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Asymptotic representations of Hamiltonian diffeomorphisms and quantization

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Abstract. We show that for a special class of geometric quantizations with "small" quantum errors, the quantum classical correspondence gives rise to an asymptotic projective unitary representation of the group of Hamiltonian diffeomorphisms. As an application, we get an obstruction to Hamiltonian actions of finitely presented groups.

1. Introduction and main results

Geometric quantization is a mathematical theory modeling the quantum classical correspondence. The latter is a fundamental physical principle stating that the quantum mechanics contains the classical mechanics in the limit when the Planck constant goes to zero. In the present paper we focus on the correspondence between Hamiltonian diffeomorphisms modeling motions of classical mechanics, and their quantum counterparts, unitary operators coming from the Schrödinger evolution. We show that for a special class of geometric quantizations with "small" quantum errors, which exist on a certain class of phase spaces (see Theorem 1.4), this correspondence gives rise to an asymptotic unitary representation of the universal cover of the group of Hamiltonian diffeomorphisms (Theorem 1.5). Interestingly enough, together with recent results from group theory [11, 17], this yields an obstruction to Hamiltonian actions of finitely presented groups (Theorem 1.10). Let us pass to precise definitions.

1.1. Hamiltonian diffeomorphisms

Let (M^{2n}, ω) be a closed symplectic manifold. Here ω is a closed differential 2-form, whose n-th power does not vanish at any point and, thus, gives rise to a volume form on M. For a function $f \in C^\infty(M)$ introduce its *Hamiltonian vector field* $\operatorname{sgrad} f$ as the unique solution of the equation $i_{\operatorname{sgrad} f}\,\omega = -df$. Given a smooth function $f : M \times [0,1] \to M$, denote $f_t(x) := f(x,t)$, and consider the time-dependent vector field $\operatorname{sgrad} f_t$. Its evolution defines a path of diffeomorphisms ϕ_t on M with $\phi_0 = \mathbf{1}$. This path is called a

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Hamiltonian path, and the diffeomorphisms f_t are called *Hamiltonian* diffeomorphisms. The latter form a group denoted by $\mathrm{Ham}(M,\omega)$ (see [20, Section 1.4] for further details).

Denote by $\widetilde{\mathrm{Ham}}(M,\omega)$ the universal cover of $\mathrm{Ham}(M,\omega)$. Its elements $\widetilde{\phi}$ are Hamiltonian paths $\{\phi_t\}$, $t \in [0,1]$ with $\phi_0 = \mathbf{1}$, considered up to a homotopy with fixed end points. We write $\phi = \phi_1$ for the projection of $\widetilde{\phi}$ to $\mathrm{Ham}(M,\omega)$. Every path $\{\phi_t\}$ is uniquely determined by a time-dependent generating Hamiltonian $f_t \in C^\infty(M)$, where the functions f_t are assumed to have zero mean: $\int_M f_t\,\omega^n = 0$ for all t.[1] We shall say that $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ is *generated* by a Hamiltonian $f \in C^\infty(M \times [0,1])$.

Let us mention that the fundamental group $\pi_1(\mathrm{Ham}(M,\omega))$ is an abelian group, and we have a central extension

$$1 \to \pi_1(\mathrm{Ham}(M,\omega)) \to \widetilde{\mathrm{Ham}}(M,\omega) \xrightarrow{\tau} \mathrm{Ham}(M,\omega) \to 1.$$

1.2. Fine quantizations

Define a fundamental operation on functions on a symplectic manifold called the *Poisson bracket*: $\{f,g\} = L_{\mathrm{sgrad}\, f}\, g$, where L stands for the Lie derivative. We write $\|f\| = \max|f|$ for the uniform norm of a function f.

In what follows we denote by $\mathcal{L}(\mathcal{H})$ the space of Hermitian operators acting on a finite-dimensional complex Hilbert space $\mathcal{H}$, and write $\mathbb{U}(\mathcal{H})$ for the unitary group of $\mathcal{H}$.

Definition 1.1. A *fine quantization* of (M,ω) consists of a sequence of positive numbers $\hbar_k$ with $\lim_{k\to\infty} k\hbar_k = 1$, a family of finite-dimensional complex Hilbert spaces $\mathcal{H}_k$ such that

$$\dim \mathcal{H}_k = \left(\frac{k}{2\pi}\right)^n \mathrm{Vol}(M,\omega) + \mathcal{O}(k^{n-1}), \tag{1}$$

and a family of $\mathbb{R}$-linear maps $Q_k : C^\infty(M) \to \mathcal{L}(\mathcal{H}_k)$ with $Q_k(1) = \mathbf{1}$, satisfying the following properties:

(P1) (*norm correspondence*) $\|Q_k(f)\|_{\mathrm{op}} = \|f\| + \mathcal{O}(k^{-1})$;

(P2) (*bracket correspondence*) $[Q_k(f), Q_k(g)] = \frac{\hbar_k}{i} Q_k(\{f,g\}) + \mathcal{O}(k^{-3})$,

where the remainder is understood in the operator norm $\|\cdot\|_{\mathrm{op}}$.

The wording "fine" is chosen in order to emphasize that the remainder in (P2) is $\mathcal{O}(k^{-3})$, as opposed to $\mathcal{O}(k^{-2})$, as it happens for a wide class of geometric quantizations. For Kähler quantizations (see Section 2 below), the order of the remainder cannot be improved to $\mathcal{O}(k^{-4})$, see [6, p. 470]. It is unknown whether the same holds true for "abstract" quantizations defined by axioms (P1) and (P2).

Recall that (M,ω) is *quantizable* if the cohomology class $[\omega]/(2\pi)$ is integral. The following conditions on the first Chern class $c_1(TM)$ and the cohomology class of symplectic form $[\omega]$ of a quantizable symplectic manifold are equivalent:

[1] The Hamiltonian dynamics does not change if one adds to the Hamiltonian any function which does not depend on the space variable $x \in M$, but possibly depends on the time t. Given any Hamiltonian, we subtract its (in general, time-dependent) mean value to get the generating Hamiltonian having zero mean.

(C1) the line $\frac{1}{2}c_1(TM) - \mathbb{R}[\omega]$ in $H^2(M, \mathbb{R})$ intersects the lattice of integral classes $H^2(M, \mathbb{Z})/\text{torsion}$;

(C2) c_1 takes even values on $\text{Ker}([\omega])$, where both c_1 and $[\omega]$ are considered as morphisms $H_2(M, \mathbb{Z})/\text{torsion} \to \mathbb{R}$.

