbf{M}_{3,8}$. They consist of rules that are copies of the rules of the sets $\mathbf{M}_{3,1}, \dots,$ $\mathbf{M}_{3,4}$, respectively.

Sets $\mathbf{M}_{3,4m-3}, \dots, \mathbf{M}_{3,4m}$. They consist of copies of the sets $\mathbf{M}_{3,1}, \dots, \mathbf{M}_{3,4}$, respectively.

Set $\mathbf{M}_{3,4m+1}$. It is a copy of set $\mathbf{M}_{3,1}$. The end configuration for set $\mathbf{M}_{3,4m+1}$, $A_3(H)$, is obtained from a copy of $A_2(H)$ by inserting the control letters according to (2.6).

The transition rules $\chi(i, i+1)$ are called χ-rules.

Lemma 2.10 ([27, Lemma 3.15]). *Let $\mathcal{C}: W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{M}_3$ with the standard base. Then for every i, there is at most one occurrence of the rules $\chi(i, i+1)^{\pm 1}$ in the history H of $\mathcal{C}$.* ∎

Lemma 2.11 ([27, Lemma 3.14 (b)]). *Let $\mathcal{C}: W_0 \to \cdots \to W_t$ be a reduced computation of $\mathbf{M}_3$ consisting of rules of one of the copies of* **LR** *or* **RL** *with standard base. Then $t \le \|W_0\| + \|W_t\| - 2$.* ∎

2.6. $\mathbf{M}_4$ and $\mathbf{M}_5$

Let B_3 be the standard base of $\mathbf{M}_3$ and B_3' be its disjoint copy. By $\mathbf{M}_4$ we denote the S-machine with standard base $B_3(B_3')^{-1}$ and rules $\theta(\mathbf{M}_4) = [\theta, \theta]$, where $\theta \in \Theta$ and Θ is the set of rules of $\mathbf{M}_3$. So the rules of $\Theta(\mathbf{M}_4)$ are the same for the $\mathbf{M}_3$-part of $\mathbf{M}_4$ and for the mirror copy of $\mathbf{M}_3$. Therefore we will denote $\Theta(\mathbf{M}_4)$ by Θ as well, although $\mathbf{M}_4$ has two mirror input sectors. The sector between the last state letter of B_3 and the first state letter of $(B_3')^{-1}$ is locked by any rule from Θ. (The 'mirror' symmetry of the base is used in [27] for the upper estimate of the Dehn function.)

The S-machine $\mathbf{M}_5$ is a *circular* analog of $\mathbf{M}_4$ defined as follows. We add one more base letter $\tilde{t}$ to the hardware of $\mathbf{M}_4$. So the standard base B of the ordinary version of $\mathbf{M}_5$ is $\{\tilde{t}\}B_3(B_3')^{-1}\{\tilde{t}\}$, where the part $\{\tilde{t}\}$ has only one letter $\tilde{t}$; but the first part $\{\tilde{t}\}$ is identified with the last part in the circular machine $\mathbf{M}_5$. It follows that the base of an admissible word can be arbitrary long for a circular machine. For example, $\{\tilde{t}\}B_3(B_3')^{-1}\{\tilde{t}\}B_3(B_3')^{-1}$ can be a base of an admissible word for $\mathbf{M}_5$. The work of $\mathbf{M}_5$ is well-defined since the sectors involving $\tilde{t}^{\pm 1}$ are locked by every rule from Θ. For $\mathbf{M}_5$, we have the start and stop words $I_5(\alpha^k, H)$ and $A_5(H)$ similar to the configurations $I_3(\alpha^k, H)$ and $A_3(H)$.

Since the machines $\mathbf{M}_4$ and $\mathbf{M}_5$ have the sets of rules $\mathbf{\Theta}$, as $\mathbf{M}_3$, they are built from machines $\mathbf{M}_{4,1}$ to $\mathbf{M}_{4,4m+1}$ and $\mathbf{M}_{5,1}$ to $\mathbf{M}_{5,4m+1}$, respectively.

2.7. The main machine M

We use the S-machine $\mathbf{M}_5$ from Section 2.6, $\mathbf{LR}_m$ from Section 2.2 and three more easy S-machines to compose the main circular S-machine $\mathbf{M}$ needed for this paper. The standard base of $\mathbf{M}$ is the same as the standard base of $\mathbf{M}_5$, i.e., $\{\tilde{t}\}B_3(B_3')^{-1}$, where B_3 has the form (2.6). We will use $\widetilde{Q}_0$ instead of Q_0, $\widetilde{R}_1$ instead of R_1 and so on to denote parts of the set of state letters since $\mathbf{M}$ has more state letters in every part of its hardware.

The rules of $\mathbf{M}$ will be partitioned into five sets (S-machines) $\mathbf{\Theta}_i$ ($i = 1, \dots, 5$) with transition rules $\theta(i, i+1)$ connecting the i-th and the $(i+1)$-st set. The state letters are also disjoint for different sets $\mathbf{\Theta}_i$. It will be clear that $\widetilde{Q}_0$ is the disjoint union of five disjoint sets including Q_0, $\widetilde{R}_1$ is the disjoint union of five disjoint sets including R_1, etc.

By default, every transition rule $\theta(i, i+1)$ of $\mathbf{M}$ locks a sector if this sector is locked by all rules from $\mathbf{\Theta}_i$ or if it is locked by all rules from $\mathbf{\Theta}_{i+1}$. It also changes the end state letters of $\mathbf{\Theta}_i$ to the start state letters of $\mathbf{\Theta}_{i+1}$, that is, the j-th part of the rule $\theta(i, i+1)$ has the form $q_j \to q_j'$ (or $q_j \stackrel{\ell}{\to} q_j'$ if the j-th sector is locked by this rule), where q_j is the state letter of the end rule of $\mathbf{\Theta}_i$, and q_j' is the state letter of the start rule of $\mathbf{\Theta}_{i+1}$. In particular, this means that the set of start state letters of $\mathbf{\Theta}_{i+1}$ is a copy of the set of end state letters of $\mathbf{\Theta}_i$ in a disjoint alphabet.

To start working, let us introduce auxiliary start state letters for $\mathbf{M}$, namely, one letter for every base letter from B_3 and B_3'. The *start configuration* W_{st} of $\mathbf{\Theta}$ is $\tilde{t}b_3(b_3')^{-1}$, where b_3 and b_3' consist of these new start state letters, i.e., the configuration W_{st} just copies the standard base $\tilde{t}B_3B_3^{-1}$ of $\mathbf{M}$. The start rule θ_1 of $\mathbf{M}$ changes the state letters from b_3 and b_3' to their copies in the single rule of $\mathbf{\Theta}_1$ defined below, and starts $\mathbf{\Theta}_1$-computations.

Set $\mathbf{\Theta}_1$. It inserts input words in the input sectors. The set contains only one positive rule inserting the letter α in the input sector next to the left of a letter p from $\widetilde{P}_1$. It also inserts a copy α^{-1} next to the right of the corresponding letter $(p')^{-1}$ (the similar mirror symmetry is assumed in the definition of all other rules.) So the positive rule of $\mathbf{\Theta}_1$ has the form

$$[t \stackrel{\ell}{\to} t, q_0 \stackrel{\ell}{\to} q_0, r_1 \to r_1, p_1 \stackrel{\ell}{\to} \alpha p_1, \dots, (p_1')^{-1} \to (p_1')^{-1}\alpha^{-1}, (r_1')^{-1} \stackrel{\ell}{\to} (r_1')^{-1}].$$

The rules of $\mathbf{\Theta}_1$ do not change state letters, so it has one state letter in each part of its hardware.

The connecting rule $\theta(12)$ changes the state letters of $\mathbf{\Theta}_1$ to their copies in a disjoint alphabet. It locks all sectors except for the input sector $\widetilde{R}_0\widetilde{P}_1$ and the mirror copy of this sector.

Set $\mathbf{\Theta}_2$. It is a copy of the S-machine $\mathbf{LR}_m$ working in the input sector and its mirror image in parallel, i.e., we identify the standard base of $\mathbf{LR}_m$ with $\widetilde{R}_0\widetilde{P}_1\widetilde{Q}_1$. The connecting rule $\theta(23)$ locks all sectors except for the input sector $\widetilde{R}_0\widetilde{P}_1$ and its mirror image.

Set $\mathbf{\Theta}_3$. It inserts history in the history sectors. This set of rules is a copy of each of the left alphabets $X_{i,l}$ of the S-machine $\mathbf{M}_2$. Every positive rule of $\mathbf{\Theta}_3$ inserts a copy of the corresponding positive letter in every history sector $\widetilde{R}_i\widetilde{P}_{i+1}$ next to the right of a state letter from $\widetilde{R}_i$.

Again, $\mathbf{\Theta}_3$ does not change the state letters, so each part of its hardware contains one letter.

The transition rule $\theta(34)$ changes the state letters to their copies in the set of rules of machine $\mathbf{M}_{5,1}$ defined at the end of Section 2.6. It locks all sectors except for the input sectors and the history sectors. The history sectors in admissible words from the domain of $\theta(34)$ have Y-letters from the left alphabets $X_{i,l}$ of the S-machine $\mathbf{M}_5$.

Set $\mathbf{\Theta}_4$. It is a copy of the S-machine $\mathbf{M}_5$. The transition rule $\theta(45)$ locks all sectors except for history ones. The admissible words in the domain of $\theta(45)$ have no letters from right alphabets.

Set $\mathbf{\Theta}_5$. The positive rules from $\mathbf{\Theta}_5$ simultaneously erase the letters of the history sectors from the right of the state letter from $\widetilde{R}_i$. That is, parts of the rules are of the form $r \to ra^{-1}$ where r is a state letter from $\widetilde{R}_i$, and a is a letter from the left alphabet of the history sector.

Finally, the accept rule θ_0 (regarded as a transition rule) from $\mathbf{M}$ can be applied when all the sectors are empty, so it locks all the sectors and changes the end state letters of $\mathbf{M}_5$ to the corresponding end state letters of $\mathbf{M}$. Thus, the main S-machine $\mathbf{M}$ has unique accept (or stop) configuration which we will denote by W_{ac}.

Lemma 2.12 ([27, Lemma 4.4]). *Let the history of a reduced computation* $\mathcal{C}\colon W_0 \to \cdots \to W_t$ *have a subword* $\chi(i-1,i)H'\chi(i,i+1)$ *(i.e., the* S*-machine* $\mathbf{M}$ *works as* $\mathbf{M}_3$ *with rules from* $\mathbf{\Theta}_4$*) or a subword* $\zeta^{(i-1,i)}H'\zeta^{(i,i+1)}$ *(i.e., it works as* $\mathbf{LR}_m$ *with rules from* $\mathbf{\Theta}_2$*). Then the base of the computation* $\mathcal{C}$ *is a reduced word and all configurations of* $\mathcal{C}$ *are uniquely defined by the history* H *and the base of* $\mathcal{C}$. ■

We say that the history H of a computation of $\mathbf{M}$ (and the computation itself) is *eligible* if it has no neighboring mutually inverse letters except possibly for the subwords $\theta(23)\theta(23)^{-1}$. (The subword $\theta(23)^{-1}\theta(23)$ is not allowed.) Considering eligible computations instead of just reduced computations is necessary for our interpretation of $\mathbf{M}$ in a group.

The history H of an eligible computation of $\mathbf{M}$ can be factorized so that every factor is either a transition rule $\theta(i,i+1)^{\pm 1}$ or a maximal non-empty product of rules of one of the sets $\mathbf{\Theta}_1$ to $\mathbf{\Theta}_5$. If, for example, $H = H'H''H'''$, where H' is a product of rules from $\mathbf{\Theta}_2$, H'' has only one rule $\theta(23)$ and H''' is a product of rules from $\mathbf{\Theta}_3$, then we say that the *step history* of the computation is $(2)(23)(3)$.

Thus the step history of a computation is a word in the alphabet

$$\{(1),(2),(3),(4),(5),(12),(23),(34),(45),(21),(32),(43),(54)\},$$

where (21) is used for the rule $\theta(12)^{-1}$ an so on. For brevity, we can omit some transition symbols, e.g. we may use (2)(3) instead of (2)(23)(3) since the only rule connecting steps 2 and 3 is $\theta(23)$.

Lemma 2.13 ([27, Lemma 4.2 (1)]). *There are no reduced computations $\mathcal{C}$ of* $\mathbf{M}$ *with standard base and step history* (34)(4)(43) *or* (54)(4)(45). ■

If the step history of a computation consists of only one letter (i), $i = 1, \dots, 5$, then we call it a *one-step computation*. The computations with step histories $(i)(i, i \pm 1)$, $(i \pm 1, i)(i)$ and $(i \pm 1, i)(i)(i, i \pm 1)$ are also considered as one-step computations. Any eligible one-step computation is always reduced by definition.

By definition, the rule $\theta(23)$ locks all history sectors of the standard base of $\mathbf{M}$ except for the input sector $\tilde{R}_0 \tilde{P}_1$ and its mirror copy. Hence every admissible word in the domain of $\theta(23)^{-1}$ has the form $W(k, k') \equiv w_1 \alpha^k w_2 (\alpha')^{-k'} w_3$, where $(\alpha')^{-1}$ is the mirror copy of α, k and k' are integers, and w_1, w_2, w_3 are fixed words in state letters; w_1 starts with $\tilde{t}$. Recall that W_{ac} is the accept word of $\mathbf{M}$.

Lemma 2.14 ([27, Lemma 4.6]).

(1) *If the word α^k is accepted by the Turing machine* $\mathbf{M}_0$*, then there is a reduced computation $W(k, k) \to \dots \to W_{\mathrm{ac}}$ of* $\mathbf{M}$ *whose history has no rules of* $\mathbf{\Theta}_1$ *and* $\mathbf{\Theta}_2$.

(2) *If the history of a computation $\mathcal{C}\colon W(k, k) \to \dots \to W_{\mathrm{ac}}$ of* $\mathbf{M}$ *has no rules of* $\mathbf{\Theta}_1$ *and* $\mathbf{\Theta}_2$*, then the word α^k is accepted by* $\mathbf{M}_0$. ■

A configuration W of $\mathbf{M}$ is called *accessible* if there is a W*-accessible computation*, i.e., either an accepting computation starting with W or a computation $W_{\mathrm{st}} \to \dots \to W$, where W_{st} is the start configuration of $\mathbf{M}$ (i.e., the configuration where all state letters are start state letters and the Y-projection is empty).

The base of a computation is called *revolving* if it starts and ends with the same letter and has no proper subwords with this property. If this base xvx is a reduced word, then it follows from the definition of admissible words that the cyclic order of letters in the word xv is the same as in the standard base, i.e., xv is a cyclic permutation of the standard base.

Lemma 2.15 ([27, Lemmas 4.8 and 4.12]). *Suppose the base xvx of an eligible computation $\mathcal{C}\colon W_0 \to \dots \to W_t$ is revolving. Then one of the following statements holds:*

(1) $\|W_j\| \le c_4 \max(\|W_0\|, \|W_t\|)$ *for every $j = 0, \dots, t$, or*

(2) *the base xvx is reduced and if xv is the standard base, then the words W_0 and W_t without the last x-letters are accessible words; the step history of $\mathcal{C}$ contains a subword* (34)(4)(45) *or a subword* (12)(2)(23). ■

Remark 2.16. By [27, Lemma 3.15], a computation with standard base and step history (34)(4)(45) has a subword $\chi(i-1, i) H' \chi(i, i+1)$, as in Lemma 2.12. Analogously, by [27, Remark 3.7], a computation with standard base and step history (12)(2)(23) has a subword $\zeta^{(i-1,i)} H' \zeta^{(i,i+1)}$.

Lemma 2.17. *Suppose $\mathcal{C}\colon W_0 \to \cdots \to W_t$ is an eligible computation, with a base xvx. Then either $(xv)^{\pm 1}$ is a power of a cyclic permutation of the standard base or*

$$|W_j|_Y \le c_4 \max(|W_0|_Y, |W_t|_Y) \quad \textit{for every } j = 0, \dots, t. \tag{2.8}$$

Proof. Note that the base of $\mathcal{C}$ has a revolving subword $yv'y$. Let $\mathcal{D}\colon V_0 \to \cdots \to V_t$ be the computation $\mathcal{C}$ restricted to this subbase. It has the same history H as $\mathcal{C}$. By Lemma 2.15, either the base of $\mathcal{D}$ is a reduced word and so yv' is a cyclic permutation of the standard base; or $|V_j|_Y \le c_4 \max(|V_0|_Y, |V_t|_Y)$ for every $j = 0, \dots, t$.

In the latter case, let us remove the subwords with the base yv', obtaining a computation $\mathcal{E} : U_0 \to \cdots \to U_t$ with a shorter base. Arguing by induction, we have either $|U_j|_Y \le c_4 \max(|U_0|_Y, |U_t|_Y)$ for every $j = 0, \dots, t$, which implies (2.8), or the base of $\mathcal{E}$ is a power of a cyclic permutation of the standard base and by Lemma 2.15, the step history of $\mathcal{C}$ contains a subword (34)(4)(45) or a subword (12)(2)(23). Then by Remark 2.16, one can apply Lemma 2.12, and since the computation $\mathcal{D}$ has the same history as $\mathcal{E}$, the base $yv'y$ must be reduced. Therefore yv' is a cyclic permutation of the standard base, and so xv is a power of a cyclic permutation of the standard base.

If $|U_j|_Y \le c_4 \max(|U_0|_Y, |U_t|_Y)$ for every j, but yv' is a cyclic permutation of the standard base, then the dual argument implies that the base of $\mathcal{E}$ and the base of $\mathcal{C}$ are reduced words. Hence xv is a power of a cyclic permutation of the standard base. ■

3. Group and diagram preliminaries

3.1. The groups

Every S-machine can be simulated by a finitely presented group (see, e.g., [24, 26, 30]). Here we present the construction from [27]. To simplify formulas, it is convenient to change the notation. From now on we shall denote by N the length of the standard base of $\mathbf{M}$.

Thus the set of state letters is $Q = \bigsqcup_{i=0}^{N-1} Q_i$ (we set $Q_N = Q_0 = \{\tilde{t}\}$), $Y = \bigsqcup_{i=1}^{N} Y_i$, and Θ is the set of rules of the S-machine $\mathbf{M}$.

The finite set of generators of the group M consists of *q-letters*, *Y-letters* and *θ-letters* defined as follows.