Indeed, (C1) yields (C2) immediately. In the opposite direction, choose a basis in $\text{Ker}([\omega])$, say $e_1, \dots, e_{m-1}$, and extend it to a basis in $H_2(M, \mathbb{Z})/\text{torsion}$ by e_0. Then $\omega(e_0) = 2\pi N$, where the number $N \in \mathbb{Z}$ is defined as an integer such that $[\omega]/(2\pi N)$ is a primitive vector. To get (C1) from (C2), we choose $\lambda = (c_1(e_0) + 2p)/(2N)$, with any integer p.

Definition 1.2. We say that (M, ω) satisfies *condition* (C) if it satisfies one of the equivalent conditions (C1) or (C2).

Condition (C) may be viewed as a generalization of the existence of metaplectic structure. It is more general: all complex projective spaces satisfy condition (C) because their second cohomology groups are one-dimensional. However, only the projective spaces with an odd complex dimension have a metaplectic structure.

Example 1.3. Take M to be $\mathbb{C}P^2$ blown up at one point. Let L, E be the basis in $H_2(M, \mathbb{Z})$ with L being the class of a general line and E of the exceptional divisor. There exists a symplectic form on M with $\omega(L) = 2\pi m$, $\omega(E) = 2\pi n$, for any integral $m > n > 0$. We have $c_1(nL - mE) = 3n - m$, and hence (C2) is satisfied if and only if $m = n \bmod 2$.

Theorem 1.4. *Every quantizable closed symplectic manifold M satisfying condition* (C) *admits a fine quantization.*

The proof is given in Section 2.

1.3. Asymptotic unitary representation

Let Q_k be a fine quantization. For a Hamiltonian f_t as above consider the unitary quantum evolution $U_k(t) : \mathcal{H}_k \to \mathcal{H}_k$ described by the Schrödinger equation

$$\dot{U}_k(t) = -\frac{i}{\hbar_k} Q_k(f_t) U_k(t), \quad U_k(0) = \mathbf{1}. \tag{2}$$

One can view the time-one map $U_k = U_k(1)$ as a quantization of the element $\widetilde{\phi}$ represented by f_t; see [15, Remark II.2.7].

Define a family of maps $\mu := \{\mu_k\}$, $k \in \mathbb{N}$,

$$\mu_k : \widetilde{\text{Ham}}(M, \omega) \to \mathbb{U}(\mathcal{H}_k),$$

as follows. For every $\widetilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ pick any Hamiltonian path joining the identity with ϕ generated by a Hamiltonian f_t, and set $\mu(\widetilde{\phi}) = U_k$, where U_k is determined by (2) as above. Let us emphasize that $\mu_k(\widetilde{\phi})$ depends on the specific choice of a Hamiltonian path in the class of paths homotopic with fixed endpoints.

Theorem 1.5. (i) *The unitaries $\mu_k(\widetilde{\phi})$ and $\mu'_k(\widetilde{\phi})$ defined via two different choices of paths homotopic with fixed endpoints representing $\phi \in \widetilde{\mathrm{Ham}}(M,\omega)$ satisfy*

$$\left\| \mu_k(\widetilde{\phi}) - \mu'_k(\widetilde{\phi}) \right\|_{\mathrm{op}} = \mathcal{O}(k^{-1}).$$

(ii) *For every $\widetilde{\phi}, \widetilde{\psi} \in \widetilde{\mathrm{Ham}}(M,\omega)$*

$$\left\| \mu_k(\widetilde{\phi})\mu_k(\widetilde{\psi}) - \mu_k(\widetilde{\phi}\widetilde{\psi}) \right\|_{\mathrm{op}} = \mathcal{O}(k^{-1}).$$

(iii) *If $\phi \neq \mathbf{1}$,*

$$\left\| \mu_k(\widetilde{\phi}) - \mathbf{1} \right\|_{\mathrm{op}} \geq 1/2 + \mathcal{O}(k^{-1}).$$

The proof is given in Section 3.2.

1.4. Constraints on subgroups of $\widetilde{\mathrm{Ham}}$

The collection of maps μ_k gives rise to an interesting algebraic object. In order to describe it, we need some preliminaries from [11,17]. For $p \geq 1$ and an operator $A : \mathcal{H} \to \mathcal{H}$ acting on a d-dimensional Hilbert space $\mathcal{H}$ denote by $\|A\|_p$ its p-th Schatten norm given by

$$\|A\|_p = \left(\mathrm{tr}\left(\left(\sqrt{A^*A}\right)^p\right)\right)^{1/p}.$$

Recall that

$$\|A\|_{\mathrm{op}} \leq \|A\|_p \leq d^{1/p}\|A\|_{\mathrm{op}}. \tag{3}$$

Definition 1.6 ([17]). A group Γ is called *p-norm approximated* if there exists a family of maps

$$\rho_k : \Gamma \to \mathbb{U}(\mathcal{H}_k),$$

where $\mathcal{H}_k$ is a sequence of Hilbert spaces of growing dimension, such that

$$\lim \|\rho_k(x)\rho_k(y) - \rho_k(xy)\|_p = 0, \quad \forall x, y \in \Gamma, \tag{4}$$

and

$$\liminf \|\rho_k(x) - \mathbf{1}\|_p > 0, \quad \forall x \in \Gamma,\ x \neq 1.$$

We call any sequence of maps ρ_k satisfying (4) an *asymptotic representation* of Γ in the sequence of unitary groups equipped with the p-norms.

Theorem 1.5 combined with estimate (3) and formula (1) immediately yields the following result.

Corollary 1.7. *Assume that a $2n$-dimensional closed symplectic manifold M admits a fine quantization. Let $\Gamma \subset \widetilde{\mathrm{Ham}}(M,\omega)$ be a finitely presented subgroup with*

$$\Gamma \cap \pi_1(\mathrm{Ham}(M,\omega)) = \{\mathbf{1}\}. \tag{5}$$

Then Γ is p-norm approximated for every $p > n$.

Existence of groups which are *not* p-norm approximated, $p > 1$, was established by Lubotzky and Oppenheim in [17]. For instance, certain finite central extensions of lattices in simple ℓ-adic Lie groups belong to this class.

Denote by $K_p \subset \pi_1(\mathrm{Ham}(M,\omega))$ the subgroup formed by elements $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(M,\omega)$ with $\lim_{k\to\infty} \|\mu_k(\widetilde{\phi}) - \mathbf{1}\|_p = 0$. Assumption (5) in Corollary 1.7 can be replaced to

$$\Gamma \cap K_p = \{\mathbf{1}\}.$$

It would be interesting to explore the subgroup K_p.

1.5. Asymptotic projective representations and constraints on Hamiltonian actions

What can we say about the restriction of the approximate representation μ_k to the fundamental group $\pi_1(\mathrm{Ham}(M,\omega)) \subset \widetilde{\mathrm{Ham}}(M,\omega)$? The following enhancement of Theorem 1.4 sheds light on this question.

Theorem 1.8. *Every quantizable Kähler closed symplectic manifold M satisfying condition* (C) *admits a fine quantization which satisfies*

$$\mu_k(\gamma) = e^{i r_k(\gamma)}\mathbf{1} + \mathcal{O}(k^{-1}),$$

where $r_k : \pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R}/(2\pi\mathbb{Z})$ is a sequence of homomorphisms.

The proof is given in Section 4. The homomorphisms r_k will be explicitly described in terms of action and Maslov invariants. The result follows from [10], which is developed in the Kähler setting. But there is no serious reason to think that the Kähler assumption is essential here.

Denote by $\mathbb{PU}(\mathcal{H}_k) = \mathbb{U}(\mathcal{H}_k)/S^1$ the projectivization of the unitary group of the Hilbert space $\mathcal{H}_k$. We equip this group with the quotient metric

$$\delta_p([A],[B]) = \inf_\theta \|A - e^{i\theta}B\|_p.$$

Let us state an analogue of Definition 1.6 for projective representations.

Definition 1.9. A group Γ is called *p-norm projectively approximated* if there exists a family of maps

$$\rho_k : \Gamma \to \mathbb{PU}(\mathcal{H}_k),$$

where $\mathcal{H}_k$ is a sequence of Hilbert spaces of growing dimension, such that

$$\lim \delta_p\big(\rho_k(x)\rho_k(y), \rho_k(xy)\big) = 0, \quad \forall x, y \in \Gamma, \tag{6}$$

and

$$\liminf \delta_p(\rho_k(x), \mathbf{1}) > 0, \quad \forall x \in \Gamma,\ x \neq 1.$$

We call any sequence of maps ρ_k satisfying (6) an *asymptotic projective representation* of Γ in the sequence of unitary groups equipped with the p-norms.

With this language, the asymptotic unitary representation μ_k from Theorem 1.8 descends to an asymptotic projective representation

$$\nu_k : \mathrm{Ham}(M,\omega) \to \mathbb{PU}(\mathcal{H}_k), \quad \phi \mapsto [\mu_k(\widetilde{\phi})],$$

where $\widetilde{\phi}$ is any lift of ϕ. Furthermore, every finitely presented subgroup of $\mathrm{Ham}(M,\omega)$ is p-norm projectively approximated. The proof is analogous to the one of Theorem 1.5, with the only extra ingredient being explained in Remark 3.2 below.

Write $\mathcal{PLO}_p$ for the class of finitely presented groups which are *not* p-norm projectively approximated. We sum up the previous discussion in the following theorem, which is the main application of our quantization-based technique to group actions on symplectic manifolds.

Theorem 1.10. *Let (M,ω) be a closed Kähler manifold of dimension $2n$ with $[\omega]/(2\pi)$ being an integral class and $c_1(TM)$ taking even values on* $\mathrm{Ker}[\omega]$. *Then every finitely presented subgroup of the group of Hamiltonian diffeomorphisms* $\mathrm{Ham}(M,\omega)$ *is p-norm projectively approximated with any $p > n$. In other words, groups from the class $\mathcal{PLO}_p$, $p > n$ do not admit a faithful Hamiltonian action on (M,ω).*

Example 1.11. This result is applicable, for instance, to quantizable closed Kähler manifolds M which are *monotone*: $c_1(TM) = \kappa[\omega]$ with some $\kappa \in \mathbb{R}$. Specific examples with $\kappa > 0$ include complex projective spaces of arbitrary dimension as well as their blow-ups with specially chosen symplectic forms (cf. Example 1.3 above). If $\kappa \leq 0$ (e.g., when M is a higher genus closed surface equipped with an area form), then the group $\mathrm{Ham}(M,\omega)$ has no torsion [1]. At the same time all currently known groups from the class $\mathcal{PLO}_p$ are finite central extensions of certain cocompact lattices and hence possess torsion. Thus, the novelty of Theorem 1.10 in the case $\kappa \leq 0$ depends on whether the class $\mathcal{PLO}_p$ contains groups without torsion, a problem which is still open (thanks to the referee for this comment).

It is also unclear to us whether Theorem 1.10 can be extended to the volume-preserving category.

Question 1.12. Can groups from the class $\mathcal{PLO}_p$ act faithfully by volume-preserving diffeomorphisms on a closed connected manifold?

An affirmative answer would highlight the symplectic nature of Theorem 1.10, while the negative one would require completely different tools.

1.6. How to construct groups from $\mathcal{PLO}_p$ (following [11, 17])

De Chiffre, Glebsky, Lubotzky, and Thom [11] and Lubotzky and Oppenheim [17] came up with a technique leading to examples of groups which are not p-norm approximated for $p > 1$. It was explained to us by Lubotzky that the same method shows that these groups are not p-norm projectively approximated, i.e., lie in $\mathcal{PLO}_p$. The argument from

[11, 17] extends *verbatim*. For the convenience of the reader we provide a brief outline of this argument adjusted to the projective case.

Fix a non-principal ultrafilter $\mathcal{U}$, and consider the ultraproduct

$$V_p := \prod_{j \to \mathcal{U}} (\mathrm{Mat}(\mathbb{C}, k_j), \|\cdot\|_p).$$

Every asymptotic projective representation of Γ yields a genuine isometric representation π_p of Γ on V_p by conjugation. The crux of the matter is that the action by conjugation is well defined since for $U_1 = e^{i\theta} U_2$, we have $U_1 A U_1^* = U_2 A U_2^*$.

Given a class of groups $\mathcal{P}$, we say that a group Γ is *residually* $\mathcal{P}$ if for every element $x \in \Gamma \setminus \{1\}$ there exists a homomorphism from Γ to a group from $\mathcal{P}$ whose kernel does not contain x. Interesting classes of groups include *linear groups* (those, admitting a faithful finite-dimensional representations) and finite groups.

Proposition 1.13 ([11]). *Let Γ be a finitely presented group with the following properties:*

(a) $H^2(\Gamma, \pi_p) = 0$*;*

(b) Γ *is not residually linear.*

Then $\Gamma \in \mathcal{PLO}_p$.

Indeed, assumption (a) enables one to apply a Newton-type process which yields a genuine representation of Γ on $\mathrm{Mat}(\mathbb{C}, k_j)$ for almost all j with respect to the ultrafilter. Moreover, every $x \neq 1$ does not lie in its kernel for almost all j. But this contradicts assumption (b).

The group Γ is constructed in two steps:

(i) Take a cocompact lattice Γ_0 in a simple Lie group G of rank ≥ 3 over ℓ-adic numbers with ℓ sufficiently large.