For every letter $q \in Q$ the set of generators of M contains L copies $q^{(i)}$ of it, $i = 1, \dots, L$, if the letter q occurs in the rules of $\mathbf{\Theta}_1$ or $\mathbf{\Theta}_2$. (The number L is one of the parameters from (2.5).) Otherwise only the letter q is included in the generating set of M.

For every letter $a \in Y$ the set of generators of M contains a and L copies $a^{(i)}$ of it, $i = 1, \dots, L$.

For every $\theta \in \Theta^+$ we have N generators $\theta_0, \dots, \theta_N$ in M (here $\theta_N \equiv \theta_0$) if θ is a rule of Θ_3 (excluding $\theta(23)$) or Θ_4, or Θ_5. For θ from Θ_1 or Θ_2 (including $\theta(23)$), we introduce LN generators $\theta_j^{(i)}$, where $j = 0, \dots, N$, $i = 1, \dots, L$ and $\theta_N^{(i)} = \theta_0^{(i+1)}$ (the superscripts are taken modulo L).

The relations of the group M correspond to the rules of the S-machine $\mathbf{M}$ as follows. For every rule $\theta = [U_0 \to V_0, \dots, U_N \to V_N] \in \Theta^+$ of sets Θ_1 or Θ_2, we have

$$U_j^{(i)} \theta_{j+1}^{(i)} = \theta_j^{(i)} V_j^{(i)}, \quad \theta_j^{(i)} a^{(i)} = a^{(i)} \theta_j^{(i)}, \quad j = 0, \dots, N,\ i = 1, \dots, L \tag{3.1}$$

for all $a \in Y_j(\theta)$, where $U_j^{(i)}$ and $V_j^{(i)}$ are obtained from U_j and V_j by adding the superscript (i) to every letter.

For $\theta = \theta(23)$, we introduce relations

$$U_j^{(i)} \theta_{j+1}^{(i)} = \theta_j^{(i)} V_j, \quad a^{(i)} \theta_j^{(i)} = \theta_j^{(i)} a, \quad j = 0, \dots, N,\ i = 1, \dots, L \tag{3.2}$$

for all $a \in Y_j(\theta)$, i.e., the superscripts are erased in the words $V_j^{(i)}$ and in the Y-letters after an application of (3.2).

For every rule $\theta = [U_0 \to V_0, \dots, U_N \to V_N] \in \Theta^+$ from Θ_3 or Θ_4, or Θ_5 and $a \in Y_j(\theta)$, we define

$$U_j \theta_{j+1} = \theta_j V_j, \quad a\theta_j = \theta_j a. \tag{3.3}$$

The first type of relations (3.1)–(3.3) will be called (θ, q)-*relations*, the second type (θ, a)-*relations*.

Finally, the required group G is given by the generators and relations of the group M and by two more additional relations, namely the *hub*-relations

$$W_{\text{st}}^{(1)} \dots W_{\text{st}}^{(L)} = 1 \quad \text{and} \quad (W_{\text{ac}})^L = 1, \tag{3.4}$$

where the word $W_{\text{st}}^{(i)}$ is a copy with superscript (i) of the start word W_{st} (of length N) of the S-machine $\mathbf{M}$ and W_{ac} is the accept word of $\mathbf{M}$.

Note that, as usual, M is a multiple HNN extension of the free group generated by all Y- and q-letters, because by Tietze transformations using (θ, q)-relations, all θ-letters, except for one for every rule θ, can be eliminated.

3.2. Van Kampen diagrams

Recall that a van Kampen *diagram* Δ over a presentation $P = \langle A \mid \mathcal{R} \rangle$ (or just over the group P) is a finite oriented connected and simply-connected planar 2-complex endowed with a *labeling function* Lab: $E(\Delta) \to A^{\pm 1}$, where $E(\Delta)$ denotes the set of oriented edges of Δ, such that $\mathrm{Lab}(e^{-1}) \equiv \mathrm{Lab}(e)^{-1}$. Given a *cell* (i.e., a 2-cell) Π of Δ, we denote by $\partial\Pi$ the boundary of Π; similarly, $\partial\Delta$ denotes the boundary of Δ. The labels of $\partial\Pi$ and $\partial\Delta$ are defined up to cyclic permutations. An additional requirement is that the label of any cell Π of Δ is equal to (a cyclic permutation of) a word $R^{\pm 1}$, where $R \in \mathcal{R}$. The label and the combinatorial length $\|\mathbf{p}\|$ of a path $\mathbf{p}$ are defined as for Cayley graphs.

The van Kampen lemma [17, 20, 29] states that a word W over the alphabet $A^{\pm 1}$ represents the identity in the group P if and only if there exists a diagram Δ over P such that $\mathrm{Lab}(\partial\Delta) \equiv W$, in particular, the combinatorial perimeter $\|\partial\Delta\|$ of Δ equals $\|W\|$ (see [17, Chapter 5, Theorem 1.1]; our formulation is closer to [20, Lemma 11.1], see also

[29, Section 5.1]). A word W representing 1 in P is freely equal to a product of conjugates of the words from $R^{\pm 1}$. The minimal number of factors in such products is called the *area* of the word W. The *area* of a diagram Δ is the number of cells in it. The proof of the van Kampen lemma [20,29] shows that $\mathrm{Area}(W)$ is equal to the area of a van Kampen diagram having the smallest number of cells among all van Kampen diagrams with boundary label $\mathrm{Lab}(\partial\Delta) \equiv W$.

The definition of *annular diagram* Δ over a group G is similar to the definition of van Kampen diagram, but the complement of Δ in the plane has two connected components. So Δ has two boundary components. By the van Kampen–Schupp lemma (see [17, Lemma 5.2] or [20, Lemma 11.2]) there is an annular diagram Δ whose boundary components $\mathbf{p}_1$ and $\mathbf{p}_2$ have clockwise labels W and W' if and only if the words W and W' are conjugate in G.

We will study diagrams over the group presentations of M and G. The edges labeled by state letters (= q-*letters*) will be called q-*edges*, the edges labeled by tape letters (= Y-*letters*) will be called Y-*edges*, and the edges labeled by θ-letters are θ-*edges*.

We denote by $|\mathbf{p}|_Y$ (resp. $|\mathbf{p}|_\theta$, $|\mathbf{p}|_q$) the Y-*length* (resp. the θ-*length*, the q-length) of a path $\mathbf{p}$, i.e., the number of Y-edges (resp. θ-edges, q-edges) in $\mathbf{p}$.

The cells corresponding to relations (3.4) are called *hubs*, the cells corresponding to (θ, q)-relations are called (θ, q)-*cells* (in particular, there are $(\theta, \tilde{t})$-*cells*), and the cells are called (θ, a)-*cells* if they correspond to (θ, a)-relations. A θ-*cell* is either a (θ, q)- or (θ, a)-cell.

A diagram is *reduced*, if it does not contain two cells (= closed 2-cells) that have a common edge $\mathbf{e}$ such that the boundary labels of these two cells are equal if one reads them starting with $\mathbf{e}$ (if such pairs of cells exist, they can be removed to obtain a diagram of smaller area and with the same boundary label(s)).

To study diagrams over the group G we shall use their simpler subdiagrams such as bands. Here we repeat one more necessary definition from [27].

Definition 3.1. Let $\mathcal{Z}$ be a subset of the set of letters in the set of generators of the group M. A $\mathcal{Z}$-band $\mathcal{B}$ is a sequence of cells $\pi_1, \dots, \pi_n$ in a reduced van Kampen diagram Δ (see Figure 1) such that:

- Every two consecutive cells π_i and π_{i+1} in this sequence have a common boundary edge $\mathbf{e}_i$ labeled by a letter from $\mathcal{Z}^{\pm 1}$.
- Each cell π_i, $i = 1, \dots, n$ has exactly two $\mathcal{Z}$-edges $\mathbf{e}_{i-1}^{-1}$ and $\mathbf{e}_i$ in the boundary $\partial\pi_i$ (i.e., edges labeled by a letter from $\mathcal{Z}^{\pm 1}$) with the requirement that either both $\mathrm{Lab}(e_{i-1})$ and $\mathrm{Lab}(e_i)$ are positive letters or both are negative ones.
- If $n = 0$, then $\mathcal{B}$ is just a $\mathcal{Z}$-edge.

The counter-clockwise boundary of the subdiagram formed by the cells $\pi_1, \dots, \pi_n$ of $\mathcal{B}$ has the factorization $\mathbf{e}^{-1}\mathbf{q}_1\mathbf{f}\mathbf{q}_2^{-1}$ where $\mathbf{e} = \mathbf{e}_0$ is a $\mathcal{Z}$-edge of π_1 and $\mathbf{f} = \mathbf{e}_n$ is a $\mathcal{Z}$-edge of π_n. We call $\mathbf{q}_1$ the *bottom* of $\mathcal{B}$ and $\mathbf{q}_2$ the *top* of $\mathcal{B}$, denoted $\mathbf{bot}(\mathcal{B})$ and $\mathbf{top}(\mathcal{B})$. If the path $\mathbf{e}^{-1}\mathbf{q}_1\mathbf{f}$ or the path $\mathbf{f}\mathbf{q}_2^{-1}\mathbf{e}^{-1}$ is a subpath of the boundary $\partial\Delta$, then $\mathcal{B}$ is called a *rim* band. Top/bottom paths and their inverses are also called the *sides* of the band. The $\mathcal{Z}$-edges $\mathbf{e}$ and $\mathbf{f}$ are called the *start* and *end* edges of the band. If $n \geq 1$ but $\mathbf{e} = \mathbf{f}$, then the $\mathcal{Z}$-band is called a $\mathcal{Z}$-*annulus*.

We consider q-*bands*, where for some j, $\mathcal{Z}$ corresponds to a part Q_j of state letters of the S-machine $\mathbf{M}$, i.e., it contains all letters q_j and $q_j^{(i)}$ ($i = 1, \dots, L$), where $q_j \in Q_j$, θ-*bands* for every $\theta \in \Theta$, and Y-*bands*, where $\mathcal{Z} = \{a, a^{(1)}, \dots, a^{(L)}\} \subseteq Y$. The convention is that Y-bands do not contain (θ, q)-cells, and so they consist of (θ, a)-cells only.

Remark 3.2. To construct the top (or bottom) path of a band $\mathcal{B}$, at the beginning one can just form a product $\mathbf{x}_1 \dots \mathbf{x}_n$ of the top paths $\mathbf{x}_i$ of the cells $\pi_1, \dots, \pi_n$ (where each π_i is a $\mathcal{Z}$-band of length 1). No θ-letter is being canceled in the word $W \equiv \mathrm{Lab}(\mathbf{x}_1) \dots \mathrm{Lab}(\mathbf{x}_n)$ if $\mathcal{B}$ is a q- or Y-band since otherwise two neighbor cells of the band would make the diagram non-reduced. The *trimmed* top/bottom label of a θ-band $\mathcal{B}$ are the maximal subwords of the top/bottom labels starting and ending with q-letters.

However a few cancellations of Y-letters are possible in W. (This can happen if one of π_i, π_{i+1} is a (θ, q)-cell and another one is a (θ, a)-cell.) We will always assume that the top/bottom label of a θ-band is a reduced form of the word W. This property is easy to achieve: by folding edges with the same labels having the same initial vertex, one can make the boundary label of a subdiagram in a van Kampen diagram reduced (see, e.g., [20, 30]).

Remark 3.3. Since $\theta_N^{(i)} = \theta_0^{(i+1)}$, we can replace $\theta_N^{(i)}$ with $\theta_0^{(i+1)}$ in (3.1) and (3.2). Thus, the superscripts in the q-letters of the same (θ, q)-relation are different if $\theta \in \mathbf{\Theta}_1 \cup \mathbf{\Theta}_2 \cup \{\theta(23)^{\pm 1}\}$ and this relation is a $(\theta, \tilde{t})$-relation. Therefore only the corresponding cells of a θ-band have different superscripts of the labels of θ-edges, and this difference is ± 1 modulo L.

We shall call a $\mathcal{Z}$-band *maximal* if it is not contained in any other $\mathcal{Z}$-band. Counting the number of maximal $\mathcal{Z}$-bands in a diagram, we will not distinguish the bands with boundaries $\mathbf{e}^{-1}\mathbf{q}_1\mathbf{f}\mathbf{q}_2^{-1}$ and $\mathbf{f}\mathbf{q}_2^{-1}\mathbf{e}^{-1}\mathbf{q}_1$, and so every $\mathcal{Z}$-edge belongs to a unique maximal $\mathcal{Z}$-band.

A $\mathcal{Z}_1$-band and a $\mathcal{Z}_2$-band *cross* if they have a common cell and $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset$.

Sometimes we specify the types of bands as follows. A q-band corresponding to one letter Q of the base is called a Q-band. For example, we will consider $\tilde{t}$-*bands* corresponding to the part $\{\tilde{t}\}$.

Lemma 3.4 ([27, Lemma 5.6]). *A reduced van Kampen diagram Δ over M has no q-annuli, no θ-annuli, and no Y-annuli. Every θ-band of Δ shares at most one cell with any q-band and with any Y-band.* ∎

If $W \equiv x_1 \dots x_n$ is a word in an alphabet X, X' is another alphabet, and $\phi\colon X \to X' \cup \{1\}$ (where 1 is the empty word) is a map, then $\phi(W) \equiv \phi(x_1) \dots \phi(x_n)$ is called a *projection* of W onto X'. We shall consider the projections of words in the generators of M onto Θ (all θ-letters map to the corresponding element of Θ, all other letters map to 1), and the projection onto the alphabet $\{Q_0 \sqcup \cdots \sqcup Q_{N-1}\}$ (every q-letter maps to the corresponding Q_i, all other letters map to 1).

Definition 3.5. The projection of the label of a side of a q-band onto the alphabet Θ is called the *history* of the band. The step history of this projection is the *step history* of the q-band. The projection of the label of a side of a θ-band onto the alphabet $\{Q_0, \ldots, Q_{N-1}\}$ is called the *base* of the band, i.e., the base of a θ-band is equal to the base of the label of its top or bottom.

As in the case of words, we will use representatives of the Q_j-s in base words.

If W is a word in the generators of M, then we denote by $W^{\emptyset}$ the projection of this word onto the alphabet of the S-machine $\mathbf{M}$; we obtain this projection after deleting all superscripts in the letters of W. In particular, $W^{\emptyset} \equiv W$, if there are no superscripts in the letters of W.

We call a word W in q-generators and Y-generators *permissible* if the word $W^{\emptyset}$ is admissible, and the letters of any 2-letter subword of W have equal superscripts (if any), except for the subwords $(q\tilde{t})^{\pm 1}$, where the letter q has some superscript (i) and $q^{\emptyset} \in Q_{N-1}$; in this case the superscript of the letter $\tilde{t}$ must be $(i+1)$ (modulo L).

Remark 3.6. It follows from the definition that if V is θ-admissible for a rule θ of $\{\theta(23)^{-1}\} \cup \mathbf{\Theta}_3 \cup \{\theta(34)\} \cup \mathbf{\Theta}_4 \cup \{\theta(45)\} \cup \mathbf{\Theta}_5$, then there is exactly one permissible word W such that $W^{\emptyset} \equiv V$, namely, $W \equiv V$. If θ is a rule of $\mathbf{\Theta}_1 \cup \{\theta(12)\} \cup \mathbf{\Theta}_2 \cup \{\theta(23)\}$, then a permissible word W with property $W^{\emptyset} \equiv V$ exists and it is uniquely defined if one chooses an arbitrary superscript for the first letter (or for any particular letter) of W.

Lemma 3.7 ([27, Lemma 5.9]). (1) *The trimmed bottom and top labels W_1 and W_2 of any reduced θ-band $\mathcal{T}$ containing at least one (θ, q)-cell are permissible and $W_2^{\emptyset} \equiv W_1^{\emptyset} \cdot \theta$.*

(2) *If W is a θ-admissible word, then for a permissible word W_1 such that $W_1^{\emptyset} \equiv W$ (given by Remark 3.6) one can construct a reduced θ-band with the trimmed bottom label W_1 and the trimmed top label W_2, where $W_2^{\emptyset} \equiv W_1^{\emptyset} \cdot \theta$.* ■

Definition 3.8. Let Δ be a reduced van Kampen diagram over M having a boundary path of the form $\mathbf{p}_1^{-1}\mathbf{q}_1\mathbf{p}_2\mathbf{q}_2^{-1}$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are sides of q-bands, and $\mathbf{q}_1$, $\mathbf{q}_2$ are maximal parts of the sides of θ-bands such that $\mathrm{Lab}(\mathbf{q}_1)$, $\mathrm{Lab}(\mathbf{q}_2)$ start and end with q-letters.

Then Δ is called a *trapezium* (see Figure 1). The path $\mathbf{q}_1$ is called the *bottom*, the path $\mathbf{q}_2$ is called the *top* of the trapezium, the paths $\mathbf{p}_1$ and $\mathbf{p}_2$ are called the *left and right sides* of the trapezium. The history (resp. step history) of the q-band whose side is $\mathbf{p}_2$ is called the *history* (resp. step history) of the trapezium; the length of the history is called the *height* of the trapezium. The base of $\mathrm{Lab}(\mathbf{q}_1)$ is called the *base* of the trapezium.

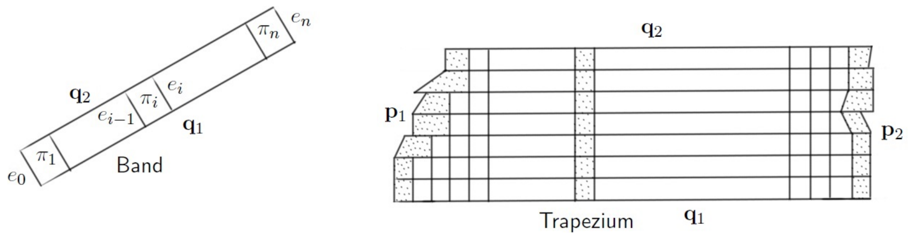

Figure 1. Band and trapezium.

Remark 3.9. Notice that the top (resp. bottom) side of a θ-band $\mathcal{T}$ does not necessarily coincide with the top (resp. bottom) side $\mathbf{q}_2$ (resp. $\mathbf{q}_1$) of the corresponding trapezium of height 1, and $\mathbf{q}_2$ (resp. $\mathbf{q}_1$) is obtained from $\mathbf{top}(\mathcal{T})$ (resp. $\mathbf{bot}(\mathcal{T})$) by trimming the first and the last Y-edges if these paths start and/or end with Y-edges.

By Lemma 3.4, any trapezium Δ of height $h \geq 1$ can be decomposed into θ-bands $\mathcal{T}_1, \dots, \mathcal{T}_h$ connecting the left and the right sides of the trapezium.

Lemma 3.10 ([27, Lemma 5.12]). (1) *Let Δ be a trapezium with history $H \equiv \theta(1)\dots\theta(d)$, $d \geq 1$. Assume that Δ has consecutive maximal θ-bands $\mathcal{T}_1, \dots \mathcal{T}_d$, and the words U_j and V_j are the trimmed bottom and the trimmed top labels of $\mathcal{T}_j$, $j = 1, \dots, d$. Then H is an eligible word, U_j, V_j are permissible words,*

$$V_1^{\emptyset} \equiv U_1^{\emptyset} \cdot \theta(1), \quad U_2 \equiv V_1, \quad \dots, \quad U_d \equiv V_{d-1}, \quad V_d^{\emptyset} \equiv U_d^{\emptyset} \cdot \theta(d).$$

Furthermore, if the first and the last q-letters of the word U_j or of the word V_j have some superscripts (i) and (i'), then $i' - i$ (modulo L) does not depend on the choice of U_j or V_j.

(2) *For every eligible computation $U \to \dots \to U \cdot H \equiv V$ of $\mathbf{M}$ with $\|H\| = d \geq 1$ there exists a trapezium Δ with bottom label U_1 (given by Remark 3.6) such that $U_1^{\emptyset} \equiv U$, top label V_d such that $V_d^{\emptyset} \equiv V$, and with history H.* ∎

Using Lemma 3.10, one can immediately derive properties of trapezia from the properties of computations obtained earlier.

If $H' \equiv \theta(i)\dots\theta(j)$ is a subword of the history H from Lemma 3.10 (1), then the bands $\mathcal{T}_i, \dots, \mathcal{T}_j$ form a subtrapezium Δ' of the trapezium Δ with the same base. A subword of the base of Δ also defines a subtrapezium with the same history.

Definition 3.11. We say that a trapezium Δ is *standard* if the base of Δ is the standard base $\mathbf{B}$, and the history of Δ (or the inverse one) contains one of the words

- $\chi(i-1,i)H'\chi(i,i+1)$ (i.e., the S-machine works as $\mathbf{\Theta}_4$), or
- $\zeta^{i-1,i}H'\zeta^{i,i+1}$ (i.e., it works as $\mathbf{\Theta}_2$).

Definition 3.12. A permissible word V is called a *disk word* if $V^{\emptyset} \equiv W^L$ for some accessible word W. (In particular, hub words are disk words.)

Lemma 3.13 ([27, Lemma 7.2]). *Every disk word V is equal to* 1 *in the group G.* ■

Lemma 3.14 ([27, Remark 7.3]). *For a disk word V, there is a reduced* disk diagram Δ *over the presentation* (3.1)–(3.4) *with boundary label V built of one hub and L trapezia corresponding to an accessible computation for the word W, where $V^{\emptyset} = W^L$.* ■

We will increase the set of relations of G by adding the (infinite) set of *disk relations* $V = 1$, one for every disk word V. So we will consider diagrams over G with *disks*, where every disk cell (or just *disk*) is labeled by such a word V. (In particular, a hub is a disk.)

Definition 3.15. We will call a reduced van Kampen or annular diagram Δ over G *minimal* if

(1) the number of disks is minimal for all diagrams with the same boundary label(s) as Δ, and

(2) Δ has minimal number of $(\theta, \tilde{t})$-cells among the diagrams with the same boundary label(s) and with minimal number of disks.

Clearly, a subdiagram of a minimal diagram is minimal itself.

The following is explained in [27, Section 7.1.2].

Lemma 3.16. *If two disks of a van Kampen diagram Δ over G are connected by at least two $\tilde{t}$-bands, then there is a diagram Δ' with the same boundary label and fewer disks in it. In particular, two disks cannot be connected by two $\tilde{t}$-bands in a minimal van Kampen diagram or by three $\tilde{t}$-bands in a minimal annular diagram.* ■

Lemma 3.16 implies the following properties. (Part (1) is [27, Lemma 7.5], the proof of part (2) is similar.)

Lemma 3.17. (1) *If a van Kampen diagram contains at least one disk and has no pairs of disks connected by at least two $\tilde{t}$-bands, then there is a disk Π in Δ such that $L-3$ consecutive maximal $\tilde{t}$-bands $\mathcal{B}_1, \dots, \mathcal{B}_{L-3}$ start on $\partial\Pi$, end on the boundary $\partial\Delta$, and for any $i \in [1, L-4]$, there are no disks in the subdiagram Γ_i bounded by $\mathcal{B}_i$, $\mathcal{B}_{i+1}$, $\partial\Pi$, and $\partial\Delta$. See Figure 2.*

(2) *If an annular diagram contains a least one disk and has no van Kampen subdiagrams with two disks connected by at least two $\tilde{t}$-bands, then there is a disk D in Δ and two non-negative integers L', L'' with $L' + L'' \geq L - 3$, such that L' (resp. L'') consecutive maximal $\tilde{t}$-bands $\mathcal{B}_1, \dots, \mathcal{B}_{L'}$ (resp. $\mathcal{C}_1, \dots, \mathcal{C}_{L''}$) start on ∂D, end on the inner (resp. outer) boundary component $\mathbf{p}'$ (resp. $\mathbf{p}''$) of Δ, and for any $i \in [1, L'-1]$ (resp. $i \in [1, L''-1]$) there are no disks in the van Kampen subdiagram Γ_i bounded by $\mathcal{B}_i$, $\mathcal{B}_{i+1}$, $\partial\Pi$, and $\mathbf{p}'$ (resp. $\mathcal{C}_i$, $\mathcal{C}_{i+1}$, $\partial\Pi$, and $\mathbf{p}''$). See Figure 3.* ■

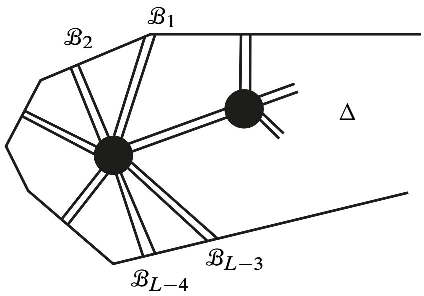

Figure 2. Lemma 3.17 (1).

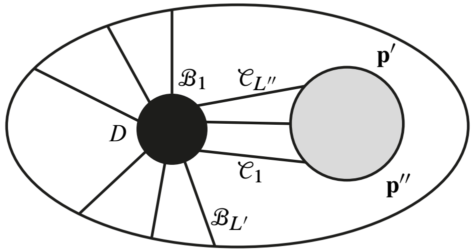

Figure 3. Lemma 3.17 (2).

A maximal q-band starting on a disk of a diagram is called a *spoke*. By induction on the number of disks, Lemma 3.17 implies the following.

Lemma 3.18 ([22, Lemma 5.19]). *If a minimal van Kampen diagram Δ has $r \geq 1$ disks, then the number of $\tilde{t}$-spokes of Δ ending on the boundary $\partial\Delta$, and therefore the number of $\tilde{t}$-edges in the boundary path of Δ, is greater than $rL/2$.* ■

Lemma 3.19 ([27, Lemma 7.7]). *Let Δ be a minimal van Kampen diagram.*

(1) *Assume that a θ-band $\mathcal{T}_0$ crosses k $\tilde{t}$-spokes $\mathcal{B}_1, \dots, \mathcal{B}_k$ starting on a disk Π, and there are no disks in the subdiagram Δ_0 bounded by $\mathcal{B}_1$, $\mathcal{B}_k$, $\mathcal{T}_0$, and Π. Then $k \leq L/2$.*

(2) *Δ contains no θ-annuli.* ■

The proof of the following lemma is given in [27, Section 7.1.3].

Lemma 3.20. (1) *Let E be a van Kampen diagram with the boundary $\mathbf{x}_1\mathbf{y}_1\mathbf{x}_2\mathbf{y}_2$ built of a disk Π with boundary $\mathbf{y}_2\mathbf{z}^{-1}$ and a rim θ-band $\mathcal{T}$ with boundary $\mathbf{x}_1\mathbf{y}_1\mathbf{x}_2\mathbf{z}$, where $\mathbf{y}_1$ and $\mathbf{z}$ are the sides of $\mathcal{T}$. Assume that the first and the last cells of $\mathcal{T}$ are different $(\theta, \tilde{t})$-cells. Then there is a diagram E' with boundary $\mathbf{x}_1'\mathbf{y}_1'\mathbf{x}_2'\mathbf{y}_2'$, built of a disk Π' with boundary $\mathbf{y}_1'(\mathbf{z}')^{-1}$ and a rim θ-band $\mathcal{T}'$, with boundary $\mathbf{x}_1'\mathbf{z}'\mathbf{x}_2'\mathbf{y}_2'$, where $\mathbf{z}'$ and $\mathbf{y}_2'$ are the sides of $\mathcal{T}'$ and* $\mathrm{Lab}(\mathbf{x}_1') \equiv \mathrm{Lab}(\mathbf{x}_1)$, $\mathrm{Lab}(\mathbf{x}_2') \equiv \mathrm{Lab}(\mathbf{x}_2)$, $\mathrm{Lab}(\mathbf{y}_1') \equiv \mathrm{Lab}(\mathbf{y}_1)$, $\mathrm{Lab}(\mathbf{y}_2') \equiv \mathrm{Lab}(\mathbf{y}_2)$. *See Figure 4.*

(2) *Let Δ be a van Kampen diagram with boundary $\mathbf{pq}$ and Δ a union of a minimal diagram Γ with $r > 0$ disks and a rim θ-band $\mathcal{T}$ with side $\mathbf{p}$. Assume that there are two $\tilde{t}$-spokes in Δ starting on a disk D and ending on $\mathbf{p}$. Then there exists a van Kampen diagram Δ' with boundary $\mathbf{p}'\mathbf{q}'$, and Δ' is a union of a minimal diagram Γ' with $r' < r$ disks and a rim θ-band $\mathcal{T}'$ having side $\mathbf{p}'$, where* $\mathrm{Lab}(\mathbf{p}') = \mathrm{Lab}(\mathbf{p})$ *in the group G, and* $\mathrm{Lab}(\mathbf{q}') \equiv \mathrm{Lab}(\mathbf{q})$. *See Figure 5.* ■

Lemma 3.21 ([27, Lemma 8.2]). *The group G has quadratic Dehn function.* ■

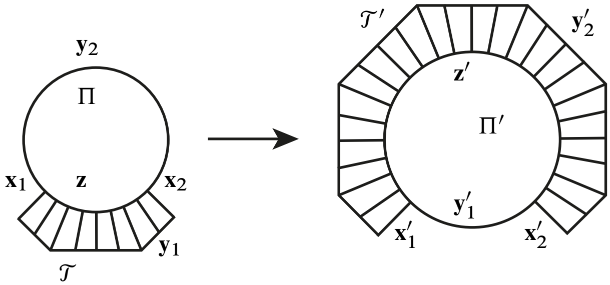

Figure 4. Lemma 3.20 (1).

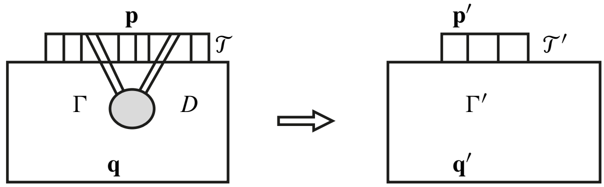

Figure 5. Lemma 3.20 (2).

4. Isomorphism problem for groups with quadratic Dehn function

In this section, we assume that the construction of the machine $\mathbf{M}$ is based on the S-machine M_1 provided by Lemma 2.3.

Lemma 4.1. *Let a reduced computation $\mathcal{C}: W \to \cdots \to W'$ of $\mathbf{M}$ with standard base have no rules of sets Θ_1 and Θ_2. If $W' \equiv W$, then the computation $\mathcal{C}$ is empty.*

Proof. Proving by contradiction, one may assume that the history H of $\mathcal{C}$ is a non-empty cyclically reduced word, because otherwise one could replace $\mathcal{C}$ with a shorter computation where the first and the last configuration are equal.

Assume first that $\mathcal{C}$ is a one-step computation. This step cannot be Θ_3 or Θ_5, since the computations with rules from these sets multiply the words in history sectors by a copy of $H^{\pm 1}$. So $\mathcal{C}$ is of type Θ_4. This assumption reformulates our problem as the same problem for the S-machine $\mathbf{M}_5$. If $\mathcal{C}$ has a χ-rule of $\mathbf{M}_5$, then one may consider the computation $W \to \cdots \to W$ with history H^2, where this rule occurs at least twice, which contradicts Lemma 2.10. Therefore there are no χ-rules in H, and so $\mathcal{C}$ is just a computation of either $\mathbf{RL}$ or $\mathbf{LR}$, or $\mathbf{M}_2$. The first and second cases contradict Lemma 2.11, because the length of powers H^s are unbounded. In the later case we restrict the computation of $\mathbf{M}_2$ to a history sector: $V \to \cdots \to V$, where a computation with any history H' multiplies the tape word from the left and from the right by copies of $H^{\pm 1}$ in disjoint alphabets. Clearly one cannot obtain a repetition, provided the word H is non-empty.

If H has at least two steps, then its step history (or a cyclic permutation of it) is a power of (3)(4)(5)(4) by Lemma 2.13. So $H^{\pm 2}$ has to contain a subword $H_1 H_2 H_3$, where H_2 has step history (43)(3)(34), H_3 and H_1^{-1} are of type (4)(45)(5)(54). Since H_2 does change history sectors but does not change the input ones, the computations with histories H_3 and H_1^{-1} start working with configurations having equal input sectors but different history sectors. Considering the subcomputations in (34)(4)(45) corresponding to the work of $\mathbf{M}_5$, $\mathbf{M}_3$, and $\mathbf{M}_2$ (as we did in the beginning of this proof), we see that the S-machine $\mathbf{M}_2$ can connect both $I_2(\alpha^k, H')$ and $I_2(\alpha^k, H'')$ with $A_2(H')$ and $A_2(H'')$, respectively, where $H' \neq H''$. It follows that there are two different reduced computations of $\mathbf{M}_1$ accepting the same input word α^k, contrary to Lemma 2.3. ■

Recall that the rule $\theta(23)$ locks all sectors of the standard base of $\mathbf{M}$ except for the input sector $\widetilde{R}_0\widetilde{P}_1$ and its mirror copy. Hence every $\theta(23)^{-1}$-admissible configuration has the form $W(k,k') \equiv w_1\alpha^k w_2(\alpha')^{-k'}w_3$, where k and k' are integers and w_1, w_2, w_3 are fixed words in state letters; w_1 starts with $\tilde{t}$.

Lemma 4.2 ([27, Lemma 8.3]). *A word $W(k,k)$ is a conjugate of the word W_{ac} in the group G (and in the group M) if and only if the subword α^k is accepted by the Turing machine $\mathbf{M}_0$.* ■

Lemma 4.3. *For arbitrary integer k, the word $W(k,k)$ has order L in G.*

Proof. Starting with the word W_{st}, a computation of set Θ_1 can insert the words α^k and $(\alpha')^{-k}$ in the input sectors. So, after the application of the connecting rule $\theta(12)$, the rules of set Θ_2 can successfully check the content of the input sectors, and the rule $\theta(23)$ gives us the word $W(k,k)$, the last configuration of this computation. Therefore the word $W(k,k)$ is accessible. Thus, the power $W(k,k)^L$ is a disk word equal to 1 in G by Lemma 3.13.

Assume that $W(k,k)^l = 1$ for a positive $l \le L/2$. Then on the one hand, the minimal diagram Δ for this equality has $l \le L/2$ $\tilde{t}$-edges in the boundary, and so it has no disks by Lemma 3.18. On the other hand, since all $\tilde{t}$-letters of the boundary label occur with the same sign, a maximal $\tilde{t}$-band of Δ cannot start and end on the boundary, and therefore the word $W(k,k)$ has no $\tilde{t}$-letters, a contradiction. ■

Lemma 4.4. *Every element of finite order in G is a conjugate of a power of some word $W(k,k)$.*

Proof. Consider a minimal diagram Δ for an equality $U^s = 1$, $s > 0$, assuming that U has minimal number of θ-letters in the conjugacy class and, under this assumption, U has minimal number of q-letters. If Δ has no disk, then it has a rim θ-band $\mathcal{T}$. The exterior side $\mathbf{y}$ of $\mathcal{T}$ cannot have length $\ge \|U\|$, since then the whole boundary of Δ has to have no θ-letters, contrary to the existence of the rim θ-band. If $\|\mathbf{y}\| = \|U\| - 1$, then the ends of $\mathcal{T}$ must have the same label, since the boundary label of Δ has period U, but this is not possible since one of these θ-letters is positive and the other one is negative for the boundary label of a band. If $\|\mathbf{y}\| \le \|U\| - 2$, then one can replace the common boundary path of $\mathcal{T}$ and $\partial\Delta$ with a path separating $\mathcal{T}$ from Δ. Hence a cyclic permutation of U can be replaced with a word, equal in G, having less θ-letters, a contradiction.

Therefore Δ contains disks. If it has θ-edges in the boundary, then there is a maximal θ-band $\mathcal{T}$ such that the van Kampen diagram bounded by $\mathcal{T}$ and a subpath $\mathbf{y}$ of $\partial\Delta$ has no θ-cells. Therefore $\mathbf{y}$ has no θ-edges, and one comes to a contradiction, as in the previous paragraph. Hence $\partial\Delta$ has no θ-edges and therefore Δ has no θ-edges by Lemma 3.19 (2).

Let us consider a disk Π provided by Lemma 3.17. Since Δ has no θ-cells, there is a common subpath $\mathbf{p}$ of $\partial\Pi$ and $\partial\Delta$ containing $L-3$ $\tilde{t}$-letters. The word U has at most $L/2$ $\tilde{t}$-letters since otherwise there is a cyclic permutation of U containing a subword of Lab($\mathbf{p}$) with $> L/2$ $\tilde{t}$-letters. So the disk relation makes U conjugate in G to a word U' with $|U'|_\theta = |U|_\theta = 0$ and with $|U'|_q < |U|_q$, a contradiction.