(ii) Take a special finite central extension Γ of Γ_0 which is not residually finite (Deligne).

The paper [11] proposes a specific example of the lattice Γ_0,

$$\Gamma_0 = \mathbb{U}(2m) \cap \mathrm{Sp}\big(2m, \mathbb{Z}[\sqrt{-1}, 1/\ell]\big)$$

considered as a cocompact lattice in $\mathrm{Sp}(2m, \mathbb{Q}_\ell)$.

The central extension $\Gamma \to \Gamma_0$, based on a technique of Deligne, is quite complicated, and we refer to [11] for details.

In order to verify assumption (a) of Proposition 1.13, the following features are used: first, the Lie group G acts on a special simplicial complex (a Bruhat–Tits building); here one uses the ℓ-adic nature of the situation. Second, the representation π_p is a particular case of an isometric representation on Banach spaces from a special class: they are obtained from Pisier's θ-Hilbertian spaces (where θ depends on p) by using quotients, l_2-sums and ultraproducts.

For verifying assumption (b) of Proposition 1.13, one uses (an immediate consequence of) the Malcev theorem: any residually linear group is residually finite. This completes our outline of the argument from [11, 17].

1.7. Stability

Another application of Theorem 1.5 deals with the following stability question: given a subgroup $\Gamma \subset \widetilde{\mathrm{Ham}}(M,\omega)$, is its quantization $\mu_k|_\Gamma : \Gamma \to \mathbb{U}(\mathcal{H}_k)$ close to a genuine representation? It follows that the answer is affirmative for the class of p-norm stable groups defined as follows [11, 17]. Here we include the case $p = \infty$, i.e. of the operator norm. Let Γ be a finitely presented group defined by finite collections of generators S and relations R, considered as subsets of the free group $\mathbb{F}_S$ generated by S. The *p-norm stability* means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite-dimensional Hilbert space $\mathcal{H}$ and every homomorphism $t : \mathbb{F}_S \to \mathbb{U}(\mathcal{H})$ with

$$\max_{r\in R} \|t(r) - \mathbf{1}\|_p \le \delta,$$

there exists a homomorphism $\rho : \Gamma \to \mathbb{U}(\mathcal{H})$ whose lift $\overline{\rho} : \mathbb{F}_S \to \mathbb{U}(\mathcal{H})$ satisfies

$$\max_{s\in S} \|t(s) - \overline{\rho}(s)\|_p < \varepsilon.$$

Let us mention that all finite groups are operator norm stable by [12, 14].

Corollary 1.14. *Assume that a $2n$-dimensional closed symplectic manifold M admits a fine quantization. Let $\Gamma = \langle S|R\rangle \subset \widetilde{\mathrm{Ham}}(M,\omega)$ be a finitely presented p-norm stable subgroup, where $p > n$. There exists a family of homomorphisms $\rho_k : \Gamma \to \mathbb{U}(\mathcal{H}_k)$ such that*

$$\max_{s\in S} \|\mu_k(s) - \rho_k(s)\|_p \to 0, \quad k \to \infty.$$

Remark 1.15. Some examples of finite subgroups of $\widetilde{\mathrm{Ham}}(M,\omega)$ come from the following construction. Let $F \subset \mathrm{Ham}(M,\omega)$ be a finite group acting in a Hamiltonian way on a closed quantizable symplectic manifold (M,ω). For instance, any unitary representation of F on a finite-dimensional complex Hilbert space V yields an action of F on the projectivization $\mathbb{P}(V)$. Denote by $\widetilde{F} \subset \widetilde{\mathrm{Ham}}(M,\omega)$ the full lift of F. If F is perfect, there exists a finite abelian extension G of F, called the universal extension [21], such that the following diagram commutes:

$$\begin{array}{ccc} G & \longrightarrow & F \\ \downarrow & & \downarrow \mathbf{1} \\ \widetilde{F} & \xrightarrow{\tau} & F. \end{array}$$

This provides a monomorphism of G into $\widetilde{\mathrm{Ham}}(M,\omega)$.

Let us note also that for any finite subgroup $F \subset \mathrm{Ham}(M,\omega)$, the restriction $\nu_k|_F$ of the asymptotic projective representation ν_k, which we constructed for quantizable Kähler manifolds satisfying condition (C), is close to a genuine projective representation, see [12].

1.8. Bibliographical and historical remarks

A few bibliographical remarks are in order. For Kähler quantization with metaplectic correction an asymptotic representation of the quantomorphisms group of a prequantum circle bundle over a closed symplectic manifold is constructed by Charles in [6]. In the present paper we generalize this result in two directions: first, we prove it for arbitrary fine quantizations, and second, for Kähler quantization, we impose condition (C) instead of the assumption that the canonical bundle admits a square root.

Charles showed in [9] that quantization enables one to reconstruct Shelukhin's quasimorphism on $\widetilde{\mathrm{Ham}}(M,\omega)$. Ioos, Kazhdan, and Polterovich [13] explored a link between quantization and almost representations of Lie algebras.

Constraints on smooth actions of finitely presented groups on closed manifolds is a classical and still rapidly developing subject. Its highlight is Zimmer's famous conjecture [22] which, roughly speaking, states that higher rank lattices in semisimple Lie groups cannot act on manifolds of sufficiently small dimension. This conjecture was recently resolved in a breakthrough work by Brown, Fisher, and Hurtado [4]. Some results on Hamiltonian actions were obtained by Polterovich, Franks, and Handel. We refer to Fisher's survey in [22] for a more detailed discussion. It would be interesting to explore potential actions of the group constructed in [11, 17] and described above, which is a finite extension of a higher rank ℓ-adic lattice with sufficiently high ℓ, along the lines of [4]. As we have learned from David Fisher, this problem is at the moment open. Furthermore, Fisher conjectured existence of constraints on actions of such groups.

Let us mention also that one of the assumptions of our Theorem 1.10 providing constraints on Hamiltonian actions of groups from the class $\mathcal{PLO}_p$ is $\dim_{\mathbb{C}} M < p$. Furthermore, for every positive integer $n < p$ there exist manifolds of complex dimension n to which the theorem is applicable (e.g., the complex projective spaces, see Example 1.11 above). This "smallness of dimension" assumption is very different from the one appearing in the Zimmer conjecture (thanks to the referee for pointing this out).

2. Constructing fine quantizations

In this section we prove Theorem 1.4 by constructing a fine quantization, which will be denoted by Op_k.