Since $L/2 < L-3$ and Lab($\mathbf{p}$) is a subword of a power of U, there is a $\tilde{t}$-letter occurring in Lab($\mathbf{p}$) at least twice. Therefore the disk word on $\partial\Pi$ has no letters with superscripts. Hence it is a power V^L, where the letters of V have no superscripts and a cyclic permutation of U is a power V^l, where $|l| < L$. It remains to show that the word V is a conjugate of some $W(k,k)$.

By Definition 3.12, the word V is accessible. Hence it can be connected by a computation $\mathcal{C}$ either with W_{st} or with W_{ac}. In the former case $\mathcal{C}$ has a rule $\theta(23)^{-1}$ since the letters of V have no superscripts. The maximal prefix $\mathcal{D}$ of $\mathcal{C}$ containing no rules $\theta(23)^{-1}$ connects V with some $\theta(23)^{-1}$-admissible word $W(k,k)$, and the configurations of $\mathcal{D}$ have no letters with superscripts. Hence the trapezium corresponding to $\mathcal{D}$ (see Lemma 3.10 (2)) has equal side labels, and so V is conjugate of $W(k,k)$, as required.

In the latter case, we may assume that the computation $\mathcal{C}$ connecting V and W_{ac} has no rules $\theta(23)^{-1}$ (otherwise one could argue as above), and so it has no rules of sets $\mathbf{\Theta}_1$ and $\mathbf{\Theta}_2$. Therefore the word V is a conjugate of W_{ac} by Lemma 3.10 (2), since the side labels of the corresponding trapezium have no superscripts and therefore are equal. In turn, by Lemma 4.2, the word W_{ac} is a conjugate of any $W(k_0,k_0)$ if the word α^{k_0} is accepted by the machine M_0. This completes the proof of the lemma. ∎

Lemma 4.5. (1) *For every k, the cyclic subgroup $\langle W(k,k)\rangle$ is malnormal in G, that is, $I = \langle W(k,k)\rangle \cap Z\langle W(k,k)\rangle Z^{-1} = \{1\}$ if $Z \notin \langle W(k,k)\rangle$. The centralizer of an element $g \in G$ of order L is equal to the cyclic subgroup $\langle g\rangle$.*

(2) *The subgroup $\langle W(k,k)\rangle$ has trivial intersection with every conjugate subgroup of $\langle W_{\text{ac}}\rangle$ provided the word α^k is not accepted by the machine $\mathbf{M}_0$.*

Proof. (1) To prove the statement about the centralizers, one may assume by Lemmas 4.3 and 4.4, that the element g of order L is represented by some word $W \equiv W(k,k)$. Therefore it suffices to prove the first claim of the lemma.

Assuming that the intersection I is non-trivial, we can find two exponents s and r such that $W^s = ZW^rZ^{-1}$ and $0 < s \le L/3$, $|r| \le L/2$ if the order of I is odd, or $s = r = L/2$ otherwise.

We should prove that the word Z is equal to a power of W in G. For this goal, we consider a minimal van Kampen diagram Δ for this equality $W^s = ZW^rZ^{-1}$ and identifying the subpaths of the boundary labeled by Z, we obtain an annular diagram Γ whose two (clockwise) boundary labels read from some vertices o and o' are W^s and W^r, and there is a simple path $\mathbf{z}$ connecting o with o' and labeled by Z. (To obtain Γ homeomorphic to a topological annulus and to make the path $\mathbf{z}$ simple, one can use 0-cells corresponding to trivial relations as in [20, Section 11].)

One can cancel out the pairs of mirror cells if Γ is not reduced. Also if Γ contains a pair of disks Π_1, Π_2 connected by two $\tilde{t}$-bands, and these disks and the bands do not surround the hole of Γ, one can replace a van Kampen subdiagram having two disks with a diagram without disks by Lemma 3.16. The obtained reduced diagram Γ' has a simple path $\mathbf{z}'$ connecting o and o', whose label is equal in G to $\mathrm{Lab}(\mathbf{z}) \equiv Z$ (see [20, Section 13.6]).

The reduced diagram Γ' has no disks. Indeed, the two boundary components of Γ' have at most $L/3 + L/2$ $\tilde{t}$-edges if $s \le L/3$, but by Lemma 3.17 (2), an annular diagram with disks has to have at least $L - 3 > \frac{5}{6}L + 1$ $\tilde{t}$-edges on the boundary. If $s = r = L/2$ we have a contradiction again since all the $\tilde{t}$-letters of W^s and W^r are positive, but the disk has to have spokes ending on both boundaries of Γ' since $L - 3 > L/2$. Hence there are no disks in Γ'.

If Γ' has no cells, then it is a diagram over the free group, and so $s = r$ and the word $\mathrm{Lab}(\mathbf{z}')$ commuting with W^s in the free group is equal to a power of W, as required. Arguing by contradiction, assume that Γ' has θ-cells. Then all maximal θ-bands of Γ' are θ-annuli surrounding the hole of Γ' by Lemma 3.4. Cutting along a side of a maximal $\tilde{t}$-band $\mathcal{T}$, we obtain a reduced van Kampen diagram Γ'' over M, which is a trapezium of height $h \ge 1$ with equal side labels. Therefore the maximal θ-bands of Γ' have no superscripts in the labels of their cells, because it follows from Lemma 3.10 (1) that the label of the right side of a trapezium with s maximal $\tilde{t}$-bands must be the $\pm s$-shift of the label of the left side of it, but $s < L$.

By Lemma 3.10 (1), a subtrapezium of Γ'' with the same history gives us a non-empty computation $W \to \cdots \to W$ without rules of sets Θ_1 and Θ_2. This contradicts Lemma 4.1.

(2) We have the same proof as in item (1) considering now a hypothetical conjugation of $W(k,k)^s$ and W_{ac}^r. Clearly these cyclically reduced words are not conjugate in the free group since they involve different q-letters. Then as above, one obtain a computation $W(k,k) \to \cdots \to W_{\mathrm{ac}}$ without rules from sets Θ_1 and Θ_2, contrary to Lemma 2.14. ■

Remark 4.6. If the machine $\mathbf{M}_0$ is chosen with non-recursive language of input words α^k, then by Lemmas 4.2 and 4.5 (2), there exists no algorithm deciding whether some non-trivial powers of the words $W(k,k)$ and W_{ac} are conjugate in G or not.

Basing on Lemmas 4.2 and 4.3, we can introduce the HNN-extension G_k of the group G for $k = 1, 2, \dots$ by adding a stable letter x to the set of generators and the relation $xW(k,k)x^{-1} = W_{\mathrm{ac}}$ to the set of defining relations of G.

We need a property of HNN-extensions similar to the property of amalgamated products obtained in [8]. Both the formulation and the proof of the first part of the following lemma were left to the reader in [8].

Lemma 4.7. *Let a function f bound from above the Dehn function of a finitely presented group A and suppose f is super-additive, i.e., $f(n_1) + f(n_2) \le f(n_1 + n_2)$ for all integers $n_1, n_2 \ge 0$, and $f(1) \ge 1$. Let B be an HNN-extension of A with finite associated subgroups $xCx^{-1} = D$. Then the Dehn function $g(n)$ of B is bounded from above by a function equivalent to $f(n)$. In particular, every group G_k has quadratic Dehn function.*

Proof. We have a finite presentation of B with one extra letter x and finitely many extra relations $xU_ix^{-1} = V_i$, where all elements of the subgroups C and D are presented by some words U_i and V_i. Let $c - 1$ be the maximum of the lengths of all U_i-s and V_i-s.

Assume that a word $W \equiv W_1x^{\pm 1}W_2x^{\pm 1}\dots$ is equal to 1 in B, where the words W_j have no x-letters. We will induct on $s = s(W) = \|W\|_B + cr$, where r is the number of x-letters in W, with trivial base $s = 0$, to show that the area Area(W) in B does not exceed $f(s)$. If W has no x-letters, then $W = 1$ in G, and therefore Area$(W) \le f(s)$.

If the word W has x-letters, then the word W has a pinch subword by Britton's lemma (see [17, Section IV.2]), i.e., a subword xW_jx^{-1} (resp. $x^{-1}W_jx$), where W_j is equal in A to some word U_i (resp. to some V_j). Therefore one can replace W_j with U_i using an auxiliary diagram of area $\le f(\|W_j\| + \|U_i\|) \le f(n_j + c - 1)$, where $n_j = \|W_j\|$. Then the application of one conjugacy by x replaces the subword xU_ix^{-1} with V_i. We obtain a word W' with $s(W') < s(W) - n_j - c$ since the number of x-letters is decreased by 2. Therefore

$$\begin{aligned}\text{Area}(W) &< f(s - n_j - c) + f(n_j + c - 1) + 1\\ &\le f(s - n_j - c) + f(n_j + c)\\ &\le f(s) \le f((c + 1)\|W\|),\end{aligned}$$

and so $g(n) \le f((c + 1)n)$ for every $n \ge 0$, which proves the first statement of the lemma. It implies the second one by Lemmas 3.21 and 4.3. ■

Remark 4.8. It is unknown if there is a finitely presented group whose Dehn function is not equivalent to a super-additive function. This problem was raised by V. S. Guba and M. V. Sapir in [14].

Lemma 4.9. *Let B be an HNN-extension of a group A with associated malnormal subgroups C and D: $xCx^{-1} = D$. Assume also that $gCg^{-1} \cap D = \{1\}$ for every element $g \in A$. Then the centralizer of any non-trivial element $h \in A$ in B is equal to the centralizer of h in A.*

Proof. Let an element z commute with h in B. Assume first that it has only one stable letter x in the normal form: $z = g_1xg_2$, where $g_1, g_2 \in A$. Then the equality

$$g_1xg_2hg_2^{-1}x^{-1}g_1^{-1} = zhz^{-1} = h \in C$$

implies that the subword $x(g_2hg_2^{-1})x^{-1}$ is a pinch, and so we have $g_2hg_2^{-1} \in C\backslash\{1\}$. Then $xg_2hg_2^{-1}x^{-1} = d \in D\backslash\{1\}$, but the conjugate in A element $g_1^{-1}dg_1 = h$ belongs to C, contrary to the assumption of the lemma.

Now assume that the normal form of z (without pinches) has at least two x-letters: $z = g_1 x^{\varepsilon} g_2 x^{\eta} g_3 \ldots$, where $\varepsilon, \eta \in \{1, -1\}$. Then the only pinch in the product $z^{-1}hz$ is $x^{-\varepsilon} g_1^{-1} h g_1 x^{\varepsilon}$.

If $g_1^{-1} h g_1 \in C$, then $\varepsilon = -1$ and $d = x g_1^{-1} h g_1 x^{-1} \in D \backslash \{1\}$. Then we have the pinch $x^{-\eta} g_2^{-1} d g_2 x^{\eta}$, where the product $g_2^{-1} d g_2$ cannot belong to C by the assumption of the lemma. Thus, it is in D, $\eta = 1$, and since D is a malnormal subgroup of A, we should have $g_2 \in D$. But this gives the pinch $x^{-1} g_2 x$ in the normal form of z, a contradiction.

If we had $g_1^{-1} h g_1 \in D$, then the same argument would give a pinch $x g_2 x^{-1}$ in the normal form of z. Therefore $z \in A$, and the lemma is proved. ■

Lemma 4.10. (1) *The group G_k has an element of order L with infinite centralizer if the word α^k is accepted by the Turing machine $\mathbf{M}_0$. For all accepted α^k the groups G_k are isomorphic with the HNN extension $\bar{G}$ of G with stable letter y and the additional relation $y W_{\mathrm{ac}} y^{-1} = W_{\mathrm{ac}}$.*

(2) *If the word α^k is not accepted by $\mathbf{M}_0$, then the centralizers of elements with order L in G_k have order L.*

Proof. (1) By Lemma 4.2, there is an element $g \in G$ such that $g W(k,k) g^{-1} = W_{\mathrm{ac}}$ in G. So in G_k, we obtain the relation

$$x^{-1} g W(k,k) g^{-1} x = W(k,k),$$

i.e., $z^{-1} W(k,k) z = W(k,k)$ for $z = g^{-1}x$ and $y W_{\mathrm{ac}} y^{-1} = W_{\mathrm{ac}}$ for $y = x g^{-1}$. Here $W(k,k)$ has order L by Lemma 4.3 and y, z have infinite order. Furthermore, one can replace the generator x by y in the presentation of G_k and obtain the presentation of $\bar{G}$.

(2) Let the element g have order L in G_k, then it is a conjugate of an element of order L from G (see [17, Theorem IV.2.4]). Therefore one may assume that $g \in G$, and its centralizer $C_G(g)$ has order L by Lemma 4.5 (1). Then the centralizer of g has the same order in $G(k)$ by Lemma 4.9, because the assumptions of that lemma are guaranteed by Lemma 4.5 (1), (2). ■

Proof of Theorem 1.1. It follows from Lemma 4.10 that the group G_k is isomorphic to the group $\bar{G}$ (which is isomorphic to every G_i with α^i accepted by the machine $\mathbf{M}_0$) if and only if the word α^k is accepted by the Turing machine M_0. So the isomorphism problem is not decidable in the set of finitely presented groups $\{G_k\}_{k=1}^{\infty}$ if the language of accepted words of $\mathbf{M}_0$ is not recursive. Hence by Lemmas 3.21 and 4.7, the isomorphism problem is algorithmically undecidable in the class of finitely presented groups with quadratic Dehn function. ■

5. Dehn functions of subgroups

In this section, we assume that the language of accepted words of the Turing machine $\mathbf{M}_0$ consists of all non-negative powers α^k in the one-letter alphabet $\{\alpha\}$, but we select $\mathbf{M}_0$ so that the time function $T_{\mathbf{M}_0}(n)$ of $\mathbf{M}_0$ grows fast. Given a recursive function $f(n)$, we can define a symmetric Turing machine $\mathbf{M}_0$ with time function satisfying the inequalities

$$T_{\mathbf{M}_0}(n) > f(n) \quad \text{for } n \geq 0. \tag{5.1}$$

Here $T_{\mathbf{M}_0}(n)$ is the time of the shortest $\mathbf{M}_0$-computation accepting the word α^n.

The next S-machine $\mathbf{M}_1^+$ depends on $\mathbf{M}_0$ only, and we will assume that it coincides with the machine $\mathbf{M}_1$ provided by Lemma 2.2 from Section 2.1, and so it coincides with machine $\mathbf{M}_1$ borrowed from [26].

The language of accepted words for $\mathbf{M}_1^+$ is $\{\alpha^k\}_{k=0}^{\infty}$ by Lemma 2.2, and we have the inequality

$$T_{\mathbf{M}_0}(n) \leq T_{\mathbf{M}_1^+}(n)$$

for the time functions by [26, Lemma 4.1]. (This lemma says that *every* computation of $\mathbf{M}_1^+$ accepting an input configuration has to simulate an accepting computation of $\mathbf{M}_0$.

By definition, the machine $\mathbf{M}_1^-$ is a copy of $\mathbf{M}_1^+$, but the language of the accepted words of $\mathbf{M}_1^-$ is $\{\alpha^k\}_{k=0}^{-\infty}$. We will assume that these two machines have disjoint sets of rules, and the common state letters of them are only the letters of the start and the accept configurations. The machine $\mathbf{M}_1$ is defined now as the union of $\mathbf{M}_1^+$ and $\mathbf{M}_1^-$, where every admissible word is admissible either for $\mathbf{M}_1^+$ or for $\mathbf{M}_1^-$. So an input configuration (resp. the accept configuration) of $\mathbf{M}_1$ is an input configuration (resp. the accept configuration) of $\mathbf{M}_1^+$ and of $\mathbf{M}_1^-$.

Lemma 5.1. *The language of accepted words of* $\mathbf{M}_1$ *is* $\{\alpha^k\}_{k=-\infty}^{\infty}$, *and for every* $n > 0$, *we have the inequality* $T_{\mathbf{M}_1}(n) \geq f(n) - C(0)$, *where* $C(0) = T_{\mathbf{M}_1}(0)$.

Proof. The first statement is obvious since $\mathbf{M}_1$ is the union of $\mathbf{M}_1^+$ and $\mathbf{M}_1^-$.

Let $\mathcal{C} : C_0 \to \cdots \to C_t$ be a shortest accepting computation of $\mathbf{M}_1$ starting with an input configuration C_0 with a non-empty input word α^n. The computation $\mathcal{C}$ has an alternating factorization $\mathcal{C} = \mathcal{C}_1 \dots \mathcal{C}_s$, where every factor belongs to either $\mathbf{M}_1^+$ or $\mathbf{M}_1^-$. Without loss of generality we assume that $\mathcal{C}_1 : C_0 \to \cdots \to C_r$ is a computation of $\mathbf{M}_1^+$.

If $n < 0$, then the computation $\mathcal{C}_1$ cannot accept or end with an input configuration by Lemma 2.2. Therefore it cannot be followed by a computation $\mathcal{C}_2$ of $\mathbf{M}_1^-$, a contradiction.

If $n > 0$ and $s = 1$, then $\mathcal{C}_1$ is an accepting computation of $\mathbf{M}_1^+$, and so

$$t \geq T_{\mathbf{M}_1^+}(n) \geq T_{\mathbf{M}_0}(n) \geq f(n).$$

If $s > 1$, then $\mathcal{C}_1$ does not accept, and so C_r is a start configuration for both $\mathbf{M}_1^+$ and $\mathbf{M}_1^-$. Let α^k be the input word in C_r. Then $k \geq 0$ by Lemma 2.2 applied to $\mathcal{C}_1^{-1}$. If $k > 0$, then $\mathcal{C}_2$ could not end working by the dual lemma applied to $\mathcal{C}_2$; hence $k = 0$.

Note that since $k = 0$, one can construct a computation $\mathcal{D}$ as computation $\mathcal{C}_1$ followed by the $\mathbf{M}_1^+$-computation of length $C(0)$ accepting the empty word, and $\mathcal{D}$ accepts α^n. Hence $T_{\mathbf{M}_0}(n) \leq T_{\mathbf{M}_1}(n) \leq r + C(0)$, and so by (5.1), we have

$$t \geq r \geq T_{\mathbf{M}_0}(n) - C(0) \geq f(n) - C(0),$$

which proves the lemma. ■

The group G is defined by $\mathbf{M}$ in Section 3.2. In this section, we also consider a 'trimmed' version of the machine $\mathbf{M}$. The set of rules of this machine $\overline{\mathbf{M}}$ is $\mathbf{\Theta}_3 \cup \mathbf{\Theta}_4 \cup \mathbf{\Theta}_5$, i.e., we now remove the sets $\mathbf{\Theta}_1$ and $\mathbf{\Theta}_2$, as well as the transition rule $\theta(23)$, from the definition of $\mathbf{M}$. The state letters occurring in the removed rules only are removed too. The words of the form $W(k,k')$ for arbitrary integers k and k' become the start configurations of the machine $\overline{\mathbf{M}}$.