In the usual Toeplitz–Kähler quantization, we consider a compact Kähler manifold (M,ω) equipped with a holomorphic Hermitian line bundle L whose Chern connection has curvature $\frac{1}{i}\omega$. The quantum space is defined as the space $\mathcal{H}_k$ of holomorphic sections of $L^k \otimes L'$, where L' is an auxiliary Hermitian holomorphic line bundle. Here, the parameter k is a positive integer. In this context, a standard way to define a quantum observable from a classical one is the *Berezin–Toeplitz* quantization: for any $f \in \mathcal{C}^\infty(M, \mathbb{R})$, we let $T_k(f)$ be the endomorphism of $\mathcal{H}_k$ such that

$$\langle T_k(f)\psi, \psi' \rangle = \langle f\psi, \psi' \rangle \tag{7}$$

for any $\psi, \psi' \in \mathcal{H}_k$. In other words, $T_k(f)\psi$ is the orthogonal projection of $f\psi$ to $\mathcal{H}_k$ with respect to the natural scalar product on the space of smooth sections of $L^k \otimes L'$. Here the scalar product of $\mathcal{C}^\infty(M, L^k \otimes L')$ is given by integrating the pointwise scalar product against the Liouville volume form.

The basic properties of these operators are the following equalities which hold for any $f, g \in \mathcal{C}^\infty(M)$:

$$\begin{aligned} T_k(fg) &= T_k(f)T_k(g) + \mathcal{O}(k^{-1}), \\ [T_k(f), T_k(g)] &= (ik)^{-1}T_k(\{f, g\}) + \mathcal{O}(k^{-2}), \\ \operatorname{tr}(T_k(f)) &= \left(\frac{k}{2\pi}\right)^n \int_M f\mu + \mathcal{O}(k^{n-1}), \end{aligned} \tag{8}$$

initially proved in [2] by using the generalized Toeplitz operators of [3]. We refer to [16, Chapter 5] for a recent detailed exposition.

These asymptotic properties of operators T_k provide a rigorous mathematical model of the *correspondence principle* from physics stating that "quantum mechanics contains classical mechanics in the *semiclassical limit* $k \to \infty$". Here the symplectic manifold (M, ω) is considered as the phase space of classical mechanics.

Furthermore, $\|T_k(f)\|_{\text{op}} = \|f\| + \mathcal{O}(k^{-1})$. Observe that in the bracket correspondence (second line of (8)), the remainder is an $\mathcal{O}(k^{-2})$, so we miss the fine quantization condition given in Definition 1.1.

The first-order correction to (8) has been computed in [5, 6]. Introduce for any $f \in \mathcal{C}^\infty(M)$, the operator

$$\operatorname{Op}_k(f) := T_k(f - (2k)^{-1}\Delta f) \tag{9}$$

where Δ is the holomorphic Laplacian of M (in complex coordinates $\Delta f = \sum G^{ij}\partial_{z_i}\partial_{\overline{z}_j}$ with (G^{ij}) the inverse of (G_{ij}) given by $\omega = i\sum G_{ij}\, dz_i \wedge dz_j$). Since $\operatorname{Op}_k(f) = T_k(f) + \mathcal{O}(k^{-1})$, the operators $\operatorname{Op}_k(f)$ satisfy (8) as well. The novelty is that we have now some explicit formulas for the first corrections:

$$\begin{aligned} \operatorname{Op}_k(f)\operatorname{Op}_k(g) &= \operatorname{Op}_k(fg) + \frac{i}{2k}\operatorname{Op}_k(\{f, g\}) + \mathcal{O}(k^{-2}), \\ [\operatorname{Op}_k(f), \operatorname{Op}_k(g)] &= (ik)^{-1}\operatorname{Op}_k\big(\{f, g\} - k^{-1}\omega_1(X_f, X_g)\big) + \mathcal{O}(k^{-3}), \\ \operatorname{tr}(\operatorname{Op}_k(f)) &= \left(\frac{k}{2\pi}\right)^n \int_M f\,\frac{(\omega + k^{-1}\omega_1)^n}{n!} + \mathcal{O}(k^{-2}), \end{aligned} \tag{10}$$

see [6, Theorem 3.4] and [5, Section 2.2]. Here $\omega_1 = i(\Theta' - \frac{1}{2}\Theta_K)$, where Θ' and Θ_K are the Chern curvature of L' and the canonical bundle K, respectively. In the complex coordinates as above, $\Theta_K = \partial\overline{\partial}\ln\det(G_{ij})$.

In the case where M has a metaplectic structure, one can choose for L' a square root of the canonical bundle, so that $\omega_1 = 0$, and we get our fine quantization. More generally, to prove the existence of fine quantizations under assumption (C), we construct a convenient auxiliary bundle L'.

Lemma 2.1. *Assume that a quantizable closed Kähler manifold (M, ω) satisfies condition* (C). *Then there exists a holomorphic Hermitian line bundle L' such that $\omega_1 = \lambda\omega$ with $\lambda \in \mathbb{Q}$.*

Proof. The basic observation we need is that for any line bundle D and integer m such that D^m is equipped with Hermitian and holomorphic structures, D has natural Hermitian and holomorphic structures inducing the ones of D^m. Furthermore, the Chern curvature of D is $1/m$ times the Chern curvature of D^m.

Now, the assumption that $\frac{1}{2}c_1^{\mathbb{R}}(K) + \mathbb{R}[\omega]$ intersects the lattice of integral classes means that there exists a line bundle L' such that $c_1^{\mathbb{R}}(L') = \frac{1}{2}c_1^{\mathbb{R}}(K) + \lambda c_1^{\mathbb{R}}(L)$. Since $c_1^{\mathbb{R}}(L) \neq 0$, we have that $\lambda = p/q$ is rational. So $(L')^{2q} = K^q \otimes L^{2p} \otimes T$ where T is a torsion line bundle, i.e. $T^m = 1$ for some $m \in \mathbb{N}$. We endow T with the Hermitian and holomorphic structures such that T^m becomes the trivial Hermitian and holomorphic line bundle, so that the Chern curvature of T is zero. Then we endow L' with the Hermitian and holomorphic structure compatible with the isomorphism $(L')^{2q} = K^q \otimes L^{2p} \otimes T$. So the Chern curvatures Θ', Θ and Θ_K of L', L and K satisfy $\Theta' = \frac{1}{2}\Theta_K + \lambda\Theta$. So $\omega_1 = i\lambda\Theta = \lambda\omega$. ∎

In the case where $\omega_1 = \lambda\omega$, the second and third equations of (10) read

$$\begin{aligned} [\mathrm{Op}_k(f), \mathrm{Op}_k(g)] &= (i(k+\lambda))^{-1}\,\mathrm{Op}_k(\{f,g\}) + \mathcal{O}(k^{-3}), \\ \mathrm{tr}(\mathrm{Op}_k(f)) &= \Big(\frac{k+\lambda}{2\pi}\Big)^n \int_M f\mu + \mathcal{O}(k^{n-2}), \end{aligned} \tag{11}$$

which proves Theorem 1.4 for a Kähler manifold with $\hbar_k = (k+\lambda)^{-1}$.

Let us generalize this to symplectic manifolds. So we start with a symplectic compact manifold (M, ω) such that $\frac{1}{2\pi}[\omega]$ is integral. We introduce a Hermitian line bundle L with Chern class $\frac{1}{2\pi}[\omega]$ and a second Hermitian line bundle L'. We denote by $\Omega_1 \in H^2(M, \mathbb{R})$ the cohomology class

$$\Omega_1 = \frac{1}{2\pi}\Big(c_1^{\mathbb{R}}(L') - \frac{1}{2}c_1^{\mathbb{R}}(K)\Big).$$

Here, the canonical bundle K is defined through any almost complex structure compatible with ω. It is well known that the Chern class of K only depends on ω. If $\mathcal{H}_k$ is a finite-dimensional subspace of $\mathcal{C}^\infty(M, L^k \otimes L')$, we can define as before the Toeplitz operators $T_k(f)$ by (7). Then we have the following results:

(1) By [7], cf. also [3,18], one can choose the family $(\mathcal{H}_k)$ so that the operators $T_k(f)$ satisfy (8).

(2) By [8], there exists a real differential operator $P : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that $\mathrm{Op}_k(f) = T_k(f) + k^{-1}T_k(Pf)$ satisfies (10) with ω_1 a representative of Ω_1.

Furthermore, by adding to P a vector field, one modifies ω_1 by an exact form. Choosing conveniently this vector field, we can obtain any representative of Ω_1.

If condition (C) holds, we can choose L' so that $\Omega_1 = \lambda[\omega]$ for some $\lambda \in \mathbb{Q}$. Choosing P so that $\omega_1 = \lambda\omega$, we obtain equations (11).

3. Quantum dynamics

3.1. The Egorov theorem for fine quantizations

We start with the Egorov theorem for fine quantizations. Let f_t be a classical Hamiltonian generating the Hamiltonian flow ϕ_t, and let $U_k(t)$ be the corresponding quantum evolution.

Theorem 3.1. *For every function $g \in C^\infty(M)$*

$$\|Q_k(g \circ \phi^{-1}) - U_k Q_k(g) U_k^{-1}\|_{\mathrm{op}} = \mathcal{O}(k^{-2}), \tag{12}$$

where the remainder depends on f and g.

This formula readily follows from [15, Proposition 2.7.1]. Let us emphasize that the quantum map U_k depends on the Hamiltonian f generating the diffeomorphism ϕ. This dependence will be analyzed later.

Proof. Recall that if ϕ_t is the Hamiltonian flow generated by a time-dependent Hamiltonian $f_t(x)$, the flow ϕ_t^{-1} is generated by $\overline{f_t} := -f_t \circ \phi_t$. It follows that for any function $g \in C^\infty(M)$

$$\frac{d}{dt} g \circ \phi^{-t} = (\phi^{-t})^*(L_{\mathrm{sgrad}\,\overline{f_t}}\, g) = (\phi^{-t})^*\{\overline{f_t}, g\} = -\{f_t, g \circ \phi^{-t}\}. \tag{13}$$

Next, turn to the analysis of the Schrödinger equation $\dot{\xi} = -\frac{i}{\hbar_k} Q_k(f_t)\xi$. Introduce the family of unitary operators

$$U(s,t) : \mathcal{H}_k \to \mathcal{H}_k, \quad \xi(s) \mapsto \xi(t)$$

which sends the solution at time s to the solution at time t. Observe that $U(0,t) = U_k(t)$ is the Schrödinger evolution, $U(t,t) = \mathbf{1}$ and $U(s,t) = U(t,s)^{-1} = U(t,s)^*$. The Schrödinger equation yields

$$\frac{\partial}{\partial s} U(t,s) = -\frac{i}{\hbar_k} Q_k(f_s) U(t,s), \quad \frac{\partial}{\partial s} U(s,t) = -\frac{i}{\hbar_k} U(s,t) Q_k(f_s). \tag{14}$$

Put now $B(s) := U(s,1) Q_k(g \circ \phi_s^{-1}) U(1,s)$, so that $B(0) = U_k Q_k(g) U_k = -1$ and $B(1) = Q_k(g \circ \phi_1^{-1})$. From (13) and (14) we get that

$$\frac{dB}{ds} = U(s,1)\Big(\frac{i}{\hbar_k}[Q_k(f_s), Q_k(g \circ \phi_s^{-1})] - Q_k(\{f_s, g \circ \phi^{-s}\})\Big) U(1,s).$$

Observe that the functions f_s and $g \circ \phi_s^{-1}$, $s \in [0,1]$ form a compact family with respect to C^∞-topology, and hence by bracket correspondence (P2) $\max_s \|dB/ds\|_{\text{op}} = \mathcal{O}(k^{-2})$. Thus

$$\|Q_k(g \circ \phi^{-1}) - U_k Q_k(g) U_k^{-1}\|_{\text{op}} = \left\| \int_0^1 dB/ds(s)\, ds \right\|_{\text{op}} = \mathcal{O}(k^{-2}),$$

as required. ∎

3.2. Proof of Theorem 1.5

Proof of Theorem 1.5. Suppose that we have two Hamiltonian paths $\gamma_0 = \phi_{t,0}$ and $\gamma_1 = \phi_{t,1}$, $t \in [0,1]$ with $\phi_{0,0} = \phi_{0,1} = \mathbf{1}$ and $\phi_{1,0} = \phi_{1,1} = \phi$, which are homotopic with fixed end points through a family $\phi_{t,s}$, $s \in [0,1]$. Denote by $U_k(\phi_{1,j})$ the time one map of the Schrödinger evolution obtained by the quantization of γ_j. We claim that

$$\|U_k(\phi_{1,1}) - U_k(\phi_{1,0})\|_{\text{op}} = \mathcal{O}(k^{-1}). \tag{15}$$