The definitions of the machines $\mathbf{M}_2$ to $\mathbf{M}_5$ and $\mathbf{M}$ depend on $\mathbf{M}_1$ only, and according to [27, Lemmas 3.10, 3.17, 3.18], $\overline{\mathbf{M}}$ accepts the same language $\mathcal{L}$ (which is equal to $\{\alpha^k\}_{k=-\infty}^{\infty}$ now). Furthermore, every computation of each of these machines accepting a word α^k must simulate the work of the previous machine accepting the same word, and so by Lemma 5.1, for every non-negative n, we have the following inequalities for the time functions:

$$f(n) - C(0) \leq T_{\mathbf{M}_1}(n) \leq T_{\mathbf{M}_2}(n) \leq T_{\mathbf{M}_3}(n).$$

By the definition of the set $\mathbf{\Theta}_4$, an accepting computation for a word $W(k,k)$ is longer than the computation of $\mathbf{M}_3$ accepting the input α^n. It follows that for every $n \geq 0$, we have

$$f(n) - C(0) \leq T_{\overline{\mathbf{M}}}(n). \tag{5.2}$$

Now we define the group $\bar{M}$ given by the generators and relations occurring in formulas (3.3) only (which correspond to the rules from $\mathbf{\Theta}_3 \cup \mathbf{\Theta}_4 \cup \mathbf{\Theta}_5$). The group $\bar{G}$ is obtained from $\bar{M}$ by imposing only one hub relation $W_{\mathrm{ac}}^L = 1$ from (3.4). In particular, the generators of the groups $\bar{M}$ and $\bar{G}$ have no superscripts (i), $i = 1, \dots, L$.

Lemma 5.2. *The canonical homomorphisms $\bar{M} \to M$ and $\bar{G} \to G$ are injective. So one may identify $\bar{M}$ and $\bar{G}$ with the subgroups of the groups M and G, respectively.*

Proof. We should prove that if a word w in the generators of $\bar{G}$ is equal to 1 in the group M (in G), then it represents 1 in $\bar{M}$ (resp. in $\bar{G}$).

Let Δ be a minimal diagram over G with boundary label w. If Δ has no disks, then by Lemma 3.4, every maximal θ-band of Δ ends on the boundary $\partial\Delta$, and so the one-letter history of it is a history of $\overline{\mathbf{M}}$ since w is a word in the generators of $\bar{M}$. It follows that Δ is a diagram over $\bar{M}$ and $w = 1$ in $\bar{M}$, as required. Thus, one may assume that Δ has at least one disk and induct on the number of disks l with base $l = 0$.

If $l > 0$, then Lemma 3.17 provides us with a disk Π and a $\tilde{t}$-band $\mathcal{B}$ connecting this disk with the boundary $\partial\Delta$. Since by Lemma 3.19 (2), Δ contains no θ-annuli, every θ-band crossing $\mathcal{B}$ ends on $\partial\Delta$ and it is a diagram over $\bar{M}$. It follows that the $\tilde{t}$-letter labelling the common edge of $\mathcal{B}$ and $\partial\Pi$ has no superscripts. Hence no letter of the accessible boundary label of Π has a superscript; this label has the form W^L.

By Lemma 3.14, we have a computation $\mathcal{C}$ of $\mathbf{M}$, connecting the word W either with W_{st} or with W_{ac}. If the history of this computation has no rules $\theta(23)^{-1}$, then it is a computation of $\overline{\mathbf{M}}$, and by Lemmas 2.14 (1) and 3.14, the disk Π can be filled in with the cells corresponding to the relations of the group $\bar{G}$ (including the hub relation W_{ac}^L).

If $\mathcal{C}$ has a rule $\theta(23)^{-1}$, then this rule is applied to a configuration $W(k,k')$. Here $k=k'$ since the word W, and so the word $W(k,k')$, is accessible. However for every integer k, the configuration $W(k,k)$ is accepted by the machine $\overline{\mathbf{M}}$ since the language of $\overline{\mathbf{M}}$-accepted input words is $\{\alpha^k\}_{k=-\infty}^{\infty}$. Therefore by Lemmas 2.14 (1) and 3.14, the subdisk with boundary label $W(k,k)^L$ can be filled in with cells corresponding to the presentation of $\overline{G}$. The same is true for the whole disk Π. Then Π can be removed from Δ along with the $\tilde{t}$-band $\mathcal{B}$, and the boundary of the remaining part Δ' of Δ is again labeled over $\overline{G}$. Since the number of disks in Δ' is $l-1$, the lemma is proved by induction. ■

Since every word $W(k,k)$ is accepted by the machine $\overline{\mathbf{M}}$, Lemma 3.14 gives us a *disk diagram* Δ with boundary label $W(k,k)^L$ built of one hub and L trapezia corresponding to a reduced accepting computation for $W(k,k)$. The boundary of the hub in Δ is labeled by W_{ac}^L, since $\overline{G}$ has only one hub relation.

One can prove that the disk diagram with boundary label $W(k,k)^L$ is minimal, but we need an estimate from below for the area of the word $W(k,k)^L$ with respect to the *finite* presentation of $\overline{G}$ (which contains a hub, but no other disks). The following statement is [23, Lemma 10.2]. (Although the machine is different in [23], the proof of Lemma 10.2 works for $\overline{G}$ without any changes.)

Lemma 5.3. *The area of a disk diagram Δ with boundary label $W(k,k)^L$ does not exceed twice the area of the disk word $W(k,k)^L$ with respect to the finite presentation of $\overline{G}$.* ■

Lemma 5.4. *The area of the word $W(k,k)^L$ with respect to the finite presentation of $\overline{G}$ is at least $L(f(k)-C(0))$.*

Proof. Let us consider the disk diagram Δ with boundary label $W(k,k)^L$ provided by Lemma 3.14. Δ contains L trapezia corresponding to an accepting computation of the machine $\overline{\mathbf{M}}$ starting with the configuration $W(k,k)$. By Lemma 3.10, the height of each trapezium Γ is at least $T_{\overline{\mathbf{M}}}(k)$, which is greater than $f(k)-C(0)$ by inequality (5.2). Hence Γ contains at least $N(f(k)-C(0))$ (θ,q)-cells, and therefore Δ has at least $NL(f(k)-C(0))$ cells. By Lemma 5.3, the area of the word $W(k,k)^L$ is at least $NL(f(k)-C(0))/2$ which proves the lemma. ■

Proof of Theorem 1.2. Note that the length of the word $W(k,k)$ is a linear function of k. Therefore to bound the Dehn function of the subgroup $\overline{G}$ from below by $f(n)$ (up to equivalence), it suffices to obtain the inequalities $\mathrm{Area}_{\overline{G}}(W(k,k)) > f(k)-C(0)$ for every $k\ge 1$. Indeed, these inequalities follow from Lemma 5.4.

Since the group $H=\overline{G}$ embeds in G by Lemma 5.2, Theorem 1.2 is proved, because by Lemma 3.21, the group G defined by $\mathbf{M}$ in Section 3.2 has quadratic Dehn function. ■

Remark 5.5. Since the word $W(k,k)$ is accepted by the machine $\overline{\mathbf{M}}$ for every integer k, it follows from Lemmas 4.2, 4.3, 4.4, 7.1, and 6.4 that the conjugacy problem is decidable for both groups G and $\overline{G}=H$ constructed in this section.

6. Conjugacy in the group G

In this section and in the next one, the construction of the machine $\mathbf{M}$ can be based on the machine $\mathbf{M}_1$ provided by either Lemma 2.2 or Lemma 2.3.

Lemma 6.1. *Let Γ be a reduced diagram over M with boundary $\mathbf{xy}$. Suppose there are neither θ-bands nor q-bands starting and ending on $\mathbf{y}$. Then:*

(1) $|\mathbf{y}|_\theta \le |\mathbf{x}|_\theta$ *and* $|\mathbf{y}|_q \le |\mathbf{x}|_q$.

(2) *If Γ is a subdiagram of a diagram Δ over G and $\mathbf{y} = \mathbf{y}_1\mathbf{y}_2\mathbf{y}_3$, where $\mathbf{y}_1$ and $\mathbf{y}_3$ are sides of q-bands, $\mathbf{y}_2$ is a side of a θ-band or a subpath of the boundary of a disk Π, then $\|\mathbf{y}_2\|$ is bounded by a quadratic function of $\|\mathbf{x}\|$. The perimeter $\|\partial\Pi\|$ is also bounded by a quadratic function of $\|\mathbf{x}\|$ if $\mathbf{y}_2$ contains at least two $\tilde{t}$-letters.*

Proof. (1) Since every maximal θ- or q-band of Γ starting on $\mathbf{y}$ has to end on $\mathbf{x}$, the inequalities follow.

(2) It follows from (1) that the numbers of maximal θ- and q-bands of Γ are bounded by a linear function of $\|\mathbf{x}\|$. Therefore the number of (θ, q)-cells of Γ is bounded by a quadratic function by Lemma 3.4. Note that the Y-lengths of the sides $\mathbf{y}_1$, $\mathbf{y}_2$ of q-bands are linearly bounded by their θ-lengths. Since every maximal Y-band starting on $\mathbf{y}_2$ has to end on a (θ, q)-cell or on $\mathbf{y}_1$, or on $\mathbf{y}_3$, or on $\mathbf{x}$, we have $|\mathbf{y}_2|_Y$ and $\|\mathbf{y}_2\|$ bounded by a quadratic function of $\|\mathbf{x}\|$. If $\mathbf{y}_2$ is a subpath of Π having at least two $\tilde{t}$-edges, we have $\|\partial\Pi\| < L\|\mathbf{y}_2\|$, and the statement (2) follows. ∎

Lemma 6.2. *There is an algorithm replacing a given word W with a word W' conjugate in G to W and having the following property. If Γ is a minimal van Kampen diagram Γ with boundary $\mathbf{pq}$, where* Lab($\mathbf{p}$) *is a subword of a cyclic permutation of the word W', then none of* (a)*,* (b)*, and* (c) *below holds.*

(a) *Γ is subdiagram of a diagram Δ over G, $\mathbf{q} = \mathbf{y}_1\mathbf{y}_2\mathbf{y}_3$, where $\mathbf{y}_1$ and $\mathbf{y}_3$ are sides of $\tilde{t}$-spokes connecting the subpath $\mathbf{y}_2$ of the boundary of a disk Π with $\mathbf{p}$, there are no disks in Γ, and Π is connected with $\mathbf{p}$ by $s > L/2$ $\tilde{t}$-bands. See Figure 6.*

(b) *$\mathbf{q}$ is a side of a rim q-band $\mathcal{C}$ of Γ starting and ending on $\mathbf{p}$. See Figure 7.*

(c) *$\mathbf{q}$ is a side of a rim θ-band $\mathcal{T}$ of Γ starting and ending on $\mathbf{p}$. See Figure 7.*

Proof. (a) Suppose that, for some word W diagrams Γ and Δ satisfying the assumption of Lemma 6.2 (a) exist. Assume that Π is connected with $\mathbf{p}$ in Γ by $r > L/2$ consecutive $\tilde{t}$-spokes $\mathcal{C}_1, \dots, \mathcal{C}_r$. By Lemma 3.19 (1), Γ contains no θ-bands connecting $\mathcal{C}_1$ and $\mathcal{C}_r$. Lemma 6.1 gives a linear bound (in terms of $\|W\|$) for the lengths of the spokes and a quadratic upper bound for the perimeter of the disk Π. So there is a subdiagram Δ' containing Π and having the boundary $\mathbf{pq}'$ with $|\mathbf{q}'|_q < |\mathbf{q}|_q$ and bounded $\|\mathbf{q}'\|$.

Replacing the subword Lab($\mathbf{p}$) with Lab$(\mathbf{q}')^{-1}$, one obtains a conjugate in G word of smaller q-length, where the length of the modified word is quadratically bounded in terms of $\|W\|$, and the search for it has effectively bounded time by Lemma 3.21. This gives an algorithm providing property (a) from the lemma.

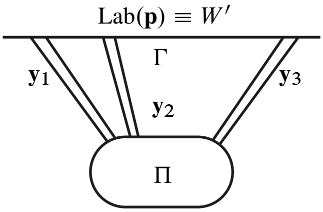

Figure 6. Diagram Γ in Lemma 6.2 (a).

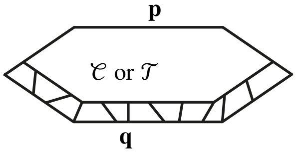

Figure 7. Diagram Γ in Lemma 6.2 (b) and (c).

(b) Consider such a diagram Γ if any exists. We may assume that Γ contains no disks. Indeed, the q-band $\mathcal{C}$ has no side q-edges, and so by Lemma 3.17 (1), the existence of a disk should imply the existence of a subdiagram already eliminated in item (a), a contradiction. Then by Lemma 3.4, every maximal θ-band starting from $\mathcal{C}$ in Γ ends on $\mathbf{p}$. Hence the length of $\mathcal{C}$ is less than $\|\mathbf{p}\|$. The side label of $\mathcal{C}$ is equal in M to $\mathrm{Lab}(\mathbf{p})$ but has q-length 0. Hence the subword $\mathrm{Lab}(\mathbf{p})$ can be replaced with a word Z, equal in M, having smaller q-length and $\|Z\|$ is linearly bounded in terms of $\|W\|$. Since the word problem is decidable in G, one can efficiently execute such a replacement. Thus, the effective procedures described in the items (a) and (b) provides us with an output satisfying both properties (a) and (b) from the formulation of the lemma.

(c) Let Γ be the diagram from the formulation of item (c). Now we may assume that the word W has properties (a) and (b). We also may assume that Γ has no maximal θ-bands except for $\mathcal{T}$ since otherwise Γ should contain a proper subdiagram with the same property. Assume that Γ has a disk. Then let Π be a disk provided by Lemma 3.17 (1). By item (a), at least two $\tilde{t}$-spokes $\mathcal{C}$ and $\mathcal{C}'$ start on Π and end on $\mathcal{T}$ since $L-5 > L/2+1$. The uniqueness of the maximal θ-band $\mathcal{T}$ in Γ implies that $\mathcal{C}$ and $\mathcal{C}'$ have length 0. Then by Lemma 3.20 (2), one can decrease the number of disks in the diagram Γ, replacing it with a diagram Γ' satisfying the same assumptions and having $\mathrm{Lab}(\mathbf{p}') \equiv \mathrm{Lab}(\mathbf{p})$. The induction on the number of disks allows us to assume that Γ is a diskless diagram, and so it coincides with $\mathcal{T}$. Replacing $\mathbf{p}$ with $\mathbf{q}^{-1}$, one decreases the θ-length of the boundary label preserving the q-length of it. Since the length of $\mathcal{T}$ does not exceed $\|W\|$, one can find and remove $\mathcal{T}$ effectively.

The algorithm terminates, because each step in paragraphs (a)–(c) of it either reduces the q-length or does not increase it but reduces the θ-length. ∎

Remark 6.3. Let us call the word W' from Lemma 6.2 an *adapted* word.

One can change the formulation of Lemma 6.2 by replacing the minimal diagram Γ over G with a reduced diagram over the group M and removing property (a). The statement remains true (the proof is a simplified proof of Lemma 6.2). We will call the words W' obtained from W according to these weaker version of Lemma 6.2 a weakly adapted word.

The following statement will be used in the next section.

Lemma 6.4. *There is an algorithm that decides whether two words U and V representing elements of infinite order in the group G are conjugate in G or not.*

Proof. We divide the proof into several steps.

Step 0. Lemma 6.2 allows us to assume that the words U and V are adapted. By the Schupp lemma, U and V are conjugate in G if and only if there is an annular diagram Δ over G with boundary components $\mathbf{p}$ and $\mathbf{q}$ labeled by U and V, respectively. If there is a recursive function f such that one always can choose Δ so that $\mathbf{p}$ and $\mathbf{q}$ can be connected by a path $\mathbf{x}$ of length $\le f(\|U\| + \|V\|)$, then the cut along $\mathbf{x}$ replaces Δ with a van Kampen diagram of bounded perimeter, and so the conjugacy problem is reduced to the word problem. Since the word problem is decidable in a group with quadratic Dehn function (see Lemma 3.21), our goal is to find such a 'short cut' under the assumption that U and V are conjugate in G.

Step 1. Let Δ be a minimal annular diagram whose boundary contours $\mathbf{p}$ and $\mathbf{q}$ are labeled by U and V, respectively. The number r of disks in Δ cannot exceed $s = |U|_q + |V|_q$ for the following reason. Let Π be the disk provided by Lemma 3.17 (2). Since the words U and V are adapted, Π is connected by spokes with both $\mathbf{p}$ and $\mathbf{q}$. Cutting out the union of Π and the diskless subdiagrams between Π and the boundary components (bounded by the spokes at Π), we get a remaining van Kampen diagram Δ' with at most $s-(L-3)+3<s$ $\tilde{t}$-edges on the boundary. Now Lemma 3.18 bounds the number of disks in Δ'.

Step 2. If Δ has a disk, then Lemma 3.17 (2) gives a disk Π connected with $\mathbf{p}$ (or with $\mathbf{q}$) by at least two $\tilde{t}$-spokes. Assuming that these two $\tilde{t}$-bands $\mathcal{C}$ and $\mathcal{C}'$ are consecutive, we consider the subdiagram Γ over the group M bounded by $\mathcal{C}$, $\mathcal{C}'$, $\partial\Pi$ and $\mathbf{p}$.

If Γ contains no θ-bands connecting $\mathcal{C}$ and $\mathcal{C}'$, then Lemma 6.1 gives a quadratic bound (in terms of $\|U\|$) for the perimeter of the disk Π and a linear bound for the length of $\mathcal{C}$.

Making a cut along the boundary of $\mathcal{C}$ and around Π, one can remove the disk Π and obtain an annular diagram Δ' with fewer disks, where the boundary label U is replaced with a word U', equal in the group G, whose length is quadratically bounded in terms of $\|U\|$.