To see this, look at the family $\phi_{t,s}$ and denote by $p_{t,s}$ the generating Hamiltonian when s is fixed, t varies, and by $q_{t,s}$ the Hamiltonian when t is fixed, s varies. All the Hamiltonians are assumed to have zero mean (cf. the footnote in Section 1.1). Then

$$\partial_s p = \partial_t q + \{p, q\}. \tag{16}$$

Put $A = \hbar_k^{-1} Q_k(p)$ and $C = \hbar_k^{-1} Q_k(q)$. Let $U(t,s)$ be the unitary evolution of

$$\partial_t U = -iAU$$

with $U(0,s) = \mathbf{1}$. Note that

$$U_k(\phi_{1,1}) = U(1,1), \quad U_k(\phi_{1,0}) = U(1,0).$$

Define B by

$$\partial_s U = -iBU. \tag{17}$$

Then

$$\partial_s \partial_t U = -iA\partial_s U - i\partial_s A U = -iABU - i\partial_s AU,$$
$$\partial_t \partial_s U = -iB\partial_t U - i\partial_t BU = -iBAU - i\partial_t BU.$$

Subtracting and rearranging, we get

$$\partial_t B = \partial_s A - i[A, B].$$

Further, by (16)

$$\partial_t C = \hbar_k^{-1} Q_k(\partial_t q) = \hbar_k^{-1} Q_k(\partial_s p) + \hbar_k^{-1} Q_k(\{p,q\}) = \partial_s A + \hbar_k^{-1} Q_k(\{p,q\}).$$

Thus

$$\partial_t (B - C) = \hbar_k^{-2} \big(-i[Q_k(p) Q_k(q)] - \hbar_k Q_k(\{p,q\})\big) = \mathcal{O}(k^{-1}),$$

by bracket correspondence (P2). Observe that $\partial_s U(0,s) = 0$, so $B(0,s) = 0$. Further, $q(0,s) = 0$, so $C(0,s) = 0$. Thus

$$\|B(1,s) - C(1,s)\|_{\mathrm{op}} = \mathcal{O}(k^{-1}).$$

But $C(1,s) = 0$ since $q(1,s) = 0$. Thus $\|B(1,s)\|_{\mathrm{op}} = \mathcal{O}(k^{-1})$ and hence by (17)

$$\|U(1,1) - U(1,0)\|_{\mathrm{op}} = \mathcal{O}(k^{-1}),$$

and (15) follows. This proves item (i) of the theorem.

Let us analyze the quantization of the product of two Hamiltonian paths. Let ϕ_t and ψ_t be two paths generated by normalized Hamiltonians f_t and g_t respectively, and denote $\theta_t = \phi_t \psi_t$. Consider the corresponding Schrödinger evolutions

$$\dot{U}_k = -i\hbar_k^{-1} Q_k(f_t) U_k, \quad U_k(0) = \mathbf{1},$$
$$\dot{V}_k = -i\hbar_k^{-1} Q_k(g_t) V_k, \quad V_k(0) = \mathbf{1}.$$

Put

$$S(t) = Q_k(f_t) + U_k(t) Q_k(g_t) U_k(t)^{-1}, \quad W_k(t) = U_k(t) V_k(t).$$

Observe that

$$\dot{W}_k = -i\hbar_k^{-1} S(t) W. \tag{18}$$

Since θ_t is generated by $h_t := f_t + g_t \circ \phi_t^{-1}$, the Egorov theorem (Theorem 3.1) yields

$$Q_k(h_t) = S(t) + \mathcal{O}(k^{-2}).$$

Denote by $Z_k(t)$ the Schrödinger evolution of θ_t, that is,

$$\dot{Z}_k = -i\hbar_k^{-1} Q_k(h_t) Z_k = \big(-i\hbar_k^{-1} S(t) + \mathcal{O}(k^{-1})\big) Z_k, \quad Z_k(0) = \mathbf{1}.$$

Comparing this equation with (18), we conclude that

$$\|U_k(1) V_k(1) - Z_k(1)\|_{\mathrm{op}} = \mathcal{O}(k^{-1}).$$

Thus μ_k is an almost-representation, which proves item (ii) of the theorem.

Finally, assume that a Hamiltonian f_t generates a Hamiltonian path ϕ_t with $\phi_1 \neq \mathbf{1}$. Thus ϕ_1 displaces an open set $Y \subset M$: $\phi_1(Y) \cap Y = \emptyset$. Take a non-vanishing function g supported in $\phi_1(Y)$. Observe that

$$\|g \circ \phi^{-1} - g\| = \|g\|. \tag{19}$$

Put $A_k := Q_k(g)$. Let U_k be the unitary operator quantizing ϕ_1. By the Egorov theorem,

$$Q_k(g \circ \phi^{-1}) = U_k A_k U_k^{-1} + \mathcal{O}(k^{-2}).$$

It follows from (19) and (P1) that $\|U_k A_k U_k^{-1} - A\|_{op} = \|A\|_{op} + \mathcal{O}(k^{-1})$. Estimating

$$\begin{aligned}\|A\|_{op} + \mathcal{O}(k^{-1}) &= \|U_k A_k U_k^* - A\|_{op} \\ &= \|U_k A U_k^* - U_k A + U_k A - A\|_{op} \\ &\le 2\|A\|_{op} \cdot \|\mathbf{1} - U_k\|_{op},\end{aligned}$$

we get that $\|\mathbf{1} - U_k\|_{op} \ge 1/2 + \mathcal{O}(k^{-1})$, which proves item (iii) of the theorem. ■

Remark 3.2. Replacing U_k by $e^{i\theta} U_k$ in the proof of (iii), we get that

$$\|U_k - e^{i\theta}\mathbf{1}\|_{op} \ge 1/2 + \mathcal{O}(k^{-1})$$

for every phase θ. This implies that the approximate projective representation ν_k appearing right after Theorem 1.8 satisfies, for every $\phi \in \mathrm{Ham}(M, \omega)$,

$$\delta_p(\nu_k(\phi), \mathbf{1}) \ge \text{const} > 0, \quad \forall k \in \mathbb{N},$$

provided $\phi \neq \mathbf{1}$.

4. Loop quantization

In this section we prove Theorem 1.8 from the introduction. A more detailed formulation of this result appears in Theorem 4.1 below.

4.1. Action and Maslov index

Let (M, ω) be a compact symplectic manifold equipped with a prequantum line bundle L and an auxiliary line bundle L' such that

$$c_1^{\mathbb{R}}(L') = \lambda c_1^{\mathbb{R}}(L) + \frac{1}{2} c_1^{\mathbb{R}}(K)$$

where K is the canonical line bundle.

Since $\frac{1}{i}\omega$ is the curvature of L, the periods of ω are multiples of 2π, so the action of any contractible periodic trajectory $\gamma(t)$, $t \in [0, T]$ of a Hamiltonian (H_t) is well defined modulo $2\pi\mathbb{Z}$ and given by the usual formula

$$A(\gamma) = \int_D \omega - \int_0^T H_t(\gamma(t))dt$$

where D is a disc with boundary γ. We can even define the action modulo 2π of any periodic trajectory, by using parallel transport in L instead of the integral of ω.