If Γ has a θ-band connecting $\mathcal{C}$ and $\mathcal{C}'$, then such a θ-band closest to Π has to share a side with $\partial\Pi$. This θ-band $\mathcal{T}$ and Π form a diagram E satisfying the assumption of Lemma 3.20 (1). Therefore Π and the subdiagram E can be replaced in Δ with a disk Π' and a diagram E'. This surgery removes the θ-band $\mathcal{T}$ from Γ and shortens the connecting $\tilde{t}$-bands $\mathcal{C}$ and $\mathcal{C}'$ in the obtained annular diagram Δ'. (We do not care about the minimality of the entire Δ'.) Then we can continue moving the disk closer to $\mathbf{p}$ until we obtain a subdiagram, where no θ-band connects $\mathcal{C}$ and $\mathcal{C}'$.

Thus, if Δ has a disk, then there is a minimal annular diagram Δ' with fewer disks and boundary labels U' and V' equal to U and V in G, where $\|U'\| + \|V'\|$ is effectively bounded in terms of $\|U\| + \|V\|$. Since the number of disks in Δ does not exceed $\|U\| + \|V\|$, the iteration of this argument provides us with a diskless annular diagram $\bar{\Delta}$ and with boundary labels $\bar{U}$ and $\bar{V}$ equal to U and V, respectively, in G. Hence an effective exhaustive search gives a finite set $S = S(U, V)$ of pairs (U_i, V_i) such that U and V are conjugate in G if and only if for some i, the words U_i and V_i are conjugate in the group M. Moreover, by Remark 6.3, the words U_i and V_i can be assumed weakly adapted. So, keeping the same notation, we may assume that the annular diagram Δ contains no disks.

Step 3. Since we may assume that the words U and V are weakly adapted, it remains to consider three options: (a) neither U nor V have θ-letters; borrowing the term from [24], the corresponding annular diagram Δ will be called a *ring*; (b) there are no q-edges in the boundary of Δ and every maximal θ-band connects $\mathbf{p}$ and $\mathbf{q}$; such a diagram is a *roll*; (c) there are q-letters and there are θ-letters in both U and V, and every maximal q- or θ-band of Δ connects $\mathbf{p}$ and $\mathbf{q}$; Δ is a *spiral*.

Step 4. Assume that Δ is a ring. Then, by Lemma 3.4, the annular diagram Δ is built of θ-annuli surrounding the hole of Δ. Different θ-annuli cannot copy each other, since otherwise one could remove some θ-annuli from Δ.

If Δ has no (θ, q)-cells, then by Lemma 3.4, every maximal Y-band connects $\mathbf{p}$ and $\mathbf{q}$, and so all the θ-annuli have the same length $|U|_a$, and the number of different θ-annuli of this length is effectively bounded. Therefore there is a path $\mathbf{x}$ of bounded length connecting $\mathbf{p}$ and $\mathbf{q}$, as desired. Therefore, we may assume that there are (θ, q)-cells in Δ, and we have $s \geq 1$ maximal q-bands $\mathcal{C}_1, \dots, \mathcal{C}_s$, each of them has the same length h and connects $\mathbf{p}$ and $\mathbf{q}$. Let the θ-annuli of Δ have boundary components with lengths l_0 and l_1, l_1 and l_2, $\dots$, l_{h-1} and l_h.

If $\max_{i=0,\dots,h} l_i \leq c_4 \max(\|U\|, \|V\|)$, then there is an effective upper bound for h since the θ-annuli cannot copy each other. So proving by contradiction, we assume that

$$\max_{i=0,\dots,h} l_i > c_4 \max(\|U\|, \|V\|). \tag{6.1}$$

We consider the 'power' Δ^L of Δ. This reduced annular diagram has boundary labels $U' = U^L$ and $V' = V^L$, respectively, and maximal q-bands $\mathcal{D}_1, \dots, \mathcal{D}_{sL}$. ($\Delta^L$ covers Δ with multiplicity L).

Cutting Δ^L along a side of $\mathcal{D}_{sL}$, we obtain a diagram over M with boundary $\mathbf{z}_1\mathbf{z}_2\mathbf{z}_3\mathbf{z}_4$, where $\mathbf{z}_2$ and $\mathbf{z}_4$ have labels U' and $(V')^{-1}$, and $\mathbf{z}_3$ is the side of the q-band $\mathcal{D}_{sL}$. Attaching to this diagram a copy $\mathcal{D}_0$ of $\mathcal{D}_{sL}$ along $\mathbf{z}_1$, we get a trapezium Γ of height h. The base of Γ has form $xv_1xv_2x \dots xv_rx$, where the letter x is the base of $\mathcal{D}_{sL}$, it does not occur in the subwords $v_1, \dots, v_r$ and r is a multiple of L. So Γ is covered by r trapezia Γ_i with bases xv_ix.

By inequality (6.1) and Lemma 2.17, the base of Γ is a power of a cyclic permutation of the standard base. Replacing U with a cyclic permutation, we may assume that the base of Γ_i is standard, it is equal to tv, where $x \equiv t$ and $v \equiv v_1 \equiv \dots \equiv v_r$. Moreover, by Lemmas 3.10 (1) and 2.15, for the top labels W_i and the bottom labels W_i' of every Γ_i, the words $W_i^{\emptyset}$ and $(W_i')^{\emptyset}$ are accessible words. Furthermore, by Remark 2.16

and Lemma 2.12, all the words $W_1, \ldots, W_r$ are equal up to the superscripts. It follows that the word U^L is permissible, and it is a power of a disk word, because r is divisible by L. So $U^L = 1$ in the group G by Lemma 3.12, contrary to the assumption that U has infinite order.

Step 5. Assume now that Δ is a roll having no (θ, q)-cells. If it has a Y-annulus $\mathcal{A}$ surrounding the hole of Δ, then it follows from the form of (θ, a)-relations, that the inner and the outer boundary components have the same boundary label. Then one can just remove $\mathcal{A}$ from Δ identifying these two sides of $\mathcal{A}$. Therefore we may assume that there are no such Y-annuli in Δ, and every maximal Y-band starts or ends on $\mathbf{p}$ or $\mathbf{q}$ by Lemma 3.4. It follows that the number of maximal Y-bands cannot exceed the sum $\|U\| + \|V\|$. To obtain an upper bound for the length of a connected path $\mathbf{x}$ from Step 1, it suffices to bound the number of (θ, a)-cells in Δ, that is, to bound from above the length of every maximal Y-band.

If a Y-band $\mathcal{A}$ starts and ends on $\mathbf{p}$ (or $\mathbf{q}$), then its length does not exceed $\|U\|$ since by Lemma 3.4, every maximal θ-band crosses $\mathcal{A}$ at most once and starts or ends on $\mathbf{p}$. Let $\mathcal{A}$ connect $\mathbf{p}$ and $\mathbf{q}$. To bound the length of $\mathcal{A}$ it suffices to bound the number of (θ, a)-cells belonging to the intersection of $\mathcal{A}$ with arbitrary maximal θ-band $\mathcal{T}$, because the number of maximal θ-bands (connecting $\mathbf{p}$ and $\mathbf{q}$) is at most $\|U\| + \|V\|$.

By Lemma 3.4, a subband of $\mathcal{A}$ cannot cross twice a subband of $\mathcal{T}$ in a van Kampen subdiagram of Δ. Therefore after the Y-band $\mathcal{A}$ crosses $\mathcal{T}$ at some cell π, it has to cross every other maximal θ-band of Δ before $\mathcal{A}$ crosses $\mathcal{T}$ again at some cell π'. So we have a convolution, i.e., the subband of $\mathcal{A}$ of length at most $\|U\| + \|V\|$ between π and π'. Hence it suffices to bound from above the number of such convolutions in $\mathcal{A}$.

The subband $\mathcal{T}'$ of $\mathcal{T}$ between π and π' can be crossed by another maximal Y-band $\mathcal{A}'$ at most once (and $\mathcal{A}'$ has to connect $\mathbf{p}$ and $\mathbf{q}$). Therefore the length of $\mathcal{T}'$ is at most $\|U\| + \|V\|$. So a side of the convolution and a side of $\mathcal{T}'$ form a loop $\mathbf{z}$ of length $O(\|U\| + \|V\|)$ surrounding the hole of Δ.

If $\mathbf{z}$ surrounds another loop $\mathbf{z}'$ of this type with $\mathrm{Lab}(\mathbf{z}') \equiv \mathrm{Lab}(\mathbf{z})$, then one can remove the diagram between $\mathbf{z}$ and $\mathbf{z}'$ and identify these two loops decreasing the number of cells in the annular diagram. Since the lengths of loops of this type are bounded, the number of them is effectively bounded as well. It follows that we have an effective upper bound for the lengths of Y-bands as desired.

Step 6. Assume now that Δ is a roll containing (θ, q)-cells. Then by Lemma 3.4, every maximal q-band of Δ is a q-annulus surrounding the hole of Δ since a roll has no q-edges in the boundary. By the same lemma, every maximal θ-band crossing a q-annulus $\mathcal{C}$ ends on $\mathbf{p}$ and $\mathbf{q}$ and cannot intersect $\mathcal{C}$ twice. Therefore the length of an arbitrary q-annulus does not exceed $\min(|U|_\theta, |V|_\theta)$. This observation effectively bounds the number of q-annuli in a minimal annular diagram Δ, because two different annuli cannot copy each other. (One could remove the diagram between them and identify such annuli, contrary to the minimality of Δ.)

If $\mathbf{p}_1, \mathbf{q}_1, \ldots, \mathbf{p}_k, \mathbf{q}_k$ are pairs of boundary components of all the q-annuli counting from $\mathbf{p}$ to $\mathbf{q}$, then the annular diagrams between $\mathbf{q}_0 = \mathbf{p}$ and $\mathbf{p}_1$, $\mathbf{q}_1$ and $\mathbf{p}_2, \ldots, \mathbf{q}_k$ and $\mathbf{p}_{k+1} = \mathbf{q}$ contain no (θ, q)-cells. Therefore there is a roll having boundary labels U and V if and only if there are at most k different q-annuli (where k and the lengths of the q-annuli are effectively bounded) and $\leq k+1$ rolls without (θ, q)-cells between $\mathbf{q}_i$ and $\mathbf{p}_{i+1}$ ($i = 0, \ldots, k$, and the lengths of all $\mathbf{p}_j$ and $\mathbf{q}_j$ are effectively bounded). So the case of rolls is effectively reduced to the special case considered in Step 5.

Step 7. It remains to assume that Δ is a spiral. It follows from Lemma 3.4 that every q-edge of Δ belongs to one of the (clockwise) consecutive maximal q-bands $\mathcal{C}_1, \ldots, \mathcal{C}_s$ connecting $\mathbf{p}$ and $\mathbf{q}$. Clearly we have $s \leq \min(|U|_q, |V|_q)$. Similarly, all maximal θ-bands $\mathcal{T}_1, \ldots, \mathcal{T}_r$ connect $\mathbf{p}$ and $\mathbf{q}$, and $r \leq \min(|U|_\theta, |V|_\theta)$.

Assume that going from $\mathbf{p}$ to $\mathbf{q}$ a θ-band $\mathcal{T} = \mathcal{T}_j$ crosses a q-band $\mathcal{C} = \mathcal{C}_i$ clockwise. Then Lemma 3.4 implies that if after this intersection, $\mathcal{T}$ crosses a q-band again, then this next intersection is with $\mathcal{C}_{i+1}$ (the indices taken modulo s) and $\mathcal{C}_{i+1}$ is crossed clockwise too. So if the number of intersections of $\mathcal{T}$ with q-bands is greater than s, it has to cross one of the bands at least twice. See Figure 8.

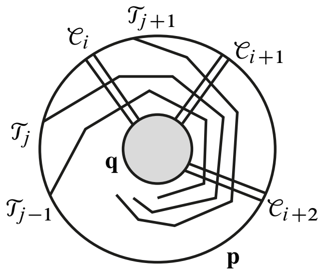

Figure 8. Spiral structure.

Our nearest goal is to bound from above the lengths of the q-bands. And so we will assume that every maximal θ-band contains more than s different (θ, q)-cells. (If there is $\mathcal{T}$ crossing q-bands at most s times, then the same has to be true for every q-band.)

The spiral structure of Δ implies the dual property: If a maximal q-band $\mathcal{C}$ (directed from $\mathbf{p}$ to $\mathbf{q}$) crosses some $\mathcal{T}_j$, then the next intersection of $\mathcal{C}$ (if any) is the intersection with $\mathcal{T}_{j-1}$ (the indices taken modulo r). It follows that the history of any q-band is periodic with a period H of length r.

Consider now a van Kampen subdiagram Γ bounded by two q-bands $\mathcal{C}$ and $\mathcal{C}'$ and parts of $\mathbf{p}$ and $\mathbf{q}$, such that Γ has no other q-bands between $\mathcal{C}$ and $\mathcal{C}'$. Let E be the maximal trapezium (possibly empty) of Γ, bounded by subbands $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{C}$ and $\mathcal{C}'$, and by θ-bands $\mathcal{S}$ and $\mathcal{S}'$ connecting $\mathcal{C}$ and $\mathcal{C}'$ in Γ. Thus, the complement of E in Γ has no θ-bands connecting $\mathcal{C}$ and $\mathcal{C}'$, and Lemma 6.1 (2) gives a quadratic bound for the lengths of $\mathcal{S}$ and $\mathcal{S}'$ in terms of $\|U\|$ and $\|V\|$. See Figure 9.

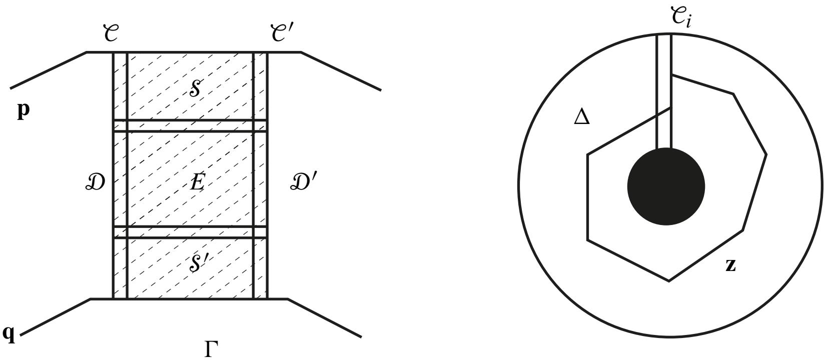

Figure 9. To Step 7 of the proof of Lemma 6.4: the subdiagram Γ (left) and the loop **z** (right).

By Lemma 3.10 (1), the trapezium E corresponds to an eligible computation with periodic history H. Therefore Lemma 2.4 gives a linear upper bound of the lengths of arbitrary θ-bands of E in terms of the lengths of the top and the bottom of E, and the length of the period H of the history. So the lengths of such θ-bands are quadratically bounded in terms of $\|U\| + \|V\|$.

If $\mathcal{T}_0$ is a part of some θ-band $\mathcal{T}$, such that $\mathcal{T}_0$ starts and ends on the same q-band $\mathcal{C}_i$ and crosses once every other maximal q-band, then the above argument provides us with a cubic upper bound for the length of $\mathcal{T}_0$. The ends of the side of $\mathcal{T}_0$ are connected by a part of length $O(r)$ of the q-band $\mathcal{C}_i$. So this side and the connecting path form a loop **z** of at most cubic length surrounding the hole of Δ. See Figure 9.

If **z** surrounds another loop $\mathbf{z}'$ of this type with $\mathrm{Lab}(\mathbf{z}') \equiv \mathrm{Lab}(\mathbf{z})$, then one can remove the diagram between **z** and $\mathbf{z}'$ and identify these two loops decreasing the number of cells in the annular diagram. Since the lengths of loops of this type are bounded, the number of them is effectively bounded as well. It follows that we have an effective upper bound for the lengths of q-bands in Δ, and so, for the length of a path connecting **p** and **q**. Lemma 6.4 is proved. ∎

Lemmas 6.4 and 4.4 prove Theorem 1.4.

7. The power conjugacy problem

Lemma 7.1. (a) *There is an algorithm such that given a word W in the generators of the group G, it decides whether the order of W in G is finite or infinite.*

(b) *To obtain an algorithm solving the power conjugacy problem in G for arbitrary pairs of words (U, V), it suffices to obtain such an algorithm under assumption that the words U and V have infinite order in G.*

Proof. (a) By Lemmas 4.3 and 4.4, W has a finite order in G if and only if $W^L = 1$ in G. Since Lemma 3.21 implies that the word problem is decidable in G, statement (a) is proved.

(b) This statement follows from (a) since some positive powers of elements having finite orders are trivial, and so are conjugate. ■

The relations (3.1)–(3.4) defining the group G immediately imply that there exists a homomorphism μ from G to the additive group $\mathbb{Z}/L\mathbb{Z}$ which sends all $\tilde{t}$-generators to 1 and all other generators to 0. To solve the power conjugacy problem in G for a given pair of words (U, V), one may replace this pair with the pair (U^L, V^L); thus, we may *assume further* that

$$\mu(U) = 0 \quad \text{and} \quad \mu(V) = 0. \tag{7.1}$$

Let U and V be two words representing elements of infinite order from the group G. Under the condition (7.1) and the assumption that the words U and V are adapted, we will also assume that some powers U^k and V^l are conjugate in G for $k, l \neq 0$. Without loss of generality we assume that $k, l > 0$, and there is no pair of positive exponents k', l' such that $U^{k'}$ is a conjugate of $V^{l'}$, where $k' < k$, $l' < l$. Thus, we will study a minimal annular diagram Δ, where the outer boundary component $\mathbf{p}$ has the clockwise label U^k and the inner boundary component $\mathbf{q}$ is labeled by V^l.

If two consecutive $\tilde{t}$-spokes $\mathcal{C}$ and $\mathcal{C}'$ start on a disk Π of Δ, end both on $\mathbf{p}$ (or both on $\mathbf{q}$), and the van Kampen subdiagram Γ, bounded by a subpath of $\mathbf{p}$ (resp. $\mathbf{q}$), a subpath of $\partial\Pi$, $\mathcal{C}$, and $\mathcal{C}'$, contains no disks (but contains the spokes $\mathcal{C}$ and $\mathcal{C}'$), then we shall call Γ a *tp-bond* at Π (resp. tq-bond). See Figure 10.