If (H_t) generates a loop $\mathcal{L} = (\phi_t,\ t \in [0, 1])$ of Hamiltonian diffeomorphisms, then our assumption on L' allows to define a mixed action-Maslov invariant as follows [19]. By Floer theory, any trajectory $\phi_t(x)$, $t \in [0, 1]$ is the boundary of a disc D. We set

$$I(\mathcal{L}) = \lambda\Big(\int_D \omega - \int_0^1 H_t(\phi_t(x))\, dt\Big) + \pi m(\psi) \tag{20}$$

where ψ is the loop of $\mathrm{Sp}(2n)$ obtained by trivializing the symplectic bundle TM over D and defining $\psi(t) := T_x\phi_t$ and $m(\psi) = 0$ or 1 depending on whether the class of ψ in $\pi_1(\mathrm{Sp}(2n)) = \mathbb{Z}$ is even or odd. One readily checks that $I(\mathcal{L})$ is well defined modulo $2\pi\mathbb{Z}$.

4.2. Quantization of a Hamiltonian loop

Assume now that (M, ω) is Kähler, that L and L' are holomorphic hermitian line bundles with Chern curvatures Θ and Θ' satisfying $\Theta = \frac{1}{i}\omega$ and $\Theta' = \lambda\Theta + \frac{1}{2}\Theta_K$. Consider the space $\mathcal{H}_k$ of holomorphic sections of $L^k \otimes L'$. For any $f \in \mathcal{C}^\infty(M, \mathbb{R})$, we define the operator $\mathrm{Op}_k(f)$ as in (9)

Let (H_t) be a Hamiltonian of M generating a loop $\mathcal{L} = (\phi_t, t \in [0, 1])$. Introduce the quantum propagator $U_{t,k}$,

$$\frac{1}{i(k+\lambda)}\partial_t U_{k,t} + \mathrm{Op}_k(H_t)U_{k,t} = 0, \quad U_{k,0} = \mathbf{1}.$$

We assume from now on that M is connected, so the periodic trajectories $(\phi_t(x), t \in [0, 1])$ have all the same action, denoted by $A(\mathcal{L})$.

Theorem 4.1. *We have* $U_{k,1} = e^{ikA(\mathcal{L})+iI(\mathcal{L})} + \mathcal{O}(k^{-1})$.

Proof. We can rewrite the Schrödinger equation as

$$\frac{1}{ik}\partial_t U_{k,t} + \Big(1 + \frac{\lambda}{k}\Big)\,\mathrm{Op}_k(H_t)U_{k,t} = 0.$$

Then, by [10, Theorem 4.2] the Schwartz kernel of $U_{k,t}$ is a Lagrangian state associated with the graph of ϕ_t. We refer to [10] for the precise definitions. What is important to us here is that since ϕ_1 is the identity, we have

$$U_{k,1} = e^{ik\theta}T_k(\sigma) + \mathcal{O}(k^{-1}) \tag{21}$$

where θ is a real number, $\sigma \in \mathcal{C}^\infty(M)$ and $T_k(\sigma)$ is the Berezin–Toeplitz operator with multiplicator σ defined as in Section 2.

Furthermore, we can compute θ and σ by introducing a half-form bundle (i.e., the square root of the canonical bundle) denoted by δ. It is possible that such a bundle does not exist on M but we only need it on the trajectory γ of a given point x. In this case we take a disc D with boundary γ and choose the square root δ which extends to D.

Then by [10, Theorem 1.1]

$$U_{k,t}(\phi_t(x), x) \sim \Big(\frac{k}{2\pi}\Big)^n e^{\frac{1}{i}\int_0^t H_r^{\mathrm{sub}}(\phi_r(x))\,dr}[\phi_t^L(x)]^{\otimes k} \otimes \mathcal{T}_t^{L_1}(x) \otimes [\mathcal{D}_t(x)]^{1/2}.$$

Here ϕ_t^L is the prequantum lift of ϕ_t to L, and $H_r^{\mathrm{sub}} = \lambda H_t$ is the subprincipal symbol of $(1 + \frac{\lambda}{k})\,\mathrm{Op}_k(H_t)$. The second term $\mathcal{T}_t^{L_1}(x) : L_1|_x \to L_1|_{\phi_t(x)}$ is the parallel transport in the line bundle $L_1 = L' \otimes \delta^{-1}$. It is the multiplication by $\exp(i\lambda \int_D \omega)$ because the curvature of L_1 is $\Theta' - \frac{1}{2}\Theta_K = \lambda\Theta = \frac{\lambda}{i}\omega$. The last term is the square root of an isomorphism $\mathcal{D}_t(x) : K_x \to K_{\phi_t(x)}$ defined by

$$\mathcal{D}_t(x)(\alpha)((T_x\phi_t)^{1,0}u) = \alpha(u), \quad \forall \alpha \in K_x,\, u \in \det T_x^{1,0} M.$$

Here the square root is chosen so as to be continuous and equal to 1 at $t = 0$.

On the other hand, by (21),

$$U_{k,1}(x,x) = \Big(\frac{k}{2\pi}\Big)^n e^{ik\theta}(\sigma(x) + \mathcal{O}(k^{-1})).$$

Now $\phi_1^L(x) = e^{iA(\mathcal{L})}$ implies that $\theta = A(\mathcal{L})$ and it remains to prove that

$$e^{\frac{1}{i}\int_0^1 H_r^{\mathrm{sub}}(\phi_r(x))\,dr}\,\mathcal{T}_1^{L_1}(x) \otimes [\mathcal{D}_1(x)]^{1/2} = e^{iI(\mathcal{L})}. \tag{22}$$

Since $T_x\phi_1$ is the identity of T_xM, $\mathcal{D}_1(x)$ is the identity of K_x so

$$(\mathcal{D}_t(x))^{1/2} = \pm \mathbf{1}_{\delta_x}.$$

To determine the sign, we trivialize TM along γ with a symplectic frame, so that $(T_x\phi_t)$ becomes a loop α of symplectic matrices based at the identity and in the corresponding trivialization of K, $\mathcal{D}_t(x)$ is the multiplication by a complex number. The sign we search depends only on the homotopy class of α. Since $\mathrm{Sp}(2n)$ deformation retracts to its subgroup $\mathrm{U}(n)$, we can assume that α is a loop of $\mathrm{U}(n)$, in which case $\mathcal{D}_t(x)$ is the complex determinant of $\alpha(t)$. Thus, our sign is positive or negative depending on whether the class of α in $\pi_1(\mathrm{Sp}(2n)) = \mathbb{Z}$ is even or odd. We conclude that each factor in (22) corresponds to a summand in (20), which completes the proof. ■

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