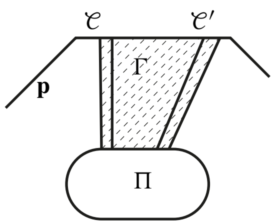

Figure 10. tp-bond Γ at Π.

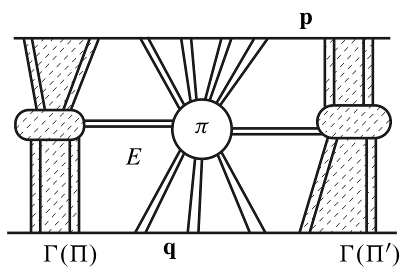

Figure 11. Subdiagram E between $\Gamma(\Pi)$ and $\Gamma(\Pi')$.

Lemma 7.2. *For every disk Π of Δ, the number of tp-bonds (tq-bonds) is greater than $L/2 - 4 > 0$. No q- or θ-band of Δ starts and ends on $\mathbf{p}$ or starts and ends on $\mathbf{q}$.*

Proof. If a θ-band connects $\mathbf{p}$ with $\mathbf{p}$, then a van Kampen subdiagram Γ of Δ is bounded by $\mathcal{T}$ and by a subpath $\mathbf{p}'$ of the cyclic path $\mathbf{p}$, and the word U must contain both positive and negative occurrences of θ-letters. Proving by contradiction, we may assume that $\mathcal{T}$ is a unique maximal θ-band of Γ since every maximal θ-band of Γ has to start and end on $\mathbf{p}'$.

Therefore the word $W \equiv \mathrm{Lab}(\mathbf{p}')$ has no θ-letters except for the first and the last ones. Hence W is a subword of a cyclic permutation of the word U. However such a diagram Γ is impossible since U is an adapted word.

Assume now that there is a q-band $\mathcal{C}$ starting and ending on $\mathbf{p}$. Then there is a van Kampen diagram Γ bounded by $\mathcal{C}$ and a subpath $\mathbf{p}'$ of $\mathbf{p}$. To obtain a contradiction, we assume that Γ has no maximal q-bands starting and ending on $\mathbf{p}$ except for $\mathcal{C}$. Suppose Γ has disks. Since the sides of $\mathcal{C}$ have no q-edges, the disk Π provided by Lemma 3.17 (2) applied to Γ has at least $L-4$ consecutive tp-bonds. Since there are no $\tilde{t}$-letters between the ends of the $\tilde{t}$-spokes defining these bonds, there is subpath $\mathbf{x}$ of $\mathbf{p}$ containing exactly $L-3$ $\tilde{t}$-edges, namely, the ends of the $\tilde{t}$-spokes defining the tp-bonds. However U has at least L $\tilde{t}$-letters by the equalities (7.1). Hence $\mathrm{Lab}(\mathbf{x})$ is a subword of a cyclic permutation of the adapted word U, and $L-3$ of these $\tilde{t}$-spokes end on $\mathbf{x}$, which contradicts Lemma 6.2 (a) since $L-3 > L/2+1$. If Γ contains no disks, the argument from the previous paragraph leads to contradiction again.

Let us prove the first statement of the lemma. Under the assumption that Δ contains disks, Lemma 3.17 (2) gives a disk Π with $L' + L'' \geq L-3$ $\tilde{t}$-spokes ending on $\mathbf{p}$ and $\mathbf{q}$. If, say, $L'' < L/2-3$, then $L' > L/2$. However U has at least L $\tilde{t}$-letters by the equalities (7.1). So by property (b) of Lemma 6.2 for an adapted word U, these $\tilde{t}$-spokes end on a subpath of $\mathbf{p}$ labeled by a subword of a cyclic permutation of U, a contradiction with property (a) of adapted words in Lemma 6.2. Thus, we have $L'' \geq L/2-3$ and similarly, $L' \geq L/2-3 > 0$.

Now consider a maximal set $\mathbf{D}$ of disks Π with the following property. Every disk Π from $\mathbf{D}$ has at least $L-5$ $\tilde{t}$-spokes ending either on $\mathbf{p}$ or on $\mathbf{q}$. Note that two neighbor $\tilde{t}$-spokes ending on $\mathbf{p}$ (or on $\mathbf{q}$) define a tp-bond at Π, i.e., the subdiagram Γ, bounded by these spokes, a part of $\mathbf{p}$, and a part of $\partial\Pi$, contains no disks, because otherwise Lemma 3.17 (1) applied to Γ would give us another disk with at least $L-4$ tp-bonds, which is impossible again since $L-4 > L/2$.

We claim that all disks of Δ belong to $\mathbf{D}$. Indeed, let $\Gamma(\Pi)$ denote the subdiagram formed by a disk Π from $\mathbf{D}$ and all the tp- and tq-bonds at Π. Consider the maximal van Kampen subdiagram E between neighboring $\Gamma(\Pi)$ and $\Gamma(\Pi')$; see Figure 11. If E contains a disk, then it has a disk π provided by Lemma 3.17 (1). It has at least $L-3$ $\tilde{t}$-spokes in E. But the number of its spokes ending either on $\mathbf{p}$ or on $\mathbf{q}$ is less than $L-5$ since π does not belong to $\mathbf{D}$. It follows that a pair of $\tilde{t}$-spokes connects π with Π or with Π' in a van Kampen subdiagram, which is impossible by Lemma 3.16. Thus, every disk Π of Δ has to belong to the set $\mathbf{D}$.

The number of $\tilde{t}$ spokes at a disk Π from $\mathbf{D}$ is L, and at most two $\tilde{t}$-spokes connect it with neighboring disks from $\mathbf{D}$. So there are at least $(L-2)-2 = L-4$ tp- and tq-bonds at Π. As above, the number of the tp-bonds at Π is less than $L/2$, whence the number of tq-bonds of it is greater than $L-4-L/2 = L/2-4 > 0$. Similarly, there are $> L/2-4$ tp-bonds at Π; and the lemma is proved. ■

Lemma 7.3. *There is a recursive function f such that the integers k and l do not exceed $f(\|U\|+\|V\|)$, provided the words U and V have no q-letters.*

Proof. By Lemma 3.17 (2) the annular diagram Δ contains no disks, and so it is a roll. Assume first that Δ has no (θ, q)-cells.

Step 1. If the words U, V have no θ-letters, then by Lemmas 7.2 and 3.4, every maximal θ-band $\mathcal{T}$ of Δ is an annulus surrounding the hole, and has side labels of the form $(U')^k$, where U' is U or, due to a superscript, a copy of U. This obviously bounds the number of such labels in terms of $\|U\|$, and since we may assume that different θ-annuli do not copy each other, every vertex of $\mathbf{p}$ can be connected with a vertex of $\mathbf{q}$ by a path of bounded length. If two such paths $\mathbf{x}_1$ and $\mathbf{x}_2$ define a van Kampen subdiagram with boundary $\mathbf{x}_1\mathbf{y}_1\mathbf{x}_2^{-1}\mathbf{y}_2^{-1}$, where

$$\mathrm{Lab}(\mathbf{x}_1) \equiv \mathrm{Lab}(\mathbf{x}_2), \quad \mathrm{Lab}(\mathbf{y}_1) = V^{l'}, \quad \mathrm{Lab}(\mathbf{y}_2) = U^{k'}$$

with $|k'| < k$ and $|l'| < l$, then we obtain a contradiction with the choice of k and l. But the absence of pairs of such cuts $\mathbf{x}_1, \mathbf{x}_2$ bounds the exponents k and l in terms of $\|U\| + \|V\|$ since the labels of such cuts belong to a bounded set.

The dual argument works if the words U, V have no Y-letters.

We may also assume that Δ has no Y-bands starting and terminating on $\mathbf{p}$ (resp. $\mathbf{q}$). Indeed otherwise there is a rim a-band, and removing it, we replace U (resp. V) with a conjugate word $\bar{U}$, such that $|\bar{U}|_a < |U|_a$ and $|\bar{U}|_\theta = |U|_\theta$; this replacement is effective.

Step 2. As in Step 1, Δ has no (θ, q)-cells, but now U contains θ-letters. By Lemma 7.2, it remains to assume that every maximal θ-band of Δ connects the contours $\mathbf{p}$ and $\mathbf{q}$. The same is true for maximal Y-bands as we noticed in the previous paragraph. It follows that $k|U|_Y = |U^k|_Y = |V^l|_Y = l|V|_Y$ and therefore

$$\frac{|U|_Y}{|V|_Y} = \frac{l}{k}. \tag{7.2}$$

Since the numbers $|U|_Y$ and $|V|_Y$ are less than $\|U\| + \|V\|$, it follows from (7.2) that k and l have a common divisor d such that $k = dk'$ and $l = dl'$, where $k', l' < \|U\| + \|V\|$. If $d = 1$, then $k, l < \|U\| + \|V\|$, i.e., we get a desired upper bound. Proving by contradiction, we assume now that $d > 1$. The words $U' \equiv U^{k'}$ and $V' \equiv V^{l'}$ have equal Y-length since their d-th powers U^k and V^l have equal Y-length.

Without loss of generality, we may assume that U starts with a Y-letter a. Let $\mathbf{p}'$ be a subpath of $\mathbf{p}$ labeled by $U'a$. Let us denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the maximal Y-bands starting with the first and the last edges of $\mathbf{p}'$. They end on $\mathbf{q}$, and we get a van Kampen subdiagram Γ bounded by $\mathbf{p}'$, by a subpath $\mathbf{q}'$ of $\mathbf{q}$ and by the sides of $\mathcal{T}_1$ and $\mathcal{T}_2$. Since all maximal Y-bands of Γ connect $\mathbf{p}'$ with $\mathbf{q}'$, we have $|\mathbf{q}'|_Y = |\mathbf{p}'|_Y$, and so $\mathrm{Lab}(\mathbf{q}') \equiv V'a$. (Here we may replace the word V' with a cyclic permutation of it.)

The boundary label of $\Gamma\backslash\mathcal{T}_2$ is $T_1 V' T_2^{-1}(U')^{-1}$, where T_1 and T_2 are side labels of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, and so we obtain

$$T_2 = (U')^{-1} T_1 V' \tag{7.3}$$

in G. Also, cutting Δ along the side of $\mathcal{T}_1$, we have in G:

$$T_1^{-1}(U')^d T_1 = (V')^d \tag{7.4}$$

because the paths $\mathbf{p}$ and $\mathbf{q}$ are labeled by $(U')^d$ and $(V')^d$, respectively.

Both diagrams Γ and Δ contain only (θ, a)-cells, and they are diagrams over the group $G_{\theta a}$ generated by Y-letters and θ-letters only, which satisfy only (θ, a)-relations from the sets (3.1)–(3.3). The form of the (θ, a)-relations implies the existence of a homomorphism ν of $G_{\theta a}$ onto the free group F generated by θ-letters: ν is identical on θ-letters and trivial on Y-letters. On the one hand, equality (7.4) gives us

$$\big(\nu(T_1)^{-1}\nu(U')\nu(T_1)\big)^d = \nu(V')^d,$$

which implies $\nu(T_1)^{-1}\nu(U')\nu(T_1) = \nu(V')$ in the free group F, and so

$$\nu(T_1) = \nu(U')^{-1}\nu(T_1)\nu(V').$$

On the other hand, we get

$$\nu(T_2) = \nu(U')^{-1}\nu(T_1)\nu(V')$$

from (7.3). Therefore we have $\nu(T_2) = \nu(T_1)$ in F, whence $T_2 = T_1$ in G, because T_1 and T_2 contain only θ-letters. Now the equalities $T_2 = T_1$ and (7.3) give us the conjugation of the words $U' = U^{k'}$ and $V' = V^{l'}$ in G, where $k' < k$ and $l' < l$, which contradicts the choice of the pair (k, l).

Step 3. If Δ contains (θ, q)-cells, then by Lemma 3.4, every maximal q-band of Δ is a q-annulus surrounding the hole of Δ since a roll has no q-edges in the boundary. By the same lemma, every maximal θ-band crossing a q-annulus $\mathcal{C}$ connects $\mathbf{p}$ and $\mathbf{q}$ and cannot intersect $\mathcal{C}$ twice. Therefore all q-annuli have length $|U|_\theta^k$, and the boundary label of each of them is a k-th power with the length of base bounded from above by $\|U\|$. Since one may assume that two different q-annuli do not copy each other, the number of q-annuli is effectively bounded. Therefore the solution of power conjugation is reduced to the annular diagrams between the annuli, where there are no (θ, q)-cells. Since the number of such annular diagrams is bounded, the problem is reduced to the case considered in Step 2, because one can use the transitiveness: if U^k is a conjugate of V^l and V^r is a conjugate of W^s, then U^{kr} is a conjugate of W^{sl}.

Lemma 7.3 is proved. ∎

For any disk Π of the diagram Δ, we have a tq-bond Γ at Π by Lemma 7.2, because $L/2 - 4 > 0$. If there is a θ-band of Γ connecting the two spokes bounding Γ, then there is such a θ-band $\mathcal{T}$ closest to Π. Let E be the subdiagram of Γ formed by Π and $\mathcal{T}$. One may apply Lemma 3.20 and replace E with a diagram E' formed by a new disk Π' and a θ-band $\mathcal{T}'$. This transformation replaces Γ with the tq-bond $\Gamma' = \Gamma \backslash \mathcal{T}$ at Π'. The iteration of such transformation replaces the tq-bond Γ with a tq-bond Γ_0 at a disk Π_0, where there are no θ-bands connecting $\tilde{t}$-spokes $\mathcal{C}$ and $\mathcal{C}'$ at Π_0.

Lemma 7.4. *The perimeter of* Π_0*, the lengths of the* $\tilde{t}$*-spokes* $\mathcal{C}, \mathcal{C}'$*, and the length* $\|\mathbf{r}\|$ *of some path* $\mathbf{r}$ *of* θ*-length* 0 *connecting* Π_0 *and* $\mathbf{q}$ *in* Γ_0 *are effectively bounded from above in terms of* $\|V\|$.

Proof. The quadratic upper bounds for the lengths $\mathcal{C}$, $\mathcal{C}'$, and $\partial\Pi_0$ in terms of the length of the subpath $\mathbf{x}$ of $\mathbf{q}$ connecting $\mathcal{C}$ and $\mathcal{C}'$ is given by Lemma 6.1. However we have $\|\mathbf{x}\| < \|V\|$ since the equality $\mu(V) = 0$ implies that the word V contains at least L $\tilde{t}$-letters, but $\mathbf{x}$ has only two $\tilde{t}$-edges by Lemma 7.2. It remains to define the path $\mathbf{r}$. This path starts from Π_0, where the q-band $\mathcal{C}$ starts, but it is a side of a maximal θ-band $\mathcal{T}_0$ of Γ_0. Then $\mathcal{T}_0$ must end on $\mathbf{q}$ by the definition of Γ_0. The length of $\mathbf{r}$ is bounded by Lemma 6.1 since the perimeter of Γ_0 is bounded and the number of cells in Γ_0 is also effectively bounded by Lemma 3.21. ■

By Lemma 7.2, all disks of Δ can be moved toward $\mathbf{q}$ in the same way we have moved Π. So we obtain an annular diagram $\tilde{\Delta}$, where by Lemma 7.4, each disk Π has effectively bounded perimeter and is connected with $\mathbf{q}$ by a path $\mathbf{r} = \mathbf{r}(\Pi)$ having effectively bounded length and $|\mathbf{r}|_\theta = 0$. The obtained annular diagram $\tilde{\Delta}$ has the same boundary labels as Δ, but it is not necessarily minimal. Every disk Π can be removed from $\tilde{\Delta}$ if one makes the cut along $\mathbf{r}^{-1}$, around Π and back along $\mathbf{r}$. After removal of all the disks, we obtain a diskless annular diagram Δ_0. See Figure 12.

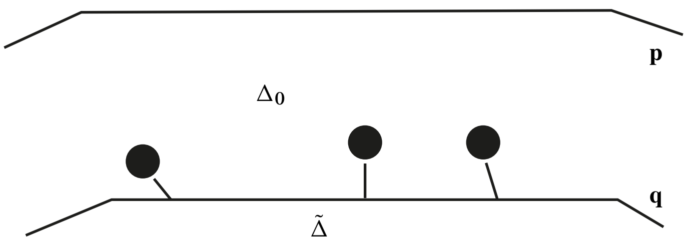

Figure 12. Δ_0 is obtained by cutting off all disks from $\tilde{\Delta}$.

We may keep notation $\mathbf{p}$ and $\mathbf{q}$ for the boundary components of $\tilde{\Delta}$, where $\mathbf{p}$ is also the outer boundary component of Δ_0. If Δ_0 is not reduced, we replace it with a reduced annular diagram with the same boundary labels. So we will assume that Δ_0 is a reduced annular diagram and $\tilde{\Delta}$ is built of Δ_0 and disks. The inner contour q_0 of Δ_0 is obtained from $\mathbf{q}$ by inserting pathes $\mathbf{z} = \mathbf{z}(\Pi)$ for every disk Π, where $|\mathbf{z}|_\theta = 0$ and the length $\|\mathbf{z}\|$ is effectively bounded in terms of $\|V\|$.

Lemma 7.5. *There is no θ-band in Δ_0 which starts and ends on $\mathbf{q}_0$ or starts and ends on $\mathbf{p}$.*

Proof. Every path $\mathbf{z} = \mathbf{z}(\Pi)$ has no θ-edges. Therefore a θ-band $\mathcal{T}$ starting and ending on $\mathbf{q}_0$ has to start and end on $\mathbf{q}$. So the word V has θ-letters, and there is van Kampen subdiagram Γ in $\widetilde{\Delta}$, where the boundary of Γ has form $\mathbf{uv}$, where $\mathbf{u}$ is a side of $\mathcal{T}$ and $\mathrm{Lab}(\mathbf{v})$ has no θ-letters except for the first and the last letter; whence $\mathrm{Lab}(\mathbf{v})$ is a subword of a cyclic permutation of $V^{\pm 1}$. The diagram Γ can be replaced with a minimal diagram with the same boundary label whose two θ-edges have to be connected by a θ-band. But this is not possible for the adapted word V. The $\mathbf{p}$-version of the lemma admits a similar proof. ∎

Lemma 7.6. *There is a recursive function f such that the integers k and l do not exceed $f(\|U\| + \|V\|)$ provided the path $\mathbf{p}$ has no θ-edges.*

Proof. It follows from Lemmas 3.4 and 7.5 that every maximal θ-band of Δ_0 is an annulus crossing every maximal q-band starting on $\mathbf{p}$ exactly once. Therefore all maximal q-bands starting on $\mathbf{p}$ have equal histories. The history and the one-letter base determine side labels of a q-band up to superscripts. If we have two maximal q-bands $\mathcal{C}$ and $\mathcal{C}'$ starting with two edges $\mathbf{e}$ and $\mathbf{e}'$ of a subpath $\mathbf{efe}'$ of $\mathbf{p}$ and the length $\|ef\|$ is a multiple of $\|U\|$, then the corresponding superscripts must be equal by Remark 3.3 since $\mu(U) = 0$ in (7.1), that is, $\mathcal{C}$ and $\mathcal{C}'$ have equal side labels. So there is a set $\mathbf{S}$ of different sides with equal labels, where $\#(\mathbf{S}) \ge k$.

An arbitrary path $\mathbf{s}$ from $\mathbf{S}$ either connects $\mathbf{p}$ and $\mathbf{q}$ or ends on a disk Π of $\widetilde{\Delta}$. In the latter case the path $\mathbf{s}$ can be extended by a subpath $\mathbf{x}$ of the path $\mathbf{z}(\Pi)$. The extension $\mathbf{s}'$ connects $\mathbf{p}$ and $\mathbf{q}$. The lengths of all $\mathbf{z}(\Pi)$ and so the lengths of the extending paths $\mathbf{x}$ were bounded in terms of $\|V\|$, i.e., by $g(\|V\|)$ for a recursive function g, in Lemma 7.4. So the number of possible labels $\mathrm{Lab}(\mathbf{x})$ is bounded by an exponential function of $g(\|V\|)$, where the base of the exponent depends on the number of generators of the group G. Hence there is a set of paths $\mathbf{S}'$ with equal labels, connecting $\mathbf{p}$ and $\mathbf{q}$, where $\#(\mathbf{S}') > c'k$, where $(c')^{-1}$ is effectively bounded from above.

An arbitrary path $\mathbf{s}' \in \mathbf{S}'$ starts with a vertex of $\mathbf{p}$, which decomposes the period U of $\mathrm{Lab}(\mathbf{p})$ as $U \equiv U_1U_2$. Similarly, the end of $\mathbf{s}'$ gives a factorization $V \equiv V_1V_2$. If two cuts $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{S}'$ define the same factorizations of the words U and V, we say that these cuts are *compatible*. Since the number of factorizations of the words U and V are bounded, there is a set of pairwise compatible paths $\mathbf{S}'' \subset S'$ with $\#(\mathbf{S}'') > c''k$, where the positive constant c'' is effectively bounded from below. However two different compatible cuts from $\mathbf{S}''$ together with parts of $\mathbf{p}$ and $\mathbf{q}$ bound a simply-connected diagram with the label $T(U')^{k'}T^{-1}(V')^{-l'}$, where T is the label of these cuts, U' and V' are cyclic permutations of the words U and V, respectively, and $k' < k, l' < l$. It follows that the powers $U^{k'}$ and $V^{l'}$ are conjugate in the group G contrary to the choice of k and l. Hence $c''k \le 1$, which effectively bounds k from above. Lemma 7.2 linearly bounds the q-length of the path $\mathbf{q}$ in terms of $|\mathbf{p}|_q$. Therefore the exponent l is also effectively bounded. ∎

Lemma 7.7. *There is a recursive function f such that the integers k and l do not exceed $f(\|U\| + \|V\|)$, provided the path* $\mathbf{p}$ *has θ-edges and q-edges.*

Proof. Step 1. Let $\mathcal{C}$ be a maximal q-band of Δ_0 starting on $\mathbf{p}$. As in Lemma 7.2, it ends on $\mathbf{q}_0$ since the word U is adapted. If a θ-band $\mathcal{T}$ starting from $\mathbf{p}$ crosses $\mathcal{C}$ from left to right, then it follows from Lemma 3.4 that it cannot also cross $\mathcal{C}$ from right to left. Also there is no other θ-band $\mathcal{T}'$ starting on $\mathbf{p}$ and crossing the q-band $\mathcal{C}$ from right to left since both $\mathcal{T}$ and $\mathcal{T}'$ cannot cross each other but both should end on $\mathbf{q}_0$. Therefore, all maximal θ-bands consequently crossing $\mathcal{C}$ and starting from $\mathbf{p}$ cross $\mathcal{C}$ from left to right (or cross it from right to left). It follows that these θ-bands start with consecutive θ-edges of $\mathbf{p}$, and so the history of $\mathcal{C}$ is a periodic word whose period is the θ-projection of U because $\mathrm{Lab}(\mathbf{p}) \equiv U^k$. Moreover, the histories of all maximal q-bands starting on $\mathbf{p}$ are periodic words with the same period H, where $0 < \|H\| < \|U\|$.

Furthermore, a side label of $\mathcal{C}$ is a periodic word with a period u, where $|u|_\theta = |U|_\theta$. To prove this, one should show that the cell π number a in $\mathcal{C}$ (counting from $\mathbf{p}$) is a copy of the cell π' having number $a + |U|_\theta$ in $\mathcal{C}$. Indeed, if a θ-band $\mathcal{T}$ (resp. $\mathcal{T}'$) starts on $\mathbf{p}$ and crosses q-bands b times (resp. b' times) before it crosses $\mathcal{C}$ at π (resp. at π'), then $b - b' = |U|_\theta$. Since $\mathcal{T}$ and $\mathcal{T}'$ have the first θ-edges labeled by the same letter, by Remark 3.3, we have equal superscripts when $\mathcal{T}$ and $\mathcal{T}'$ cross $\mathcal{C}$ at π and π', respectively, because $\mu(U) = 0$ in (7.1).

Step 2. As in Lemma 7.6, a side $\mathbf{y}$ of every maximal q-band admits a continuation $\mathbf{x} = \mathbf{yz}$ in $\tilde{\Delta}$, where the length of $\mathbf{z}$ is bounded, and we have a set $\mathbf{S}$ of such compatible cutting paths $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r$ ($\mathbf{x}_i = \mathbf{y}_i\mathbf{z}_i$), starting with different vertices of $\mathbf{p}$, and so, all the beginnings $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_r$ are the side labels of q-bands $\mathcal{C}_1, \dots, \mathcal{C}_r$ starting with the edges of $\mathbf{p}$ with the same base letter q_0. We add the additional requirement that the prefixes of length $\|H\|$ of all words $\mathrm{Lab}(\mathbf{y}_1), \dots, \mathrm{Lab}(\mathbf{y}_r)$ are equal (say, the histories of the corresponding q-bands $\mathcal{C}_i$ start with H), and still have $r > ck$, where the positive constant c^{-1} is recursively bounded from above in terms of $\|U\| + \|V\|$.

Since the side label of the q-bands $\mathcal{C}_i$ are compatible and $\mu(U) = 0$, we have $\mathrm{Lab}(\mathbf{x}) \equiv u^s v$, for every $\mathbf{x} \in \mathbf{S}$, where $s = s(\mathbf{x})$ and the word v has bounded length. So changing the constant c effectively, one may assume that the suffixes v are the same for every $\mathbf{x} \in \mathbf{S}$. Then it follows that we have sufficiently many different pairs of different paths $(\mathbf{x}', \mathbf{x}'')$ from $\mathbf{S}$, the origins $(x')_-$ and $(x'')_-$ of which are 'close' to each other; more precisely, the number of disjoint pairs $(\mathbf{x}', \mathbf{x}'') \in \mathbf{S}^2$, where the subpath of $\mathbf{p}$ connecting $(x')_-$ and $(x'')_-$ has length $\le 3c^{-1}$ is greater than $r/4$. Let $\mathbf{P}$ be the set of such pairs.

Step 3. We want to bound from above the lengths $\|\mathbf{x}'\|, \|\mathbf{x}''\|$ for arbitrary pair $(\mathbf{x}', \mathbf{x}'') \in \mathbf{P}$. Thereby the number of different labels of the paths from such pair will be effectively bounded. However two compatible cutting paths from $\mathbf{S}$ cannot have equal labels, since as in Lemma 7.6, this would lead to a contradiction with the minimality of the pair (k, l).

Let E be a van Kampen subdiagram of $\tilde{\Delta}$ with boundary path $\mathbf{x}'\mathbf{q}'(\mathbf{x}'')^{-1}(\mathbf{p}')^{-1}$, where $(\mathbf{x}', \mathbf{x}'') \in \mathbf{P}$, $\mathbf{p}'$ and $\mathbf{q}'$ are subpaths of $\mathbf{p}$ and $\mathbf{q}$, respectively, and so $\|\mathbf{p}'\| \le 3c^{-1}$ and

$|\mathbf{p}'|_q \le 3c^{-1}$. This implies that $|\mathbf{q}'|_q \le c^{-1}L$ since every maximal q-band starting on $\mathbf{q}$ ends either on $\mathbf{p}$ or on a disk, which is also connected to $\mathbf{p}$ by q-bands.

Replacing the words U and V with cyclic permutations, we may assume that

$$\mathrm{Lab}(\mathbf{p}') \equiv U^a \quad \text{and} \quad \mathrm{Lab}(\mathbf{q}') \equiv V^b \quad \text{for some } a, b > 0.$$

Step 4. Recall that $\mathbf{x}' = \mathbf{y}'\mathbf{z}'$, where the length of $\mathbf{z}'$ is bounded and $\mathbf{y}'$ is a side of a maximal q-band $\mathcal{C}'$ stating on $\mathbf{p}$. Similarly, we have $\mathcal{C}''$ and $\mathbf{x}'' = \mathbf{y}''\mathbf{z}''$. If E has a θ-band connecting $\mathcal{C}'$ and $\mathcal{C}''$, we have a trapezium Γ of maximal height formed by such θ-bands and parts of $\mathcal{C}'$ and $\mathcal{C}''$. Two components of $E\backslash\Gamma$ (just one if Γ is empty) have maximal q-subbands of bounded lengths since maximal θ-bands crossing them have at least one end on $\mathbf{p}'$ or $\mathbf{q}'$. Thus, it remains to bound the height h of Γ.

By Lemma 3.10 (1), the top and the bottom labels W_0 and W_h of Γ are the first and the last permissible words of a computation $\mathcal{W} : W_0^{\emptyset} \to \cdots \to W_h^{\emptyset}$ with periodic history having period H. Therefore by Lemma 2.5 the height h is recursively bounded in terms of $\|W_0\|$, $\|W_h\|$, and $\|H\|$, provided there is no subcomputation $W_i^{\emptyset} \to \cdots \to W_j^{\emptyset}$ of $\mathcal{W}$ with history H and with $W_i^{\emptyset} \equiv W_j^{\emptyset}$. Then it follows that h is also effectively bounded in terms of $\|U\| + \|V\|$, as desired. Thus to complete the proof by contradiction, we assume now that $\mathcal{W}$ contains a subcomputation $W_i^{\emptyset} \to \cdots \to W_j^{\emptyset}$ with history H and with $W_i^{\emptyset} \equiv W_j^{\emptyset}$.

Step 5. It follows from Lemma 3.10 (2) that the trapezium Γ contains a subtrapezium Γ' corresponding to the subcomputation $\mathcal{W}' : W_i^{\emptyset} \to \cdots \to W_j^{\emptyset}$. Since $W_i^{\emptyset} \equiv W_j^{\emptyset}$, we have $W_i \equiv W_j$, because Γ is bounded by subbands of $\mathcal{C}$ and $\mathcal{C}'$, which are copies of each other, and so the corresponding letters of W_i and W_j have equal superscripts by Remark 3.3.

Consider now the following auxiliary surgery. Since $W_i \equiv W_j$, one can make a cut along a side of a θ-band of Γ labeled by W_i and insert a trapezium $\Gamma^{(n)}$ with history H^n, where $n > 1$. The obtained trapezium Γ_n has the same top/bottom labels, has H-periodic history, but $h_n - h = n\|H\|$, where h_n is the height of Γ_n. This surgery also replaces the diagram $E = E_0$ with a diagram E_n. See Figure 13.

Recall that by the definition of the set of cuts $\mathbf{S}$, both words $\mathrm{Lab}(\mathbf{x}')$ and $\mathrm{Lab}(\mathbf{x}'')$ are equal to $u^s v$ and $u^t v$ with bounded length of v, $\mathrm{Lab}(\mathbf{p}') \equiv U^a$, $\mathrm{Lab}(\mathbf{q}') \equiv V^b$. Since

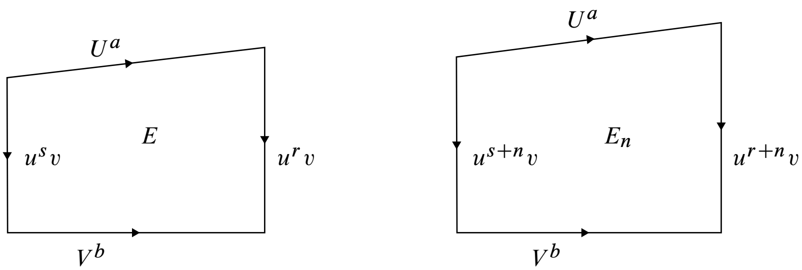

Figure 13. Subdiagram E and diagram E_n.

$a < k$ and $b < l$, we have $t \neq s$, and without loss of generality, we may assume that $t > s$. Thus, the boundary label of E gives us the equality $u^t v = U^{-a} u^s v V^b$ in G. For $n = (t - s)$, the diagram E_n provides us with the equality $u^{t+n} v = U^{-a} u^{s+n} v V^b$, i.e., $u^s v = U^{-a} u^{s+n} v V^b$. Similarly, from $E_{2n}, E_{3n}, \dots,$ we obtain

$$\begin{aligned} u^s v &= U^{-a} u^{s+n} v V^b, \\ u^{s+n} v &= U^{-a} u^{s+2n} v V^b, \\ &\vdots \\ u^{s+(l-1)n} v &= U^{-a} u^{s+nl} v V^b. \end{aligned}$$

On the one hand, it follows that

$$u^s v = U^{-a} u^{s+n} v V^b = U^{-2a} u^{s+2n} v V^{2b} = \cdots = U^{-la} u^{s+ln} v V^{lb} \tag{7.5}$$

in G. On the other hand, cutting Δ along the path $\mathbf{x}'$, we obtain a diagram whose boundary label gives us $u^s v = U^{-k} u^s v V^l$ in G, whence $u^s v = U^{-kb} u^s v V^{lb}$, which together with (7.5) gives

$$u^{ln} = U^{la-kb}. \tag{7.6}$$

Step 6. To obtain the final contradiction, it remains to show that the equality (7.6) is impossible in G.

The word u is a label of a side of a reduced q-band. Therefore its label is a word with non-empty cyclically reduced ν-projection onto the free group generated by θ-letters. If $kb - la = 0$, then by Lemmas 3.18 and 3.4, the minimal diagram for the equality $u^{ln} = 1$ has neither disks nor (θ, q)-cells. So it is a diagram over a group generated by θ- and Y-letters. Then the homomorphism ν gives the equality $\nu(u)^{ln} = 1$ in the free group, a contradiction.

If $kb - la \neq 0$, then the van Kampen diagram Δ' corresponding to (7.6) has no disks. Indeed, otherwise by Lemma 3.17 (1), we have a disk with $s \geq L - 3$ consecutive $\tilde{t}$-spokes $\mathcal{C}_1, \dots, \mathcal{C}_s$ ending on the boundary subpath labeled by U^{kb-la}, because u has no q-letters. If there are no other disks between neighbor $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ $(i = 1, \dots, s-1)$, then we have a contradiction with the property that U is an adapted word. If there is a disk in a diagram Γ_i, between some $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$, then again Lemma 3.17 (1) provides us with a disk π in Γ_i, contrary to the definition of adapted word. Every maximal q-band of Δ' has to start and end on the boundary subpath labeled by the power of U, and so there is a q-band starting and ending on a subpath labeled by a cyclic permutation of $U^{\pm 1}$, which is impossible since the word U is adapted. Hence U cannot contain q-letters, contrary to the assumption of the lemma. ■

Proof of Theorem 1.3. (1) To decide if some powers U^k and V^l with non-zero exponents are conjugate in G, we may assume by Lemma 7.1 that the words U and V represent elements of infinite order. Also it can be assumed that equality (7.1) holds and that the words U and V are adapted according to Lemma 6.2. If both U and V have no q-letters, then the exponents k, l can be effectively bounded in terms of $\|U\| + \|V\|$ by Lemma 7.3. Otherwise the recursive bounds for k and l are given by Lemmas 7.6 and 7.7. This reduces the power conjugacy to the conjugacy of words of bounded length. Since the conjugacy problem for pairs of words of infinite order is decidable by Theorem 1.4, the power conjugacy problem is decidable in G. The group G has undecidable conjugacy problem and quadratic Dehn function by Lemmas 4.2 and 3.21 if the machine $\mathbf{M}_0$ is chosen with non-recursive language of accepted input words. Thus, Theorem 1.3 (1) is proved.

(2) Let us start with McCool's group

$$\Pi_2 = \langle y_n, z_n \ (n = 1, 2, \dots) \mid y_n z_n = z_n y_n,\ y_{\phi(n)} = z^n_{\phi(n)} \ (n = 1, 2, \dots) \rangle,$$

where ϕ is a recursive one-to-one function with a non-recursive range. This group has decidable word problem [19], and so it has decidable conjugacy problem, being a free product of abelian groups. It follows from the relations that some powers of y_i and z_i are conjugate if and only if they are equal, and we can obtain such an equality if and only if i belongs to the range of the function ϕ. Since this range is not recursive, the power conjugacy problem is undecidable in the group Π_2.

By [25, Theorem 3], the countable group Π_2 with decidable conjugacy problem embeds in a 2-generated group K with decidable conjugacy problem. Moreover, by [25, Lemma 8 (6)], this embedding has the Frattini property, i.e., two elements from the subgroup Π_2 are conjugate in K if and only if they are conjugate in Π_2. Hence the power conjugation problem is undecidable in K too.

Finally, by [24, Theorem 1.1] the finitely generated group K having decidable conjugacy problem Frattini embeds in a finitely presented group with decidable conjugacy problem. Thus, the power conjugacy problem is undecidable in H too, and Theorem 1.3 (2) is proved. ∎

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